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Spectral gaps of almost Mathieu operators in the exponential regime

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Abstract. For almost Mathieu operator

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(\theta + n\alpha)u_n,$$

the dry version of the Ten Martini Problem predicts that the spectrum $\Sigma_{\lambda,\alpha}$ of $H_{\lambda,\alpha,\theta}$ has all gaps open for all $\lambda \neq 0$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Avila and Jitomirskaya prove that $\Sigma_{\lambda,\alpha}$ has all gaps open for Diophantine α and $0 < |\lambda| < 1$. In the present paper, we show that $\Sigma_{\lambda,\alpha}$ has all gaps open for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with small λ.

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Contents

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1. Introduction and main results

The almost Mathieu operator (AMO) is the (discrete) quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha)u_n, \quad \text{with } v(\theta) = 2\cos 2\pi\theta,$$

where λ is the coupling, α is the frequency, and θ is the phase.

The AMO is a tight binding model for the Hamiltonian of an electron in a one-dimensional lattice or in a two-dimensional lattice, subject to a perpendicular (uniform) magnetic field (through a Landau gauge) [19, 32]. This model also describes a square lattice with anisotropic nearest neighbor coupling or anisotropic coupling to the nearest neighbors on a triangular lattice [11, 34]. In addition, in 1980 the discovery by von Klitzing [31] of the integer Quantum Hall Effect leaded to a beautiful theory by Thouless, Kohmoto, Nightingale, and den Nijs. Central to their theory is the use of the AMO as a model for Bloch electrons in a magnetic field. For more applications in physics, we refer the reader to [26] and the references therein.

Besides its application to some fundamental problems in physics, the AMO itself is also fascinating because of its remarkable richness of the related spectral theory. In Barry Simon's list of Schrödinger operator problems for the twenty-first century [33], there are three problems about the AMO. The spectral theory of AMO has attracted many authors, for example, Avila and Damanik [3], Avila and Jitomirskaya [4, 5], Avron and Simon [9, 10], Bourgain [13], Jitomirskaya and Last [22], and so on.

Here we are concerned with the topological structure of the spectrum, which is heavily related to the arithmetic properties of frequency α. If $\alpha = p/q$ is rational, it is well known that the spectrum consists of the union of q intervals called *bands,* possibly touching the endpoints. When $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\lambda \neq 0$, the spectrum set $\Sigma_{\lambda,\alpha}$ of $H_{\lambda,\alpha,\theta}$ (in this case the spectrum of $H_{\lambda,\alpha,\theta}$ is independent of θ) has been conjectured for a long time to be a Cantor set. This conjecture is named after the Ten Martini Problem.[1] It has been solved by Avila and Jitomirskaya completely [4] by Anderson localization (i.e., only pure point spectrum with exponentially decaying eigenfunctions) of $H_{\lambda,\alpha,\theta}$ when $|\lambda| > e^{\frac{16\beta}{9}}$, where

$$\beta = \beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}, \tag{1.1}$$

[1] The Ten Martini Problem is the fourth problem in [33].

and $\frac{p_n}{q_n}$ be the continued fraction approximants (§2.5) to α. See [4] for more historic backgrounds about the Ten Martini Problem. Recently, the condition $|\lambda| > e^{\frac{16\beta}{9}}$ has been refined by the authors of this paper [28][2] to $|\lambda| > e^{\frac{3\beta}{2}}$.

About the topological structure of the spectrum of the AMO, a stronger conjecture is the so-called "dry version" of the Ten Martini Problem. In order to state it, we introduce the integrated density of states $N_{\lambda,\alpha}(E)$ (see (2.9)) of AMO, which is a continuous non-decreasing surjective function with $N_{\lambda,\alpha}\colon \mathbb{R} \mapsto [0,1]$. For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, the basic relation between $\Sigma_{\lambda,\alpha}$ and $N_{\lambda,\alpha}$ is that $E \notin \Sigma_{\lambda,\alpha}$ if and only if $N_{\lambda,\alpha}$ is constant in a neighborhood of E. Each connected component of $\mathbb{R}\backslash\Sigma_{\lambda,\alpha}$ is called a *gap* of $\Sigma_{\lambda,\alpha}$. If E is an endpoint of some gap, then $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ (combining [20] with [24]). The dry version of the Ten Martini Problem predicts the converse is also true. Concretely, $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ with $E \in \Sigma_{\lambda,\alpha}$ implies E is an endpoint of some gap for all $\lambda \neq 0$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ (this obviously implies the Ten Martini Problem). For convenience, we say that all gaps of $\Sigma_{\lambda,\alpha}$ are *open* if $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ and $E \in \Sigma_{\lambda,\alpha}$ implies E is an endpoint of some gap. Equivalently, *the dry version of the Ten Martini Problem predicts $\Sigma_{\lambda,\alpha}$ has all gaps open for all $\lambda \neq 0$ and*[3] $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

In proving the dry version of the Ten Martini Problem, much progress has been recently achieved by many authors. The proofs depend on whether $\beta(\alpha) > 0$ or $\beta(\alpha) = 0$. One usually calls a set of the kind $\{\alpha \in \mathbb{R}\backslash\mathbb{Q}\colon \beta(\alpha) > 0\}$ *exponential regime* and a set of the kind $\{\alpha \in \mathbb{R}\backslash\mathbb{Q}\colon \beta(\alpha) = 0\}$ *sub-exponential regime.*

In the exponential regime ($\beta(\alpha) > 0$), for any $\varepsilon > 0$, one has

$$\left|\frac{p_n}{q_n} - \alpha\right| \le \frac{1}{q_n q_{n+1}} \le \frac{1}{e^{(\beta-\varepsilon)q_n}}$$

if n is large enough (see (2.12)). This means rational number $\frac{p_n}{q_n}$ is exponentially close to α, thus the gaps of spectrum $\Sigma_{\lambda,\alpha}$ can be rational approximated by the gaps[4] of $\Sigma_{\lambda,\frac{p_n}{q_n}}$. Choi, Elliott, and Yui in [15] considered the 1/3-Hölder continuity of the spectrum for $\lambda = 1$, i.e., $\mathrm{Dist}(\Sigma_{1,\alpha_1}, \Sigma_{1,\alpha_2}) < C|\alpha_1 - \alpha_2|^{1/3}$, and give a good estimate for the gaps of $\Sigma_{1,\frac{p_n}{q_n}}$, where $\mathrm{Dist}(K_1, K_2)$ means Hausdorff distance between two subsets $K_1 \subset \mathbb{R}$ and $K_2 \subset \mathbb{R}$. Then they prove that if $\lambda = 1$, then $\Sigma_{\lambda,\alpha}$ has all gaps open for $\beta > 9\ln 2 + 3\ln 3$. In [7], Avron, Mouche

[2] After submitting the present paper, we learned of that Avila, You, and Zhou [8] claimed they extended the result to regime $|\lambda| > e^{\beta}$ in different way. (Their preprint is not available yet.)

[3] After submitting the present paper, we learned of that Avila-You-Zhou claimed solution of the dry version of the Ten Martini Problem for all values of parameters except $\beta(\alpha) = 0$ and $|\lambda| = 1$ (Their preprint is not available yet).

[4] For $\alpha \in \mathbb{Q}$, $\Sigma_{\lambda,\alpha} = \bigcup_\theta \Sigma_{\lambda,\alpha,\theta}$, where $\Sigma_{\lambda,\alpha,\theta}$ is the spectrum of $H_{\lambda,\alpha,\theta}$.

and Simon considered the $1/2$-Hölder continuity of the spectrum. Combining [7] with [15], Avila and Jitomirskaya [4] obtained that $\Sigma_{\lambda,\alpha}$ has all gaps open for $0 < \beta \leq \infty$ and $e^{-\beta} < |\lambda| < e^{\beta}$. In particular, for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that $\beta = \infty$, $\Sigma_{\lambda,\alpha}$ has all gaps open if $\lambda \neq 0$. Thus in the present paper, unless stated otherwise, we always assume that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies $\beta = \beta(\alpha) < \infty$ and $\lambda \neq 0$.

Now let us return to the sub-exponential regime ($\beta(\alpha) = 0$). Typical of this case is the well-known Diophantine number. We say $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies *a Diophantine condition* $\mathrm{DC}(\kappa, \tau)$ with $\kappa > 0$ and $\tau > 0$, if

$$|q\alpha - p| > \kappa |q|^{-\tau} \quad \text{for any } (p, q) \in \mathbb{Z}^2, q \neq 0.$$

Let

$$\mathrm{DC} = \bigcup_{\kappa>0,\tau>0} \mathrm{DC}(\kappa, \tau).$$

We say α satisfies the *Diophantine condition,* if $\alpha \in \mathrm{DC}$. Note that the set DC is a proper subset of the sub-exponential regime, i.e., $\mathrm{DC} \subsetneqq \{\alpha\colon \beta(\alpha) = 0\}$. For $\alpha \in \mathrm{DC}$, Puig in [29, 30] developed a method to estimate the gaps via establishing reducibility (§2.1). He proved that, for $\alpha \in \mathrm{DC}$, if $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ with $E \in \Sigma_{\lambda,\alpha}$ and cocycle $S_{\lambda,E}$ (see (2.7) for the definition) is analytically reducible, then E is an endpoint of some gap. In [5], Avila and Jitomirskaya developed a quantitative version of Aubry duality and used it to obtain a sharp estimate of the rotation number $\rho(\alpha, A)$ (§2.2) with $A = S_{\lambda,E}$ for $\alpha \in \mathrm{DC}$. As a result, they established that the cocycle $S_{\lambda,E}$ is reducible when $0 < |\lambda| < 1$ with $E \in \Sigma_{\lambda,\alpha}$ and $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z}+\mathbb{Z}$. Combining with Puig's discussion, they proved that $\Sigma_{\lambda,\alpha}$ has all gaps open if $\alpha \in \mathrm{DC}$ and $0 < |\lambda| < 1$.

In conclusion, we give a list for the unsolved cases about the dry version of the Ten Martini Problem:[5]

$$\alpha \in \{\alpha\colon 0 < \beta(\alpha) < \infty\}, \quad 0 < |\lambda| \leq e^{-\beta}; \tag{1}$$

$$\alpha \in \mathrm{DC}, \quad |\lambda| = 1; \tag{2}$$

$$\alpha \in \{\alpha\colon \beta(\alpha) = 0\} \setminus \mathrm{DC}, \quad 0 < |\lambda| \leq 1. \tag{3}$$

In the present paper, we prove the following theorem.

Theorem 1.1 (main theorem). *For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that $\beta = \beta(\alpha) < \infty$, there exists a absolute constant C, such that $\Sigma_{\lambda,\alpha}$ has all gaps open if $0 < |\lambda| < e^{-C\beta}$.*

[5] By Aubry duality, it suffices to discuss $0 < |\lambda| \leq 1$.

Remark 1.1. The main contribution in the present paper is that the unsolved regime $|\lambda| \in (0, e^{-\beta}]$ in Case 1 is shrunk to $|\lambda| \in [e^{-C\beta}, e^{-\beta}]$. We should point out that the constant $C > 0$ is very large. Therefore there is a long way to decrease it to $C < 1$ such that the problem is solved completely. The unsolved Case 3 is now solved by letting $\beta = 0$ in Theorem 1.1, except $|\lambda| = 1$. Actually, Case 3 is solved by careful checking the proofs of [5] and [30].

The paper is organized as follows. In §2, we give some preliminary notions and facts which are taken from [4]. In §3, we obtain the strong localization estimate of the Aubry dual model $\widehat{H}_{\lambda,\alpha,\theta}$ in the exponential regime (i.e., $\beta(\alpha) > 0$). In §4, we set up sharp estimate of the rotation number (Theorem 4.7) for resonant phase by developing the quantitative version of Aubry duality in exponential regime. This process is the same as to set up almost reducibility for cocycles $S_{\lambda,E}$. In §5, we obtain the analytic reducibility in a strip domain for non-resonant phase (Theorem 5.2) by constructing a new reducible matrix in $\mathrm{PSL}(2,\mathbb{R})$ (by Lemma 5.1 and Theorem 5.1). Combining with the sharp estimate of rotation number in §4, we set up the reducibility for cocycle $S_{\lambda,E}$ when $E \in \Sigma_{\lambda,\alpha}$ and E satisfies $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ (Theorem 5.3). In §6, in order to use Puig's method, we generalize his result to exponential regime by KAM iteration (Theorem 6.1). In the end, we give a summary about the dry version of the Ten Martini Problem (Theorem 6.2).

2. Preliminaries

2.1. Cocycles. Denote by $\mathrm{SL}(2,\mathbb{C})$ the all complex 2×2-matrixes with determinant 1. We say a function $f \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{C})$ if f is well defined in $\mathbb{R}/\mathbb{Z}$, i.e., $f(x+1) = f(x)$ and f is analytic in a neighbor of $\operatorname{Im} x = 0$. The definitions of $\mathrm{SL}(2,\mathbb{R})$ and $C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ are similar to those of $\mathrm{SL}(2,\mathbb{C})$ and $C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{C})$, respectively, except that the involved matrixes are real and the functions are real analytic. A C^ω-cocycle in $\mathrm{SL}(2,\mathbb{C})$ is a pair $(\alpha, A) \in \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z},\mathrm{SL}(2,\mathbb{C}))$, where $A \in C^\omega(\mathbb{R}/\mathbb{Z},\mathrm{SL}(2,\mathbb{C}))$ means $A(x) \in \mathrm{SL}(2,\mathbb{C})$ and the elements of A are in $C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{C})$. Sometimes, we say A a C^ω-cocycle for short, if there is no ambiguity. Note that all functions and cocycles in the present paper are analytic. Thus we often do not mention the analyticity, for instance, we call A a "cocycle" instead of a "C^ω-cocycle."

The Lyapunov exponent for the cocycle A is given by

$$L(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx, \tag{2.1}$$

where

$$A_n(x) = A(x + (n-1)\alpha)A(x + (n-2)\alpha)\dots A(x). \tag{2.2}$$

By Corollary 2 in [16] (since irrational rotations are uniquely ergodic)

$$L(\alpha, A) = \lim_{n\to\infty} \sup_{x\in\mathbb{R}/\mathbb{Z}} \frac{1}{n} \ln \|A_n(x)\|, \tag{2.3}$$

that is, the convergence in (2.3) is uniform with respect to $x \in \mathbb{R}$. In detail, for all $\varepsilon > 0$,

$$\|A_n(x)\| \le e^{(L(\alpha,A)+\varepsilon)n}, \quad \text{for } n \text{ large enough.} \tag{2.4}$$

Given two cocycles (α, A) and (α, A'), a conjugacy between them is a cocycle $B \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ such that

$$B(x+\alpha)^{-1}A(x)B(x) = A'. \tag{2.5}$$

The notion of real conjugacy (between real cocycles) is the same as before, except that we ask for $B \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$, i.e., $B(x+1) = \pm B(x)$ and $\det B = 1$. We say that cocycle (α, A) is reducible if it is conjugate to a constant cocycle.

