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On the Fourier dimension and a modification

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Abstract. We give a sufficient condition for the Fourier dimension of a countable union of sets to equal the supremum of the Fourier dimensions of the sets in the union, and show by example that the Fourier dimension is not countably stable in general. A natural approach to finite stability of the Fourier dimension for sets would be to try to prove that the Fourier dimension for measures is finitely stable, but we give an example showing that it is not in general. We also describe some situations where the Fourier dimension for measures is stable or is stable for all but one value of some parameter. Finally we propose a way of modifying the definition of the Fourier dimension so that it becomes countably stable, and show that for each s there is a class of sets such that a measure has modified Fourier dimension greater than or equal to s if and only if it annihilates all sets in the class.

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1. Introduction

Let A be a Borel subset of $\mathbf{R}^d$. One way to prove a lower bound for the Hausdorff dimension of A is to consider integrals of the form

$$I_s(\mu) = \iint |x-y|^{-s} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y);$$

if μ is a Borel measure such that $\mu(A) > 0$ and $I_s(\mu) < \infty$ for some s, then $\dim_{\mathrm{H}} A \geq s$. For a finite Borel measure μ, the *Fourier transform* is defined as

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} \, \mathrm{d}\mu(x),$$

where $\xi \in \mathbf{R}^d$ and $\cdot$ denotes the Euclidean inner product. It can be shown [13, Lemma 12.12] that if μ has compact support then

$$I_s(\mu) = \text{const.}(d,s) \int |\hat{\mu}(\xi)|^2 |\xi|^{s-d} \, \mathrm{d}\xi$$

for $0 < s < d$, and thus $I_{s_0}(\mu)$ is finite if $\hat{\mu}(\xi) \lesssim |\xi|^{-s/2}$ for some $s > s_0$ (here and in the remainder, $f(\xi) \lesssim g(\xi)$ means that there exists a constant C such that $|f(\xi)| \le C|g(\xi)|$ for all ξ). This motivates defining the *Fourier dimension* of A as

$$\dim_{\mathrm{F}} A = \sup\{ s \in [0,d];\ \hat{\mu}(\xi) \lesssim |\xi|^{-s/2},\ \mu \in \mathcal{P}(A) \},$$

where $\mathcal{P}(A)$ denotes the set of Borel probability measures on $\mathbf{R}^d$ that give full measure to A (it would not make any difference if the supremum was taken only over such measures with compact support, for if $\hat{\mu}(\xi) \lesssim |\xi|^{-s/2}$ then there is a compactly supported probability measure ν that is absolutely continuous with respect to μ and satisfies $\hat{\nu}(\xi) \lesssim |\xi|^{-s/2}$ by Lemma 1 below). Thus the Fourier dimension is a lower bound for the Hausdorff dimension. The Fourier dimension of a finite Borel measure μ on $\mathbf{R}^d$ is defined as

$$\dim_{\mathrm{F}} \mu = \sup\{ s \in [0,d];\ \hat{\mu}(\xi) \lesssim |\xi|^{-s/2} \},$$

or equivalently

$$\dim_{\mathrm{F}} \mu = \min\left(d,\ \liminf_{|\xi|\to\infty} \frac{-2\log|\hat{\mu}(\xi)|}{\log|\xi|} \right),$$

so that

$$\dim_{\mathrm{F}} A = \sup\{ \dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(A) \}.$$

If $A \subset B$ then $\mathcal{P}(A) \subset \mathcal{P}(B)$ and hence

$$\dim_{\mathrm{F}}(A) = \sup\{ \dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(A) \} \le \sup\{ \dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(B) \} = \dim_{\mathrm{F}}(B),$$

showing that the Fourier dimension is monotone. It seems not to be previously known whether the Fourier dimension is stable under finite or countable unions, that is, whether

$$\dim_{\mathrm{F}} \Big(\bigcup_k A_k \Big) = \sup_k \dim_{\mathrm{F}} A_k, \tag{1}$$

where $\{A_k\}$ is a finite or countable family of sets. The inequality $\ge$ follows from the monotonicity, but there might be sets for which the inequality is strict. In Section 2 we show that (1) holds if for each n the intersection $A_n \cap \overline{\bigcup_{k \neq n} A_k}$ has small "modified Fourier dimension" (defined below), and in particular if all such intersections are countable. We also give an example of a countably infinite family of sets such that (1) does *not* hold.

This still leaves open the question of *finite* stability. The most straightforward approach would be to prove a corresponding stability for the Fourier dimension of measures, namely that

$$\dim_{\mathrm{F}}(\mu + \nu) = \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu). \tag{2}$$

From this one could derive the finite stability for sets, using that any probability measure on $A \cup B$ is a convex combination of probability measures on A and B. The inequality $\geq$ always holds in (2) since the set of functions that are $\lesssim |\xi|^{-s/2}$ is closed under finite sums, but we give an example in Section 3 showing that strict inequality can occur. We also describe some situations in which (2) *does* hold – this seems to be the typical case.

To achieve countable stability, we consider the following modification of the Fourier dimension.

Definition. The *modified Fourier dimension* of a Borel set $A \subset \mathbf{R}^d$ is defined as

$$\dim_{\mathrm{FM}} A = \sup\{\dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(\mathbf{R}^d),\ \mu(A) > 0\},$$

and the modified Fourier dimension of a finite Borel measure μ is defined as

$$\dim_{\mathrm{FM}} \mu = \sup\{\dim_{\mathrm{F}} \nu;\ \nu \in \mathcal{P}(\mathbf{R}^d),\ \mu \ll \nu\},$$

where $\ll$ denotes absolute continuity.

Thus

$$\dim_{\mathrm{FM}} A = \sup\{\dim_{\mathrm{FM}} \mu;\ \mu \in \mathcal{P}(A)\}.$$

In Section 4, we investigate some basic properties of the modified Fourier dimension, and give examples to show that it is different from the usual Fourier dimension and the Hausdorff dimension.

In Section 5, we show that if μ annihilates all the common null sets for the measures that have modified Fourier dimension greater than or equal to s, then $\dim_{\mathrm{FM}} \mu \geq s$. Other classes of measures that can be characterised by their null sets in this way are the measures that are absolutely continuous to some fixed measure, and, less trivially, the measures $\mu \in \mathcal{P}([0, 1])$ such that $\lim_{|\xi|\to\infty} \hat{\mu}(\xi) = 0$ (see [12]). A necessary condition for such a characterisation to be possible is that the class of measures be a *band*, meaning that any measure that is absolutely continuous to some measure in the class lies in the class. The definition of the modified Fourier dimension is natural from this point of view, since the class of measures that have modified Fourier dimension greater than or equal to s is the smallest band that includes the measures that have (usual) Fourier dimension greater than or equal to s.

1.1. Previous work. Here we mention briefly some of the previous work related to the Fourier dimension that we are aware of. A *Salem set* is a set whose Fourier dimension equals its Hausdorff dimension.

Salem [17] showed that for each $s \in (0,1)$ there is a compact Salem subset of $[0,1]$ of dimension s (although this was shown for the restriction of the Fourier transform to the integers). An *explicit* example of a Salem set of any prescribed dimension in $(0,1)$ is given by the set of α*-well approximable numbers*, namely the set

$$E(\alpha) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ x \in [0,1];\ \|kx\| \le k^{-(1+\alpha)} \},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. By a theorem of Jarník [7] and Besicovitch [1] the set $E(\alpha)$ has Hausdorff dimension $2/(2+\alpha)$ for $\alpha > 0$, and Kaufman [11] showed that there is a measure in $\mathcal{P}(E(\alpha))$ with Fourier dimension $2/(2+\alpha)$ (see also Bluhm's paper [2]).

It was shown by Kaufman [10] that for any C^2-curve Γ in $\mathbf{R}^2$ with positive curvature and any $s \subset (0,1)$, there is a compact Salem set $S \subset \Gamma$ of dimension s. From this it can be deduced [5, Proposition 1.1] that for any $s \in [0,1]$ there is a continuous function $[0,1] \to \mathbf{R}$ whose graph has Fourier dimension s. Fraser, Orponen and Sahlsten [5] proved that the graph of any function $[0,1] \to \mathbf{R}$ has *compact* Fourier dimension (defined below) less than or equal to 1, and that the set of continuous functions $[0,1] \to \mathbf{R}$ whose graphs have Fourier dimension 0 is residual with respect to the supremum norm among all continuous functions $[0,1] \to \mathbf{R}$.

