

Connectedness locus for pairs of affine maps and zeros of power series

Boris Solomyak¹

Abstract. We study the connectedness locus \mathcal{N} for the family of iterated function systems of pairs of affine-linear maps in the plane (the non-self-similar case). First results on the set \mathcal{N} were obtained in joint work with P. Shmerkin [11]. Here we establish rigorous bounds for the set \mathcal{N} based on the study of power series of special form. We also derive some bounds for the region of “*-transversality” which have applications to the computation of Hausdorff measure of the self-affine attractor. We prove that a large portion of the set \mathcal{N} is connected and locally connected, and conjecture that the entire connectedness locus is connected. We also prove that the set \mathcal{N} has many zero angle “cusp corners,” at certain points with algebraic coordinates.

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1. Introduction

Here we set the notation and discuss earlier results on the set \mathcal{N} . This section has some overlap with the introductory part of [11]. Let $E = E(T, \mathbf{b})$ be the attractor of the IFS $\{T\mathbf{x}, T\mathbf{x} + \mathbf{b}\}$, *i.e.*, the unique nonempty compact set in \mathbb{R}^d satisfying

$$E = TE \cup (TE + \mathbf{b}). \quad (1.1)$$

Observe that

$$E(T, \mathbf{b}) = \left\{ \sum_{n=0}^{\infty} a_n T^n \mathbf{b} : a_n \in \{0, 1\} \right\} \quad (1.2)$$

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since the right-hand side is well-defined (that is, the sums converge because T is a contraction, and the set is compact and non-empty) and satisfies (1.1).

We can assume that all the eigenvalues of T have spectral (geometric) multiplicity one, and \mathbf{b} is a cyclic vector for T , that is, $H := \text{Span}\{T^k \mathbf{b} : k \geq 0\} = \mathbb{R}^d$. There is no loss of generality in making this assumption, since otherwise we can replace T by the restriction of T to H and consider the corresponding IFS on H .

It is well-known (see [9]) that the set $E = E(T, \mathbf{b})$ is connected if and only if $TE \cap (TE + \mathbf{b}) \neq \emptyset$. This easily implies the following criterion for connectedness. Denote

$$\mathcal{B} = \left\{ 1 + \sum_{n=1}^{\infty} b_n z^n : b_n \in \{-1, 0, 1\} \right\}.$$

The symbol \mathbb{D} stands for the open unit disk.

Proposition 1.1 ([11]). *Let T be a linear contraction with (possibly complex) eigenvalues λ_j , for $j = 1, \dots, m$, having algebraic multiplicities $k_j \geq 1$, and geometric multiplicities equal to one. Let \mathbf{b} be a cyclic vector for T . Then $E(T, \mathbf{b})$ is connected if and only if there exists $f \in \mathcal{B}$ such that*

$$f(\lambda_j) = \dots = f^{(k_j-1)}(\lambda_j) = 0, \quad j = 1, \dots, m. \tag{1.3}$$

In particular, connectedness does not depend on \mathbf{b} .

From now on, we restrict ourselves to the case $d = 2$. Applying an invertible linear transformation as a conjugacy, we can assume without loss of generality that T is one of the following:

- (i)
$$T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$
- (ii)
$$T = \begin{bmatrix} \gamma & 0 \\ 0 & \lambda \end{bmatrix},$$
- (iii)
$$T = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where λ, γ, a, b are real, $|\lambda|, |\gamma| < 1$, and $a^2 + b^2 < 1$. Note that $\lambda \neq \gamma$ by the assumption that T has a cyclic vector. The following corollary is immediate from Proposition 1.1.

Corollary 1.2 ([11]). *Let $E(T, \mathbf{b})$ be the attractor of the IFS $\{T\mathbf{x}, T\mathbf{x} + \mathbf{b}\}$ where T is of the form (i), (ii), or (iii), and let \mathbf{b} be a cyclic vector for T .*

(a) *In the case (i), the self-affine set $E(T, \mathbf{b})$ is connected if and only if there exists $f \in \mathcal{B}$ such that $f(a + ib) = 0$.*

(b) *In the case (ii), the self-affine set $E(T, \mathbf{b})$ is connected if and only if there exists $f \in \mathcal{B}$ such that $f(\lambda) = f(\gamma) = 0$.*

(c) *In the case (iii), the self-affine set $E(T, \mathbf{b})$ is connected if and only if there exists $f \in \mathcal{B}$ such that $f(\lambda) = f'(\lambda) = 0$.*

Each of the cases leads to a set which we call the *connectedness locus* for the corresponding family of self-affine sets. Let

$$\mathcal{M} := \{z = a + ib \in \mathbb{D} : \text{there exists } f \in \mathcal{B}, f(z) = 0\},$$

$$\mathcal{N} := \{(\gamma, \lambda) \in (-1, 1)^2 : \text{there exists } f \in \mathcal{B}, f(\gamma) = f(\lambda) = 0\},$$

$$\mathcal{O} := \{\lambda \in (-1, 1) : \text{there exists } f \in \mathcal{B}, f(\lambda) = f'(\lambda) = 0\}.$$

Thus, \mathcal{M} , \mathcal{N} , and \mathcal{O} are essentially the sets of parameters for which the attractors in cases (i), (ii), (iii) are connected. The only difference is that we allow $b = 0$ in \mathcal{M} and $\gamma = \lambda$ in \mathcal{N} to ensure that these sets are relatively closed in the unit disk.

The set \mathcal{M} has been extensively studied as the *Mandelbrot set for the pair of linear maps*, see e.g. [4], [5], [1], [15], [14] and references therein.

Note that in case (i) the attractors are self-similar, which simplifies some of the considerations.

This paper is devoted to the study of the set \mathcal{N} , or rather, $\mathcal{N} \cap (0, 1)^2$. (By symmetry, we can assume that $\lambda > 0$. However, the case of $\gamma < 0$ does not reduce to the case of λ and γ having the same sign and we leave it for a future study.) It is easy to see ([11]) that

$$\{(\lambda, \gamma) \in [0.5, 1)^2 : \lambda\gamma \geq 0.5\} \subset \mathcal{N} \cap (0, 1)^2 \subset [0.5, 1)^2.$$

A picture of the set \mathcal{N} is shown in Figure 1 (which also appears in [11]). It is created by a program of Christoph Bandt, similar to the one used in [1] to draw the set \mathcal{M} . The set $\{(\lambda, \gamma) \in [0.5, 1)^2 : \lambda\gamma \geq 0.5\}$ is shaded gray. The algorithm rigorously checks that a point is outside \mathcal{N} and paints it “white.” The points that are not declared to be “white” after a certain number of iterations are declared to be in \mathcal{N} and painted “black.” Thus the figure should be viewed as an “outward approximation” for \mathcal{N} . However, this is not completely accurate; for instance, the apparent disconnected pieces of \mathcal{N} are a computing artifact, as we show below.

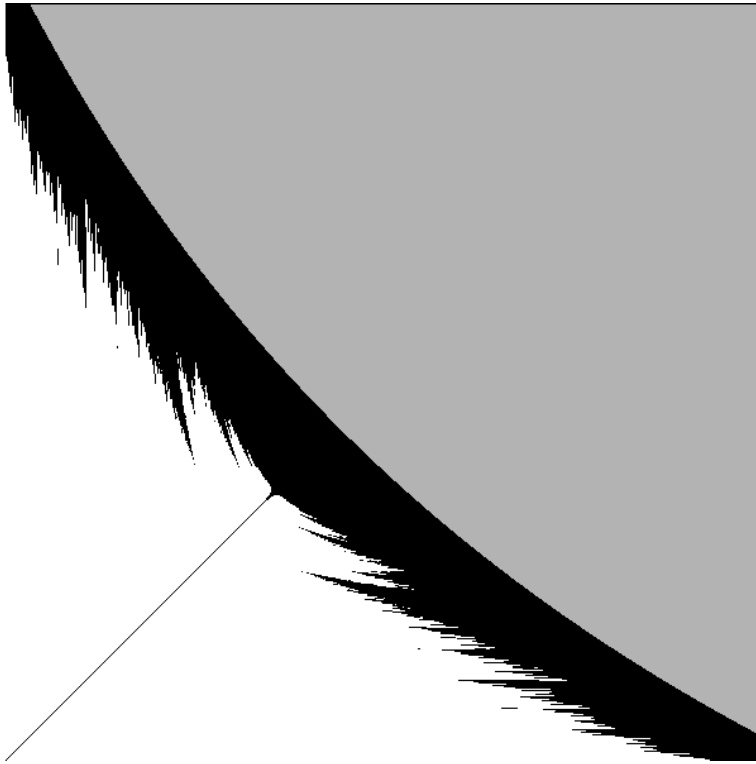


Figure 1. Connectedness locus for the family of self-affine sets.

Another remark is that the computation is very time-consuming near the diagonal, so the picture is not accurate there.