2.2. The rotation number. Let

$$A(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ c(\theta) & d(\theta) \end{pmatrix},$$

we define the map

$$T_{\alpha,A}\colon (\theta,\varphi) \in \mathbb{T} \times \frac{1}{2}\mathbb{T} \longmapsto (\theta + \alpha, \varphi_{\alpha,A}(\theta,\varphi)) \in \mathbb{T} \times \frac{1}{2}\mathbb{T},$$

with

$$\varphi_{\alpha,A} = \frac{1}{2\pi} \arctan\Big(\frac{c(\theta) + d(\theta)\tan 2\pi\varphi}{a(\theta) + b(\theta)\tan 2\pi\varphi}\Big),$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Assume now that

$$A\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

is homotopic to the identity. Then $T_{\alpha,A}$ admits a continuous lift

$$\widetilde{T}_{\alpha,A}\colon (\theta,\varphi) \in \mathbb{R}\times\mathbb{R} \longrightarrow (\theta+\alpha, \tilde{\varphi}_{\alpha,A}(\theta,\varphi)) \in \mathbb{R}\times\mathbb{R}$$

such that

$$\tilde{\varphi}_{\alpha,A}(\theta,\varphi) \mod \frac{1}{2}\mathbb{Z} = \varphi_{\alpha,A}(\theta,\varphi)$$

and $\tilde{\varphi}_{\alpha,A}(\theta,\varphi)-\varphi$ is well defined on $\mathbb{T}\times\frac{1}{2}\mathbb{T}$. The number

$$\rho(\alpha,A) = \limsup_{n\to\infty} \frac{1}{n}(p_2 \circ \widetilde{T}^n_{\alpha,A}(\theta,\varphi)-\varphi) \mod \frac{1}{2}\mathbb{Z},$$

does not depend on the choices of θ and φ, where

$$p_2(\theta,\varphi)=\varphi,$$

is the *rotation number* of (α, A), see [20, 25].

It follows from the definition that (cf. [5, p. 8])

$$\|\rho(\alpha,A)-\theta\|_{\mathbb{R}/2\mathbb{Z}} < C \sup_{x\in\mathbb{R}} \|A(x)-R_\theta\|, \tag{2.6}$$

where

$$\|x\|_{\mathbb{R}/2\mathbb{Z}} = \min_{\ell\in\mathbb{Z}} \left|x-\frac{\ell}{2}\right|,$$

$\|\cdot\|$ is any Euclidean norm, and

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

If we take

$$A, A'\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

and

$$B\colon \mathbb{R}/2\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

(note that $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ implies $B\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$) such that A is homotopic to the identity and

$$B(x+\alpha)^{-1}A(x)B(x) = A',$$

then A' is homotopic to the identity and

$$2\rho(\alpha,A)-2\rho(\alpha,A') = k\alpha \mod \mathbb{Z},$$

where k is the degree of B (denoted by $\deg(B)$), i.e., $x \mapsto B(x)$ is homotopic to $x \mapsto R_{\frac{kx}{2}}$.

2.3. Almost Mathieu cocycles and the integrated density of states. For the almost Mathieu operators $\{H_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{R}}$, the spectrum of operator $H_{\lambda,\alpha,\theta}$ does not depend on θ, denoted by $\Sigma_{\lambda,\alpha}$. Indeed, shift is an unitary operator on $\ell^2(\mathbb{Z})$, thus $\Sigma_{\lambda,\alpha,\theta} = \Sigma_{\lambda,\alpha,\theta+\alpha}$, where $\Sigma_{\lambda,\alpha,\theta}$ is the spectrum of $H_{\lambda,\alpha,\theta}$. By the minimality of $\theta \mapsto \theta + \alpha$ and continuity of spectrum $\Sigma_{\lambda,\alpha,\theta}$ with respect to θ, the statement follows.

Let

$$S_{\lambda,E} = \begin{pmatrix} E - 2\lambda\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.7}$$

We call $(\alpha, S_{\lambda,E})$ *almost Mathieu cocycle*. It's easy to see that the almost Mathieu cocycle is homotopic to the identity, and let $\rho_{\lambda,\alpha}(E) \in [0, \frac{1}{2}]$ be the rotation number of the almost Mathieu cocycle $(\alpha, S_{\lambda,E})$.

Next we will give the definition of the integrated density of states $N_{\lambda,\alpha}$, which has been mentioned in §1.

Let H be a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$. Then $(H-z)^{-1}$ is analytic in $\mathbb{C}\backslash\Sigma(H)$, where $\Sigma(H)$ is the spectrum of H, and we have for $f \in \ell^2$

$$\operatorname{Im}\langle (H-z)^{-1} f, f\rangle = \operatorname{Im} z \cdot \|(H-z)^{-1} f\|^2,$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\ell^2(\mathbb{Z})$. Thus

$$\phi_f(z) = \langle (H-z)^{-1} f, f\rangle$$

is an analytic function on the upper half plane with $\operatorname{Im}\phi_f \geq 0$ (ϕ_f is a so-called "Herglotz function").

Therefore one has a representation

$$\phi_f(z) = \langle (H-z)^{-1} f, f\rangle = \int_{\mathbb{R}} \frac{1}{x-z} d\mu^f(x) \tag{2.8}$$

where μ^f is the spectral measure associated to f.

Fix an almost Mathieu operator $H_{\lambda,\alpha,\theta}$. Denote by $\mu^f_{\lambda,\alpha,\theta}$ the spectral measure of operator $H_{\lambda,\alpha,\theta}$ and vector f as before. The *integrated density of states* (IDS) $N_{\lambda,\alpha}$ is obtained by averaging the spectral measure $\mu^{\delta_0}_{\lambda,\alpha,\theta}$ with respect to θ, i.e.,

$$N_{\lambda,\alpha}(E) = \int_{\mathbb{R}/\mathbb{Z}} \mu^{\delta_0}_{\lambda,\alpha,\theta}(-\infty, E]d\theta, \tag{2.9}$$

where δ_0 is the normal vector in $\ell^2(\mathbb{Z})$ with 0th component being 1, others being 0.

Between the integrated density of states $N_{\lambda,\alpha}(E)$ and the rotation number $\rho_{\lambda,\alpha}(E)$, there is the relation [23]

$$N_{\lambda,\alpha}(E) = 1 - 2\rho_{\lambda,\alpha}(E). \tag{2.10}$$

In particular, $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ is equivalent to $2\rho_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$.

Let

$$L_{\lambda,\alpha}(E) = L(\alpha, S_{\lambda,E})$$

be the Lyapunov exponent of $S_{\lambda,E}$. In [14] Bourgain and Jitomirskaya obtain the accurate value of Lyapunov exponent when $E \in \Sigma_{\lambda,\alpha}$.

Theorem 2.1 ([14]). *For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \in \mathbb{R}$ and $E \in \Sigma_{\lambda,\alpha}$, one has*

$$L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}.$$

2.4. Classical Aubry duality. Let

$$\widehat{H}_{\lambda,\alpha,\theta} = \lambda H_{\lambda^{-1},\alpha,\theta}.$$

If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then the spectrum of $\widehat{H}_{\lambda,\alpha,\theta}$ is exactly $\Sigma_{\lambda,\alpha}$, see [17]. $\widehat{H}_{\lambda,\alpha,\theta}$ is called Aubry dual model of $H_{\lambda,\alpha,\theta}$. Classical Aubry duality expresses an algebraic relation between the families of operators $\{\widehat{H}_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ and $\{H_{\lambda,\alpha,x}\}_{x\in\mathbb{R}}$ by Bloch waves, i.e., if

$$u\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$$

is an L^2 function whose Fourier coefficients $\hat{u}$ satisfy

$$\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u},$$

then

$$U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$$

satisfies

$$S_{\lambda,E}(x) \cdot U(x) = e^{2\pi i\theta}U(x+\alpha).$$

2.5. Continued fraction expansion. Define, as usual, for $0 \le \alpha < 1$,

$$a_0 = 0, \quad \alpha_0 = \alpha,$$

and inductively, for $k > 0$,

$$a_k = \lfloor \alpha_{k-1}^{-1} \rfloor, \quad \alpha_k = \alpha_{k-1}^{-1} - a_k,$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal t.

We define

$$p_0 = 0, \quad q_0 = 1,$$
$$p_1 = 1, \quad q_1 = a_1,$$

and, inductively,

$$p_k = a_k p_{k-1} + p_{k-2},$$
$$q_k = a_k q_{k-1} + q_{k-2}.$$

Recall that $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of best denominators of irrational number α, since it satisifies

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \quad \text{for all } 1 \leq k < q_{n+1}, \tag{2.11}$$

where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{\ell\in\mathbb{Z}} |x-\ell|$. Moreover, we also have the following estimate,

$$\frac{1}{2q_{n+1}} \leq \Delta_n \triangleq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{q_{n+1}}. \tag{2.12}$$

3. Strong localization estimate for $0 < \beta(\alpha) < \infty$

Given $\theta \in \mathbb{R}$ and $\epsilon_0 > 0$, we say k is an ϵ_0-*resonance* for θ if

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\epsilon_0|k|}$$

and

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j|\leq|k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

Clearly, $0 \in \mathbb{Z}$ is an ϵ_0-resonance. We order the ϵ_0-resonances

$$0 = |n_0| < |n_1| \leq |n_2| < \cdots.$$

We say θ is ϵ_0-*resonant* if the set of ϵ_0-resonances is infinite. If θ is non-resonant, with the set of resonances $\{n_0, n_1, \ldots, n_{j_\theta}\}$, we set

$$n_{j_\theta+1} = \infty.$$

Note that if $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = 0$ for some $k \in \mathbb{Z}$, then k is an resonance for θ, and θ is not ϵ_0-resonant.

Below, C is a large absolute constant and c is a small absolute constant, which may change through the arguments, even when appear in the same formula. However, their dependence on other parameters will be explicitly indicated. For instance, we denote by $C(\alpha)$ a large constant depending on α.

Before starting our main work in this part, we firstly give some simple facts.

Lemma 3.1. *Assume* $0 < \beta(\alpha) < \infty$, *then*

$$\inf_{0<|j|\le k} \|j\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge c(\alpha)e^{-2\beta k}, \tag{3.1}$$

and[6]

$$\inf_{0<|j|\le k} \|j\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge e^{-3\beta k}, \quad \text{for } k > k(\alpha). \tag{3.2}$$

Proof. By (1.1) and (2.12) there exists some $n_0 > 0$ such that for $n > n_0(\alpha)$,

$$\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{1}{2}q_{n+1}^{-1} \ge e^{-2\beta q_n}. \tag{3.3}$$

Let

$$c(\alpha) = \inf_{0<|j|\le q_{n_0+1}} \|j\alpha\|_{\mathbb{R}/\mathbb{Z}} > 0.$$

Assume $0 < |j| \le k$. If $|j| \ge q_{n_0+1}$, select $q_n \le |j| < q_{n+1}$ with $n \ge n_0 + 1$. By (2.11) and (3.3)

$$\begin{aligned} \|j\alpha\|_{\mathbb{R}/\mathbb{Z}} &\ge \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\ge e^{-2\beta q_n} \\ &\ge e^{-2\beta k} \\ &\ge c(\alpha)e^{-2\beta k}. \end{aligned} \tag{3.4}$$

If $|j| < q_{n_0+1}$, by the definition of $c(\alpha)$,

$$\|j\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge c(\alpha) \ge c(\alpha)e^{-2\beta k}.$$

This implies (3.1). For (3.2), note that $c(\alpha) > e^{-\beta k}$ for $k > k(\alpha)$. □

Remark 3.1. In particular, $\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge c(\alpha)e^{-2\beta|k|}$ for all $k \in \mathbb{Z}\backslash\{0\}$. This is a small divisor condition when we solve the homological equation (see Theorem 5.2 or Theorem 6.1).

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[6] In (3.2), $k > k(*)$ means k is large enough depending on $*$.

Lemma 3.2. *If $\epsilon_0 = C_1\beta > 0$, C_1 is a large absolute constant. Then there exists $k_0(\alpha) > 0$ such that if $|k| > k_0(\alpha)$ and $\|2\theta - k\alpha\| \leq e^{-\epsilon_0|k|}$, then k is an ϵ_0-resonance for θ.*

Proof. It suffices to prove

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j|\leq|k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

If $|j| \leq |k|$ and $j \neq k$, by (3.2) there exists some $k_0(\alpha)$ such that

$$\begin{aligned}\|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}} &\geq \|(k-j)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\geq e^{-6\beta|k|} - e^{-\epsilon_0|k|} \\ &> e^{-\epsilon_0|k|} \\ &\geq \|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}}\end{aligned} \tag{3.5}$$

for $k > k_0(\alpha)$. It follows that k is an ϵ_0-resonance for θ. □

Definition 3.1. We say that $\widehat{H}_{\lambda,\alpha,\theta}$ satisfies a *strong localization estimate* if there exists $C_0 > 0$, $\epsilon_0 > 0$ and $\epsilon_1 > 0$ such that for any solution $\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ with $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, where E in the spectrum of $\widehat{H}_{\lambda,\alpha,\theta}$, i.e., $E \in \Sigma_{\lambda,\alpha}$, we have

$$|\hat{u}_k| \leq C(\hat{u})e^{-\epsilon_1|k|} \quad \text{for } C_0|n_j| < |k| < C_0^{-1}|n_{j+1}|.$$

Lemma 3.3 ([4, Lemma 9.7]). *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ and $0 \leq \ell_0 \leq q_n - 1$ be such that*

$$|\sin\pi(x + \ell_0\alpha)| = \inf_{0\leq\ell\leq q_n-1} |\sin\pi(x + \ell\alpha)|,$$

then for some absolute constant $C > 0$,

$$-C\ln q_n \leq \sum_{\ell=0,\ell\neq\ell_0}^{q_n-1} \ln|\sin\pi(x+\ell\alpha)| + (q_n - 1)\ln 2 \leq C\ln q_n, \tag{3.6}$$

where q_n is given in §2.5.

The next theorem is our main work in this section.

Theorem 3.1. *Fix $\epsilon_0 = C_1\beta > 0$, where C_1 is large enough so that it is much larger than any absolute constant C, c^{-1} emerging in the present paper. Then there exists some constant C_2 such that, for $0 < |\lambda| < e^{-C_2\beta}$, $\widehat{H}_{\lambda,\alpha,\theta}$ satisfies a strong localization estimate with parameters $C_0 = 3$, $\epsilon_0 = C_1\beta$ and $\epsilon_1 = \frac{-\ln|\lambda|}{64}$.*

Remark 3.2. Refering to Lemma 4.2 in the next section, it follows that

$$|n_{j+1}| > \frac{C_1}{8}|n_j|.$$

Thus there exists k such that

$$3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$$

if C_1 is large enough.

By Aubry duality $\widehat{H}_{\lambda,\alpha,\theta} = \lambda H_{\lambda^{-1},\alpha,\theta}$, thus to prove Theorem 3.1, we only need prove $\breve{H}_{\lambda,\alpha,\theta} \triangleq H_{\lambda^{-1},\alpha,\theta}$ satisfies the strong localization estimate instead. Since this does not change any of the statements, sometimes the dependence of parameters $E, \lambda, \alpha, \theta$ will be ignored in the following. Assume $\breve{H}\phi = E\phi$ with $\phi(0) = 1$ and $|\phi(k)| \leq 1 + |k|$. Our objective is to prove

$$|\phi(y)| \leq C(\phi)e^{-\frac{L}{64}|y|}.$$

Without loss of generality, assume $0 < \lambda < 1$ (for $\lambda < 0$, note that $\breve{H}_{\lambda,\alpha,\theta} = \breve{H}_{-\lambda,\alpha,\theta+\frac{1}{2}}$). By Theorem 2.1, the Lyapunov exponent of $S_{\lambda^{-1},E}$ satisfies $L = -\ln\lambda$, where $E \in \Sigma_{\lambda^{-1},\alpha}$.