Kahane showed that images of compact sets under Brownian motion and fractional Brownian motion are almost surely Salem sets, see [9]. It was shown by Fouché and Mukeru [4] that the level sets of fractional Brownian motion are almost surely Salem sets (in the special case of Brownian motion this follows from a result of Kahane, see [4, Section 3.2]).

Jordan and Sahlsten [8] showed that Gibbs measures of Hausdorff dimension greater than $1/2$ (satisfying a certain condition on the Gibbs potential) for the Gauss map $x \mapsto 1/x \pmod 1$ have positive Fourier dimension.

Wolff's book [18] about harmonic analysis discusses some applications of the Fourier transform to problems in geometric measure theory.

1.2. Some remarks. It is not so difficult to see that the Fourier dimension for measures is invariant under translations and invertible linear transformations, and thus the Fourier dimension and modified Fourier dimension for sets are invariant as well.

For any finite Borel measure μ on $\mathbf{R}^d$,

$$\lim_{T\to\infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} |\hat{\mu}(\xi)|^2 \, \mathrm{d}\xi = \sum_{x\in\mathbf{R}^d} \mu(\{x\})^2$$

(this is a variant of Wiener's lemma). If μ has an atom it is thus not possible that $\lim_{|\xi|\to\infty} \hat{\mu}(\xi) = 0$, so $\dim_{\mathrm{F}} \mu = 0$, and also $\dim_{\mathrm{FM}} \mu = 0$ since ν has an atom whenever $\mu \ll \nu$. It follows that $\dim_{\mathrm{F}} A = \dim_{\mathrm{FM}} A = 0$ for any countable set $A \subset \mathbf{R}^d$.

Suppose next that A is a countable union of k-dimensional hyperplanes in $\mathbf{R}^d$ with $k < d$. If μ gives positive measure to A, then there must be a hyperplane P such that $\mu(P) > 0$. But then the projection of μ onto any line L that goes through the origin and is orthogonal to P has an atom, so $\hat{\mu}$ does not decay along L. This shows that $\dim_{\mathrm{F}} A = \dim_{\mathrm{FM}} A = 0$. Thus for example a line segment in $\mathbf{R}^2$ has Fourier dimension 0 even though an interval in $\mathbf{R}$ has Fourier dimension 1.

From a special case of a theorem by Davenport, Erdős and LeVeque [3], it can be derived [15, Corollary 7.4] that if μ is a probability measure on $\mathbf{R}$ such that $\hat{\mu}(\xi) \lesssim |\xi|^{-\alpha}$ for some $\alpha > 0$, then μ-a.e. x is normal to any base (meaning that $(b^k x)_{k=0}^{\infty}$ is uniformly distributed mod 1 for any $b \in \{2, 3, \ldots\}$). Thus if $A \subset \mathbf{R}$ does not contain any number that is normal to all bases, then $\dim_{\mathrm{F}} A = \dim_{\mathrm{FM}} A = 0$. In particular this applies to the middle-third Cantor set, since it consists of numbers that do not have any 1 in their ternary decimal expansion and hence are not normal to base 3.

1.3. Other variants of the Fourier dimension. One alternative way of defining the Fourier dimension of a Borel set $A \subset \mathbf{R}^d$ is to require the measure in the definition to give full measure to a compact subset of A, rather than to A itself. This variant, which will here be called the *compact Fourier dimension*, is thus defined by

$$\dim_{\mathrm{FC}} A = \sup\{s \in [0, d];\ \hat{\mu}(\xi) \lesssim |\xi|^{-s/2},\ \mu \in \mathcal{P}(K),\ K \subset A \text{ is compact}\}.$$

The anonymous referee of this paper provided an argument showing that the compact Fourier dimension is countably stable whenever all the sets in the union are closed (see Proposition 5 below), and pointed out that this can be used to deduce that the Fourier dimension and the compact Fourier dimension are not the same.

Inspired by this, we then found an example that shows that the compact Fourier dimension is not in general finitely stable.

The Hausdorff dimension is *inner regular* in the sense that

$$\dim_{\mathrm{H}} A = \sup_{\substack{K \subset A \\ K \text{ compact}}} \dim_{\mathrm{H}} K$$

for any Borel set $A \subset \mathbf{R}^d$ (this follows from [16, Theorem 48]), and the same is true of the modified Fourier dimension by inner regularity of finite Borel measures on $\mathbf{R}^d$. Another way of expressing the fact that $\dim_{\mathrm{FC}}$ is different from $\dim_{\mathrm{F}}$ is to say that $\dim_{\mathrm{F}}$ is not inner regular.

One might consider to define the Fourier dimension and the modified Fourier dimension of any $B \subset \mathbf{R}^d$, by taking the supremum over all measures in $\mathcal{P}(\mathbf{R}^d)$ that give full or positive measure to some Borel set $A \subset B$, but then $\dim_{\mathrm{F}}$ and $\dim_{\mathrm{FM}}$ are not even finitely stable. For there is a construction by Bernstein (using the well ordering theorem for sets with cardinality $\mathfrak{c}$) of a set $B \subset \mathbf{R}$ such that any closed subset of B or B^c is countable [14, Theorem 5.3]. Thus any non-atomic measure $\mu \in \mathcal{P}(\mathbf{R})$ gives measure 0 to any compact subset of B or B^c, and by inner regularity to any Borel subset of B or B^c. It follows that B and B^c have Fourier dimension and modified Fourier dimension 0, but $B \cup B^c = \mathbf{R}$ has dimension 1.

This can be modified slightly to produce Lebesgue measurable sets $C_1, C_2 \subset \mathbf{R}$ that would violate the finite stability. For each natural number n, let A_n be a Salem set of dimension $1 - 1/n$ and let

$$C_1 = B \cap \bigcup_{n=1}^{\infty} A_n, \qquad C_2 = B^c \cap \bigcup_{n=1}^{\infty} A_n.$$

Then C_1 and C_2 are Lebesgue measurable since each A_n has Lebesgue measure 0, and since they are subsets of B and B^c respectively they would have Fourier dimension and modified Fourier dimension 0. On the other hand,

$$\dim_{\mathrm{F}}(C_1 \cup C_2) = \dim_{\mathrm{F}} \Big(\bigcup_{n=1}^{\infty} A_n \Big) = 1,$$

and thus also $\dim_{\mathrm{FM}}(C_1 \cup C_2) = 1$.

2. Stability of the Fourier dimension for sets

In this section it is shown that the Fourier dimension is stable under finite or countable unions of sets that satisfy a certain intersection condition, and that the compact Fourier dimension is stable under finite or countable unions of closed sets.

Then examples are given, showing that the Fourier dimension is not countably stable, that the compact Fourier dimension is different from the Fourier dimension and that the compact Fourier dimension is not finitely stable.

The following lemma is used in the proofs of Theorem 2 and Proposition 5, and also in Section 3 and Section 5.