Next we recall for completeness the results on the set \mathcal{N} obtained in [11]. Denote

$$\text{Diag}(F) := \{(\lambda, \lambda) : \lambda \in F\}$$

for $F \subset \mathbb{R}$. We see that the set \mathcal{N} has an “antenna” $\Gamma(\mathcal{N})$, defined in [11] as the connected component of $\text{Diag}([\frac{1}{2}, 1]) \setminus \text{Clos}(\mathcal{N} \setminus \text{Diag}(\mathbb{R}))$ containing $(\frac{1}{2}, \frac{1}{2})$. In fact, we can consider the set of points on the diagonal which are limit points of $\mathcal{N} \setminus \text{Diag}(\mathbb{R})$. By definition, this set consists of those (λ, λ) for which there exist $f_n \in \mathcal{B}$ with real zeros $\gamma_n < \lambda_n$ such that $\gamma_n \rightarrow \lambda$, $\lambda_n \rightarrow \lambda$. By compactness of \mathcal{B} it follows that there exists $f \in \mathcal{B}$ with a double zero at λ , that is, $\lambda \in \mathcal{O}$. Conversely, a point (λ, λ) , where $\lambda \in (0, 2^{-1/2})$ is a double zero of some power series $f \in \mathcal{B}$, is in the closure of \mathcal{N} if f has infinitely many coefficients not equal to -1 , since we can then make an arbitrarily small *negative* perturbation of f staying in \mathcal{B} , which will result in a pair of real zeros close to λ . (Here we use the

fact that λ is necessarily a local minimum of f . It cannot be triple zero, since the smallest triple zero of $f \in \mathcal{B}$ is at least $0.72 > 2^{-1/2}$ by [3, Theorem 2]; see more about this in the next section.) It follows from [11] that $\text{Clos}(\mathcal{N} \setminus \text{Diag}(\mathbb{R})) \cap \text{Diag}(\mathbb{R})$ is disconnected, and it is conjectured to have infinitely many connected components. The “tip” of the antenna, that is, $(\beta, \beta) \in \Gamma$ such that β is maximal, is found in [11, Cor.2.10] with high accuracy: $\beta = .6684756 \pm 10^{-7}$.

Another interesting question concerns the topological structure of \mathcal{N} . By analogy with the Bandt’s conjecture from [1], we expect that $\mathcal{N} \setminus \text{Diag}(\mathbb{R})$ is contained in the closure of the set of its interior points. It is not obvious even that there exist interior points in the nontrivial part of $\mathcal{N} \setminus \mathcal{N}_t$, where

$$\mathcal{N}_t = \left\{ (\gamma, \lambda) \in (-1, 1)^2 : |\gamma||\lambda| \geq \frac{1}{2} \right\}.$$

However, in [11] it was shown that a small, but explicitly given, disk around $(2^{-1/2}, 2^{-1/2})$ is contained in \mathcal{N} .

2. Statement of results

There is another method, which does not involve much computing, to show that certain regions are disjoint from \mathcal{N} . It is based on the idea that it is much easier to estimate zeros of power series with “convex” restrictions on the coefficients and uses so-called $(*)$ -functions, first introduced in [12]. We will also need their generalizations from [3].

Definition 2.1. A power series

$$h(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$$

is called an $(m*)$ -function if there exist integers $1 \leq \ell_1 < \ell_2 < \dots < \ell_m < \infty$ such that a_{ℓ_k} are any real numbers for $k = 1, \dots, m$, and

$$\begin{aligned} a_n &= -1, & 1 \leq n \leq \ell_1 - 1, \\ a_n &= (-1)^k, & \ell_{k-1} + 1 \leq n \leq \ell_k - 1, \quad k = 2, \dots, m, \\ a_n &= (-1)^{m+1}, & n \geq \ell_m + 1. \end{aligned}$$

Moreover, we require that h has exactly $(m + 1)$ coefficient sign changes. (It is clear from the assumptions on a_n that the number of sign changes is at most $(m + 1)$, however, it could potentially be less, if for some j we have $\ell_{j+1} = \ell_j + 1$, $\ell_{j+2} = \ell_j + 2$.) A $(1*)$ -function will be called a $(*)$ -function, and a $(2*)$ -function will be called a $(**)$ -function.

Let

$$\begin{aligned} \mathcal{N}_+ &:= \mathcal{N} \cap \{(\gamma, \lambda) \in (0, 1)^2 : \gamma < \lambda\} \\ &= \{(\gamma, \lambda) : 0 < \gamma < \lambda < 1 \text{ and there is } f \in \mathcal{B}, f(\gamma) = f(\lambda) = 0\}. \end{aligned}$$

Further, consider

$$\mathcal{B}_{[-1,1]} := \left\{ 1 + \sum_{n=1}^{\infty} a_n z^n : a_n \in [-1, 1] \right\} \supset \mathcal{B}$$

and

$$\Omega_+ := \{(\gamma, \lambda) : 0 < \gamma < \lambda < 1 \text{ and there is } f \in \mathcal{B}_{[-1,1]}, f(\gamma) = f(\lambda) = 0\}.$$

By definition, $\mathcal{N}_+ \subset \Omega_+$. For a power series f with bounded real coefficients, let $\xi_1(f) \leq \xi_2(f) \leq \dots$ denote its positive zeros ordered by magnitude and counted with multiplicity (for convenience we let $\xi_k(f) = 1$ if there are fewer than k positive zeros). In [3, Theorem 3 and Section 2] it is proved that for any $k \geq 2$, the smallest k -th order zero α_k of a power series in $\mathcal{B}_{[-1,1]}$ is algebraic, and the corresponding power series is a (k^*) -function. In particular, $\alpha_2 \approx .649138$ is the positive zero of $2x^5 - 8x^2 + 11x - 4$.

Proposition 2.2. (a) *The function*

$$\phi : \gamma \mapsto \min\{\xi_2(f) : f \in \mathcal{B}_{[-1,1]}, f(\gamma) = 0\}$$

is well-defined on $(.5, \alpha_2)$. It is continuous, decreasing, and satisfies

$$\lim_{t \rightarrow .5^+} \phi(t) = 1, \quad \lim_{t \rightarrow \alpha_2^-} \phi(t) = \alpha_2.$$

(b) *For every $\gamma \in (.5, \alpha_2)$ we have*

$$\gamma < \lambda < \phi(\gamma) \implies (\gamma, \lambda) \notin \Omega_+.$$

(c) *For every $\gamma \in (.5, \alpha_2)$ there exists a unique function in $\mathcal{B}_{[-1,1]}$ which vanishes at γ and $\phi(\gamma)$. Moreover, it is a $(^*)$ -function*

$$h_k^{(a)}(x) = 1 - x - \dots - x^{k-1} + ax^k + \frac{x^{k+1}}{1-x} \in \mathcal{B}_{[-1,1]} \tag{2.1}$$

such that

$$h_k^{(a)}(\gamma) = h_k^{(a)}(\phi(\gamma)) = 0.$$

Moreover,

$$(h_k^{(a)})'(\gamma) < 0 \quad \text{and} \quad (h_k^{(a)})'(\phi(\gamma)) > 0.$$

The following table contains some values of the function ϕ (rounded-off in such a way that the actual values are slightly larger):

γ	.51	.52	.53	.54	.55	.56	.57	.58	.59
$\phi(\gamma)$.862	.831	.811	.79	.77	.755	.742	.728	.716

γ	.6	.61	.62	.63	.64
$\phi(\gamma)$.703	.691	.68	.67	.658

Note that the $(*)$ -function $h_k^{(a)}$ is in \mathcal{B} if and only if $a \in \{-1, 0, 1\}$. Thus, we get a countable set of points which belong to $\partial\Omega_+ \cap \mathcal{N}_+$. It turns out that the set \mathcal{N} has “cusp corners” at these points, as we show in our first main theorem. This property distinguishes \mathcal{N} from the set \mathcal{M} , which has spiral points and no corners with interior angle less than $2\pi/3$ (conjecturally, none at all), see [16].

Theorem 2.3. *Suppose that $\frac{1}{2} < \gamma_0 < \lambda_0 < 1$ and (γ_0, λ_0) is such that there is a unique function $h \in \mathcal{B}$ which vanishes at γ_0 and λ_0 , and moreover, all the coefficients of h are eventually $+1$ and $h'(\gamma_0) < 0$, $h'(\lambda_0) > 0$. Then $z_0 = (\gamma_0, \lambda_0)$ is a “tip of a corner” of the set \mathcal{N}_+ , with zero interior angle. More precisely, there exist $\delta > 0$ and positive constants C_1 and C_2 , such that*

$$B_\delta(z_0) \cap \mathcal{N}_+ \subset \{(\gamma, \lambda) : C_1(\gamma_0 - \gamma)^\alpha < \lambda - \lambda_0 < C_2(\gamma_0 - \gamma)^\alpha\}$$

where $\alpha = \frac{\log \lambda_0}{\log \gamma_0}$. In fact, we can take

$$C_1 = \frac{2|h'(\gamma_0)|^\alpha(1 - \gamma_0)^\alpha}{2^\alpha 3h'(\lambda_0)} \quad \text{and} \quad C_2 = \frac{3^\alpha 2|h'(\gamma_0)|^\alpha}{2^\alpha(1 - \lambda_0)h'(\lambda_0)}.$$

In particular, these conditions are satisfied if h is a $(*)$ -function, i.e.

$$h(x) = 1 - x - \dots - x^{k-1} + ax^k + \frac{x^{k+1}}{(1-x)}$$

for some $k \geq 1$ and $a \in \{-1, 0, 1\}$.