Define

$$H_I = R_I \breve{H} R_I,$$

where

$$R_I = \text{coordinate restriction to } I = [x_1, x_2] \subset \mathbb{Z},$$

and denote by

$$G_I = (\breve{H}_I - E)^{-1}$$

the associated Green function, if $\breve{H}_I - E$ is invertible. Denote by $G_I(x, y)$ the matrix elements of the Green function G_I.

Definition 3.2. Fix $m > 0$ and $1/10 < \delta < 1/2$. A point $y \in \mathbb{Z}$ will be called (m, k)-regular with δ if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$ such that

$$|G_{[x_1,x_2]}(y, x_i)| < e^{-m|y-x_i|} \quad \text{and} \quad \text{dist}(y, x_i) \geq \delta k \quad \text{for } i = 1, 2; \tag{3.7}$$

otherwise, y will be called (m, k)-singular with δ.

It is easy to check that [13, p. 61]

$$\phi(x) = -G_{[x_1,x_2]}(x_1, x)\phi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\phi(x_2 + 1), \tag{3.8}$$

where $x \in I = [x_1, x_2] \subset \mathbb{Z}$.

Lemma 3.4. *For any $m > 0$ and any δ with $1/10 < \delta < 1/2$, 0 is (m, k)-singular with δ if $k > k(m)$.*

Proof. Otherwise, 0 is (m, k)-regular with some $1/10 < \delta < 1/2$, i.e.,

$$|G_{[x_1,x_2]}(0, x_i)| < e^{-m|y-x_i|} \le e^{-\frac{m}{10}k} \quad \text{for } i = 1, 2, \tag{3.9}$$

since $|y - x_i| > \frac{k}{10}$. In (3.8), let $x = 0$ and recall that

$$\phi(x_1 - 1) \le 1 + |x_1 - 1| \le 1 + k$$

and

$$\phi(x_2 + 1) \le 1 + |x_2 + 1| \le 1 + k.$$

Thus

$$\begin{aligned} |\phi(0)| &= |G_{[x_1,x_2]}(x_1, 0)\phi(x_1 - 1) + G_{[x_1,x_2]}(0, x_2)\phi(x_2 + 1)| \\ &\le 2(1 + k)e^{-\frac{m}{10}k}. \end{aligned} \tag{3.10}$$

This implies $|\phi(0)| < 1$ if $k > k(m)$, which is contradicted to the hypothesis $\phi(0) = 1$. □

Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(\check{H}_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}),$$

and

$$A = S_{\lambda^{-1},E}.$$

Then the k-step transfer-matrix $A_k(\theta)$ given by (2.2) can be written as [13, p. 14]

$$A_k(\theta) = \begin{pmatrix} P_k(\theta) & -P_{k-1}(\theta + \alpha) \\ P_{k-1}(\theta) & -P_{k-2}(\theta + \alpha) \end{pmatrix}. \tag{3.11}$$

By Cramer's rule [13, p. 15] for given x_1 and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has

$$|G_I(x_1, y)| = \left| \frac{P_{x_2-y}(\theta + (y + 1)\alpha)}{P_k(\theta + x_1\alpha)} \right| \tag{3.12}$$

and

$$|G_I(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \tag{3.13}$$

The numerators in (3.12) and (3.13) can be bounded uniformly with respect to θ by (2.4) and (3.11), i.e., for any $\varepsilon > 0$,

$$|P_k(\theta)| \le \|A_k(\theta)\| \le e^{(L+\varepsilon)k} \quad \text{for sufficiently large } k \text{ and all } \theta. \tag{3.14}$$

In fact, (3.14) can be also uniform with respect to $E \in \Sigma_{\lambda^{-1},\alpha}$ by the compactness of $\Sigma_{\lambda^{-1},\alpha}$ and subadditivity of $\ln \|A_k\|$ (see the proof of Theorem 4.2).

Following [21], $P_k(\theta)$ is an even function of $\theta + \frac{1}{2}(k-1)\alpha$ and can be written as a polynomial of order k in $\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)$:

$$\begin{aligned} P_k(\theta) &= \sum_{j=0}^{k} c_j \cos^j 2\pi(\theta + \frac{1}{2}(k-1)\alpha) \\ &\triangleq Q_k(\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)). \end{aligned} \tag{3.15}$$

Let

$$A_{k,r} = \{\theta \in \mathbb{R} : |Q_k(\cos 2\pi\theta)| \le e^{(k+1)r}\}$$

with $k \in \mathbb{N}$ and $r > 0$.

Definition 3.3. We say that the set $\{\theta_1, \dots, \theta_{k+1}\}$ is γ-uniform if

$$\max_{x\in[-1,1]} \max_{i=1,\dots,k+1} \prod_{j=1, j\ne i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{k\gamma}. \tag{3.16}$$

The next two lemmas are from [4], for self-contain we give the proof.

Lemma 3.5 ([4, Lemma 9.2]). *Suppose that $y \in \mathbb{Z}$ is $(L-\rho, k)$-singular with $1/10 < \delta < 1/2$, then for any $\varepsilon > 0$ and any $x \in \mathbb{Z}$ such that $y - (1-\delta)k \le x \le y - \delta k$, we have that $\theta + (x + \frac{1}{2}(k-1))\alpha$ belongs to $A_{k,L-\rho\delta+\varepsilon}$ for k large enough.*

Proof. Otherwise, there exist $\varepsilon > 0$ and x_1 satisfying $y-(1-\delta)k \le x_1 \le y-\delta k$ and $\theta + (x_1 + \frac{1}{2}(k-1))\alpha \notin A_{k,L-\rho\delta+\varepsilon}$, i.e., $P_k(\theta + x_1\alpha) > e^{(k+1)(L-\rho\delta+\varepsilon)}$ by (3.15). Let $I = [x_1, x_2]$ with $x_2 = x_1 + k - 1$, then $y \in I$ and $\operatorname{dist}(y, x_i) \ge \delta k$ for $i = 1, 2$. By (3.12), (3.13), and (3.14), we have

$$\begin{aligned} |G_I(y, x_i)| &\le e^{(L+\varepsilon)(k-|y-x_i|)-(k+1)(L-\rho\delta+\varepsilon)} \\ &< e^{-(L-\rho)|y-x_i|} \quad \text{for } i = 1, 2. \end{aligned} \tag{3.17}$$

This implies y is $(L-\rho, k)$-regular, contradicting to the hypothesis. ∎

Lemma 3.6 ([4, Lemma 9.3]). *Let* $\gamma_1 < \gamma$. *If* $\theta_1, \ldots, \theta_{k+1} \in A_{k,L-\gamma}$, *then* $\{\theta_1, \ldots, \theta_{k+1}\}$ *is not* γ_1*-uniform for* $k > k(\gamma, \gamma_1, \lambda)$.

Proof. If Lemma 3.6 is false, then

$$\max_{x\in[-1,1]} \prod_{j=1, j\neq i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{k\gamma_1}, \quad i = 1, 2, \ldots, k+1.$$

By (3.15), we can write polynomial $Q_k(x)$ in the Lagrange interpolation form at points $\cos 2\pi\theta_i$, $i = 1, 2, \ldots, k+1$, thus

$$\begin{aligned} |Q_k(x)| &= \left| \sum_{i=1}^{k+1} Q_k(\cos 2\pi\theta_i) \frac{\prod_{j\neq i}(x - \cos 2\pi\theta_j)}{\prod_{j\neq i}(\cos 2\pi\theta_i - \cos 2\pi\theta_j)} \right| \\ &\leq (k+1)e^{(k+1)(L-\gamma)} e^{k\gamma_1} \\ &= e^{kL}(k+1)e^{-k(\gamma-\gamma_1)+L-\gamma} \\ &< e^{kL} \end{aligned}$$

for all $x \in [-1, 1]$ and $k > k(\gamma, \gamma_1, \lambda)$. By (3.15) again, $|P_k(x)| < e^{kL}$ for all $x \in \mathbb{R}$. However, by Herman's subharmonic function methods (see [13, p. 16], or [20, p. 461]),

$$\int_{\mathbb{R}/\mathbb{Z}} \ln |P_k(x)| dx \geq kL.$$

This is impossible. □

Without loss of generality, assume

$$3|n_j| < y < \frac{|n_{j+1}|}{3}.$$

Select n such that

$$q_n \leq \frac{y}{8} < q_{n+1}$$

and let s be the largest positive integer satisfying

$$sq_n \leq \frac{y}{8}.$$

Set $I_1, I_2 \subset \mathbb{Z}$ as

$$I_1 = [-2sq_n + 1, 0] \quad \text{and} \quad I_2 = [y - 2sq_n + 1, y + 2sq_n], \tag{3.18}$$

if $n_j < 0$, and

$$I_1 = [0, 2sq_n - 1] \qquad \text{and} \quad I_2 = [y - 2sq_n + 1, y + 2sq_n], \tag{3.19}$$

if $n_j \geq 0$. In either case, the total number of elements in $I_1 \cup I_2$ is $6sq_n$. Let

$$\theta_{j'} = \theta + j'\alpha \quad \text{for } j' \in I_1 \cup I_2.$$

Lemma 3.7. *Under the condition of Theorem 3.1, the set $\{\theta_{j'}\}_{j' \in I_1 \cup I_2}$ constructed as (3.18) or (3.19) is $C\epsilon_0$-uniform for $y > y(\alpha)$ (or equivalently $n > n(\alpha)$).*

Proof. Firstly we estimate the numerator in (3.16). In (3.16), let $x = \cos 2\pi a$ and take the logarithm. We have

$$\begin{aligned}
&\sum_{j' \in I_1 \cup I_2, j' \neq i} \ln |\cos 2\pi a - \cos 2\pi\theta_{j'}| \\
&= \sum_{j' \in I_1 \cup I_2, j' \neq i} \ln |\sin \pi(a + \theta_{j'})| \\
&\quad + \sum_{j' \in I_1 \cup I_2, j' \neq i} \ln |\sin \pi(a - \theta_{j'})| + (6sq_n - 1) \ln 2 \\
&= \Sigma_+ + \Sigma_- + (6sq_n - 1) \ln 2,
\end{aligned} \tag{3.20}$$

where

$$\Sigma_+ = \sum_{j' \in I_1 \cup I_2, j' \neq i} \ln |\sin \pi(a + \theta_{j'})|, \tag{3.21}$$

and

$$\Sigma_- = \sum_{j' \in I_1 \cup I_2, j' \neq i} \ln |\sin \pi(a - \theta_{j'})|. \tag{3.22}$$

Both Σ_+ and Σ_- consist of $6s$ terms of the form of (3.6), plus 6s terms of the form

$$\ln \min_{j'=0,1,\ldots,q_n-1} |\sin \pi(x + j'\alpha)|, \tag{3.23}$$

minus $\ln |\sin \pi(a \pm \theta_i)|$. Since there exists a interval of length q_n in sum of (3.21) and (3.22) containing i, thus the minimum over this interval is not more than $\ln |\sin \pi(a \pm \theta_i)|$ (by the minimality). Thus, by (3.6) one has

$$\sum_{j' \in I_1 \cup I_2, j' \neq i} \ln |\cos 2\pi a - \cos 2\pi\theta_{j'}| \leq -6sq_n \ln 2 + Cs \ln q_n. \tag{3.24}$$

The estimate of the denominator of (3.16) requires a bit more work. In (3.20), let $a = \theta_i$, we obtain

$$\sum_{j' \in I_1 \cup I_2, j' \neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_{j'}| = \Sigma_+ + \Sigma_- + (6sq_n - 1)\ln 2, \quad (3.25)$$

where

$$\Sigma_+ = \sum_{j' \in I_1 \cup I_2, j' \neq i} \ln|\sin \pi(2\theta + (i + j')\alpha)|, \quad (3.26)$$

and

$$\Sigma_- = \sum_{j' \in I_1 \cup I_2, j' \neq i} \ln|\sin \pi(i - j')\alpha|. \quad (3.27)$$

Firstly, Σ_- consists of $6s$ terms of the form of (3.6) plus $6s - 1$ minimum terms like (3.23) (since there exists a interval of length q_n containing i, the sum over this interval is exactly of the form (3.6)). By (2.11) and (3.3),

$$\min_{0<|j'|<q_{n+1}} \|j'\alpha\|_{\mathbb{R}/\mathbb{Z}} = \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-2\beta q_n},$$

for $n > n(\alpha)$. Therefore, for $n > n(\alpha)$,

$$\max\{\ln|\sin x|, \ln|\sin(x + \pi j'\alpha)|\} \geq -C\beta q_n, \text{ for } x \in \mathbb{R} \text{ and } 0 < |j'| < q_{n+1}. \quad (3.28)$$

By known condition $sq_n < q_{n+1}$, then there exist at most 6 minimum terms smaller than $-C\beta q_n$. Next we estimate the minimum terms. Obviously, $|i - j'| < Csq_n$ for $i, j' \in I_1 \cup I_2$. By (3.2),

$$\min_{j' \in I_1 \cup I_2, j' \neq i} \ln|\sin \pi(i - j')\alpha| \geq -Csq_n\beta \text{ for } n > n(\alpha). \quad (3.29)$$

By (3.6), (3.28), and (3.29), we obtain

$$\Sigma_- \geq -6sq_n \ln 2 - Csq_n\beta. \quad (3.30)$$

Similarly, Σ_+ consist of $6s$ terms of the form of (3.6) plus 6s minimum terms and minus $\ln|\sin 2\pi\theta_i|$, and there exist at most 6 minimum terms smaller than $-C\beta q_n$ by (3.28). Thus we only need estimate the minimum term. By the definition of I_1 and I_2, one easily verifies $i + j' \neq -n_j$ and $|i + j'| < |n_{j+1}|$. By Lemma 3.8 below, one has

$$\min_{j' \in I_1 \cup I_2, j' \neq i} \|2\theta + (i + j')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-Csq_n\epsilon_0}. \quad (3.31)$$

Replacing (3.29) with (3.31), and following the discussion of Σ_-, we have

$$\Sigma_+ \geq -6sq_n \ln 2 - Csq_n\epsilon_0, \tag{3.32}$$

for $n > n(\alpha)$ or $y > y(\alpha)$. Putting (3.24), (3.30), and (3.32) together,

$$\max_{i\in I_1\cup I_2} \prod_{j'\in I_1\cup I_2, j'\neq i} \frac{|x-\cos 2\pi\theta_{j'}|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_{j'}|} < e^{C6sq_n\epsilon_0}, \tag{3.33}$$

for $y > y(\alpha)$. □

Lemma 3.8. *Under the condition of Lemma 3.7, suppose $i + j' \neq -n_j$ and $|i + j'| < |n_{j+1}|$, where $i, j' \in I_1 \cup I_2$, then*

$$\|2\theta + (i + j')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-Csq_n\epsilon_0}, \tag{3.34}$$

for $n > n(\alpha)$ (or equivalently $y > y(\alpha)$).

Proof. Let $|k_0| \leq |i + j'|$ be such that

$$\|2\theta + k_0\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|k|\leq|i+j'|} \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

We have two cases.

CASE 1. $k_0 \neq i + j'$. If $\|2\theta + k_0\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-Csq_n\epsilon_0}$, by the minimality of k_0, we have

$$\|2\theta + (i + j')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \|2\theta + k_0\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-Csq_n\epsilon_0}.$$

If $\|2\theta + k_0\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-Csq_n\epsilon_0}$, by (3.1)

$$\begin{aligned} \|2\theta + (i + j')\alpha\|_{\mathbb{R}/\mathbb{Z}} &\geq \|(i + j' - k_0)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta + k_0\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\geq c(\alpha)e^{-2\beta|i+j'-k_0|} - e^{-Csq_n\epsilon_0} \\ &\geq e^{-Csq_n\beta}, \end{aligned} \tag{3.35}$$

for $n > n(\alpha)$, since $|i + j' - k_0| < Csq_n$.