Lemma 1. *Let μ be a finite Borel measure on $\mathbf{R}^d$ and let f be a non-negative C^m-function with compact support, where $m = \lceil 3d/2 \rceil$. Define the measure ν on $\mathbf{R}^d$ by $\mathrm{d}\nu = f\,\mathrm{d}\mu$. Then*

$$\hat{\mu}(\xi) \lesssim |\xi|^{-s/2} \implies \hat{\nu}(\xi) \lesssim |\xi|^{-s/2}$$

for all $s \in [0, d]$, and in particular

$$\dim_{\mathrm{F}} \nu \geq \dim_{\mathrm{F}} \mu .$$

Proof. Since f is of class C^m and has compact support, there is a constant M such that

$$|\hat{f}(t)| \leq \frac{M}{1 + |t|^m}$$

for all $t \in \mathbf{R}^d$. In particular $\hat{f}$ is Lebesgue integrable, and from this it also follows that the Fourier inversion formula holds pointwise everywhere for f. Thus

$$\begin{aligned}
\hat{\nu}(\xi) &= \int e^{-2\pi i \xi\cdot x} f(x)\,\mathrm{d}\mu\,(x) \\
&= \int e^{-2\pi i \xi\cdot x} \left(\int e^{2\pi i t\cdot x} \hat{f}(t)\,\mathrm{d}t \right) \mathrm{d}\mu\,(x) \\
&= \int \left(\int e^{-2\pi i (\xi - t)\cdot x}\,\mathrm{d}\mu\,(x) \right) \hat{f}(t)\,\mathrm{d}t \\
&= \int \hat{\mu}(\xi - t)\hat{f}(t)\,\mathrm{d}t\,.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{\{|\xi - t| < |\xi|/2\}} |\hat{\mu}(\xi - t)\hat{f}(t)|\,\mathrm{d}t &\leq \int_{\{|t| \geq |\xi|/2\}} \frac{\mu(\mathbf{R}^d)M}{1 + |t|^m}\,\mathrm{d}t \\
&\leq \mu(\mathbf{R}^d)M \int_{\{|t| \geq |\xi|/2\}} |t|^{-m}\,\mathrm{d}t \\
&= \mathrm{const.} \cdot |\xi|^{d-m} \\
&\lesssim |\xi|^{-d/2},
\end{aligned}$$

and if $\hat{\mu}(\xi) \le C|\xi|^{-s/2}$ for all $\xi \in \mathbf{R}^d$ then

$$\int_{\{|\xi-t|\ge|\xi|/2\}} |\hat{\mu}(\xi-t)\hat{f}(t)|\,\mathrm{d}t \le \frac{C2^{s/2}}{|\xi|^{s/2}} \int |\hat{f}(t)|\,\mathrm{d}t \lesssim |\xi|^{-s/2}.$$

Thus

$$\hat{\nu}(\xi) \lesssim |\xi|^{-d/2} + |\xi|^{-s/2}$$

whenever $\hat{\mu}(\xi) \lesssim |\xi|^{-s/2}$, which proves the lemma. □

Theorem 2. *Let $\{A_k\}$ be a finite or countable family of Borel subsets of $\mathbf{R}^d$ such that*

$$\sup_n \dim_{\mathrm{FM}} \Big(A_n \cap \overline{\bigcup_{k\ne n} A_k}\Big) < \dim_{\mathrm{F}} \Big(\bigcup_k A_k\Big).$$

Then

$$\dim_{\mathrm{F}} \Big(\bigcup_k A_k\Big) = \sup_k \dim_{\mathrm{F}} A_k.$$

Proof. The inequality $\ge$ is immediate from the monotonicity of $\dim_{\mathrm{F}}$. To see the other inequality, take an arbitrary $\mu \in \mathcal{P}(\bigcup_k A_k)$ such that

$$\sup_n \dim_{\mathrm{FM}} \Big(A_n \cap \overline{\bigcup_{k\ne n} A_k}\Big) < \dim_{\mathrm{F}} \mu$$

and let n be such that $\mu(A_n) > 0$. Then by the definition of the modified Fourier dimension,

$$\mu\Big(A_n \cap \overline{\bigcup_{k\ne n} A_k}\Big) = 0 \quad \text{and thus} \quad \mu\Big(A_n \setminus \overline{\bigcup_{k\ne n} A_k}\Big) > 0.$$

Let f be a non-negative C^∞-function that is 0 on $\overline{\bigcup_{k\ne n} A_k}$ and positive everywhere else. Then $\mu(f) > 0$, and there is a non-negative C^∞-function g with compact support such that $\mu(fg) > 0$ as well. The measure ν defined by

$$\mathrm{d}\nu = \frac{fg}{\mu(fg)}\,\mathrm{d}\mu$$

is a probability measure on A_n, so

$$\sup_k \dim_{\mathrm{F}} A_k \ge \dim_{\mathrm{F}} A_n \ge \dim_{\mathrm{F}} \nu \ge \dim_{\mathrm{F}} \mu,$$

where the last inequality is by Lemma 1. Taking supremum on the right over all $\mu \in \mathcal{P}(\bigcup_k A_k)$ gives the inequality $\le$ in the statement. □

Corollary 3. *Let $\{A_k\}$ be a finite or countable family of Borel subsets of $\mathbf{R}^d$ such that*

$$\sup_n \dim_{\mathrm{FM}} \Big(A_n \cap \overline{\bigcup_{k \neq n} A_k} \Big) \leq \sup_k \dim_{\mathrm{F}} A_k .$$

Then

$$\dim_{\mathrm{F}} \Big(\bigcup_k A_k \Big) = \sup_k \dim_{\mathrm{F}} A_k .$$

Proof. If the conclusion does not hold then neither does the assumption, since then

$$\sup_k \dim_{\mathrm{F}} A_k < \dim_{\mathrm{F}} \Big(\bigcup_k A_k \Big) \leq \sup_n \dim_{\mathrm{FM}} \Big(A_n \cap \overline{\bigcup_{k \neq n} A_k} \Big),$$

where the second inequality is by Theorem 2. □

Corollary 4. *Let $\{A_k\}$ be a finite or countable family of Borel subsets of $\mathbf{R}^d$ such that*

$$A_n \cap \overline{\bigcup_{k \neq n} A_k}$$

is countable for all n. Then

$$\dim_{\mathrm{F}} \Big(\bigcup_k A_k \Big) = \sup_k \dim_{\mathrm{F}} A_k .$$

Proof. This follows from Corollary 3 since any countable set has modified Fourier dimension 0. □

The proof of the following proposition was provided by the anonymous referee, who also noted that the statement can be applied to the sets constructed in Example 7 below to show that the compact Fourier dimension is different from the Fourier dimension. Another application is Example 8 below, which shows that the compact Fourier dimension is not finitely stable.

Proposition 5. *Let $\{A_k\}$ be a finite or countable family of closed subsets of $\mathbf{R}^d$. Then*

$$\dim_{\mathrm{FC}} \Big(\bigcup_k A_k \Big) = \sup_k \dim_{\mathrm{FC}} A_k .$$

Proof. The inequality $\geq$ holds since $\dim_{\mathrm{FC}}$ is monotone. To see the other inequality, let μ be any Borel probability measure whose topological support K is a subset of $\bigcup_k A_k$. By Baire's category theorem the complete metric space K cannot be expressed as a countable union of nowhere dense sets, and since

$$K = \bigcup_k K \cap A_k$$

it follows that there must be some n such that $K \cap A_n = \overline{K \cap A_n}$ has non-empty interior in K. Thus there is an open subset V of $\mathbf{R}^d$ such that

$$\emptyset \neq K \cap V \subset K \cap A_n.$$

Let f be a non-negative C^∞-function that is 0 outside of V and satisfies $\mu(f) > 0$, and define the probability measure ν by

$$\mathrm{d}\nu = \frac{f}{\mu(f)} \,\mathrm{d}\mu\,.$$

Then

$$\dim_{\mathrm{F}} \mu \leq \dim_{\mathrm{F}} \nu \leq \dim_{\mathrm{FC}} A_n \leq \sup_k \dim_{\mathrm{FC}} A_k,$$

where the first inequality is by Lemma 1 and second inequality holds since ν gives full measure to A_n. Taking the supremum over all probability measures μ such that $\operatorname{supp} \mu \subset \bigcup_k A_k$ completes the proof. □

Example 7 below shows that the Fourier dimension is not countably stable, and also that the strict inequality in the assumption of Theorem 2 cannot be changed to a non-strict inequality. The following lemma is used in Example 7 and Example 8.

Lemma 6. *For any* $\varepsilon \in (0, 1]$,

$$\inf_\mu \sup_{j \geq 1} |\hat{\mu}(j)| \geq \frac{\pi \varepsilon}{8 + 2\pi\varepsilon} \quad \Big(\geq \frac{\varepsilon}{5}\Big),$$

where the infimum is over all $\mu \in \mathcal{P}([\varepsilon, 1])$ *and the supremum is over all positive integers* j.