Remarks. 1. Note that all the points described in the theorem are algebraic, since the function h is rational over \mathbb{Z} . The first “tips of the corners” to which the theorem applies, for an appropriate $(*)$ -function, are as follows (given with 5-6 digit accuracy):

- (0.618034, 0.68232), which is a pair of zeros of

$$1 - x - x^2 - x^3 + \frac{x^5}{(1-x)}$$

(incidentally, the reciprocals of this pair are the golden ratio and the 4th Pisot number);

- (0.550607, 0.7691), which is a pair of zeros of

$$1 - x - x^2 - x^3 - x^4 + \frac{x^5}{(1-x)};$$

- (0.532958, 0.804916), which is a pair of zeros of

$$1 - x - x^2 - x^3 - x^4 + \frac{x^6}{(1-x)};$$

- (0.519703, 0.83221), which is a pair of zeros of

$$1 - x - x^2 - x^3 - x^4 - x^5 + \frac{x^6}{(1-x)};$$

- (0.513951, 0.85068), which is a pair of zeros of

$$1 - x - x^2 - x^3 - x^4 - x^5 + \frac{x^7}{(1-x)}.$$

2. The “cusp corners” obtained from (*)-functions are only the “most outward” cusp corners of \mathcal{N}_+ . There are many others visible in Figures 1 and 2, which are probably pairs of zeros of power series $h \in \mathcal{B}$ with all but finitely many coefficients equal to 1, as in Theorem 2.3. For instance, it appears that there is a corner at (0.645200, 0.68232) (with the second zero again the reciprocal of the 4th Pisot number), which is a pair of zeros of $h(x) = 1 - x - x^2 - x^3 + x^4 + x^6 + x^8 / (1 - x)$. In order to prove this rigorously, one only needs to check that h is the unique function in \mathcal{B} with this pair of zeros, but we haven’t done this.

Our second main result is concerned with connectedness properties of the set \mathcal{N} . Let

$$\widetilde{\mathcal{N}}_+ := \{(\gamma, \lambda) \in \mathcal{N}_+ : \text{there exists } f \in \mathcal{B}, f(\gamma) = f(\lambda) = 0, \xi_3(f) \leq \lambda\}.$$

Recall that $\mathcal{N}_t = \{(\gamma, \lambda) \in (-1, 1)^2 : |\gamma\lambda| \geq \frac{1}{2}\}$ is the “trivial” part of \mathcal{N} .



Figure 2. The set $\mathcal{N}_+ \cap [0.647, 0.661] \times [0.677, 0.691]$, with several prominent “cusp corners”.

Theorem 2.4. *The set $\mathcal{N}_+ \setminus (\tilde{\mathcal{N}}_+ \cup \mathcal{N}_t)$ is locally connected. Moreover, there is no connected component of \mathcal{N}_+ that is disjoint from $\tilde{\mathcal{N}}_+ \cup \mathcal{N}_t$.*

We were not able to prove the connectedness of the entire set \mathcal{N} , but conjecture that this is the case. The next proposition shows that the last theorem is non-vacuous, in fact, $\mathcal{N}_+ \setminus (\tilde{\mathcal{N}}_+ \cup \mathcal{N}_t)$ contains a substantial portion of the set $\mathcal{N}_+ \setminus \mathcal{N}_t$. In particular, the set \mathcal{N} is connected near the “cusp corners” from Theorem 2.3. Let

$$\tilde{\Omega}_+ := \{(\gamma, \lambda) \in \Omega_+ : \text{there exists } f \in \mathcal{B}_{[-1,1]}, f(\gamma) = f(\lambda) = 0, \xi_3(f) \leq \lambda\}.$$

Clearly, $\tilde{\mathcal{N}}_+ \subset \tilde{\Omega}_+$. Recall that α_3 denotes the smallest triple zero of a power series in $\mathcal{B}_{[-1,1]}$; in [3, Section 2] it is shown that $\alpha_3 \approx .727883$ is a zero of a polynomial with integer coefficients of degree 12.

Proposition 2.5. (a) *The function*

$$\psi : \gamma \mapsto \min\{\xi_3(f) : f \in \mathcal{B}_{[-1,1]}, f(\gamma) = 0\}$$

is well-defined on $(.5, \alpha_3)$. It is continuous, decreasing, and satisfies

$$\lim_{t \rightarrow .5^+} \psi(t) = 1, \quad \lim_{t \rightarrow \alpha_3^-} \psi(t) = \alpha_3.$$

(b) *For every $\gamma \in (.5, \alpha_3)$ we have*

$$\gamma < \lambda < \psi(\gamma) \implies (\gamma, \lambda) \notin \tilde{\Omega}_+.$$

(c) *For every $\gamma \in (.5, \alpha_3)$ there exists a unique $(**)$ -function*

$$H_{k,\ell}^{(a,b)}(x) = 1 - \sum_{i=1}^{k-1} x^i + ax^k + \sum_{i=k+1}^{\ell-1} x^i + bx^\ell - \frac{x^{\ell+1}}{1-x} \in \mathcal{B}_{[-1,1]} \quad (2.2)$$

such that

$$H_{k,\ell}^{(a,b)}(\gamma) = H_{k,\ell}^{(a,b)}(\psi(\gamma)) = (H_{k,\ell}^{(a,b)})'(\psi(\gamma)) = 0.$$

The following table contains some values of the function ψ (rounded-off in such a way that the actual values of ψ are slightly larger):

γ	.53	.55	.57	.59	.61	.63
$\psi(\gamma)$.877	.85	.832	.815	.799	.785
γ	.65	.67	.69	.71	.7278	
$\psi(\gamma)$.771	.759	.747	.736	.7278	

Propositions 2.2 and 2.5 are illustrated in Figure 3, which shows the region obtained from the tables. For (γ, λ) in the region below the lower broken line, the corresponding self-affine set is totally disconnected. Theorem 2.4 implies that the part of \mathcal{N}_+ between the broken lines and outside \mathcal{N}_t is locally connected and there are no components of \mathcal{N} entirely contained between the broken lines and the diagonal. We also show several points which are known to belong to \mathcal{N}_+ (these are some of the ‘‘cusp corners’’ from Theorem 2.3).

Another application of the bounds on the set $\tilde{\mathcal{N}}_+$ comes from the paper by P. Shmerkin [10]. Following [10, Definition 4.10], we say that \mathcal{R} is a region of $*$ -transversality if for all $(\gamma, \lambda) \in \mathcal{R}$ there is $f \in \mathcal{B}$ such that $f(\gamma) = f(\lambda) = 0$ but $f'(\gamma) \neq 0, f'(\lambda) \neq 0$. It is clear $\mathcal{N}_+ \setminus \tilde{\mathcal{N}}_+$ is a region of $*$ -transversality. It is proved in [10] that for Lebesgue-a.e. (λ, γ) with $\gamma\lambda < 1/2 < \gamma$ the self-affine attractor $K_{\gamma,\lambda}$ has Hausdorff dimension $1 + \log(2\lambda)/\log(1/\gamma)$, and if \mathcal{R} is a region of $*$ -transversality contained in $\{(\gamma, \lambda) : \gamma\lambda < 1/2\}$, then $K_{\gamma,\lambda}$ has zero Hausdorff measure in its dimension for Lebesgue-a.e. $(\gamma, \lambda) \in \mathcal{R}$.

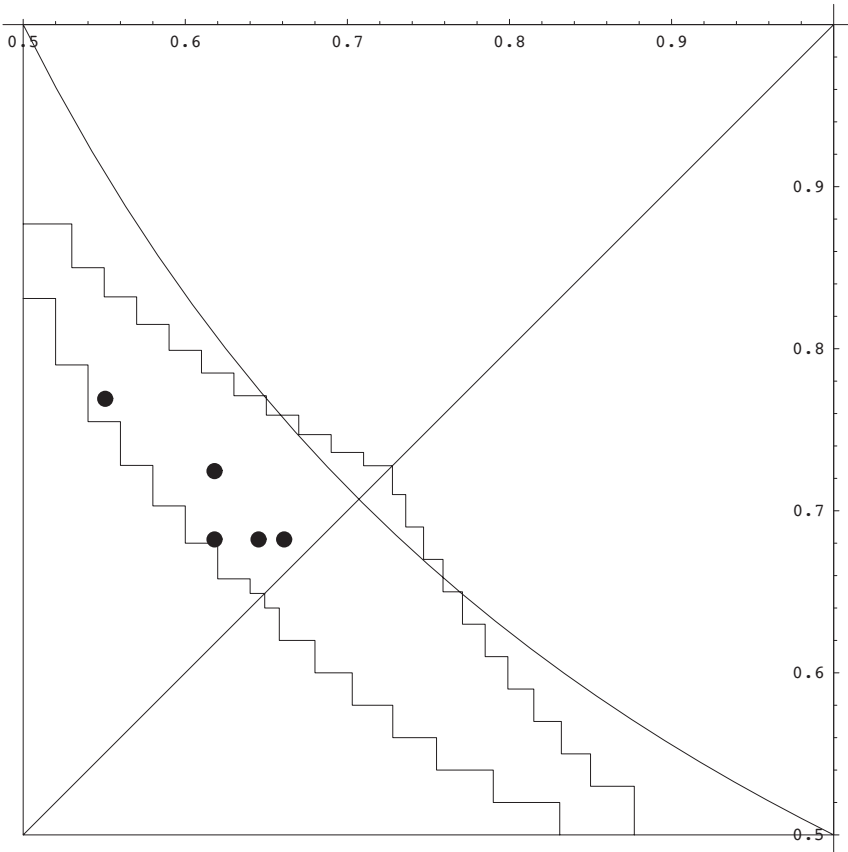


Figure 3

3. Proofs

Proof of Proposition 2.2. Consider the family of $(*)$ -functions $h_k^{(a)}$ with $k \geq 1$ and $a \in [-1, 1]$, given by (2.1), and equip it with a total order as follows:

$$h_k^{(u)} > h_\ell^{(v)} \quad \text{if } k < \ell \text{ or } k = \ell, u > v.$$