CASE 2. $k_0 = i + j'$. If $-k_0$ is not an resonance for θ, then by the definition of resonance

$$\|2\theta + (i + j')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-\epsilon_0|k_0|} \geq e^{-Csq_n\epsilon_0}.$$

If $-k_0$ is an resonance for θ, therefore $|k_0| \leq |n_j|$ (otherwise $-k_0 = n_{j+1}$). Next we discuss $\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-Csq_n\epsilon_0}$ and $\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-Csq_n\epsilon_0}$ respectively. Following the proof of case 1, we also have, for $n > n(\alpha)$

$$\|2\theta + (i + j')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-Csq_n\epsilon_0}. \tag{3.36}$$

Putting all cases together, we complete the proof of this lemma. □

Remark 3.3. Note that (3.34) holds if n is large enough, which only depends on α, does not depend on θ. By the way, all estimates in the present paper is uniform with respect to θ and $E \in \Sigma_{\lambda,\alpha}$. This is important.

By Lemma 3.6 and 3.7, there exists at least one of θ_{j_0} with $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{6sq_n-1,L-C\epsilon_0}$. We will prove that for all $j' \in I_1$, $\theta_{j'} \in A_{6sq_n-1,L-C\epsilon_0}$ if $\lambda < e^{-C_2\beta}$ with C_2 large enough, thus there exists some $j_0 \in I_2$ such that $\theta_{j_0} \notin A_{6sq_n-1,L-C\epsilon_0}$.

Lemma 3.9. *There exists some absolute constant C_2 such that for all $j' \in I_1$, $\theta_{j'} \in A_{6sq_n-1,L-C\epsilon_0}$ if $0 < \lambda < e^{-C_2\beta}$ and $n > n(\lambda,\alpha)$.*

Proof. Recall that by Lemma 3.4, $y = 0$ is (m,k)-singular with any δ satisfying $\frac{1}{10} < \delta < \frac{1}{2}$ if k is large enough. In Lemma 3.5, let $y = 0$, $\delta = \frac{99}{600}$, $\rho = \frac{99}{100}L$, $\varepsilon = \frac{1}{100}L$ and $k = 6sq_{n-1} - 1$. One easily checks that for all $j' \in I_1$, $\theta_{j'} \in A_{6sq_n-1,\frac{50799}{60000}L}$. Obviously, $\frac{50799}{60000}L < L - C\epsilon_0$ if $0 < \lambda < e^{-C_2\beta}$ because of $L = -\ln\lambda$. □

Proof of Theorem 3.1. Let $j_0 \in I_2$ be such that $\theta_{j_0} \notin A_{6sq_n-1,L-C\epsilon_0}$. Set

$$I = [j_0 - 3sq_n + 1, j_0 + 3sq_n - 1] = [x_1, x_2].$$

Let $\varepsilon = \epsilon_0$ in (3.14), combining with (3.12) and (3.13),

$$\begin{aligned}|G_I(y,x_i)| &\le e^{(L+\epsilon_0)(6sq_n-1-|y-x_i|)-6sq_n(L-C\epsilon_0)} \\ &\le e^{-L|y-x_i|+Csq_n\epsilon_0} \quad \text{for } i = 1,2.\end{aligned}$$

By a simple computation $|y - x_i| \ge sq_n - 2 \ge \frac{y}{16}$. Recall that $L = -\ln\lambda$, thus

$$|G_I(y,x_i)| \le e^{-\frac{y}{16}(L-C\epsilon_0)} \le e^{-\frac{L}{32}y} \quad \text{for } i = 1,2, \tag{3.37}$$

if $|\lambda| < e^{-C_2\beta}$ with C_2 large enough. By (3.8), we obtain that for $y > y(\lambda,\alpha)$, $|\phi(y)| \le e^{-\frac{L}{64}y}$ with y satisfying $3|n_j| < y < |n_{j+1}|/3$. This implies $|\phi(y)| \le C(\lambda,\alpha)e^{-\frac{L}{64}y}$ for all y with $3|n_j| < y < |n_{j+1}|/3$. For $y < 0$, the proof is similar. □

We actually have proved a slightly more precise version of Theorem 3.1.

Theorem 3.2. *Let $\epsilon_0 = C_1\beta$ and $|\lambda| \in (0, e^{-C_2\beta})$ where C_1, C_2 are the constants in Theorem 3.1, and let $\hat{u}$ be a solution of the equation $\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ satisfying $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, where $E \in \Sigma_{\lambda,\alpha}$. Then we have that $|\hat{u}_k| \leq e^{-\frac{L}{64}|k|}$ if $3|n_j| < |k| < 3^{-1}|n_{j+1}|$ and $|k| > C(\lambda,\alpha)$, or equivalently, that $|\hat{u}_k| \leq C(\lambda,\alpha)e^{-\frac{L}{64}|k|}$ for all k satisfying $3|n_j| < |k| < 3^{-1}|n_{j+1}|$, where set $\{n_j\}$ is the ϵ_0-resonance for θ.*

Remark 3.4. If θ is not ϵ_0-resonant, and a solution $\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ satisfying $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, then by Theorem 3.2, $|\hat{u}_k| \leq C(\lambda,\alpha)e^{-\frac{L}{64}|k|}$ with $|k| > 3|n_{j_\theta}|$, since $n_{j_\theta+1} = \infty$, where $L = -\ln\lambda$.

4. The estimate of rotation number for resonant phase

It is well known that for almost every $E \in \Sigma_{\lambda,\alpha}$, there exists a solution $\hat{u}$ of the equation $\widehat{H}\hat{u} = E\hat{u}$ with $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq (1 + |k|)^C$.See for the proof of continuous-time Schrödinger operator. The proof of discrete Schrödinger operator is similar, see [27]. Generally, it does not hold for every $E \in \Sigma_{\lambda,\alpha}$. Such exclusion is inherent to Gelfand–Maurin theorem. Avila and Jitomirskaya in [5] conquer this difficulty by changing the phase θ. This is a starting point of the quantitative version of Aubry duality.

Lemma 4.1 ([5, Theorem 3.3]). *If $E \in \Sigma_{\lambda,\alpha}$, then there exists $\theta \in \mathbb{R}$ and a bounded solution of $\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ with $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1$.*

Fix α such that $0 < \beta(\alpha) < \infty$, and fix C_1 in Theorem 3.1. Without loss of generality, assume $\lambda > 0$. By Theorem 3.1 or Theorem 3.2, there exists an absolute constant C_2 such that, for $0 < \lambda < e^{-C_2\beta}$, $\widehat{H}_{\lambda,\alpha,\theta}$ satisfies a strong localization estimate with parameters ϵ_0, $\epsilon_1 = 2\pi h$ and C_0, where $\epsilon_0 = C_1\beta$, $h = C_1\epsilon_0$ and $C_0 = 3$. This is because $2\pi h < \frac{-\ln\lambda}{64}$ in view of $0 < \lambda < e^{-C_2\beta}$ with C_2 large enough. Given $E \in \Sigma_{\lambda,\alpha}$, let $\theta = \theta(E)$ and $\hat{u}_k$ be given by Lemma 4.1. In this section, assume $\theta(E)$ is ϵ_0-resonant with the infinite set of ϵ_0-resonances $\{n_j\}_{j=1}^{\infty}$. Let $\|\cdot\|$ be the Euclidean norms, and denote

$$\|f\|_\eta = \sup_{|\operatorname{Im} x|<\eta} \|f(x)\|$$

and

$$\|f\|_0 = \sup_{x\in\mathbb{R}} \|f(x)\|.$$

Below, unless stated otherwise, set

$$A = S_{\lambda,E} = \begin{pmatrix} E - 2\lambda\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}$$

with $0 < \lambda < e^{-C_2\beta}$ and $E \in \Sigma_{\lambda,\alpha}$. Note that in Section 3, $A = S_{\lambda^{-1},E}$.

Lemma 4.2. *For $|n_j|$ large enough (depending on α),*

$$\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-8\beta|n_{j+1}|}, \tag{4.1}$$

in particular,

$$|n_{j+1}| > \frac{C_1}{8}|n_j|. \tag{4.2}$$

Proof. By (3.2),

$$\begin{aligned} \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} &\geq \|(n_{j+1} - n_j)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - n_{j+1}\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\geq e^{-6\beta|n_{j+1}|} - e^{-\epsilon_0|n_{j+1}|} \\ &\geq e^{-8\beta|n_{j+1}|} \end{aligned} \tag{4.3}$$

This implies (4.1). Combining with the fact $\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\epsilon_0|n_j|}$, one has (4.2). ☐

We will say that a trigonometrical polynomial

$$p\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$$

has *essential degree at most k* if its Fourier coefficients outside an interval I of length k (for $I = [a,b]$, $k = b - a$) are vanishing.

Lemma 4.3 ([5, Theorem 6.1]). *Let $1 \leq r \leq \lfloor q_{n+1}/q_n \rfloor$. If p has essential degree at most $k = rq_n - 1$ and $x_0 \in \mathbb{R}/\mathbb{Z}$, then*

$$\|p\|_0 \leq C q_{n+1}^{Cr} \sup_{0 \leq j \leq k} |p(x_0 + j\alpha)|. \tag{4.4}$$

In the present paper, under condition

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n},$$

equation (4.4) becomes

$$\begin{aligned} \|p\|_0 &\leq C e^{Cr\ln q_{n+1}} \sup_{0\leq j\leq k} |p(x_0 + j\alpha)| \\ &\leq e^{C\beta k} \sup_{0\leq j\leq k} |p(x_0 + j\alpha)|, \end{aligned} \tag{4.5}$$

for $n > n(\alpha)$ or equivalently $k > k(\alpha)$.

For any n with $9|n_j| < n < \frac{1}{9}|n_{j+1}|$ of the form $n = rq_k - 1 < q_{k+1}$ (by Lemma 4.2, there exists such n if $|n_j|$ is large enough depending on α), let

$$u^{I_1}(x) = \sum_{k \in I_1} \hat{u}_k e^{2\pi i k x}$$

with

$$I_1 = \Big[-\Big[\frac{n}{2}\Big], n - \Big[\frac{n}{2}\Big]\Big] = [x_1, x_2].$$

Recall that $\hat{u}_k$ is given by Lemma 4.1 and satisfies the estimate in Theorem 3.2. Define

$$U^{I_1}(x) = \begin{pmatrix} e^{2\pi i\theta} u^{I_1}(x) \\ u^{I_1}(x-\alpha) \end{pmatrix},$$

by direct computation

$$AU^{I_1}(x) = e^{2\pi i\theta} U^{I_1}(x+\alpha) + e^{2\pi i\theta} \begin{pmatrix} g(x) \\ 0 \end{pmatrix}, \tag{4.6}$$

and the Fourier coefficients of $g(x)$ satisfy

$$\hat{g}_k = \chi_{I_1}(k)(E - 2\cos 2\pi(\theta + k\alpha))\hat{u}_k - \lambda \sum_{j \in \{-1,1\}} \chi_{I_1}(k-j)\hat{u}_{k-j}, \tag{4.7}$$

where χ_I is the characteristic function of I. Since $\widehat{H}\hat{u} = E\hat{u}$, we also have

$$-\hat{g}_k = \chi_{\mathbb{Z}\backslash I_1}(k)(E - 2\cos 2\pi(\theta + k\alpha))\hat{u}_k - \lambda \sum_{j \in \{-1,1\}} \chi_{\mathbb{Z}\backslash I_1}(k-j)\hat{u}_{k-j}. \tag{4.8}$$

By (4.7) and (4.8), $\hat{g}_k \neq 0$, only at four points x_1, x_2, $x_1 - 1$ and $x_2 + 1$. By the strong localization estimate

$$|\hat{u}_k| \leq C(\lambda,\alpha) e^{-2\pi h|k|},$$

it is easy to see $\|g\|_{\frac{h}{3}} \leq C(\lambda,\alpha)e^{-3hn}$, in particular $\|g\|_{\frac{h}{3}} \leq e^{-2hn}$ for $n > n(\lambda,\alpha)$, since $C(\lambda,\alpha) < e^{hn}$ for $n > n(\lambda,\alpha)$.

Lemma 4.4 ([2, Theorem 10]). *If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \neq 0$, $E \in \mathbb{R}$ and $\epsilon \geq 0$, then*

$$L(\alpha, A^{E,\epsilon}) = \max\{L(\alpha, A^E), \ln|\lambda| + 2\pi\epsilon\},$$

where

$$A^{E,\epsilon} = \begin{pmatrix} E - 2\lambda\cos 2\pi(x + i\epsilon) & -1 \\ 1 & 0 \end{pmatrix},$$

and $A^E = A^{E,0}$.

Corollary 4.1. *If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $|\lambda| < 1$ and $\frac{\ln|\lambda|}{2\pi} \le \epsilon \le \frac{-\ln|\lambda|}{2\pi}$, then*

$$L(\alpha, A^{E,\epsilon}) = 0 \quad \textit{for } E \in \Sigma_{\lambda,\alpha}.$$

Proof. By Theorem 2.1, if $|\lambda| < 1$ and $E \in \Sigma_{\lambda,\alpha}$, then $L(\alpha, A^E) = 0$. If we suppose $0 \le \epsilon \le \frac{-\ln|\lambda|}{2\pi}$, i.e., $\ln|\lambda| + 2\pi\epsilon \le 0$, then $L(\alpha, A^{E,\epsilon}) = 0$ by Lemma 4.4. By symmetry $L(\alpha, A^{E,\epsilon}) = 0$ for $\frac{\ln|\lambda|}{2\pi} \le \epsilon \le \frac{-\ln|\lambda|}{2\pi}$. □

Next we will set up the priori estimate of transfer matrix, precisely,

$$\|A_n(x)\| = e^{o(n)}$$

through band $|\operatorname{Im} x| < \frac{-\ln|\lambda|}{2\pi}$ uniformly, where $A = S_{\lambda,E}$ and A_n is given by (2.2). This can be done by Furman's uniquely ergodic theorem and vanishing Lyapunov exponent (Corollary 4.1).

Theorem 4.1 ([16, Theorem 1]). *Let $\{f_n\}$ be a continuous subadditive cocycle on a uniquely ergodic system (X, μ, T), i.e., X is a compact metric space, $T : X \mapsto X$ is a homeomorphism with μ being the unique T-invariant probability measure on X, and $f_n \in C(X)$ with $f_{n+m}(x) \le f_n(x) + f_m(T^n x)$ for all $x \in X$. Then, for every $x \in X$ and uniformly on X,*

$$\limsup_{n\to\infty} \frac{1}{n} f_n(x) \le \lim_{n\to\infty} \frac{1}{n} \int f_n d\mu. \tag{4.9}$$

Theorem 4.1 is usually called *Furman's uniquely ergodic theorem.*

Theorem 4.2. *For all $E \in \Sigma_{\lambda,\alpha}$,*

$$\|A_k^E(x)\|_\eta \le C(\lambda, \alpha) e^{\beta k},$$

where $\eta = \frac{-\ln|\lambda|}{2\pi}$.