Proof. Fix $\varepsilon > 0$ and take any $\mu \in \mathcal{P}([\varepsilon, 1])$. If φ is a real-valued continuous function supported on $[0, \varepsilon]$ such that

$$\int \varphi(x) \,\mathrm{d}x = 1 \quad \text{and} \quad \sum_{k=-\infty}^{\infty} |\hat{\varphi}(k)| < \infty$$

then

$$0 = \mu(\varphi) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)\overline{\hat{\mu}(k)} = 1 + 2\,\mathrm{Re}\Big(\sum_{k=1}^{\infty} \hat{\varphi}(k)\overline{\hat{\mu}(k)}\Big),$$

and thus

$$\frac{1}{2} \leq \sum_{k=1}^{\infty} |\hat{\varphi}(k)|\,|\hat{\mu}(k)| \leq \Big(\sum_{k=1}^{\infty} |\hat{\varphi}(k)|\Big) \sup_{j \geq 1} |\hat{\mu}(j)|.$$

Now let χ be the indicator function of $[0, \varepsilon/2]$ and take φ to be the triangle pulse

$$\varphi = \Big(\frac{2\chi}{\varepsilon}\Big) * \Big(\frac{2\chi}{\varepsilon}\Big),$$

where $*$ denotes convolution. Then

$$|\hat{\varphi}(k)| = \left|\frac{2\hat{\chi}(k)}{\varepsilon}\right|^2 = \operatorname{sinc}^2\Big(\frac{k\pi\varepsilon}{2}\Big) \le \min\Big(1, \frac{4}{k^2\pi^2\varepsilon^2}\Big),$$

so that

$$\sum_{k=1}^{\infty} |\hat{\varphi}(k)| \le \left\lceil \frac{2}{\pi\varepsilon} \right\rceil + \frac{4}{\pi^2\varepsilon^2} \sum_{\lceil \frac{2}{\pi\varepsilon}\rceil + 1}^{\infty} \frac{1}{k^2} \le \frac{2}{\pi\varepsilon} + 1 + \frac{4}{\pi^2\varepsilon^2} \int_{\frac{2}{\pi\varepsilon}}^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \frac{4 + \pi\varepsilon}{\pi\varepsilon}.$$

It follows that

$$\sup_{j \ge 1} |\hat{\mu}(j)| \ge \frac{1}{2} \cdot \frac{\pi\varepsilon}{4 + \pi\varepsilon}. \qquad \square$$

Example 7. Let $(l_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers such that

$$\lim_{k\to\infty} \frac{l_k}{k} = \infty.$$

Define the compact sets

$$A_k = \overline{\{x \in [0,1];\ x_{l_k+1} \ldots x_{l_k+k} \ne 0^k\}},$$

where $x = 0.x_1x_2\ldots$ is the binary decimal expansion of x, and let

$$B_n = \bigcap_{k=n}^{\infty} A_k.$$

Take any measure $\mu \in \mathcal{P}(B_n)$ and let μ_k be the image of μ under the map $x \mapsto 2^{l_k}x \pmod 1$. If $k \ge n$ then μ_k gives full measure to $[2^{-k}, 1]$, so by Lemma 6 there is some $j_k \ge 1$ such that

$$\hat{\mu}(2^{l_k} j_k) = \hat{\mu}_k(j_k) \ge \frac{2^{-k}}{5}.$$

Thus for any $s > 0$,

$$\limsup_{\xi\to\infty} |\hat{\mu}(\xi)|\,|\xi|^{s/2} \ge \lim_{k\to\infty} |\hat{\mu}(2^{l_k} j_k)|(2^{l_k} j_k)^{s/2} \ge \lim_{k\to\infty} \frac{2^{sl_k/2-k}}{5} = \infty.$$

It follows that $\dim_{\mathrm{F}}(B_n) = 0$ for all n.

Let λ be Lebesgue measure on $[0,1]$. Then for each n

$$\lambda\Big(\bigcup_{n=1}^{\infty} B_n\Big) \geq \lambda(B_n) \geq 1 - \sum_{k=n}^{\infty} 2^{-k} = 1 - 2^{-(n-1)},$$

so that $\lambda\left(\bigcup_{n=1}^{\infty} B_n\right) = 1$ and hence $\dim_{\mathrm{F}}\left(\bigcup_{n=1}^{\infty} B_n\right) = 1$. This shows that the Fourier dimension is not countably stable, and that $\dim_{\mathrm{FC}}$ is different from $\dim_{\mathrm{F}}$ since $\dim_{\mathrm{FC}}\left(\bigcup_{n=1}^{\infty} B_n\right) = 0$ by Proposition 5.

For $d \geq 2$, the sets $\{B_n \times [0,1]^{d-1}\}$ give a counterexample to countable stability of the Fourier dimension in $\mathbf{R}^d$. Moreover, these sets do not satisfy the conclusion of Theorem 2, but they would satisfy the assumption if the strict inequality was replaced by a non-strict inequality. Thus it is not possible to weaken the assumption of Theorem 2 in that way.

The following variation of Example 7 shows that the compact Fourier dimension is not finitely stable.

Example 8. Let $s \in (\sqrt{3}-1, 1)$ and choose b such that

$$\frac{1-s}{s} < b < \frac{s}{2}$$

(this is possible since $s > \sqrt{3}-1$). Let $(l_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let $m_k = \lceil b l_k \rceil$. Define the compact sets

$$A_k = \overline{\{x \in [0,1];\ x_{l_k+1} \dots x_{l_k+m_k} \neq 0^{m_k}\}},$$

where $x = 0.x_1 x_2 \dots$ is the binary decimal expansion of x, and let

$$B_n = \bigcap_{k=n}^{\infty} A_k.$$

Take any measure $\mu \in \mathcal{P}(B_n)$ and let μ_k be the image of μ under the map $x \mapsto 2^{l_k} x \pmod 1$. If $k \geq n$ then μ_k gives full measure to $[2^{-m_k}, 1]$, so by Lemma 6 there is some $j_k \geq 1$ such that

$$\hat{\mu}(2^{l_k} j_k) = \hat{\mu}_k(j_k) \geq \frac{2^{-m_k}}{5}.$$

Thus

$$\limsup_{\xi\to\infty} |\hat{\mu}(\xi)||\xi|^{s/2} \geq \lim_{k\to\infty} |\hat{\mu}(2^{l_k} j_k)|(2^{l_k} j_k)^{s/2} \geq \lim_{k\to\infty} \frac{2^{sl_k/2-m_k}}{5} = \infty,$$

and it follows that $\dim_{\mathrm{F}}(B_n) \leq s$ for all n. Consequently $\dim_{\mathrm{FC}}\left(\bigcup_{n=1}^{\infty} B_n\right) \leq s$ by Proposition 5.

Next consider the complement C of $\bigcup_{n=1}^{\infty} B_n$ in $[0,1]$, that is,

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} ([0,1] \setminus A_k).$$

For each k the set $[0,1] \setminus A_k$ is a union of 2^{l_k} intervals of length $2^{-(l_k+m_k)}$, and thus

$$\mathcal{H}^s_\delta([0,1] \setminus A_k) \leq 2^{l_k} \cdot 2^{-s(l_k+m_k)},$$

whenever k is so large that $l_k + m_k \geq -\log_2 \delta$. Thus for all large enough n,

$$\mathcal{H}^s_\delta(C) \leq \mathcal{H}^s_\delta\Big(\bigcup_{k=n}^{\infty}([0,1] \setminus A_k)\Big) \leq \sum_{k=n}^{\infty} 2^{(1-s)l_k - sm_k}.$$

The sum converges by the choice of (m_k), and since n can be taken arbitrarily large it follows that

$$\mathcal{H}^s_\delta(C) = 0.$$

for all $\delta > 0$. Thus

$$\dim_{\mathrm{FC}} C \leq \dim_{\mathrm{H}} C \leq s,$$

so that $[0,1]$ is the union of two sets with compact Fourier dimension strictly less than 1.

3. Stability of the Fourier dimension for measures

As mentioned in the introduction, finite stability of the Fourier dimension for sets would follow if it could be shown that

$$\dim_{\mathrm{F}}(\mu + \nu) = \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu) \tag{3}$$

for all finite Borel measures μ and ν. The inequality $\geq$ always holds, but Example 10 below shows that strict inequality is possible. The following lemma is used in that example.