Obviously,

$$h_k^{(u)} > h_\ell^{(v)} \implies h_k^{(u)}(x) > h_\ell^{(v)}(x) \quad \text{for all } x \in (0, 1).$$

The set $\mathcal{B}_{[-1,1]}$ is a normal family of analytic functions in the unit disk; therefore, it is compact in the uniform topology on any compact subset of $(0, 1)$. We can also identify $\mathcal{B}_{[-1,1]}$ with the infinite product $[-1, 1]^\infty$ equipped with the product

topology. Observe that $u \mapsto h_k^{(u)}$ is a continuous function from $[-1, 1]$ to $\mathcal{B}_{[-1,1]}$ for all $k \geq 1$. It is strictly decreasing in the order defined above, and moreover,

$$h_k^{(-1)} = h_{k+1}^{(1)} \quad \text{for all } k \geq 1.$$

By [12, Section 3] and [3, Section 2], there is a $(*)$ -function $h_4^{(b)}$ having a double zero at α_2 , with $b \approx .0875294$.

It is easy to see that every $h_k^{(u)} < h_4^{(b)}$ has exactly two distinct positive zeros. Indeed, since $h_k^{(u)}(0) = 1$, $\lim_{x \rightarrow 1^-} h_k^{(u)}(x) = +\infty$, and $h_k^{(u)}(\alpha_2) < h_4^{(b)}(\alpha_2) = 0$, $h_k^{(u)}$ has at least two positive zeros. On the other hand, the derivative $(h_k^{(u)})'(x)$ has only one coefficient sign change, so it has at most one positive zero by the Descartes Rule of Signs, hence $h_k^{(u)}$ has at most two positive zeros. Clearly, the zeros (when they exist) continuously depend on u .

Claim 1. *For every $\gamma \in (0.5, \alpha_2)$ there exists a $(*)$ -function $h_k^{(u)}$ such that*

$$h_k^{(u)}(\gamma) = 0.$$

Indeed, $h_4^{(b)}(\gamma) > 0$, and for k sufficiently large we have $h_k^{(1)}(\gamma) < 0$, since $\lim_{k \rightarrow \infty} h_k^{(1)}(\gamma) = 1 - \gamma - \gamma^2 - \dots = \frac{1-2\gamma}{1-\gamma}$. By continuity, there exist k and u such that $h_k^{(u)}(\gamma) = 0$, and of course, $h_k^{(u)} < h_4^{(b)}$. Since $h_k^{(u)}(\alpha_2) < 0$, there is another zero $\lambda > \alpha_2 > \gamma$. The claim is proved.

Claim 2. *We have*

$$\phi(\gamma) = \lambda,$$

and $h_k^{(u)}$ is the unique function in $\mathcal{B}_{[-1,1]}$ with zeros at γ and λ .

Indeed, suppose $f \in \mathcal{B}_{[-1,1]}$ is such that $f(\gamma) = 0$ and $f \not\equiv h_k^{(u)}$. Consider $g(x) = f(x) - h_k^{(u)}(x)$. Then $g(x)$ is a power series with at most one coefficient sign change and $g(\gamma) = 0$. It follows that γ is the only positive zero of g . The first nonzero coefficient of g is positive, so it is positive for small positive x . It follows that $g(x) < 0$ for all $x > \gamma$, hence $f(x) = g(x) + h_k^{(u)}(x) < 0$ for all $x \in (\gamma, \lambda]$. So, for all $f \not\equiv h_k^{(u)} \in \mathcal{B}_{[-1,1]}$, $\xi_2(f) > \lambda$, and the claim is proved.

The claims show that the function ϕ is well-defined on $(\frac{1}{2}, \alpha_2)$. The remaining statements of part (a) are now easy to derive. In fact, one can obtain sharp asymptotics for $\phi(t)$ as $t \rightarrow \frac{1}{2}+$ and $t \rightarrow 1-$, but we do not pursue this.

(b) This statement is immediate from the definition of ϕ .

(c) The formula for the “optimal” function and its uniqueness are already proved. The statement about the derivative of $h_k^{(a)}$ is also clear: as already mentioned, $(h_k^{(a)})'$ has only one sign change and its zero (which the minimum of $h_k^{(a)}$) must lie in $(\gamma, \phi(\gamma))$. \square

The table of ϕ values is obtained from Claims 1 and 2 above. See the [appendix](#) for details.

Proof of Theorem 2.3. Suppose that $z_n = (\gamma_n, \lambda_n) \in \mathcal{N}_+$ are such that

$$z_n \mapsto z_0 = (\gamma_0, \lambda_0).$$

Consider functions (maybe non-unique) $h_n \in \mathcal{B}$ such that

$$h_n(\gamma_n) = h_n(\lambda_n) = 0.$$

Since \mathcal{B} is compact, there is a subsequence of h_n converging to some $\tilde{h} \in \mathcal{B}$, with $\tilde{h}(\gamma_0) = \tilde{h}(\lambda_0) = 0$. By the assumption of uniqueness of such a function, we have $\tilde{h} = h$. Since convergence in \mathcal{B} is coefficientwise, it follows that for any $N \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that h_n agrees with h in the first N terms for all $n \geq n_0$.

This already implies that z_0 is a “corner” with interior angle at most $\pi/2$. Indeed, if h_n agrees with h in the first $N \geq N_0$ terms, where $N_0 - 1$ is the last term of h with a coefficient different from $+1$, then $h - h_n$ has only non-negative coefficients and hence $h_n(x) < h(x)$ for all $x \in (0, 1)$. Since $h(x) \leq 0$ for $x \in [\gamma_0, \lambda_0]$ we obtain that the zeros of h_n must satisfy $\gamma_n < \gamma_0$, $\lambda_n > \lambda_0$. For the more delicate estimate we need the following lemma:

Lemma 3.1. *Suppose that $\frac{1}{2} < \gamma_0 < \lambda_0 < 1$ are such that γ_0 and λ_0 are zeros of $h \in \mathcal{B}$, as in the statement of Theorem 2.3, and let $f(x) = h(x) - x^N R(x)$, where R is a power series with coefficients 0, 1. Then for N sufficiently large, f has zeros $\tilde{\gamma}$ and $\tilde{\lambda}$ satisfying*

$$\frac{2\gamma_0^N R(\gamma_0)}{3|h'(\gamma_0)|} \leq \gamma_0 - \tilde{\gamma} \leq \frac{2\gamma_0^N R(\gamma_0)}{|h'(\gamma_0)|},$$

and

$$\frac{2\lambda_0^N R(\lambda_0)}{3h'(\lambda_0)} \leq \tilde{\lambda} - \lambda_0 \leq \frac{2\lambda_0^N R(\lambda_0)}{h'(\lambda_0)}.$$

Recall that $h'(\gamma_0) < 0$ and $h'(\lambda_0) > 0$ by assumption.