Proof. By Corollary 4.1, $L(\alpha, A^{E,\epsilon}) = 0$ for any $-\eta \le \epsilon \le \eta$ and $E \in \Sigma_{\lambda,\alpha}$. In Theorem 4.1, let $f_n = \ln\|A_n^{E,\epsilon}\|$, $X = \mathbb{R}/\mathbb{Z}$, $Tx = x + \alpha$ and μ is Lebesgue measure. Since irrational rotations are uniquely ergodic, then there exists some $k_0(\lambda, \alpha, E, \epsilon)$ such that

$$\ln\|A_k^E(x)\| < \beta k$$

for all x satisfying $\operatorname{Im} x = \epsilon$ with $|\epsilon| \le \frac{-\ln|\lambda|}{2\pi}$ and $k \ge k_0(\lambda, \alpha, E, \epsilon)$.

By continuity and compactness of $\mathbb{R}/\mathbb{Z}$, there exists $\delta(\lambda, \alpha, E, \epsilon)$ such that if $|E' - E| < \delta$ and $|\operatorname{Im} x' - \epsilon| < \delta$, then

$$\ln \|A_k^{E'}(x')\| < \beta k \tag{4.10}$$

for every $k_0(\lambda, \alpha, E, \epsilon) \leq k \leq 2k_0(\lambda, \alpha, E, \epsilon) + 1$.

For any $k > 2k_0(\lambda, \alpha, E, \epsilon)$, let $k = sk_0 + r$, where $k_0 \leq r < 2k_0$, then by subadditivity,

$$\ln \|A_k^{E'}(x')\| \leq s \max_{|\operatorname{Im} x_1 - \epsilon| < \delta} \ln \|A_{k_0}^{E'}(x_1)\| + \max_{|\operatorname{Im} x_1 - \epsilon| < \delta} \ln \|A_r^{E'}(x_1)\| < \beta k.$$

Thus (4.10) holds for all $k \geq k_0(\lambda, \alpha, E, \epsilon)$. By the compactness of $\{|\epsilon| \leq \eta\}$ and $\Sigma_{\lambda,\alpha}$, there exists $k_0(\lambda, \alpha)$, such that

$$\ln \|A_k^E(x)\| < \beta k$$

for every x satisfying $|\operatorname{Im} x| \leq \eta$, $E \in \Sigma_{\lambda,\alpha}$ and $k > k_0(\lambda, \alpha)$. It follows that

$$\|A_k^E(x)\|_\eta \leq C(\lambda, \alpha) e^{\beta k}. \qquad \square$$

Remark 4.1. In fact, our proof suggests that for any $\delta > 0$,

$$\|A_k^E(x)\|_\eta \leq C(\delta, \lambda, \alpha) e^{\delta k}$$

with $\eta = -\frac{1}{2\pi} \ln |\lambda|$. This verifies a claim by Avila in [1, footnote 5].

For more subtle estimate of the transfer matrix, a couple of lemmata and theorems are necessary.

Theorem 4.3. *For $n > n(\lambda, \alpha)$,*

$$\inf_{|\operatorname{Im} x| < \frac{h}{3}} \|U^{I_1}(x)\| \geq e^{-C\beta n}. \tag{4.11}$$

Proof. If (4.11) is false, then let x_0 with $\operatorname{Im} x_0 = t$ and $|t| < \frac{h}{3}$ such that

$$\|U^{I_1}(x_0)\| \leq e^{-C\beta n}.$$

By (4.6) and Theorem 4.2,

$$\|U^{I_1}(x_0 + j\alpha)\| \leq e^{-C\beta n}, \quad 0 \leq j \leq n,$$

since $\|g\|_{\frac{h}{3}} < e^{-2hn}$ for $n > n(\lambda, \alpha)$. This implies, for $n > n(\lambda, \alpha)$,

$$|u^{I_1}(x_0 + j\alpha)| \leq e^{-C\beta n}, \quad 0 \leq j \leq n.$$

Thus

$$\|u_t^{I_1}\|_0 \le e^{-C\beta n}$$

by (4.5), contradicting to

$$\int u_t^{I_1}(x)dx = 1$$

(since $\hat{u}_0 = 1$), where $u_t^{I_1}(x) = u^{I_1}(x + ti)$. □

Theorem 4.4 ([1, Theorem 2.6]). *Let $U : \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2$ be analytic in $|\operatorname{Im} x| < \eta$. Assume that $\delta_1 < \|U(x)\| < \delta_2^{-1}$ holds for all x satisfying $|\operatorname{Im} x| < \eta$. Then there exists*

$$B : \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{SL}(2, \mathbb{C})$$

being analytic in $|\operatorname{Im} x| < \eta$ with first column U and

$$\|B\|_\eta \le C\delta_1^{-2}\delta_2^{-1}(1 - \ln(\delta_1\delta_2)).$$

Lemma 4.5.

$$\max_{x\in\mathbb{R}} \|A_m(x)\| \le C(\lambda, \alpha)m^C. \tag{4.12}$$

Proof. The estimate $|\hat{u}_k| \le 1$ implies $\|U^{I_1}\|_\beta < e^{C\beta n}$. Let $B(x) \in \mathrm{SL}(2, \mathbb{C})$ be the matrix, whose first column is $U^{I_1}(x)$, given by Theorem 4.4 with $\eta = \beta$, then $\|B\|_\beta \le e^{C\beta n}$ for $n > n(\lambda, \alpha)$. Combining with (4.6), one easily verifies

$$B(x+\alpha)^{-1}A(x)B(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_2(x) & \beta_3 x \end{pmatrix}, \tag{4.13}$$

where $\|b\|_\beta < e^{C\beta n}$, and $\|\beta_1\|_\beta, \|\beta_2\|_\beta, \|\beta_3\|_\beta < e^{-hn}$, since $\|g\|_{\frac{h}{3}} \le e^{-2hn}$. Taking $\Phi = DB(x)^{-1}$, where

$$D = \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$$

with $d = e^{-\frac{hn}{4}}$, we get

$$\Phi(x+\alpha)A(x)\Phi(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + H(x), \tag{4.14}$$

where $\|H\|_\beta < e^{-\frac{hn}{4}}$ and $\|\Phi\|_\beta < e^{hn}$. Thus

$$\sup_{0\le s\le e^{\frac{hn}{4}}} \|A_s(x)\|_\beta \le e^{3hn}. \tag{4.15}$$

If $m > C(\lambda, \alpha)$, we can select n with

$$C \frac{\ln m}{h} < n < C^2 \frac{\ln m}{h}$$

of the form

$$n = rq_k - 1 < q_{k+1}$$

and

$$9|n_j| < n < \frac{1}{9}|n_{j+1}|,$$

thus

$$\|A_m\|_\beta < m^C$$

by (4.15). That is

$$\|A_m\|_\beta < C(\lambda, \alpha) m^C \quad \text{for all } m. \qquad \square$$

Fix some $n = |n_j|$ and let $N = |n_{j+1}|$. Construct new function $u^{I_2}(x)$ with

$$I_2 = \left[-\left[\frac{N}{9}\right], \left[\frac{N}{9}\right] \right]$$

and a vector-valued function

$$U^{I_2}(x) = \begin{pmatrix} e^{2\pi i \theta} u^{I_2}(x) \\ u^{I_2}(x - \alpha) \end{pmatrix}$$

as before.

Theorem 4.5. *For $n > n(\lambda, \alpha)$,*

$$\inf_{|\operatorname{Im} x| < \frac{h}{3}} \|U^{I_2}(x)\| \geq e^{-C\beta n}. \tag{4.16}$$

Proof. Let rq_k be the minimal such that $rq_k > 9|n_j|$ and $rq_k - 1 < q_{k+1}$, and let

$$J = \left[-\left[\frac{rq_k}{2}\right], rq_k - 1 - \left[\frac{rq_k}{2}\right] \right].$$

Define $U^J(x)$ as before. By the estimates $|\hat{u}_k| \leq e^{-2\pi h|k|}$ for $3n < |k| < \frac{N}{3}$ and $|u_k| \leq 1$ for others (since $n > n(\lambda, \alpha)$), we have

$$\|U^{I_2} - U^J\|_{\frac{h}{3}} \leq e^{-hn}.$$

By (4.11) and a simple fact $rq_k \leq Cn$, one has

$$\inf_{|\operatorname{Im} x| < \frac{h}{3}} \|U^J(x)\| \geq e^{-C\beta n}.$$

This implies

$$\inf_{|\operatorname{Im} x|<\frac{h}{3}} \|U^{I_2}(x)\| \geq e^{-C\beta n}. \tag{4.17}$$

We finish the proof of the theorem. □

Let

$$\widetilde{U}(x) = e^{\pi i n_j x} U^{I_2}(x)$$

and

$$\tilde{\theta} = \theta - \frac{n_j \alpha}{2}.$$

Note that $\widetilde{U}(x)$ depends on I_2, for simplicity we drop the dependence, since below the interval is always

$$I_2 = \Big[-\Big[\frac{N}{9}\Big], \Big[\frac{N}{9}\Big]\Big].$$

Let $B(x)$ be the matrix with columns $\widetilde{U}(x)$ and $\overline{\widetilde{U}(x)}$, where $\overline{\widetilde{U}(x)}$ is the complex conjugate of $\widetilde{U}(x)$, and let

$$P^{-1} = \|2\theta - n_j \alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

By the same arguments of (4.6)–(4.8), for $n > n(\lambda, \alpha)$,

$$A\widetilde{U}(x) = e^{2\pi i \tilde{\theta}} \widetilde{U}(x+\alpha) + \begin{pmatrix} g(x) \\ 0 \end{pmatrix} \quad \text{with } \|g\|_{\frac{h}{3}} < e^{-chN}. \tag{4.18}$$

By the definition of resonance and Lemma 4.2,

$$e^{\epsilon_0 n} \leq P \leq e^{8\beta N} \quad \text{for } n > n(\alpha). \tag{4.19}$$

Theorem 4.6. *For $n > n(\lambda, \alpha)$,*

$$\inf_{x \in \mathbb{R}/\mathbb{Z}} |\det B(x)| \geq P^{-C}. \tag{4.20}$$

Proof. By the proof of [5, Lemma 8.1], for any complex matrix M with columns V and W,

$$|\det M| = \|V\| \min_{\lambda \in \mathbb{C}} \|W - \lambda V\| \tag{4.21}$$

and the minimizing λ satisfies $\|\lambda V\| \leq \|W\|$. Suppose (4.20) would not hold. By Theorem 4.5, $\inf_{x\in\mathbb{R}} \|\widetilde{U}(x)\| \geq e^{-C\beta n}$; then there exists $x_0 \in \mathbb{R}$ and $\lambda_0 \in \mathbb{C}$ ($|\lambda_0| \leq 1$) such that

$$|\overline{\widetilde{U}(x_0)} - \lambda_0 \widetilde{U}(x_0)| \leq P^{-C}.$$

By (4.12) and (4.18), we have

$$|e^{-2\pi ij\tilde{\theta}}\overline{\widetilde{U}(x_0+j\alpha)}-e^{2\pi ij\tilde{\theta}}\lambda_0\widetilde{U}(x_0+j\alpha)|\leq P^{-C}, 0\leq j\leq P. \tag{4.22}$$

That is

$$|\overline{\widetilde{U}(x_0+j\alpha)}-e^{4\pi ij\tilde{\theta}}\lambda_0\widetilde{U}(x_0+j\alpha)|\leq P^{-C}, 0\leq j\leq P. \tag{4.23}$$

Note the simple fact that

$$|e^{4\pi ij\tilde{\theta}}-1|< C\|2j\tilde{\theta}\|_{\mathbb{R}/\mathbb{Z}}< P^{-c}, \quad \text{for } 0\leq j\leq P^{1-c},$$

since $\|2\tilde{\theta}\|_{\mathbb{R}/\mathbb{Z}}=P^{-1}$. Combining with (4.23) and noting $\|\widetilde{U}\|_0\leq C(\lambda,\alpha)n$ by the strong localization estimate, one has

$$|\overline{\widetilde{U}(x_0+j\alpha)}-\lambda_0\widetilde{U}(x_0+j\alpha)|\leq P^{-c}, 0\leq j\leq P^{1-c}. \tag{4.24}$$

Let $\widetilde{U}_k(x)$ obtained by truncating the Fourier coefficients of $\widetilde{U}(x)$ at scale $k=\frac{c}{\beta}\ln P$. By (4.19), one has $9n<k<\frac{1}{9}N$. By the strong localization estimate in Theorem 3.2 and the definition of $\widetilde{U}(x)$,

$$\|\widetilde{U}-\widetilde{U}_k\|_0\leq e^{-\frac{c}{\beta}h\ln P}\leq P^{-c}. \tag{4.25}$$

Therefore, we may assume the essential degree of $\widetilde{U}$ is $\frac{c}{\beta}\ln P$. By (4.5) and (4.24), we have (first replacing $\widetilde{U}(x)$ with $\widetilde{U}(2x)$ so that $\widetilde{U}(2x)$ is well defined in $\mathbb{R}/\mathbb{Z}$)

$$\sup_{x\in\mathbb{R}/\mathbb{Z}}|\overline{\widetilde{U}(x)}-\lambda_0\widetilde{U}(x)|\leq e^{C\beta\frac{c}{\beta}\ln P}P^{-c}\leq P^{-c}. \tag{4.26}$$

In (4.23), let

$$j=[\frac{P}{4}].$$

We get

$$|i\overline{\widetilde{U}(x_1)}+i\lambda_0\widetilde{U}(x_1)|\leq P^{-c}, \tag{4.27}$$

where

$$x_1=x_0+[\frac{P}{4}]\alpha.$$

By (4.26) and (4.27), $|\widetilde{U}(x_1)|\leq P^{-c}$. Recall that $\inf_{x\in\mathbb{R}}\|\widetilde{U}(x)\|\geq e^{-C\beta n}$, thus we get $P\leq e^{C\beta n}$. This contradicts to (4.19) $P\geq e^{\epsilon_0 n}$, since $\epsilon_0=C_1\beta$ and we assume C_1 is much larger than any absolute constant C emerging in this paper. □

The following theorem gives a sharp estimate of the rotation number if phase $\theta(E)$ is ϵ_0-resonant.

Theorem 4.7. *Fix $n = |n_j|$ (large enough depending on λ and α) and $N = |n_{j+1}|$, then there exists m_j with $|m_j| \leq Cn$ such that*

$$\|2\rho(\alpha, A) - m_j\alpha \pm (2\theta - n_j\alpha)\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-chN}.$$

Proof. Let $S = \operatorname{Re}\widetilde{U}$, $T = \operatorname{Im}\widetilde{U}$, and let $\widetilde{W}$ be the matrix with columns S and $\pm T$ so that $\det\widetilde{W} > 0$. Then by (4.18)

$$A\widetilde{W}(x) = \widetilde{W}(x+\alpha)\cdot R_{\mp\tilde{\theta}} + O(e^{-chN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{4.28}$$

Let $W(x) = |\frac{\det B(x)}{2}|^{-1/2}\widetilde{W}(x)$, it is easy to verify $\det W = 1$. By Theorem 4.6,

$$AW(x) = \frac{|\det B(x+\alpha)|^{1/2}}{|\det B(x)|^{1/2}}W(x+\alpha)\cdot R_{\mp\tilde{\theta}} + O(e^{-chN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{4.29}$$

By (4.18) and $\det A = 1$, $|\det B(x+\alpha)| - |\det B(x)| = O(e^{-chN})$, thus we have

$$AW(x) = W(x+\alpha)\cdot R_{\mp\tilde{\theta}} + O(e^{-chN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{4.30}$$

Since $\det W = 1$ and

$$W(x) = \left|\frac{\det B(x)}{2}\right|^{-1/2}|\widetilde{W}(x), \qquad \text{for } x \in \mathbb{R}/\mathbb{Z},$$

we have

$$\|W^{-1}\| \leq P^C.$$

Then

$$W(x+\alpha)^{-1}AW(x) = R_{\mp\tilde{\theta}} + O(e^{-chN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{4.31}$$

Since $W(x)$ is well defined in $\mathbb{R}/2\mathbb{Z}$, combing with (2.6),

$$\|2\rho(\alpha, A) - m\alpha \pm 2\tilde{\theta}||_{\mathbb{R}/\mathbb{Z}} \leq e^{-chN},$$

where $m = \deg(W)$. Thus, to prove this theorem, we only need prove

$$|\deg(W)| \leq Cn.$$

Next we will estimate the degree of W. The degree of W is the same as the degree of any of its columns.[7] It is enough to estimate the degree of $\frac{M(x)}{\|M(x)\|}$ for $M = S$or $M = T$. Note that

$$\|\int_{\mathbb{R}/\mathbb{Z}} e^{-\pi i n_j x}(S(x) + iT(x))dx\| = \sqrt{2}.$$

[7] Let $S\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathbb{R}^2\backslash\mathbf{0}$, we say degree of S is k, denoted by $\deg(S) = k$, if S is homotopic to $\binom{\cos kx/2}{\sin kx/2}$.