Lemma 9. *Let α and β be two distinct real numbers. Then*

$$\left|\int_0^1 e^{2\pi i\alpha x}\sin(2\pi\beta x)\,\mathrm{d}x\right| \le \frac{1}{\big||\alpha|-|\beta|\big|}.$$

Proof. For any $\gamma > 0$,

$$\begin{aligned}\left|\int_0^1 e^{2\pi i\gamma x}\,\mathrm{d}x\right| &= \left|\int_0^{\frac{\lfloor\gamma\rfloor}{\gamma}} e^{2\pi i\gamma x}\,\mathrm{d}x + \int_{\frac{\lfloor\gamma\rfloor}{\gamma}}^1 e^{2\pi i\gamma x}\,\mathrm{d}x\right| \\ &= \left|\int_{\frac{\lfloor\gamma\rfloor}{\gamma}}^1 e^{2\pi i\gamma x}\,\mathrm{d}x\right| \le 1-\frac{\lfloor\gamma\rfloor}{\gamma} \le \frac{1}{\gamma}.\end{aligned}$$

From this together with the identity

$$e^{2\pi i\alpha x}\sin(2\pi\beta x) = \frac{i}{2}\left(e^{2\pi i(\alpha-\beta)x} - e^{2\pi i(\alpha+\beta)x}\right),$$

it follows that

$$\begin{aligned}\left|\int_0^1 e^{2\pi i\alpha x}\sin(2\pi\beta x)\,\mathrm{d}x\right| &\le \frac{1}{2}\left|\int_0^1 e^{2\pi i|\alpha-\beta|x}\,\mathrm{d}x\right| + \frac{1}{2}\left|\int_0^1 e^{2\pi i|\alpha+\beta|x}\,\mathrm{d}x\right| \\ &\le \frac{1}{2}\Big(\frac{1}{|\alpha-\beta|}+\frac{1}{|\alpha+\beta|}\Big) \\ &\le \frac{1}{|\,|\alpha|-|\beta|\,|}.\end{aligned}$$

□

Example 10. This example shows that the Fourier dimension for measures is not in general finitely stable.

Let

$$g(x) = 1 + \sum_{k=1}^{\infty} 2^{-k}\sin(2\pi\cdot 2^{k^2}x);$$

this is a continuous non-negative function. Define the probability measure μ on $[0,1]$ by $\mathrm{d}\mu = g\,\mathrm{d}x$. Using that

$$\begin{aligned}\int_0^1 e^{-2\pi i l x}\sin(2\pi l x)\,\mathrm{d}x &= \int_0^1 \cos(2\pi l x)\sin(2\pi l x)\,\mathrm{d}x - i\int_0^1 \sin^2(2\pi l x)\,\mathrm{d}x \\ &= \frac{-i}{2}\end{aligned}$$

for $l \in \mathbf{N}$, one sees that for $n \geq 1$

$$
\begin{aligned}
|\hat{\mu}(2^{n^2}) + i \cdot 2^{-(n+1)}| &\leq \sum_{\substack{k \geq 1 \\ k \neq n}} 2^{-k} \left| \int_0^1 e^{-2\pi i \cdot 2^{n^2} x} \sin(2\pi \cdot 2^{k^2} x) \, \mathrm{d}x \right| \\
&\overset{*}{\leq} \sum_{\substack{k \geq 1 \\ k \neq n}} \frac{2^{-k}}{|2^{n^2} - 2^{k^2}|} \\
&\leq \frac{\sum_{k=1}^{\infty} 2^{-k}}{2^{n^2} - 2^{(n-1)^2}} \\
&\leq \frac{2}{2^{n^2}},
\end{aligned}
$$

where the inequality at $*$ is by Lemma 9. Thus for any $s > 0$,

$$
\begin{aligned}
\limsup_{|\xi| \to \infty} |\hat{\mu}(\xi)| \, |\xi|^{s/2} &\geq \limsup_{n \to \infty} |\hat{\mu}(2^{n^2})| \cdot 2^{sn^2/2} \\
&\geq \lim_{n \to \infty} \left(2^{-(n+1)} - \frac{2}{2^{n^2}} \right) \cdot 2^{sn^2/2} \\
&= \infty,
\end{aligned}
$$

and it follows that $\dim_{\mathrm{F}} \mu = 0$.

Next, let

$$
h(x) = 1 - \sum_{k=1}^{\infty} 2^{-k} \sin(2\pi \cdot 2^{k^2} x)
$$

and define the probability measure ν on $[0, 1]$ by $\mathrm{d}\nu = h \, \mathrm{d}x$. Then $\dim_{\mathrm{F}} \nu = 0$ as well, but $\mu + \nu$ is twice Lebesgue measure, which has Fourier dimension 1.

Even though (3) does not hold in general, it does hold if μ and ν have different Fourier dimensions. For suppose that, say, $\dim_{\mathrm{F}} \mu < \dim_{\mathrm{F}} \nu$. Then for every $s \in (\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu)$ there is a sequence (ξ_k) with $|\xi_k| \to \infty$ such that

$$
\lim_{k \to \infty} |\hat{\mu}(\xi_k)| \, |\xi_k|^{s/2} = \infty \quad \text{and} \quad \lim_{k \to \infty} |\hat{\nu}(\xi_k)| \, |\xi_k|^{s/2} = 0,
$$

so that

$$\limsup_{|\xi|\to\infty} |\hat{\mu}(\xi) + \hat{\nu}(\xi)|\, |\xi|^{s/2} = \infty$$

and hence $\dim_F(\mu + \nu) \le s$. Thus

$$\dim_F(\mu + \nu) \le \inf\{s \in (\dim_F \mu, \dim_F \nu)\} = \dim_F \mu \le \dim_F(\mu + \nu).$$

For the same reason, any convex combination of μ and ν satisfies

$$\dim_F((1-\lambda)\mu + \lambda\nu) = \min(\dim_F \mu, \dim_F \nu).$$

Next suppose that $\dim_F \mu = \dim_F \nu = s$ and that there is some $\lambda_0 \in [0, 1]$ such that

$$\dim_F((1-\lambda_0)\mu + \lambda_0\nu) > s.$$

Then for any $\lambda \in [0, 1] \setminus \{\lambda_0\}$, the measure $(1-\lambda)\mu + \lambda\nu$ is a convex combination of $(1-\lambda_0)\mu + \lambda_0\nu$ and one of μ, ν, so it has Fourier dimension s. Thus there is at most one convex combination of μ and ν that has Fourier dimension greater than s.

The results in the rest of this section describe situations where (3) holds, or where it fails for at most one value of some parameter.

Proposition 11. *Let μ and ν be finite Borel measures on $\mathbf{R}^d$ whose supports are compact and disjoint. Then*

$$\dim_F(\mu + \nu) = \min(\dim_F \mu, \dim_F \nu).$$

Proof. Let f be a non-negative, smooth and compactly supported function that has the value 1 on $\operatorname{supp}\mu$ and the value 0 on $\operatorname{supp}\nu$. Then $f \, \mathrm{d}(\mu + \nu) = \mathrm{d}\mu$, so

$$\dim_F \mu \ge \dim_F(\mu + \nu)$$

by Lemma 1. Similarly,

$$\dim_F \nu \ge \dim_F(\mu + \nu),$$

and thus

$$\dim_F(\mu + \nu) \le \min(\dim_F \mu, \dim_F \nu).$$

The proposition now follows since the opposite inequality always holds. □

Proposition 12. *Let μ be a finite Borel measure on $\mathbf{R}^d$ with compact support and let μ_t be the translation of μ by $t \in \mathbf{R}^d$. Then*

$$\dim_{\mathrm{F}}(\mu + \mu_t) = \dim_{\mathrm{F}} \mu.$$

Proof. Since the Fourier dimension is translation invariant,

$$\dim_{\mathrm{F}}(\mu + \mu_t) \geq \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \mu_t) = \dim_{\mathrm{F}} \mu.$$

The opposite inequality clearly holds if $t = 0$, so assume that $t \neq 0$ and let n be an odd integer so large that $\operatorname{supp} \mu \cap \operatorname{supp} \mu_{nt} = \emptyset$. Note that

$$|\mu \hat{+} \mu_t(\xi)| = |(1 + e^{-2\pi i t\cdot\xi})\hat{\mu}(\xi)| = 2\,|\cos(\pi t \cdot \xi)||\hat{\mu}(\xi)|,$$

and similarly

$$|\mu \hat{+} \mu_{nt}(\xi)| = 2\,|\cos(\pi n t \cdot \xi)|\,|\hat{\mu}(\xi)|.$$