First we deduce the theorem, assuming the lemma. The argument at the beginning of the proof shows that for any $N \in \mathbb{N}$ there exists $\delta > 0$ such that for all $z = (\gamma, \lambda) \in \mathcal{N}_+ \cap B_\delta(z_0)$ (where $z_0 = (\gamma_0, \lambda_0)$), if $f \in \mathcal{B}$ is such that $f(\gamma) = f(\lambda) = 0$, then f agrees with h in the first N coefficients. (Note that $z \in \mathcal{N}_+$ implies there does indeed exist $f \in \mathcal{B}$ such that $f(\gamma) = f(\lambda) = 0$.) Let $N_0 \in \mathbb{N}$ be such that h has only coefficients equal to $+1$ starting from N_0 . Let $\delta > 0$ be so small that $N \geq N_0$. Then $f(z) = h(z) - z^N R(z)$ for some power series R with coefficients $0, 1$ and we obtain from Lemma 3.1:

$$\frac{2\gamma_0^N R(\gamma_0)}{3|h'(\gamma_0)|} \leq \gamma_0 - \gamma \leq \frac{2\gamma_0^N R(\gamma_0)}{|h'(\gamma_0)|},$$

and

$$\frac{2\lambda_0^N R(\lambda_0)}{3h'(\lambda_0)} \leq \lambda - \lambda_0 \leq \frac{2\lambda_0^N R(\lambda_0)}{h'(\lambda_0)}.$$

Let

$$\alpha = \frac{\log \lambda_0}{\log \gamma_0}.$$

Note that $\gamma_0^\alpha = \lambda_0$, so

$$\left(\frac{2}{3}\right)^\alpha \frac{\lambda_0^N R(\gamma_0)^\alpha}{|h'(\gamma_0)|^\alpha} \leq (\gamma_0 - \tilde{\gamma})^\alpha \leq \frac{2^\alpha \lambda_0^N R(\gamma_0)^\alpha}{|h'(\gamma_0)|^\alpha}.$$

Thus we have that

$$\frac{2R(\lambda_0)|h'(\gamma_0)|^\alpha}{2^\alpha 3R(\gamma_0)^\alpha h'(\lambda_0)} \leq \frac{\tilde{\lambda} - \lambda_0}{(\gamma_0 - \tilde{\gamma})^\alpha} \leq \frac{3^\alpha 2R(\lambda_0)|h'(\gamma_0)|^\alpha}{2^\alpha R(\gamma_0)^\alpha h'(\lambda_0)}.$$

Now,

$$1 \leq R(\gamma_0) \leq \frac{1}{1 - \gamma_0}$$

and

$$1 \leq R(\lambda_0) \leq \frac{1}{1 - \lambda_0},$$

whence

$$\frac{2|h'(\gamma_0)|^\alpha (1 - \gamma_0)^\alpha}{2^\alpha 3h'(\lambda_0)} \leq \frac{\tilde{\lambda} - \lambda_0}{(\gamma_0 - \tilde{\gamma})^\alpha} \leq \frac{3^\alpha 2|h'(\gamma_0)|^\alpha}{2^\alpha (1 - \lambda_0)h'(\lambda_0)},$$

as desired. The claim that the conditions on h are satisfied whenever h is a $(*)$ -function is immediate from definitions and Proposition 2.2. \square

Proof of Lemma 3.1. This is standard, but we provide the argument for completeness. We will only prove the estimate for $\tilde{\lambda}$, since the one for $\tilde{\gamma}$ is obtained in exactly the same way. We will need an easy inequality:

$$|g''(x)| \leq 2(1-x)^{-3} \quad \text{for all } g \in \mathcal{B} \text{ and } x \in (0, 1). \tag{3.1}$$

Recall that $h(\lambda_0) = 0$ and $h'(\lambda_0) > 0$, so $h(x) > 0$ to the right of λ_0 . Thus, it is clear that for large N there will be a zero of $f(x) = h(x) - x^N R(x)$ in a small neighborhood $(\lambda_0, \lambda_0 + t]$. Since the claim is local, we can assume that $\lambda_0 + t \leq 1 - \delta < 1$ for some $\delta > 0$ (independent of N , e.g. we can take $\delta = (1 - \lambda_0)/2$).

We have

$$f(\lambda_0 + t) = h(\lambda_0 + t) - (\lambda_0 + t)^N R(\lambda_0 + t).$$

Recall that $f(\lambda_0) < 0$, and we want to make sure that $f(\lambda_0 + t) \geq 0$. By Taylor's formula,

$$h(\lambda_0 + t) \geq h'(\lambda_0)t - \frac{C_2 t^2}{2},$$

where, in view of (3.1),

$$C_2 := 2(1 - \delta)^{-3} \geq \max\{|h''(x)| : x \in [\lambda_0, \lambda_0 + t]\}.$$

We can assume that N is large enough, so that

$$t := \frac{4\delta^{-1}\lambda_0^N}{h'(\lambda_0)} < \frac{4h'(\lambda_0)}{C_2}.$$

Then

$$h'(\lambda_0)t - \frac{1}{2}C_2 t^2 > \frac{1}{2}h'(\lambda_0)t.$$

We claim that

$$\frac{1}{2}h'(\lambda_0)t \geq (\lambda_0 + t)^N R(\lambda_0 + t)$$

for N sufficiently large. By the definition of t ,

$$\begin{aligned} (\lambda_0 + t)^N R(\lambda_0 + t) &\leq \left(\lambda_0 + \frac{4\delta^{-1}\lambda_0^N}{h'(\lambda_0)}\right)^N R(1 - \delta) \\ &\leq \lambda_0^N \left(1 + \frac{4\delta^{-1}\lambda_0^{N-1}}{h'(\lambda_0)}\right)^N \delta^{-1}. \end{aligned}$$

Since

$$\lim_{N \rightarrow \infty} (1 + 4\delta^{-1}\lambda_0^{N-1}/h'(\lambda_0))^N = 1,$$

we conclude that

$$(\lambda_0 + t)^N R(\lambda_0 + t) \leq 2\delta^{-1}\lambda_0^N = \frac{1}{2}h'(\lambda_0)t$$

for N sufficiently large, as desired. Thus we have shown that f has a zero $\tilde{\lambda} \in (\lambda_0, \lambda_0 + t]$, where $t = 4\delta^{-1}\lambda_0^N/h'(\lambda_0)$ and N is large enough.

By the Mean Value Theorem, there exists $c \in (\lambda_0, \tilde{\lambda})$ such that

$$\lambda_0^N R(\lambda_0) = f(\tilde{\lambda}) - f(\lambda_0) = (\tilde{\lambda} - \lambda_0)f'(c),$$

so that

$$\tilde{\lambda} - \lambda_0 = \frac{\lambda_0^N R(\lambda_0)}{f'(c)}. \quad (3.2)$$

In view of (3.1),

$$|f'(c) - f'(\lambda_0)| \leq |c - \lambda_0| \cdot 2\delta^{-3} < t \cdot 2\delta^{-3} = \frac{8\delta^{-4}\lambda_0^N}{h'(\lambda_0)}.$$

Thus we can choose N sufficiently large, so that

$$|f'(c) - f'(\lambda_0)| < \frac{1}{4}h'(\lambda_0).$$

Next,

$$|f'(\lambda_0) - h'(\lambda_0)| = |\lambda_0^N R'(\lambda_0) + N\lambda_0^{N-1}R(\lambda_0)| \leq \lambda_0^N \delta^{-2} + N\lambda_0^{N-1}\delta^{-1},$$

which is also less than $\frac{1}{4}h'(\lambda_0)$ for N sufficiently large. Then

$$|f'(c) - h'(\lambda_0)| < \frac{1}{2}h'(\lambda_0),$$

whence $f'(c) \in (\frac{1}{2}h'(\lambda_0), \frac{3}{2}h'(\lambda_0))$, and so we have from (3.2):

$$\frac{2\lambda_0^N R(\lambda_0)}{3h'(\lambda_0)} \leq \frac{\lambda_0^N R(\lambda_0)}{f'(c)} = \tilde{\lambda} - \lambda_0 \leq \frac{2\lambda_0^N R(\lambda_0)}{h'(\lambda_0)},$$

as desired. □

Proof of Theorem 2.4. This proof is a modification of the argument by Bandt [1, Section 11], which is, in turn, based on [5]. Let

$$\Delta := \{(\gamma, \lambda) \in (0, 1)^2 : \gamma < \lambda, \gamma\lambda < 1/2\}$$

and consider the quotient space

$$\mathcal{X} := \text{Clos}(\Delta \setminus \widetilde{\mathcal{N}}_+) / \partial(\Delta \setminus \widetilde{\mathcal{N}}_+) \tag{3.3}$$

with induced topology. Denote by ω the point corresponding to the contracted boundary.

Recall that \mathcal{B} is the set of all power series of the form

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n,$$

with $a_n \in \{-1, 0, 1\}$. We can identify \mathcal{B} with the space $\{-1, 0, 1\}^{\mathbb{N}}$ equipped with the product topology. Observe that this topology coincides with the topology of uniform convergence on compact subsets of the unit disk.

Claim 1. $\widetilde{\mathcal{N}}_+$ is relatively closed in Δ .

Indeed, let $(\gamma_n, \lambda_n) \rightarrow (\gamma, \lambda) \in \Delta$ and $(\gamma_n, \lambda_n) \in \widetilde{\mathcal{N}}_+$. Then there exist $f_n \in \mathcal{B}$ with $f_n(\gamma_n) = f_n(\lambda_n) = 0$ and $\xi_3(f_n) \leq \lambda_n$. This means that there exist $\alpha_n \leq \lambda_n$ such that $f_n(\alpha_n) = 0$ (if α_n is equal to γ_n or λ_n this is understood as having the corresponding zero of multiplicity 2). By compactness, without loss of generality, we can assume that $f_n \rightarrow f \in \mathcal{B}$ and $\alpha_n \rightarrow \alpha$. Then $f(\gamma) = f(\lambda) = f(\alpha) = 0$ and $\alpha \leq \lambda$ (again using our convention concerning double zeros). Thus $(\gamma, \lambda) \in \widetilde{\mathcal{N}}_+$, and Claim 1 is proved.