Without loss of generality, assume

$$\int_{\mathbb{R}/\mathbb{Z}} \|S(x)\| dx \geq \sqrt{2}/2.$$

By (4.28),

$$AS(x) = S(x+\alpha)\cos 2\pi\tilde{\theta} \mp T(x+\alpha)\sin 2\pi\tilde{\theta} + O(e^{-chN}), \quad x \in \mathbb{R}/\mathbb{Z}. \tag{4.32}$$

Combining with $\|2\tilde{\theta}\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\epsilon_0 n}$, we have

$$AS(x) = S(x+\alpha) + O(e^{-c\epsilon_0 n}),$$

or

$$AS(x) = -S(x+\alpha) + O(e^{-c\epsilon_0 n}).$$

Following the proof of Theorem 4.5, we have the similar estimate

$$\inf_{x\in\mathbb{R}} \|S(x)\| \geq e^{-C\beta n}. \tag{4.33}$$

Denote $\widetilde{S}(x)$ by truncating the Fourier series of S at scale Cn, then

$$\|\widetilde{S}(x) - S(x)\| \leq e^{-Chn} < \frac{\|S(x)\|}{2}$$

for $x \in \mathbb{R}/2\mathbb{Z}$ and $n > n(\lambda, \alpha)$. Thus the degree of S is equal to the degree of $\widetilde{S}$.

Now we estimate the degree of $\widetilde{S}(x)$. Let

$$\widetilde{S}(2x) = \begin{pmatrix} \widetilde{S}_1(x) \\ \widetilde{S}_2(x) \end{pmatrix}.$$

Then $\widetilde{S}_1(x)$, $\widetilde{S}_2(x)$ only have Fourier series at scale Cn. Note that $\widetilde{S}_1(x) + i\widetilde{S}_2(x)$ can be written as a polynomial of z and z^{-1}, where $z = e^{2\pi i x}$. More precisely, there exists a polynomial $f(z)$ of order less than Cn and $k \in \mathbb{N}$ such that

$$\frac{f(e^{2\pi i x})}{e^{2\pi i k x}} = \widetilde{S}_1(x) + i\widetilde{S}_2(x), \quad \text{where } k < Cn.$$

It is a well known fact that the degree of $\widetilde{S}(x)$ is equal to the zeros of $f(z)$ in disk $\mathbb{D} = \{z : |z| \leq 1\}$ minus k. Then $|\deg \widetilde{S}| \leq Cn$, i.e., $|\deg W| \leq Cn$. □

Remark 4.2. From (4.31), it is easy to see that $S_{\lambda,E}$ is almost reducible to $R_{\pm\theta}$, if $\theta = \theta(E)$ given by Lemma 4.1 is ϵ_0-resonant. Combining with Theorem 5.2 in the next section, we have for every $E \in \Sigma_{\lambda,\alpha}$, $S_{\lambda,E}$ is almost reducible.

5. Reducibility for non-resonant phase

In §4, we obtain sharp estimate of the rotation number $\rho(\alpha, A)$ when $\theta(E)$ is ϵ_0-resonant. In this section, we will set up reducibility for $A = S_{\lambda,E}$ with $E \in \Sigma_{\lambda,\alpha}$ when $\theta(E)$ is not ϵ_0-resonant.

Lemma 5.1. *Let $W\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathbb{C}^2$ be an real analytic vector in $|\operatorname{Im} x| < \eta$. Assume that $\inf_{|\operatorname{Im} x|<\eta} \|W(x)\| > \delta$ with some $\delta > 0$, then there exists*

$$B\colon \mathbb{R}/2\mathbb{Z} \longmapsto \mathrm{SL}(2,\mathbb{R})$$

being real analytic in $|\operatorname{Im} x| < \eta$ with first column W.

Proof. Let

$$W(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}.$$

By Theorem 4.4 there exist b_1 and b_2 being analytic in $|\operatorname{Im} x| < \eta$ such that

$$w_1 b_1 - w_2 b_2 = 1.$$

Let[8]

$$\tilde{w}_1(z) = \frac{b_1(z) + \bar{b}_1(z)}{2}$$

and

$$\tilde{w}_2(z) = \frac{b_2(z) + \bar{b}_2(z)}{2}.$$

Then

$$B = \begin{pmatrix} w_1 & \tilde{w}_2 \\ w_2 & \tilde{w}_1 \end{pmatrix} : \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$$

is real analytic in $|\operatorname{Im} x| < \eta$. □

Remark 5.1. Given a non-zero real analytic vector-valued function

$$W(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}$$

with

$$W(x+1) = \pm W(x),$$

[8] $\bar{a}(z)$ is defined by $\bar{a}(z) = \sum \overline{a_n} z^n$, if $a(z) = \sum a_n z^n$. Note that $\overline{a(z)}$ is the complex conjugate of $a(z)$, however $\bar{a}(z)$ is not.

all of Avila, Jitomirskaya, Puig and so on construct B as follows:

$$B(x) = \frac{1}{w_1^2 + w_2^2} \begin{pmatrix} w_1(x) & -w_2(x) \\ w_2(x) & w_1(x) \end{pmatrix}.$$

Since both w_1 and w_2 are real analytic, $w_1^2 + w_2^2 > 0$ for $x \in \mathbb{R}$. By continuity, $w_1^2 + w_2^2 \neq 0$ in a neighbor of real axis and $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ is real analytic in a neighbor of real axis (this process is a key step to set up reducibility for cocycle $A = S_{\lambda,E}$. See the proof of Theorem 5.2). Usually, $B(x)$ is not real analytic in the given strip. In the present paper, since $W(x)$ is well defined in $\mathbb{R}/2\mathbb{Z}$, we can use Lemma 5.1 to construct a cocycle $B\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$ with first column W so that B is real analytic in the given strip. However, we do not have a map $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ in general. Fortunately, the following theorem suggests that it does not matter whether $B\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$ or $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ in defining reducibility.

Theorem 5.1. *If*

$$B\colon \mathbb{R}/2\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

is analytic in $|\operatorname{Im} x| < \eta$ *and* $B(x+\alpha)^{-1}A(x)B(x)$ *is constant, then there exists*

$$B'\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

being analytic in $|\operatorname{Im} x| < \eta$ *such that* $B'(x+\alpha)^{-1}A(x)B'(x)$ *is constant.*

Proof. Step 1. We will prove that there exists

$$B_1\colon \mathbb{R}/4\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

being analytic in $|\operatorname{Im} x| < \eta$ such that

$$B_1^{-1}(x+\alpha)A(x)B_1(x) = V$$

and

$$B_1(x+1)^{-1}B_1(x) = D,$$

where V, D are constant and commute (i.e., $VD = DV$).

By hypothesis there exists $B\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$ such that

$$B^{-1}(x+\alpha)A(x)B(x) = V_1,$$

with V_1 being constant. Let

$$D_1(x) = B(x+1)^{-1}B(x).$$

Then

$$D_1(x+2) = D_1(x)$$

and

$$\begin{aligned} V_1 D_1(x) V_1^{-1} &= B(x+1+\alpha)^{-1} A(x+1) B(x+1) B(x+1)^{-1} \\ &\quad\ B(x) B(x)^{-1} A(x)^{-1} B(x+\alpha) \\ &= B(x+1+\alpha)^{-1} B(x+\alpha) \\ &= D_1(x+\alpha). \end{aligned} \tag{5.1}$$

Assume that V_1 is not conjugate to a rotation R_θ with $2\theta \in \frac{1}{2}\alpha\mathbb{Z}+\mathbb{Z}$. Write $D_1(x)$ in the Fourier series (note that $D_1(x)$ is well defined in $\mathbb{R}/2\mathbb{Z}$)

$$D_1(x) = \sum_{k\in\mathbb{Z}} \widehat{D}_1(k) e^{\pi i k x}, \quad \widehat{D}_1(k) \in M(2,\mathbb{C}), \tag{5.2}$$

then

$$\widehat{D}_1(k) e^{\pi i k \alpha} = V_1 \widehat{D}_1(k) V_1^{-1}. \tag{5.3}$$

If $\widehat{D}_1(k) \neq 0$ for some $k \neq 0$, then $e^{\pi i k \alpha}$ is an eigenvalue of

$$\mathrm{Ad}(V_1)\colon M(2,\mathbb{C}) \longrightarrow M(2,\mathbb{C}),$$

where

$$\mathrm{Ad}(V_1)\cdot F = V_1 F V_1^{-1} \quad \text{for } F \in M(2,\mathbb{C}).$$

This implies that V_1 is conjugate to some rotation R_θ with $2\theta = \pm\frac{k\alpha}{2} + \ell$ for some $\ell \in \mathbb{Z}$ (see Lemma 5.2 below), contradicting to our assumption. Thus we deduce that $D_1(x) = \widehat{D}_1(0)$ is a constant. Let $B_1(x) = B(x)$, $D = D_1$ and $V = V_1$. Then

$$VD = DV$$

by (5.1) and

$$B_1^{-1}(x+\alpha) A(x) B_1(x) = V.$$

Assume that V_1 is conjugate to some rotation R_θ with $2\theta = \frac{k\alpha}{2} + \ell$, where $k, \ell \in \mathbb{Z}$, i.e., $V_1 = U R_\theta U^{-1}$ with $U \in \mathrm{SL}(2,\mathbb{R})$. Let

$$B_1(x) = B(x) U R_{\frac{k}{4}x} U^{-1}.$$

Then

$$B_1(x+4) = B_1(x),$$

and

$$B_1(x+\alpha)^{-1} A(x) B_1(x) = \pm \mathrm{I}, \tag{5.4}$$

where I is the identity of 2×2 matrix.

Let

$$D_2(x) = B_1(x+1)^{-1} B_1(x).$$

As in (5.1), we have

$$D_2(x+\alpha) = D_2(x).$$

By the minimality of $x \mapsto x+\alpha$, D_2 is constant. Let

$$V = \pm \mathrm{I} \quad \text{and} \quad D = D_2.$$

Then

$$VD = DV$$

and

$$B_1^{-1}(x+\alpha)A(x)B_1(x) = V$$

by (5.4).

Step 2. Let

$$d = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathrm{I} - \varepsilon D)^{-1} \ln \lambda \, d\lambda$$

where Γ is a closed cure in complex plane, contains all spectra of εD and $0 \notin \Gamma$, and $\varepsilon \in \{-1, 1\}$ ($\varepsilon = 1$ if the spectra of D are positive, otherwise $\varepsilon = -1$). It is easy to check that $D = \varepsilon e^d$ and $d \in \mathrm{sl}(2, \mathbb{R})$ commutes with V and D, where $d \in \mathrm{sl}(2, \mathbb{R})$ means the trace of matrix d (denote $\mathrm{tr} d$) is 0. Let

$$B'(x) = B_1(x) e^{xd}.$$

then

$$B'(x+1)^{-1} B'(x) = \varepsilon \, \mathrm{I} \,,$$

i.e., $B' \colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2, \mathbb{R})$. Moreover, $B'(x+\alpha)^{-1} A(x) B'(x) = e^{-\alpha d} V$ is constant. □

Lemma 5.2. *If for some $k \in \mathbb{Z} \backslash \{0\}$ and 2×2 matrix $D \neq 0$, the following holds,*

$$D e^{\pi i k \alpha} = V D V^{-1}, \tag{5.5}$$

where V is a real constant cocycle. Then V is conjugate to a rotation R_θ with $2\theta = \pm \frac{k\alpha}{2} + \ell$ for some $\ell \in \mathbb{Z}$.

Proof. Without loss of generality, assume V is the form of

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{with } t \neq \pm 1 \text{ and } t \in \mathbb{R},$$

or

$$\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix} \quad \text{with } a \neq 0 \text{ and } a \in \mathbb{R},$$

or

$$\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \quad \text{with } \theta \in \mathbb{R},$$

since $\det V = 1$.

If

$$V = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}.$$

Write

$$D = (D_{ij})_{i,j=1,2},$$

by a simple computation in (5.5), we have

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} e^{\pi i k\alpha} = \begin{pmatrix} D_{11} & e^{4\pi i\theta} D_{12} \\ e^{-4\pi i\theta} D_{21} & D_{22} \end{pmatrix}. \tag{5.6}$$

Thus,

$$D_{11}, D_{22}, D_{21} = 0 \quad \text{and} \quad e^{4\pi i\theta} = e^{\pi i k\alpha}$$

and

$$D_{11}, D_{22}, D_{12} = 0 \quad \text{and} \quad e^{-4\pi i\theta} = e^{\pi i k\alpha}.$$

In either case

$$2\theta = \pm\frac{k\alpha}{2} + \ell \quad \text{for some } \ell \in \mathbb{Z}.$$

For

$$V = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{with } t \neq \pm 1,$$

or

$$V = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix} \quad \text{with } a \neq 0,$$

we can prove that those two cases can not happen by a similar discussion as the above. □

Remark 5.2. By Theorem 5.1, it does not matter whether $B\colon \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$ or $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ in the definition of reducibility. The basic idea of the proof in Theorem 5.1 is due to Avila and Krikorian [6], where they deal with another problem [6, Lemma 4.3].

Lemma 5.3. *Cocycle $A = S_{\lambda,E}$ can not be analytically reducible to* $\pm\mathrm{I}$.

Proof. Otherwise, without loss of generality, we can assume that there exists an analytic function

$$B\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

such that

$$B(x+\alpha)^{-1}A(x)B(x) = \mathrm{I}.$$

Since $B(x) \in \mathrm{PSL}(2,\mathbb{R})$,

$$B(x+1) = \pm B(x).$$

When $B(x+1) = B(x)$ the proof is simpler, see Remark 5.3. Here we give the proof only for $B(x+1) = -B(x)$. Since

$$B(x+\alpha)^{-1}A(x)B(x) = \mathrm{I},$$

it is easy to see that B must be with the form

$$B(x) = \begin{pmatrix} u_1(x) & u_2(x) \\ u_1(x-\alpha) & u_2(x-\alpha) \end{pmatrix}$$

and

$$(E - 2\lambda\cos 2\pi x)u_1(x) - u_1(x-\alpha) = u_1(x+\alpha),$$

$$(E - 2\lambda\cos 2\pi x)u_2(x) - u_2(x-\alpha) = u_2(x+\alpha).$$

By comparing the Fourier coefficients (note that both u_1 and u_2 are well defined in $\mathbb{R}/2\mathbb{Z}$), we obtain

$$(E - 2\cos(\pi k\alpha))\hat{u}_1(k) = \lambda(\hat{u}_1(k+2) + \hat{u}_1(k-2)), \tag{5.7}$$

$$(E - 2\cos(\pi k\alpha))\hat{u}_2(k) = \lambda(\hat{u}_2(k+2) + \hat{u}_2(k-2)), \tag{5.8}$$

where $\hat{u}_i(k)$ is the Fourier coefficients of u_i, $i = 1, 2$.