Since $\cos(nx)/\cos x$ is bounded, this gives

$$|\mu \hat{+} \mu_{nt}(\xi)| = 2\left|\frac{\cos(\pi n t \cdot \xi)}{\cos(\pi t \cdot \xi)}\right|\,|\cos(\pi t \cdot \xi)|\,|\hat{\mu}(\xi)| \lesssim |\mu \hat{+} \mu_t(\xi)|.$$

Thus

$$\dim_{\mathrm{F}}(\mu + \mu_t) \leq \dim_{\mathrm{F}}(\mu + \mu_{nt}) = \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \mu_{nt}) = \dim_{\mathrm{F}} \mu,$$

where the first equality is by Proposition 11. □

Proposition 13. *Let μ and ν be finite Borel measures on $\mathbf{R}^d$ with compact supports, and for $t \in \mathbf{R}^d$ let ν_t be the translation of ν by t. Then there is at most one t such that*

$$\dim_{\mathrm{F}}(\mu + \nu_t) > \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu).$$

Proof. It shall be shown that

$$\min(\dim_{\mathrm{F}}(\mu + \nu_{t_1}), \dim_{\mathrm{F}}(\mu + \nu_{t_2})) \leq \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu)$$

whenever $t_1 \neq t_2$. By the translation invariance of $\dim_{\mathrm{F}}$, this is equivalent to

$$\min(\dim_{\mathrm{F}}(\kappa + \nu), \dim_{\mathrm{F}}(\kappa + \nu_{\Delta})) \leq \min(\dim_{\mathrm{F}} \kappa, \dim_{\mathrm{F}} \nu),$$

where κ is the translation of μ by $-t_1$ and $\Delta = t_2 - t_1$. Suppose that s is less than the expression on the left – then $s < d$ and

$$\begin{cases} \hat{\kappa}(\xi) + \qquad\qquad \hat{\nu}(\xi) \lesssim |\xi|^{-s/2} \\ \hat{\kappa}(\xi) + e^{-2\pi i \Delta\cdot\xi}\, \hat{\nu}(\xi) \lesssim |\xi|^{-s/2}. \end{cases} \tag{4}$$

Let n be an integer so large that $\operatorname{supp}\kappa \cap \operatorname{supp}\nu_{n\Delta} = \emptyset$. Subtracting the second relation in (4) from the first gives

$$\sin(\pi\Delta\cdot\xi)\hat{\nu}(\xi) \lesssim (1 - e^{-2\pi i\Delta\cdot\xi})\hat{\nu}(\xi) \lesssim |\xi|^{-s/2},$$

and since $\sin(nx)/\sin x$ is bounded it follows that

$$(1 - e^{-2\pi i n\Delta\cdot\xi})\hat{\nu}(\xi) \lesssim \frac{\sin(\pi n\Delta\cdot\xi)}{\sin(\pi\Delta\cdot\xi)} \sin(\pi\Delta\cdot\xi)\hat{\nu}(\xi) \lesssim |\xi|^{-s/2}.$$

Subtracting this from the first relation in (4) then gives

$$\hat{\kappa}(\xi) + e^{-2\pi i n\Delta\cdot\xi}\, \hat{\nu}(\xi) \lesssim |\xi|^{-s/2},$$

and thus

$$s \le \dim_{\mathrm{F}}(\kappa + \nu_{n\Delta}) = \min(\dim_{\mathrm{F}}\kappa, \dim_{\mathrm{F}}\nu_{n\Delta}) = \min(\dim_{\mathrm{F}}\kappa, \dim_{\mathrm{F}}\nu),$$

where the first equality is by Proposition 11. □

Lemma 14. *Let $B \in \mathbf{R}^{d\times d}$ be an invertible matrix such that $|\lambda| \neq 1$ for all eigenvalues λ of B, and let $f : \mathbf{R}^d \to \mathbf{R}$ be a function such that $\lim_{|\xi|\to\infty} f(\xi) = 0$ and*

$$|f(\xi) - f(B\xi)| \lesssim |\xi|^{-\alpha}$$

for some $\alpha > 0$. Then

$$f(\xi) \lesssim |\xi|^{-\alpha}.$$

Proof. Using that $|B^{-1}\xi| \ge \|B\|^{-1}|\xi|$ one sees that

$$|f(B^{-1}\xi) - f(\xi)| \lesssim |B^{-1}\xi|^{-\alpha} \lesssim |\xi|^{-\alpha},$$

and thus there is a constant C such that

$$|f(\xi) - f(B\xi)| \le C|\xi|^{-\alpha} \quad \text{and} \quad |f(\xi) - f(B^{-1}\xi)| \le C|\xi|^{-\alpha}$$

for all ξ.

Let

$$V_u = \bigoplus_{|\lambda|>1} E_\lambda \quad \text{and} \quad V_s = \bigoplus_{|\lambda|<1} E_\lambda,$$

where E_λ denotes the (generalised) eigenspace of B corresponding to the eigenvalue λ. Then there is some $c > 1$ and an m such that if $k \geq m$ then $|B^k \xi| \geq c^k |\xi|$ for all $\xi \in V_u$ and $|B^{-k}\xi| \geq c^k |\xi|$ for all $\xi \in V_s$ (this need not be true with $m = 1$, for instance if B is a large Jordan block with diagonal entries slightly larger than 1).

Take any $\xi \in \mathbf{R}^d \setminus \{0\}$ and write it as $\xi = \xi_u + \xi_s$ with $\xi_u \in V_u$ and $\xi_s \in V_s$. Suppose first that $|\xi_u| \geq |\xi_s|$, or equivalently that $|\xi_u| \geq |\xi|/\sqrt{2}$. Then for any $n \geq m$,

$$\begin{aligned}
|f(\xi) - f(B^n\xi)| &\leq \sum_{k=0}^{n-1} |f(B^k\xi) - f(B^{k+1}\xi)| \\
&\leq C \sum_{k=0}^{n-1} |B^k \xi|^{-\alpha} \\
&\leq C \sum_{k=0}^{n-1} |B^k \xi_u|^{-\alpha} \\
&\leq C \Big(\sum_{k=0}^{m-1} \|B^{-1}\|^{k\alpha} + \sum_{k=m}^{\infty} c^{-k\alpha} \Big) |\xi_u|^{-\alpha} \\
&\leq C \Big(\sum_{k=0}^{m-1} \|B^{-1}\|^{k\alpha} + \sum_{k=m}^{\infty} c^{-k\alpha} \Big) 2^{\alpha/2}\, |\xi|^{-\alpha}.
\end{aligned}$$

Letting $n \to \infty$ shows that $|f(\xi)| \leq D|\xi|^{-\alpha}$ with

$$D = 2C \Big(\max \Big(\sum_{k=0}^{m-1} \|B^{-1}\|^{k\alpha}, \sum_{k=0}^{m-1} \|B\|^{k\alpha} \Big) + \sum_{k=m}^{\infty} c^{-k\alpha} \Big) 2^{\alpha/2}.$$

Similarly, if $|\xi_s| \geq |\xi_u|$ then for any $n \geq m$,

$$|f(\xi) - f(B^{-n}\xi)| \leq \cdots \leq C \Big(\sum_{k=0}^{m-1} \|B\|^{k\alpha} + \sum_{k=m}^{\infty} c^{-k\alpha} \Big) 2^{\alpha/2}\, |\xi|^{-\alpha},$$

and letting $n \to \infty$ shows that $|f(\xi)| \leq D|\xi|^{-\alpha}$ in this case as well. □