We will also need the following fact (see [3, Theorem 2] and [10, Th. 2.4]): if $f \in \mathcal{B}$ and $\alpha_1, \dots, \alpha_k$ are (some) complex roots of f in the unit disk, counted with multiplicity, then $|\alpha_1 \cdots \alpha_k| \geq (1 + k^{-1})^{-k/2} (k + 1)^{-1/2}$. Taking $k = 4$, we obtain

$$|\alpha_1 \alpha_2 \alpha_3 \alpha_4| \geq 16 \cdot 5^{-5/2} > 1/4. \tag{3.4}$$

Let $\phi: \mathcal{B} \rightarrow \mathcal{X}$ be the function defined as follows: If $f \in \mathcal{B}$ is such that $\gamma = \xi_1(f) < \xi_2(f) = \lambda$ and $(\gamma, \lambda) \in \Delta \setminus \widetilde{\mathcal{N}}_+$, then $\phi(f) := (\gamma, \lambda)$; otherwise, $\phi(f) := \omega$.

Claim 2. $\phi: \mathcal{B} \rightarrow \mathcal{X}$ is continuous.

Indeed, if $\phi(f) = (\gamma, \lambda)$, then f has simple zeros at γ and λ , hence a small perturbation of f will result in a small perturbation of these zeros. Suppose that $\phi(f) = \omega$. We need to show that if \tilde{f} is a small perturbation of f , then $\phi(\tilde{f})$ is close to ω . We have the following possibilities:

- (a) f has no positive zeros;
- (b) f has one simple positive zero;
- (c) $\xi_1(f) < \xi_2(f) < \xi_3(f)$ but $(\xi_1(f), \xi_2(f)) \in \tilde{\mathcal{N}}_+$;
- (d) $\xi_1(f) = \xi_2(f) < \xi_3(f)$;
- (e) $\xi_1(f) < \xi_2(f) = \xi_3(f)$;
- (f) $\xi_1(f) = \xi_2(f) = \xi_3(f)$, that is, f has a triple zero.

It is not hard to see that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if the distance from \tilde{f} to f in \mathcal{B} is less than δ , then every zero of \tilde{f} in $(0, 1 - 2\varepsilon)$ is ε -close to a zero of f . Another general fact is useful: for a small perturbation of a real-analytic function, new real zeros cannot appear; zeros can only disappear (i.e. become non-real).

In cases (a) and (b), either \tilde{f} has the same property, hence $\phi(\tilde{f}) = \omega$, or new zeros appear near 1, which could result in $\phi(\tilde{f}) = (\gamma, \lambda)$ with λ near 1, that is, (γ, λ) is close to ω in the topology of \mathcal{X} . In the case (c), a small perturbation \tilde{f} will have $\phi(\tilde{f}) = \omega$ or $\phi(\tilde{f}) = (\xi_1(\tilde{f}), \xi_2(\tilde{f}))$, which is close to $(\xi_1(f), \xi_2(f)) \in \tilde{\mathcal{N}}_+$, hence close to ω in the topology of \mathcal{X} . Suppose that case (d) holds. If the double zero at $\xi_1(f) = \xi_2(f)$ doesn't disappear, then we either still have a double zero for \tilde{f} , or two real zeros close to each other. In the former case we have $\phi(\tilde{f}) = \omega$, and in the latter case $\phi(\tilde{f}) = (\xi_1(\tilde{f}), \xi_2(\tilde{f}))$ is close to the diagonal, that is, close to ω in the topology of \mathcal{X} . If, on the other hand, the double zero disappears (becomes non-real) and $\phi(\tilde{f}) \neq \omega$, then $\phi(\tilde{f}) = (\xi_1(\tilde{f}), \xi_2(\tilde{f}))$, which is close to (γ, λ) where $\gamma \leq \lambda$ are zeros of f and $\xi_1(f) = \xi_2(f) < \gamma$. However, in the latter case we have $\gamma\lambda > 1/2$ by (3.4), hence $(\gamma, \lambda) \notin \Delta$ and so $(\xi_1(\tilde{f}), \xi_2(\tilde{f}))$ is still close to ω . If (e) or (f) holds, we have a similar argument: assuming $\phi(\tilde{f}) = (\tilde{\gamma}, \tilde{\lambda}) \neq \omega$, this point is either close to one in $\tilde{\mathcal{N}}$, or close to the diagonal, or close to $\partial\Delta$, in view of (3.4). This concludes the proof of Claim 2.

Now we essentially repeat the argument from [5]. For a finite word $u = u_1 \dots u_n$ in the alphabet $\{-1, 0, 1\}$ let

$$\mathcal{B}_u = \left\{ 1 + u_1x + \dots + u_nx^n + \sum_{k=n+1}^{\infty} b_kx^k \in \mathcal{B} \right\}.$$

Denote by u, α the word of length $n + 1$ obtained by adding the symbol α to u . Let

$$f = 1 + u_1x + \dots + u_nx^n \in \mathcal{B}_{u,0}.$$

We have $f(1 - x^{n+1})^{-1} \in \mathcal{B}_{u,1}$, $f(1 - x^{n+1})^{-1} \in \mathcal{B}_{u,-1}$. Observe that these three functions have the same set of zeros in $(0, 1)$, hence they are mapped into the same point by ϕ . It follows that

$$\phi(\mathcal{B}_{u,-1}) \cap \phi(\mathcal{B}_{u,0}) \cap \phi(\mathcal{B}_{u,1}) \neq \emptyset$$

for an arbitrary u . This property is called *recursive connectedness* in [5]. It is proved in [5] (and quite easy to see) that this property implies that $\phi(\mathcal{B})$ is connected and locally connected in \mathcal{X} . Note that

$$\phi(\mathcal{B}) = \pi(\Delta \cap \mathcal{N}_+ \setminus \widetilde{\mathcal{N}}_+) \cup \{\omega\}$$

where

$$\pi : \text{Clos}(\Delta \cap \mathcal{N}_+ \setminus \widetilde{\mathcal{N}}_+) \longrightarrow \mathcal{X}$$

is the natural projection associated with the quotient map. (We know that $\omega \in \phi(\mathcal{B})$ since, for instance, there are power series in \mathcal{B} with no positive zeros.) This immediately implies the statement of the theorem. \square

Proof of Proposition 2.5. By [11, Cor. 2.5.], for any $\gamma > \frac{1}{2}$ there exists $f \in \mathcal{B} \subset \mathcal{B}_{[-1,1]}$ such that $f(\gamma) = 0$ and f has at least three zeros in $(0, 1)$ (for example, we may take the zeros of f to be γ with multiplicity one, and $\sqrt{\frac{1}{2\gamma}}$ with multiplicity two). Thus the set in the definition of the function ψ is non-empty; it has a minimum by compactness of the class $\mathcal{B}_{[-1,1]}$. The statement (b) is immediate from the definitions. The remaining statements of (a) will easily follow from part (c). Its proof is divided into several lemmas. \square

Lemma 3.2. *Suppose that h is a $(**)$ -function such that $h'(x_0) = 0$ and $h(t) > 0$, $h'(t) < 0$ for all $t \in (0, x_0)$. Then there is no $f \in \mathcal{B}_{[-1,1]}$ such that $\xi_3(f) = x_0$, unless $h = f$, and x_0 is a triple zero of $f = h$.*

Proof. Suppose that $f \in \mathcal{B}_{[-1,1]}$ violates the assertion of the lemma. Let

$$g(x) = f(x) - h(x).$$

By the definition of a (**)-function, we have

$$g(x) = A_1(x) - A_2(x) + A_3(x),$$

where $A_1(x)$ and $A_2(x)$ are polynomials and $A_3(x)$ is a power series, all three with non-negative coefficients, such that the highest power in A_i is less than the lowest power in A_{i+1} . Thus, g and g' have at most two coefficient sign changes each.

Since x_0 is the third zero of f and $f(0) = 1$, we have $f'(x_0) \leq 0$. (Indeed, otherwise f is negative in a left neighborhood of x_0 , but on an interval where a real-analytic function changes its sign it must have an odd number of zeros, counting with multiplicities.) Thus, $g'(x_0) = f'(x_0) - h'(x_0) = f'(x_0) \leq 0$. Observe that there must be a zero ζ_1 of f' between the first and second zeros of f (if these two zeros of f coincide, that is, it is a double zero, which is equal to ζ_1). We have $f(\zeta_1) \leq 0$, hence $g(\zeta_1) = f(\zeta_1) - h(\zeta_1) < 0$, and $g'(\zeta_1) = -h'(\zeta_1) > 0$. By the Descartes Rule of Signs, g' can have at most two positive zeros. There has to be a zero of g' in (ζ_1, x_0) . There also have to be another zero of g' in $(0, \zeta_1)$, since $g(0) = 0$, $g(\zeta_1) < 0$, and $g'(\zeta_1) > 0$. Thus, g' has exactly two coefficient sign changes, hence $A_1(x) \neq 0$. But then g increases sufficiently close to zero, whence g' must have at least two zeros in $(0, \zeta_1)$. This is a contradiction. \square

Lemma 3.3. $\alpha_3 = \min\{\xi_3(f) : f \in \mathcal{B}_{[-1,1]}\}$.