Let τ be a new self-adjoint operator on $\ell^2(\mathbb{Z})$, with

$$(\tau f)(k) = f(k+2) + f(k-2) + \frac{2}{\lambda}\cos(\pi k\alpha) f(k), \quad \text{for all } f \in \ell^2(\mathbb{Z}). \tag{5.9}$$

After a simple computation

$$\sum_{j=m}^{n} (f\tau g - g\tau f)(j) = W_n(f,g) - W_{m-1}(f,g), \tag{5.10}$$

where

$$\begin{aligned} W_n(f,g) = {} & f(n)g(n+2) + f(n-1)g(n+1) \\ & - g(n)f(n+2) - g(n-1)f(n+1). \end{aligned} \tag{5.11}$$

In (5.10), let

$$f = \{\hat{u}_1(k)\}_{k\in\mathbb{Z}} \quad \text{and} \quad g = \{\hat{u}_2(k)\}_{k\in\mathbb{Z}}.$$

Combining with (5.7) and (5.8), one has

$$W_n(\hat{u}_1, \hat{u}_2) = W_m(\hat{u}_1, \hat{u}_2). \tag{5.12}$$

Since u_i is analytic,

$$\lim_{n\to\infty} \hat{u}_i(n) = 0 \quad \text{for } i = 1, 2$$

and

$$\lim_{m\to\infty} W_m(\hat{u}_1, \hat{u}_2) = 0.$$

By (5.12),

$$\begin{aligned} W_n(\hat{u}_1, \hat{u}_2) &= \hat{u}_1(n)\hat{u}_2(n+2) + \hat{u}_1(n-1)\hat{u}_2(n+1) \\ &\quad - \hat{u}_2(n)\hat{u}_1(n+2) - \hat{u}_2(n-1)\hat{u}_1(n+1) \\ &= 0. \end{aligned} \tag{5.13}$$

Moreover, $\hat{u}_i(k) = 0$ for even k because of

$$u_i(x+1) = -u_i(x), \quad i = 1, 2.$$

In (5.13), let $n = 2k$, we have

$$\hat{u}_1(2k-1)\hat{u}_2(2k+1) - \hat{u}_2(2k-1)\hat{u}_1(2k+1) = 0. \tag{5.14}$$

This implies $\hat{u}_1$ and $\hat{u}_2$ are linear related, contradicting to $\det B = 1$. □

Remark 5.3. For another case $B(x+1) = B(x)$, i.e., $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{SL}(2,\mathbb{R})$, the proof is simper. We only need replace (5.9) with

$$(\tau f)(k) = f(k+1) + f(k-1) + \frac{2}{\lambda}\cos(2\pi k\alpha) f(k)$$

and (5.11) with

$$W_n(f,g) = f(n)g(n+1) - g(n)f(n+1).$$

If $E \in \Sigma_{\lambda,\alpha}$ such that $\theta(E)$ is not ϵ_0-resonant, by Remark 3.4, there exists a non-zero exponentially decaying solution of $\widehat{H}\hat{u} = E\hat{u}$. Next we will set up the reducibility of cocycle $A = S_{\lambda,E}$ via constructing reducible matrix.

Theorem 5.2. *Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\theta \in \mathbb{R}$ and $E \in \Sigma_{\lambda,\alpha}$, suppose there exists a non-zero exponentially decaying eigenfunction $\hat{u} = \{\hat{u}_k\}_{k\in\mathbb{Z}}$, i.e., $\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ with $|\hat{u}_k| \leq e^{-2\pi\eta|k|}$ for k large enough, then the following hold.*

(1) *If $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$, then there exists*

$$B\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

being analytic in $|\operatorname{Im} x| < \eta$, such that

$$B(x+\alpha)^{-1}A(x)B(x) = R_{\pm\theta},$$

i.e., (α, A) is analytically reducible in strip $|\operatorname{Im} x| < \eta$, where $A = S_{\lambda,E}$. In this case

$$\rho(\alpha, A) = \pm\theta + \frac{m}{2}\alpha \mod \mathbb{Z},$$

where[9] *$m = \deg(B)$ and $2\rho(\alpha, A) \notin \alpha\mathbb{Z} + \mathbb{Z}$.*

(1) *If $2\theta - k\alpha \in \mathbb{Z}$ for some $k \in \mathbb{Z}$ and $\eta > 8\beta(\alpha)$, then there exists*

$$B\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

being analytic in $|\operatorname{Im} x| < \frac{\eta}{4}$, such that

$$B(x+\alpha)^{-1}A(x)B(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix},$$

with $a \neq 0$, i.e., (α, A) is analytically reducible in strip $|\operatorname{Im} x| < \frac{\eta}{4}$. In this case

$$\rho(\alpha, A) = m\alpha \mod \mathbb{Z},$$

where $m = \deg(B)$, i.e., $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$.

[9] Since B is well defined in $\mathbb{R}/\mathbb{Z}$, $m = \deg(B)$ must be even.

Proof. Since $|\hat{u}_k| \le e^{-2\pi\eta|k|}$ for k large enough, $u(x) = \sum \hat{u}_k e^{2\pi i k x}$ is analytic in $|\operatorname{Im} x| < \eta$. Let

$$U(x) = \begin{pmatrix} e^{2\pi i\theta} u(x) \\ u(x-\alpha) \end{pmatrix}.$$

Then (see §2.4)

$$A(x) \cdot U(x) = e^{2\pi i\theta} U(x+\alpha). \tag{5.15}$$

Let $\widetilde{B}(x)$ be a matrix with columns $U(x)$ and $\bar{U}(x)$, i.e.,

$$\widetilde{B}(x) = (U(x), \bar{U}(x)).$$

Note that $\bar{U}(x)$ is given by footnote 6. By the minimality of $x \mapsto x+\alpha$ and (5.15), $\det \widetilde{B}$ is a constant.

Case A. If $\det \widetilde{B} \neq 0$, we have

$$\widetilde{B}(x+\alpha)^{-1} A(x) \widetilde{B}(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}.$$

It is easy to see that $\det \widetilde{B} = \pm ci$ for some $c > 0$. If we take

$$B = \frac{1}{(2c)^{1/2}} \widetilde{B} \begin{pmatrix} 1 & \pm i \\ 1 & \mp i \end{pmatrix},$$

then

$$B(x+\alpha)^{-1} A(x) B(x) = R_{\pm\theta},$$

and

$$\rho(\alpha, A) = \pm\theta + \frac{m}{2}\alpha \mod \mathbb{Z},$$

where $m = \deg(B)$.

Now we are in position to prove $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$. Otherwise, there exists some $k \in \mathbb{Z}$ such that $2\theta - k\alpha \in \mathbb{Z}$. Let $B'(x) = B(x) R_{\pm\frac{kx}{2}}$, we have

$$B'(x+\alpha)^{-1} A(x) B'(x) = \mathrm{I}$$

or

$$B'(x+\alpha)^{-1} A(x) B'(x) = -\mathrm{I}.$$

This is impossible by Lemma 5.3.

Case B. If $\det \widetilde{B} = 0$. By the minimality of $x \mapsto x + \alpha$ and (5.15), $U(x) \neq 0$ for all x with $|\operatorname{Im} x| < \eta$. Thus we have $U(x) = \psi(x) W(x)$ with $W(x+1) = \pm W(x)$ and $W(x)$ being real analytic in $|\operatorname{Im} x| < \eta$, and $|\psi(x)| = 1$ for $x \in \mathbb{R}$ (see Lemma 5.4 below).

There exists $\delta > 0$ such that $\|W(x)\| > \delta$ in $|\operatorname{Im} x| < \frac{\eta}{2}$, since $W(x) \neq 0$ for all x with $|\operatorname{Im} x| < \eta$. Let B_1 be given by Lemma 5.1 with first column W, then $B_1 : \mathbb{R}/2\mathbb{Z} \mapsto \mathrm{SL}(2, \mathbb{R})$ is analytic in $|\operatorname{Im} x| < \frac{\eta}{2}$, and

$$B_1(x+\alpha)^{-1} A(x) B_1(x) = \begin{pmatrix} d(x) & \kappa(x) \\ 0 & d(x)^{-1} \end{pmatrix},$$

with

$$d(x) = \frac{\psi(x+\alpha)}{\psi(x)} e^{2\pi i \theta}.$$

Since $|d(x)| = 1$ and $d(x)$ is real for $x \in \mathbb{R}$, $d(x) = \pm 1$, i.e.,

$$B_1(x+\alpha)^{-1} A(x) B_1(x) = \begin{pmatrix} \pm 1 & \kappa(x) \\ 0 & \pm \end{pmatrix}.$$

Moreover, $2\rho(\alpha, A) = m_1 \alpha \mod \mathbb{Z}$ since the degree of $\left(\begin{smallmatrix} \pm 1 & \kappa(x) \\ 0 & \pm 1 \end{smallmatrix}\right)$ is 0, where $m_1 = \deg(B_1)$.

If $\eta > 8\beta$, we can further conjugate A to a constant parabolic matrix by solving (comparing Fourier coefficients) the homological equation

$$\pm \phi(x+\alpha) \mp \phi(x) = \kappa(x) - \int_0^2 \kappa(x)\, dx$$

in $\mathbb{R}/2\mathbb{Z}$ with $\hat{\phi}_0 = 0$. More precisely,

$$\hat{\phi}_k = \mp \frac{\hat{\kappa}_k}{1 - e^{\pi i k \alpha}}, \quad k \neq 0,$$

thus ϕ is analytic in $|\operatorname{Im} x| < \frac{\eta}{4}$ because of $\kappa(x)$ being analytic in $|\operatorname{Im} x| < \frac{\eta}{2}$ and small divisor condition (3.1). Let

$$B_2(x) = B_1(x) \begin{pmatrix} 1 & \phi(x) \\ 0 & 1 \end{pmatrix}.$$

Thew get

$$B_2(x+\alpha)^{-1} A(x) B_2(x) = \begin{pmatrix} \pm 1 & \int_0^2 \kappa(x) dx \\ 0 & \pm 1 \end{pmatrix},$$

and B_2 is well defined in $\mathbb{R}/2\mathbb{Z}$.

By Theorem 5.1 (let $B = B_2$ in Theorem 5.1), there exists

$$B_3 \colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{PSL}(2, \mathbb{R})$$

such that $B_3(x+\alpha)^{-1}A(x)B_3(x)$ is a constant cocycle C. We will prove that C is conjugate to $\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$ with a a constant. Otherwise, C is conjugate to rotation $R_{\theta'}$ with $2\theta' \in \alpha\mathbb{Z} + \mathbb{Z}$ (since $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$), this is impossible by the discussion in Case A; or C is conjugate to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ with $t \neq \pm 1$, this is impossible since $E \in \Sigma_{\lambda,\alpha}$ ($S_{\lambda,E}$ is not uniformly hyperbolic[10] for $E \in \Sigma_{\lambda,\alpha}$, see [24]). Therefore, there exists a cocycle U such that

$$U^{-1}CU = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}.$$

Let $B(x) = B_3(x)U$, then

$$B(x+\alpha)^{-1}A(x)B(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}.$$

This implies that $2\rho(\alpha, A) = m\alpha \mod \mathbb{Z}$, where $m = \deg(B)$. Note that $a \neq 0$ by Lemma 5.3.

Now we prove that

$$2\theta = k\alpha \mod \mathbb{Z}.$$

Since

$$d = \pm 1 \quad \text{and} \quad d(x) = \frac{\psi(x+\alpha)}{\psi(x)} e^{2\pi i \theta},$$

we get

$$\psi(x+\alpha)e^{2\pi i\theta} = \pm\psi(x).$$

This implies (comparing Fourier coefficients) that $\psi(x) = e^{-\pi i k x}$ (note that ψ is well defined in $\mathbb{R}/2\mathbb{Z}$) and $e^{2\pi i\theta} = \pm e^{\pi i k \alpha}$ for some $k \in \mathbb{Z}$, that is $2\theta = k\alpha \mod \mathbb{Z}$.

Putting case A and B together, we finish the proof. $\square$

Remark 5.4. In above discussion, we have proven that if (α, A) is reducible and $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$, where $A = S_{\lambda,E}$ with $E \in \Sigma_{\lambda,\alpha}$, then (α, A) must be conjugate to $\begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$, with $a \neq 0$.

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[10] We say that the cocycle (α, A) is *uniformly hyperbolic* if there exist constants $c > 0$, $\gamma > 1$ such that $\|A_n(x)\| \geq c\gamma^n$ for every $x \in \mathbb{R}$ and $n > 0$.

Lemma 5.4. *Under the notation of Theorem 5.2, if* $\det \widetilde{B} = 0$*, we have*

$$U(x) = \psi(x)W(x)$$

with $W(x)$ *being real analytic in* $|\operatorname{Im} x| < \eta$ *and*

$$W(x+1) = \pm W(x),$$

and

$$|\psi(x)| = 1 \quad \text{for } x \in \mathbb{R}.$$

Proof. Let

$$U(z) = \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix},$$

$|\operatorname{Im} z| < \eta$. By condition $\det \widetilde{B} = 0$, then there exists $k(z)$ such that

$$u_1(z) = k(z)\bar{u}_1(z) \text{ and } u_2(z) = k(z)\bar{u}_2(z). \tag{5.16}$$

By minimality of $z \mapsto z + \alpha$ and (5.15), $U(z) \neq 0$ for $|\operatorname{Im} z| < \eta$. Thus $k(z) \neq 0$ for all $|\operatorname{Im} z| < \eta$. Moreover,

$$k(z) = \frac{u_1(z)}{\bar{u}_1(z)} \quad \text{or} \quad k(z) = \frac{u_2(z)}{\bar{u}_2(z)},$$

which implies $k(z)$ can be selected so that $k(z)$ is analytic in $|\operatorname{Im} z| < \eta$ and $|k(x)| = 1$ for $x \in \mathbb{R}$.