Proposition 15. *Let μ be a finite Borel measure on $\mathbf{R}^d$ with $\lim_{|\xi|\to\infty} \hat{\mu}(\xi) = 0$, and let $A \in \mathbf{R}^{d\times d}$ be an invertible matrix such that $|\lambda| \neq 1$ for all eigenvalues λ of A. Then*

$$\dim_{\mathrm{F}}(\mu + A\mu) = \dim_{\mathrm{F}} \mu.$$

Proof. Since the Fourier dimension is invariant under invertible linear transformations,

$$\dim_{\mathrm{F}}(\mu + A\mu) \geq \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}}(A\mu)) = \dim_{\mathrm{F}} \mu,$$

and in particular the lemma is true if $\dim_{\mathrm{F}}(\mu + A\mu) = 0$. To see the opposite inequality when $\dim_{\mathrm{F}}(\mu + A\mu) > 0$, take any $s \in (0, \dim_{\mathrm{F}}(\mu + A\mu))$ and let $B = A^T$, so that

$$\hat{A\mu}(\xi) = \hat{\mu}(A^T\xi) = \hat{\mu}(B\xi).$$

Then

$$\big||\hat{\mu}(\xi)| - |\hat{\mu}(B\xi)|\big| \leq |\hat{\mu}(\xi) + \hat{\mu}(B\xi)| \lesssim |\xi|^{-s/2},$$

so Lemma 14 applied to $f(\xi) = |\hat{\mu}(\xi)|$ says that $\hat{\mu}(\xi) \lesssim |\xi|^{-s/2}$ and therefore $s \leq \dim_{\mathrm{F}} \mu$. □

Proposition 16. *Let μ, ν be finite Borel measures on $\mathbf{R}^d$ such that $\lim_{|\xi|\to\infty} \hat{\mu}(\xi) = \lim_{|\xi|\to\infty} \hat{\nu}(\xi) = 0$, and let $A \in \mathbf{R}^{d\times d}$ be a matrix such that $\mathrm{Re}\,\lambda \neq 0$ for all eigenvalues λ of A. For $t \in \mathbf{R}$, let $\nu_t = \exp(tA)\nu$. Then there is at most one t such that*

$$\dim_{\mathrm{F}}(\mu + \nu_t) > \min(\dim_{\mathrm{F}} \mu, \dim_{\mathrm{F}} \nu).$$

Proof. The statement is trivially true if μ and ν have different Fourier dimensions, so assume that $\dim_{\mathrm{F}} \mu = \dim_{\mathrm{F}} \nu$. Take any distinct t_1, t_2 and suppose that

$$\begin{aligned} s &< \min(\dim_{\mathrm{F}}(\mu + \nu_{t_1}), \dim_{\mathrm{F}}(\mu + \nu_{t_2})) \\ &= \min(\dim_{\mathrm{F}}(\kappa + \nu), \dim_{\mathrm{F}}(\kappa + \nu_{t_2 - t_1})), \end{aligned}$$

where $\kappa = \exp(-t_1 A)\mu$. Then $s < d$ and

$$\begin{cases} \hat{\kappa}(\xi) + \hat{\nu}(\xi) \lesssim |\xi|^{-s/2} \\ \hat{\kappa}(\xi) + \hat{\nu}(B\xi) \lesssim |\xi|^{-s/2}, \end{cases}$$

where $B = \exp((t_2 - t_1)A)^T$, and subtracting the second relation from the first gives

$$\big||\hat{\nu}(\xi)| - |\hat{\nu}(B\xi)|\big| \leq |\hat{\nu}(\xi) - \hat{\nu}(B\xi)| \lesssim |\xi|^{-s/2}.$$

The matrix B has no eigenvalue on the unit circle, so $\hat{\nu}(\xi) \lesssim |\xi|^{-s/2}$ by Lemma 14 applied to $f(\xi) = |\hat{\nu}(\xi)|$. Thus $s \leq \dim_{\mathrm{F}} \nu$, which concludes the proof. □

4. The modified Fourier dimension

Recall that the modified Fourier dimension of a Borel set $A \subset \mathbf{R}^d$ is defined by

$$\dim_{\mathrm{FM}} A = \sup\{\dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(\mathbf{R}^d),\ \mu(A) > 0\}.$$

Theorem 17. *The modified Fourier dimension is monotone and countably stable, and satisfies* $\dim_{\mathrm{F}} A \le \dim_{\mathrm{FM}} A \le \dim_{\mathrm{H}} A$ *for any Borel set* $A \subset \mathbf{R}^d$.

Proof. If $A \subset B$ then $\{\mu \in \mathcal{P}(\mathbf{R}^d);\ \mu(A) > 0\} \subset \{\mu \in \mathcal{P}(\mathbf{R}^d);\ \mu(B) > 0\}$, so

$$\begin{aligned}\dim_{\mathrm{FM}} A &= \sup\{\dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(\mathbf{R}^d),\ \mu(A) > 0\}\\ &\le \sup\{\dim_{\mathrm{F}} \mu;\ \mu \in \mathcal{P}(\mathbf{R}^d),\ \mu(B) > 0\}\\ &= \dim_{\mathrm{FM}} B.\end{aligned}$$

Thus $\dim_{\mathrm{FM}}$ is monotone.

Let $\{A_k\}$ be a finite or countable family of Borel sets. For any $\mu \in \mathcal{P}(\mathbf{R}^d)$ such that $\mu(\bigcup A_k) > 0$ there must be some n such that $\mu(A_n) > 0$, and thus

$$\sup_k \dim_{\mathrm{FM}} A_k \ge \dim_{\mathrm{FM}} A_n \ge \dim_{\mathrm{F}} \mu.$$

Taking the supremum on the right over $\{\mu \in \mathcal{P}(\mathbf{R}^d);\ \mu(\bigcup A_k) > 0\}$ shows that

$$\sup_k \dim_{\mathrm{FM}} A_k \ge \dim_{\mathrm{FM}} \Big(\bigcup_k A_k\Big).$$

The opposite inequality holds by monotonicity.

It is obvious that $\dim_{\mathrm{F}} A \le \dim_{\mathrm{FM}} A$ since any $\mu \in \mathcal{P}(\mathbf{R}^d)$ that gives full measure to A in particular gives positive measure to A. The proof outlined in the introduction of the inequality $\dim_{\mathrm{F}} A \le \dim_{\mathrm{H}} A$ works without modification if $\dim_{\mathrm{F}}$ is replaced by $\dim_{\mathrm{FM}}$. □

The following two examples show that $\dim_{\mathrm{FM}}$ is not the same as either of $\dim_{\mathrm{F}}$ and $\dim_{\mathrm{H}}$.

Example 18. The sets B_n defined in Example 7 were shown to have Fourier dimension 0 but positive Lebesgue measure, and hence modified Fourier dimension 1.

Example 19. The middle-third Cantor set has modified Fourier dimension 0 (see the introduction), but Hausdorff dimension $\log 2/\log 3$.

5. Null sets of s-dimensional measures

Let

$$\mathcal{M}_s = \{\mu \in \mathcal{P}(\mathbf{R}^d);\ \dim_{\mathrm{FM}} \mu \geq s\}.$$

In this section it will be shown that $\mathcal{M}_s$ is characterised by its class of common null sets, or more precisely that

$$\mu \in \mathcal{M}_s \iff \mu(E) = 0 \quad \text{for all } E \in \mathcal{E}_s, \tag{5}$$

where

$$\mathcal{E}_s = \{E \in \mathcal{B}(\mathbf{R}^d);\ \mu(E) = 0 \text{ for all } \mu \in \mathcal{M}_s\}.$$

For $\mathcal{C} \subset \mathcal{P}(\mathbf{R}^d)$ and $\mathcal{E} \subset \mathcal{B}(\mathbf{R}^d)$ let

$$\mathcal{C}^{\perp} = \{E \in \mathcal{B}(\mathbf{R}^d);\ \mu(E) = 0 \text{ for all } \mu \in \mathcal{C}\}$$

$$\mathcal{E}^{\perp} = \{\mu \in \mathcal{P}(\mathbf{R}^d);\ \mu(E) = 0 \text{ for all } E \in \mathcal{E}\}.$$

Then $\mathcal{E}_s = \mathcal{M}_s^{\perp}$ and the condition (5) can be expressed as $\mathcal{M}_s = \mathcal{E}_s^{\perp}$, or equivalently as $\mathcal{M}_s^{\perp\perp} = \mathcal{M}_s$.