Proof. Suppose there exists $f \in \mathcal{B}_{[-1,1]}$ such that

$$\lambda := \xi_3(f) < \alpha_3.$$

It is proved in [3, Section 2] that there is a (**)-function $H = H_{k,\ell}^{(u,v)}$ such that

$$H(\alpha_3) = H'(\alpha_3) = H''(\alpha_3) = 0.$$

(In fact, $k = 4$ and $\ell = 10$.) Consider the function

$$h(x) = H(x) + sx^\ell, \quad \text{with } s = -\frac{H'(\lambda)}{\ell\lambda^{\ell-1}}.$$

This is a (**)-function, though not necessarily in $\mathcal{B}_{[-1,1]}$, since the x^ℓ -coefficient may exceed 1 in absolute value. We have $H(x) > 0$, $H'(x) < 0$ for $x \in (0, \alpha_3)$, hence $h(x) > 0$ for all $x \in (0, \lambda)$ and $h'(\lambda) = 0$ by definition. We claim that $h'(x) < 0$ for all $x \in (0, \lambda)$. Indeed, h' has two coefficient sign changes, hence at most two positive zeros. We know that h' is negative near zero, $h'(\lambda) = 0$, $h'(\alpha_3) = s\ell\alpha_3^{\ell-1} > 0$, and h' is negative sufficiently close to 1. It follows that h'

has a zero in $(\alpha_3, 1)$, so it does not vanish in $(0, \lambda)$, implying the claim. Thus, h is a $(**)$ -function satisfying the assumptions of Lemma 3.2 for $x_0 = \lambda$, so the existence of f is a contradiction. \square

Lemma 3.3 implies that $\psi(\gamma) \geq \alpha_3 > \gamma$ for $\gamma \in (\frac{1}{2}, \alpha_3)$. Fix $\gamma \in (\frac{1}{2}, \alpha_3)$. Recall that $\psi(\gamma)$ is well-defined, which means that there exists a function $f \in \mathcal{B}_{[-1,1]}$ such that $\psi(\gamma) = \xi_3(f)$. Such a function will be called “optimal” (for a given γ).

Lemma 3.4. *An optimal function f for $\gamma \in (\frac{1}{2}, \alpha_3)$ has a double zero at $\lambda = \xi_3(f)$, that is,*

$$f(\lambda) = f'(\lambda) = 0.$$

Proof. Suppose $f'(\lambda) \neq 0$. Since $f(0) = 1$ and λ is the third positive zero of a real analytic function, f is strictly decreasing in a neighborhood of λ . By Descartes Rule of Signs, f has at least three coefficient sign changes. Therefore, we can find integers $0 < \ell_1 < \ell_2 < \ell_3$ such that $a_{\ell_1} < 0$, $a_{\ell_2} > 0$, and $a_{\ell_3} < 0$, where a_{ℓ_i} is the coefficient of x^{ℓ_i} in f . Consider

$$\tilde{f}(x) := f(x) + \varepsilon(\gamma^{\ell_2 - \ell_1} x^{\ell_1} - x^{\ell_2}).$$

Then $\tilde{f} \in \mathcal{B}_{[-1,1]}$ for sufficiently small $\varepsilon > 0$. Moreover, $\tilde{f}(\gamma) = f(\gamma) = 0$ and $\tilde{f}(x) < f(x)$ for $x \in (\gamma, 1)$. Thus, for sufficiently small $\varepsilon > 0$, the function \tilde{f} has a zero close to λ which is less than λ . We claim that this zero is $\xi_3(\tilde{f})$, which contradicts $\lambda = \psi(\gamma)$. Indeed, if the first two positive zeros of f are distinct (and they are smaller than λ), this property will persist for \tilde{f} . If γ is a double zero, then \tilde{f} has a second zero γ' close to γ . This proves the claim, and the lemma follows. \square

Lemma 3.5. *The optimal function f for $\gamma \in (\frac{1}{2}, \alpha_3)$ is unique; it is a $(**)$ -function $h_{k,\ell}^{(a,b)}$ for some $1 \leq k < \ell < \infty$ and $a, b \in [-1, 1]$.*

Proof. Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ be optimal, and suppose that it is not a $(**)$ -function (see (2.2)). Let $\ell_1 \geq 1$ be minimal such that $a_{\ell_1} > -1$. Then choose $\ell_2 > \ell_1$ minimal such that $a_{\ell_2} < 1$ (note that ℓ_2 exists since f must have at least three coefficient sign changes). If f is not a $(**)$ -function, then we can find $\ell_3 > \ell_2$ such that $a_{\ell_3} > -1$. Let $c_2, c_3 \in \mathbb{R}$ be such that

$$g(x) := -x^{\ell_1} + c_2 x^{\ell_2} + c_3 x^{\ell_3}$$

satisfies $g(\gamma) = g(\lambda) = 0$. (This is a linear system of equations with determinant $\gamma^{\ell_2} \lambda^{\ell_3} - \gamma^{\ell_3} \lambda^{\ell_2} \neq 0$, so there is a unique solution.) Notice that $c_2 > 0$ and $c_3 < 0$,

since there must be two coefficient sign changes in g . Clearly, λ is a simple zero for g , and it is a double zero for f by Lemma 3.4. Thus, there exist $b_1, b_2 > 0$ such that

$$|f(x)| \leq b_1|x - \lambda|^2, \quad |g(x)| \geq b_2|x - \lambda|$$

for x near λ . Consider

$$\tilde{f}(x) := f(x) + \varepsilon g(x).$$

Then $\tilde{f} \in \mathcal{B}_{[-1,1]}$ for sufficiently small $\varepsilon > 0$. Observe that $\tilde{f}(\gamma) = \tilde{f}(\lambda) = 0$ by construction. Recall that $f(0) = 1$, $f(\gamma) = 0$, and $f(\lambda) = f'(\lambda) = 0$, hence $\min_{[\gamma, \lambda]} f < 0$. We can make sure that $\varepsilon > 0$ is so small that $\min_{[\gamma, \lambda]} \tilde{f} < 0$. On the other hand,

$$\tilde{f}\left(\lambda - \frac{\varepsilon}{n}\right) = f\left(\lambda - \frac{\varepsilon}{n}\right) + \varepsilon g\left(\lambda - \frac{\varepsilon}{n}\right) \geq -b_1\left(\frac{\varepsilon}{n}\right)^2 + \varepsilon b_2\left(\frac{\varepsilon}{n}\right) > 0$$

provided that $n > b_1/b_2$. Then \tilde{f} has a zero in (γ, λ) which implies that $\xi_3(\tilde{f}) \leq \lambda$. Since $\lambda = \psi(\gamma)$, we have $\xi_3(\tilde{f}) = \lambda$, so \tilde{f} is optimal for γ (as well as f). This contradicts Lemma 3.5 since λ is not a double zero of \tilde{f} . It remains to verify that the optimal function is unique. Assuming that we have two distinct optimal functions, we take their difference, which has at most two coefficient sign changes, since both are (**)-functions. This leads to a contradiction since the difference has at least three positive zeros.

This concludes the proof of the lemma and of the claim (c) in Proposition 2.5. The remaining statements of the proposition follow easily. \square

4. Appendix: how to compute the functions ϕ and ψ

We first explain how the function ϕ was computed, using Mathematica. Consider the (*)-function

$$h(x) = 1 - x - \dots - x^{k-1} + ax^k + \frac{x^{k+1}}{1-x},$$

where $a \in [-1, 1]$. First, we fix k . The algorithm takes γ as an input. Then, $a = F(\gamma)$ is determined so that $h(\gamma) = 0$. We must check that $-1 \leq a \leq 1$, so that h is indeed a member of $\mathcal{B}_{[-1,1]}$. Next, we find the second root of h using the FindRoot command with an appropriate starting point. We choose the starting point to guarantee that we find λ rather than γ (FindRoot uses Newton's method to find the root of a function. It will find the root closest to the starting point. Recall the shape of the (*)-function h . We must choose a starting point to the right of the minimum of h to guarantee that Mathematica finds λ rather than γ . We know

our choice of starting point works as long as the output of FindRoot is not equal to our input γ).

Now, as was seen in [12, 3], there is a $(*)$ -function $h_4^{(b)}$ having a double zero at α_2 . Therefore we begin by fixing $k = 4$. Consider the $(*)$ -function

$$h_1(x) = 1 - x - x^2 - x^3 - F(\gamma)x^4 + \frac{x^5}{1-x}.$$

We solve for $F(\gamma)$ to ensure that $h_1(\gamma) = 0$ and obtain

$$F(\gamma) = \frac{1 - 2\gamma + \gamma^4 + \gamma^5}{\gamma^4 - \gamma^5}.$$

We find that 0.8 works as a starting point for FindRoot. Using NSolve, we find that $|a| = |F(\gamma)| \leq 1$ for $\gamma \in (0.550607, 0.7691)$, approximately. However, recall that

$$\phi: (0.5, \alpha_2) \longrightarrow (0, 1).$$

Thus we are only interested in looking at $\gamma \in (0.550607, \alpha_2 = 0.649138)$. Figure 4 shows a plot of $\phi(\gamma)$ for $\gamma \in (0.550607, 0.649138)$.