We will prove that there exists φ being analytic in $|\operatorname{Im} z| < \eta$ such that $\varphi^2 = k$ and φ is well defined in $\mathbb{R}/2\mathbb{Z}$ with $|\varphi(x)| = 1$ for $x \in \mathbb{R}$ (i.e., $\bar{\varphi}\varphi = 1$). Fix a point $z_0 \in \mathbb{R}$, and solve

$$p'(z) = \frac{k'(z)}{k(z)} \quad \text{with } p(z_0) = \ln k(z_0)$$

(selecting a branch). We have $p(z)$ is analytic in $|\operatorname{Im} z| < \eta$ and $e^{p(z)} = k(z)$. Let

$$\varphi(z) = e^{\frac{1}{2}p(z)}.$$

Then $\varphi^2 = k$. By the uniqueness theorem of analytic function in Complex Analysis, it's easy to verify φ is well defined in $\mathbb{R}/2\mathbb{Z}$ and $|\varphi(x)| = 1$ for $x \in \mathbb{R}$. Combining with (5.16), for $x \in \mathbb{R}$,

$$\bar{\varphi}(x)u_i(x) = \varphi(x)\overline{u_i}(x), \qquad i = 1, 2,$$

which implies both $\bar{\varphi}u_1$ and $\bar{\varphi}u_2$ are real analytic in $|\operatorname{Im} z| < \eta$. Letting

$$W = \begin{pmatrix} \bar{\varphi}u_1 \\ \bar{\varphi}u_2 \end{pmatrix} \quad \text{and} \quad \psi = \varphi,$$

we prove the lemma. □

Theorem 5.3. *For* $0 < \beta(\alpha) < \infty$ *and* $|\lambda| < e^{-C_2\beta}$, *let* $A = S_{\lambda,E}$ *with* $E \in \Sigma_{\lambda,\alpha}$. *If* $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$ *(i.e.,* $N_{\lambda,\alpha}(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ *by (2.10)), then there exists*

$$B\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{PSL}(2, \mathbb{R}),$$

analytically extending to $|\operatorname{Im} x| < \frac{h}{4}$, *such that*

$$B(x+\alpha)^{-1}A(x)B(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix},$$

with $a \neq 0$.

Proof. Let $E \in \Sigma_{\lambda,\alpha}$, we first prove that if $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$, then $\theta(E)$ given by Lemma 4.1 is not ϵ_0-resonant. Otherwise, by Theorem 4.7, there exists m_j such that

$$|m_j| < C|n_j|$$

and

$$\|2\rho(\alpha, A) - m_j\alpha \pm (2\theta - n_j\alpha)\|_{\mathbb{R}/\mathbb{Z}} < e^{-ch|n_{j+1}|}.$$

By (4.3),

$$\begin{aligned} \|2\rho(\alpha, A) - m_j\alpha\|_{\mathbb{R}/\mathbb{Z}} &\geq \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} - e^{-ch|n_{j+1}|} \\ &> e^{-8\beta|n_{j+1}|} - e^{-ch|n_{j+1}|} \\ &> 0, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \|2\rho(\alpha, A) - m_j\alpha\|_{\mathbb{R}/\mathbb{Z}} &\leq \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} + e^{-ch|n_{j+1}|} \\ &\leq e^{-\epsilon_0|n_j|} + e^{-ch|n_{j+1}|} \\ &\leq e^{-c\epsilon_0|m_j|}. \end{aligned} \tag{5.18}$$

It follows from (5.18) that $\rho(\alpha, A)$ has a $c\epsilon_0$-resonance at m_j if $|m_j|$ is large enough by Lemma 3.2. If the set of $c\epsilon_0$-resonance for $\rho(\alpha, A)$ is finite, i.e., $\{m_j\}$ is finite, by (5.17), there exists some $\delta > 0$ such that

$$\|2\rho(\alpha, A) - m_j\alpha\| > \delta \quad \text{for all } j,$$

which is contradicted to the fact

$$\|\rho(\alpha, A) - m_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \longrightarrow 0 \quad \text{as } j \to \infty$$

by the second inequality in (5.18). Thus $\rho(\alpha, A)$ is $c\epsilon_0$-resonant, this is impossible because of $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$.

Now that $\theta(E)$ is not ϵ_0-resonant, by Remark 3.4 there exists a non-zero exponentially decaying solution $\hat{u}$ of $\widehat{H}_{\lambda,\alpha,\theta}\hat{u} = E\hat{u}$ with $|u_k| \le e^{-2\pi h|k|}$ for $|k|$ large enough, where $h = C_1^2\beta$ by our hypothesis in the beginning of §4. Combining with Theorem 5.2, we finish the proof. $\square$

Theorem 5.4 ([4, Theorem 4.1]). *For $\beta(\alpha) = 0$ and $|\lambda| < 1$, let $A = S_{\lambda,E}$ with $E \in \Sigma_{\lambda,\alpha}$, there exists a small constant $c(\lambda,\alpha)$ such that, if $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$ then there exists $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ being analytic in $|\operatorname{Im} x| < c(\lambda,\alpha)$ such that $B(x+\alpha)^{-1}A(x)B(x)$ is constant.*

Remark 5.5. Avila and Jitomirskaya prove Theorem 5.4 only for $\alpha \in \mathrm{DC}$, in fact, their proof suggests it holds for all $\beta(\alpha) = 0$ (after carefully checking their proof).

6. Proof of the main theorem

Theorem 6.1. *If $E_0 \in \Sigma_{\lambda,\alpha}$ such that $2\rho(\alpha, A_{E_0}) \in \alpha\mathbb{Z} + \mathbb{Z}$, and (α, A_{E_0}) is analytically reducible in $|\operatorname{Im} x| < \eta$ with $\eta > 6\beta(\alpha)$ $(0 \le \beta(\alpha) < \infty)$, where $A_{E_0} = S_{\lambda,E_0}$, then E_0 is an endpoint of some gap.*

Proof. Here we only give the proof if $0 < \beta(\alpha) < \infty$. For α with $\beta(\alpha) = 0$, the proof is similar. Let

$$B\colon \mathbb{R}/\mathbb{Z} \longrightarrow \mathrm{PSL}(2,\mathbb{R})$$

be analytic in $|\operatorname{Im} x| < \eta$ such that $B(x+\alpha)^{-1}A_{E_0}(x)B(x)$ is a constant cocycle. Since $2\rho(\alpha, A_{E_0}) \in \alpha\mathbb{Z} + \mathbb{Z}$, combining with Remark 5.4, we have

$$B(x+\alpha)^{-1}A_{E_0}(x)B(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}, \tag{6.1}$$

with $a \neq 0$. Without loss of generality, assume

$$B(x+\alpha)^{-1}A_{E_0}(x)B(x) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \triangleq Z \quad \text{with } a < 0.$$

Writing $B = (B_{ij})_{i,j=1,2}$, one easily obtains

$$B_{21}(x+\alpha) = B_{11}(x) \quad \text{and} \quad B_{22}(x+\alpha) = B_{12}(x) - aB_{21}(x+\alpha). \tag{6.2}$$

Below, let $\varepsilon > 0$ be small. After carefully computing,

$$B(x+\alpha)^{-1}A_{E_0+\varepsilon}(x)B(x) = Z + \varepsilon P, \tag{6.3}$$

where

$$P = \begin{pmatrix} B_{11}B_{12} - aB_{11}^2 & -aB_{11}B_{12} + B_{12}^2 \\ -B_{11}^2 & -B_{11}B_{12} \end{pmatrix}. \tag{6.4}$$

We will prove that for an appropriate cocycle $B_1 : \mathbb{R}/\mathbb{Z} \mapsto \mathrm{SL}(2, \mathbb{R})$, one has

$$B_1(x + \alpha)^{-1}(Z + \varepsilon P(x))B_1(x) = Z + \varepsilon[P] + O(\varepsilon^2), \tag{6.5}$$

where $[\cdot]$ denotes the average of a matrix-valued function over $\mathbb{R}/\mathbb{Z}$. This can be done by a step KAM iteration (or averaging theory). Refer to [18]. Namely, we will look for a cocycle B_1 with the form of

$$B_1 = e^{\varepsilon Y}, \quad \text{where } Y : \mathbb{R}/\mathbb{Z} \mapsto \mathrm{sl}(2, \mathbb{R})$$

(i.e., $Y(x + 1) = Y(x)$ and $tr(Y(x)) = 0$). Clearly,

$$\begin{aligned} &B_1(x + \alpha)^{-1}(Z + \varepsilon P(x))B_1(x) \\ &\quad = (\mathrm{I} - \varepsilon Y(x + \alpha) + O(\varepsilon^2))(Z + \varepsilon P)(\mathrm{I} + \varepsilon Y + O(\varepsilon^2)) \\ &\quad = Z + \varepsilon(ZY(x) + P(x) - Y(x + \alpha)Z) + O(\varepsilon^2). \end{aligned} \tag{6.6}$$

Let

$$T(x) = Z^{-1}P(x) - \frac{\mathrm{tr}(Z^{-1}P)}{2}\mathrm{I}$$

and solve the homological equation

$$Y(x + \alpha)Z - ZY(x) = Z(T(x) - \widehat{T}(0)) \text{ in } \mathbb{R}/\mathbb{Z} \tag{6.7}$$

with $\widehat{Y}(0) = 0$. We get, for $k \neq 0$,

$$\widehat{Y}_{11}(k) = \frac{*}{(1 - e^{2\pi i k\alpha})^2},$$

$$\widehat{Y}_{12}(k) = \frac{*}{(1 - e^{2\pi i k\alpha})^3},$$

$$\widehat{Y}_{21}(k) = \frac{*}{(1 - e^{2\pi i k\alpha})},$$

$$\widehat{Y}_{22}(k) = \frac{*}{(1 - e^{2\pi i k\alpha})^2},$$

where $\widehat{Y}_{ij}(k)$ is the Fourier coefficients of matrix elements Y_{ij} of Y, $i, j = 1, 2$, and $*$ may be different. Using small divisor condition (3.1), Y is analytic if $\eta > 6\beta$. Since Y is a solution of the equation

$$Y(x + \alpha) - ZY(x)Z^{-1} = Z(T(x) - \widehat{T}(0))Z^{-1},$$

we deduce that

$$\operatorname{tr}(Y(x+\alpha)) - \operatorname{tr}(Y(x)) = \operatorname{tr}(T(x) - \widehat{T}(0)) = 0,$$

i.e., $\operatorname{tr} Y(x)$ is constant for $x \in \mathbb{R}/\mathbb{Z}$. Note that $\widehat{Y}(0) = 0$, then $\operatorname{tr}(Y(x)) = 0$ for $x \in \mathbb{R}/\mathbb{Z}$, i.e., $B_1 = e^{\varepsilon Y}$ is indeed a cocycle.

By (6.3) $\det(Z + \varepsilon P) = 1$, it is straightforward to compute

$$\operatorname{tr}(Z^{-1}P) = -\varepsilon \det P,$$

thus the coefficients of ε in (6.6) satisfy

$$ZY(x) + P(x) - Y(x+\alpha)Z = [P] + O(\varepsilon), \tag{6.8}$$

which implies (6.5).

Moreover,

$$Z + \varepsilon[P] + O(\varepsilon^2) = \exp(Z_0 + \varepsilon Z_1 + O(\varepsilon^2)), \tag{6.9}$$

where

$$Z_0 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z_1 = \begin{pmatrix} [B_{11}B_{12}] - \frac{a}{2}[B_{11}^2] & -a[B_{11}B_{12}] + [B_{12}^2] \\ -[B_{11}^2] & -[B_{11}B_{12}] + \frac{a}{2}[B_{11}^2] \end{pmatrix}. \tag{6.10}$$

Let

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & -d_1 \end{pmatrix} = Z_0 + \varepsilon Z_1,$$

whose determinant is

$$d = -d_1^2 - d_2 d_3 = a\varepsilon[B_{11}^2] + O(\varepsilon^2) < 0$$

for small $\varepsilon > 0$, since $[B_{11}^2] > 0$ (otherwise $B_{11} = 0$, by (6.2) $B_{21} = 0$, this is impossible). Now we let

$$F = \begin{pmatrix} d_2 & d_2 \\ -d_1 + \sqrt{-d} & -d_1 - \sqrt{-d} \end{pmatrix},$$

which has determinant $-2a\sqrt{-a\varepsilon[B_{11}^2]} + O(\varepsilon)$, then

$$\|F\| = O(1), \quad \|F^{-1}\| = O(\varepsilon^{-1/2}),$$

and

$$F^{-1}DF = \begin{pmatrix} \sqrt{-d} & 0 \\ 0 & -\sqrt{-d} \end{pmatrix} \triangleq H.$$

Moreover,

$$\begin{aligned}\exp(Z_0 + \varepsilon Z_1 + O(\varepsilon^2)) &= \exp(F(H + O(\varepsilon^{3/2}))F^{-1}) \\ &= F \exp(H + O(\varepsilon^{3/2}))F^{-1}.\end{aligned} \tag{6.11}$$

Note that

$$H + O(\varepsilon^{3/2}) = \sqrt{-a\varepsilon[B_{11}^2]}\left(\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} + O(\varepsilon)\right).$$

Therefore, if ε is small enough, the cocycle $A_{E_0+\varepsilon}$ has an exponential dichotomy (i.e., $A_{E_0+\varepsilon}$ is uniformly hyperbolic), which implies $E_0 + \varepsilon \notin \Sigma_{\lambda,\alpha}$, i.e., E_0 is an endpoint of some gap. □

Remark 6.1. In [30], Puig proves Theorem 6.1 for $\alpha \in \mathrm{DC}$, we extend his result to all α with $\beta(\alpha) < \infty$.

Combining with Avila and Jitomirskaya's work [4, 5], we give a summary of the dry version of the Ten Martini Problem.

Theorem 6.2. *For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, let $\beta(\alpha)$ be given by* (1.1)*, then the following statements hold*:

(1) *if $\beta(\alpha) = \infty$, then $\Sigma_{\lambda,\alpha}$ has all gaps open for all $\lambda \neq 0$*;

(2) *if $0 < \beta(\alpha) < \infty$, then $\Sigma_{\lambda,\alpha}$ has all gaps open for $0 < |\lambda| < e^{-C_2\beta}$, or $e^{-\beta} < |\lambda| < e^{\beta}$, or $|\lambda| > e^{C_2\beta}$, where C_2 is a large absolute constant*;

(3) *if $\beta(\alpha) = 0$, then $\Sigma_{\lambda,\alpha}$ has all gaps open if $\lambda \neq 0, -1, 1$.*

Proof. If $\beta(\alpha) = \infty$, this case has already been proved by Avila and Jitomirskaya [4, Theorem 8.2].

If $0 < \beta(\alpha) < \infty$, Avila and Jitomirskaya [4, Theorem 8.2] have proved that $\Sigma_{\lambda,\alpha}$ has all gaps open for $e^{-\beta} < |\lambda| < e^{\beta}$. Fix $\epsilon_0 = C_1\beta$, $h = C_1\epsilon_0$, where C_1 is a large absolute constant given in Theorem 3.1. Let C_2 be a large absolute constant also given in the beginning of §4. If $|\lambda| < e^{-C_2\beta}$, by Theorem 5.3, for any spectrum E_0 satisfying $N_{\lambda,\alpha}(E_0) \in \alpha\mathbb{Z}+\mathbb{Z}$, i.e., $2\rho(\alpha, A_{E_0}) \in \alpha\mathbb{Z}+\mathbb{Z}$, there exists a morphism $B\colon \mathbb{R}/\mathbb{Z} \mapsto \mathrm{PSL}(2,\mathbb{R})$ being analytic in $|\operatorname{Im} x| < \frac{h}{4}$ such that $B(x+\alpha)^{-1}A_{E_0}(x)B(x)$ is constant. Note that $\frac{h}{4} > 6\beta$, since C_1 is large. By Theorem 6.1, E_0 is an endpoint of some gap. For $|\lambda| > e^{C_2\beta}$, note that $\Sigma_{\lambda^{-1},\alpha} = \lambda^{-1}\Sigma_{\lambda,\alpha}$ and $N_{\lambda^{-1},\alpha}(\lambda^{-1}E) = N_{\lambda,\alpha}(E)$ (Aubry duality).

If $\beta(\alpha) = 0$, we only need replace Theorem 5.3 with Theorem 5.4. □

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