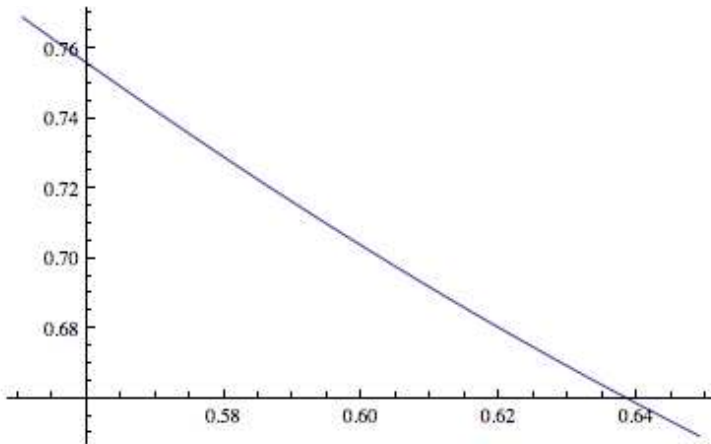


Figure 4. $\phi(\gamma)$ from h_1 .

Now, note that when $\gamma = 0.550607$, $F(\gamma) \approx 1$. Thus at $\gamma = 0.550607$, the coefficient of x^4 is -1 . Thus this is a “switching point,” that is, at this point, h_1 switches to a $(*)$ -function with $k = 5$. This is one of the points we are interested in, because it will be in the set \mathcal{N} .

Next we consider the $(*)$ -function

$$h_2(x) = 1 - x - x^2 - x^3 - x^4 - G(\gamma)x^5 + \frac{x^6}{1-x},$$

so that $k = 5$. We solve for $G(\gamma)$ so that γ is indeed a root of $h_2(x)$, and find

$$G(\gamma) = \frac{1 - 2\gamma + \gamma^5 + \gamma^6}{\gamma^5 - \gamma^6}.$$

We again check the range for which $|G(\gamma)| \leq 1$, and find that the inequality holds for $\gamma \in (0.519703, 0.832218)$.

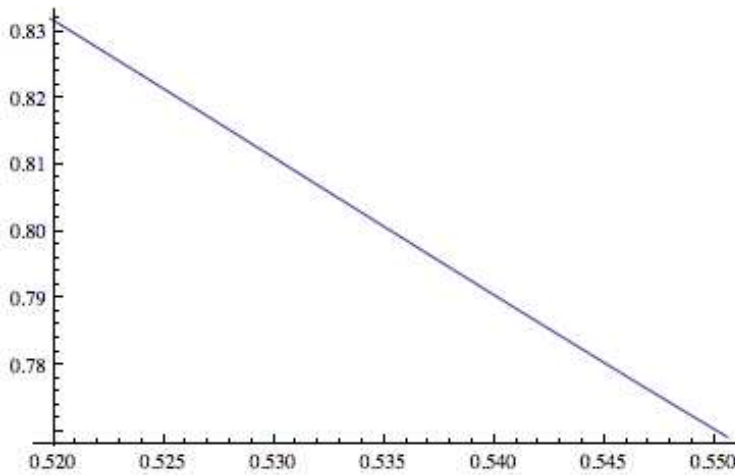


Figure 5. $\phi(\gamma)$ from h_2 .

Figure 5 shows a plot of $\phi(\gamma)$ for $\gamma \in (0.519703, 0.550607)$. Note that $G(0.529703) \approx 1$, so that at $\gamma = 0.529703$ h_2 becomes a $(*)$ -function with $k = 6$. Thus we continue similarly by setting $k = 6$. Let

$$h_3(x) = 1 - x - x^2 - x^3 - x^4 - x^5 - K(\gamma)x^6 + \frac{x^7}{1-x},$$

and solve for $K(\gamma)$ so that $h_3(\gamma) = 0$ to obtain

$$K(\gamma) = \frac{1 - 2\gamma + \gamma^6 + \gamma^7}{\gamma^6 - \gamma^7}.$$

Using NSolve, we find that $|K(\gamma)| \leq 1$ for $\gamma \in (0.508831, 0.866368)$. We may continue in this manner in order to obtain $\phi(\gamma)$ for $\gamma \rightarrow 0.5$. Note, however, that the process does not terminate.

Now we explain how the function ψ may be computed. Recall that

$$\psi : (0.5, \alpha_3) \longrightarrow [0, 1],$$

where $\alpha_3 \approx 0.727883$. Consider the (**)-function

$$H_{k,\ell}(x) = 1 - \sum_{i=1}^{k-1} x^i + ax^k + \sum_{i=k+1}^{\ell-1} x^i + bx^\ell - \frac{x^{\ell+1}}{1-x},$$

where $a, b \in [-1, 1]$. Recall that we would like to find $H_{k,\ell}(x)$ such that

$$H_{k,\ell}(\gamma) = H_{k,\ell}(\lambda) = H'_{k,\ell}(\lambda) = 0.$$

Since we have two unknowns a and b , for this algorithm we will start with λ and obtain $a = F_a(\lambda)$ and $b = F_b(\lambda)$ such that $H_{k,\ell}(\lambda) = H'_{k,\ell}(\lambda) = 0$, and use FindRoot to find γ such that $H_{k,\ell}(\gamma) = 0$. In [3] it was proved that there is a (**)-function $H=H_{4,10}$ such that

$$H(\alpha_3) = H'(\alpha_3) = H''(\alpha_3) = 0.$$

Thus we begin by considering the function

$$F(x) = 1 - x - x^2 - x^3 + x^5 + x^6 + x^7 + x^8 + x^9 - \frac{x^{11}}{1-x}.$$

Then we let

$$H_1(x) = F(x) + F_a(\lambda)x^4 + F_b(\lambda)x^{10}.$$

Note that

$$F'(x) = -1 - 2x - 3x^2 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 - \frac{11x^{10}}{1-x} - \frac{x^{11}}{(1-x)^2}.$$

We solve the system of equations

$$\begin{aligned} H_1(\lambda) &= F(\lambda) + F_a(\lambda)\lambda^4 + F_b(\lambda)\lambda^{10} = 0 \\ H'_1(\lambda) &= F'(\lambda) + 4F_a(\lambda)\lambda^3 + 10F_b(\lambda)\lambda^9 = 0 \end{aligned}$$

for $F_a(\lambda)$ and $F_b(\lambda)$ and find that

$$F_a(\lambda) = \frac{\lambda F'(\lambda) - 10F(\lambda)}{6\lambda^4}$$

and

$$F_b(\lambda) = \frac{4F(\lambda) - \lambda F'(\lambda)}{6\lambda^{10}}.$$

In this case, we need to have both $|F_a(\lambda)| \leq 1$ and $|F_b(\lambda)| \leq 1$. We find that

$$|F_a(\lambda)| \leq 1 \quad \text{for } \lambda \in (0.606471, 0.83611),$$

(note that $F_a(0.606471) = F_a(0.83611) = -1$) and

$$|F_b(\lambda)| \leq 1 \quad \text{for } \lambda \in (0.692945, \alpha_3)$$

(where $F_b(0.692945) = -1$). So, the first coefficient that “switches” is at $k = 10$, when $F_b(\lambda) = -1$. Next we use FindRoot to find γ . So this algorithm takes λ as an input and outputs γ , so this is effectively ψ^{-1} .

Next, we use our “switching point.” We let

$$F_1(x) = F(x) + x^{10} + x^{11}$$

and

$$G_1(x) = F_1(x) + F_{a_1}(\lambda)x^4 + F_{b_1}(\lambda)x^{11}$$

and solve for $F_{a_1}(\lambda)$ and $F_{b_1}(\lambda)$ so that

$$G_1(\lambda) = G_1'(\lambda) = 0.$$

We find that

$$F_{a_1}(\lambda) = \frac{\lambda F_1'(\lambda) - 11F_1(\lambda)}{7\lambda^4}$$

and

$$F_{b_1}(\lambda) = \frac{-\lambda F_1'(\lambda) + 4F_1(\lambda)}{7\lambda^{11}}.$$

Again we check for which λ we have $|F_{a_1}(\lambda)| \leq 1$ and $|F_{b_1}(\lambda)| \leq 1$ and continue in this way.

Update (April 2015). After the paper was submitted, we became aware of [6], where it is shown that for all $1 < \lambda^{-1} < \gamma^{-1} < 1.05$ the self-affine attractor $K_{\gamma,\lambda}$ contains $(0, 0)$ in its interior. This was improved by K. Hare and N. Sidorov [7], who demonstrated the same under the bounds $1 < \lambda^{-1} < \gamma^{-1} < 1.202$ and obtained other interesting results on the family $K_{\gamma,\lambda}$ and the connectedness locus \mathcal{N} . In particular, they proved that \mathcal{N} is not simply connected. Their research of these sets was continued in [8].

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Boris Solomyak, Bar-Ilan University, Ramat-Gan 5290002, Israel

e-mail: bsolom3@gmail.com