It is also natural to consider the sets

$$\mathcal{C}_s = \{\mu \in \mathcal{P}(\mathbf{R}^d);\ \text{there exists } \nu \in \mathcal{P}(\mathbf{R}^d) \text{ such that } \mu \ll \nu \text{ and } \hat{\nu}(\xi) \lesssim |\xi|^{-s/2}\}.$$

For $s \in (0, d]$ they are related to $\mathcal{M}_s$ by

$$\mathcal{M}_s = \bigcap_{t<s} \mathcal{C}_t$$

(this is also true for $s = 0$ if one allows negative t:s in the intersection).

The goal of this section is to prove the following theorem.

Theorem 20. *The sets $\mathcal{C}_s$ and $\mathcal{M}_s$ satisfy*

$$\mathcal{C}_s^{\perp\perp} = \mathcal{C}_s, \quad \text{for } 0 \leq s,$$
$$\mathcal{M}_s^{\perp\perp} = \mathcal{M}_s, \quad \text{for } 0 \leq s \leq d.$$

The theorem is trivial for $s = 0$ since

$$\mathcal{C}_0 = \mathcal{M}_0 = \mathcal{P}(\mathbf{R}^d)$$

and

$$\mathcal{C}_0^{\perp} = \mathcal{M}_0^{\perp} = \{\emptyset\}.$$

For general s, the first step in the proof is to reduce the problem using some properties of $\perp$ that are collected in the next lemma.

Lemma 21. *Let $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\{\mathcal{D}_\alpha\}_{\alpha\in I}$ be subsets of either $\mathcal{P}(\mathbf{R}^d)$ or $\mathcal{B}(\mathbf{R}^d)$. Then*

(i) $\mathcal{D} \subset \mathcal{D}^{\perp\perp}$;

(ii) $\mathcal{D}_1 \subset \mathcal{D}_2 \implies \mathcal{D}_2^\perp \subset \mathcal{D}_1^\perp$;

(iii) $\mathcal{D}^{\perp\perp\perp} = \mathcal{D}^\perp$;

(iv) $\bigcup_{\alpha\in I} \mathcal{D}_\alpha^\perp \subset \left(\bigcap_{\alpha\in I} \mathcal{D}_\alpha\right)^\perp$;

(v) $\bigcap_{\alpha\in I} \mathcal{D}_\alpha^\perp = \left(\bigcup_{\alpha\in I} \mathcal{D}_\alpha\right)^\perp$.

Property (iii) implies that $\mathcal{D}^{\perp(k+2)} = \mathcal{D}^{\perp k}$ for any $k \geq 0$, as illustrated by the following diagram.

$$\begin{array}{ccccccccc} \mathcal{D} & & \subset & & \mathcal{D}^{\perp\perp} & & = & & \mathcal{D}^{\perp\perp\perp\perp} & & \cdots \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\ & & \mathcal{D}^{\perp} & & = & & \mathcal{D}^{\perp\perp\perp} & & = & & \cdots \end{array}$$

Proof. Let $\mathcal{X}$ be the space that the $\mathcal{D}$:s are subsets of and let $\mathcal{Y}$ be the "dual" space, that is,

$$\mathcal{Y} = \begin{cases} \mathcal{B}(\mathbf{R}^d) & \text{if } \mathcal{X} = \mathcal{P}(\mathbf{R}^d), \\ \mathcal{P}(\mathbf{R}^d) & \text{if } \mathcal{X} = \mathcal{B}(\mathbf{R}^d). \end{cases}$$

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then the equation $\{x, y\} = \{\mu, E\}$ determines uniquely a measure μ and a set E. Thus it is possible to define

$$(x, y) = \mu(E), \quad \text{where } \{x, y\} = \{\mu, E\}.$$

(i) Let $x \in \mathcal{D}$. Then $(x, y) = 0$ for any $y \in \mathcal{D}^\perp$, which by definition of $\mathcal{D}^{\perp\perp}$ means that $x \in \mathcal{D}^{\perp\perp}$.

(ii) Let $y \in \mathcal{D}_2^\perp$. Then $(x, y) = 0$ for all $x \in \mathcal{D}_2$, and thus $(x, y) = 0$ for all $x \in \mathcal{D}_1$ since $\mathcal{D}_1 \subset \mathcal{D}_2$. Hence $y \in \mathcal{D}_1^\perp$.

(iii) Applying (ii) to the statement of (i) shows that $\mathcal{D}^{\perp\perp\perp} \subset \mathcal{D}^\perp$ and applying (i) to $\mathcal{D}^\perp$ shows that $\mathcal{D}^\perp \subset \mathcal{D}^{\perp\perp\perp}$.

(iv) By (ii) it follows that

$$\mathcal{D}_\alpha^\perp \subset \Big(\bigcap_{\alpha' \in I} \mathcal{D}_{\alpha'} \Big)^\perp$$

for any $\alpha \in I$, and hence

$$\bigcup_{\alpha \in I} \mathcal{D}_\alpha^\perp \subset \Big(\bigcap_{\alpha' \in I} \mathcal{D}_{\alpha'} \Big)^\perp.$$

(v) The definition of $\mathcal{D}^\perp$ can be expressed as

$$\mathcal{D}^\perp = \bigcap_{x \in \mathcal{D}} \{ y \in \mathcal{Y};\ (x, y) = 0 \} = \bigcap_{x \in \mathcal{D}} \{x\}^\perp,$$

and thus

$$\Big(\bigcup_{\alpha \in I} \mathcal{D}_\alpha \Big)^\perp = \bigcap_{x \in \bigcup_{\alpha \in I} \mathcal{D}_\alpha} \{x\}^\perp = \bigcap_{\alpha \in I} \bigcap_{x \in \mathcal{D}_\alpha} \{x\}^\perp = \bigcap_{\alpha \in I} \mathcal{D}_\alpha^\perp. \qquad \square$$

Once it is proved that $\mathcal{C}_s^{\perp\perp} = \mathcal{C}_s$ it follows by Lemma 21 that $\mathcal{M}_s^{\perp\perp} = \mathcal{M}_s$ as well, for then

$$\mathcal{M}_s^{\perp\perp} = \Big(\bigcap_{t<s} \mathcal{C}_t \Big)^{\perp\perp} \subset \Big(\bigcup_{t<s} \mathcal{C}_t^\perp \Big)^\perp = \bigcap_{t<s} \mathcal{C}_t^{\perp\perp} = \bigcap_{t<s} \mathcal{C}_t = \mathcal{M}_s$$

by (ii), (iv), and (v), and the opposite inclusion holds by (i). Moreover, if $\mathcal{D}$ has the form $\mathcal{D} = \mathcal{D}'^{\perp\perp}$ then (iii) gives

$$\mathcal{D}^{\perp\perp} = \mathcal{D}'^{\perp\perp\perp\perp} = \mathcal{D}'^{\perp\perp} = \mathcal{D}.$$

To prove Theorem 20, it thus suffices to show that $\mathcal{C}_s = \mathcal{C}_s'^{\perp\perp}$, where

$$\mathcal{C}_s' = \{ \nu \in \mathcal{P}(\mathbf{R}^d);\ \hat{\nu}(\xi) \lesssim |\xi|^{-s/2} \}.$$

It is easy to see that $\mathcal{C}_s \subset \mathcal{C}_s'^{\perp\perp}$ (see the first part of the proof of Theorem 20, on page 335), but the other inclusion takes a bit of work. The idea is to take an arbitrary measure $\mu \in \mathcal{C}_s'^{\perp\perp}$ and decompose it as $\mu = \mu_1 + \mu_2$ such that μ_1 is absolutely continuous to some measure in $\mathcal{C}_s'$ (thus $\mu_1 \in \mathcal{C}_s$) and μ_2 is singular to all measures in $\mathcal{C}_s'$, and then show that $\mu_2 = 0$ so that $\mu = \mu_1 \in \mathcal{C}_s$.

5.1. Decomposition of μ with respect to $\mathcal{C}'_s$

Definition. A set $\mathcal{C} \subset \mathcal{P}(\mathbf{R}^d)$ is *countably quasiconvex* if for any finite or infinite sequence (ν_k) in $\mathcal{C}$ there is a sequence (p_k) of positive numbers such that

$$\sum_k p_k = 1$$

and

$$\sum_k p_k \nu_k \in \mathcal{C}.$$

Thus any countable convex combination of measures in a countably quasiconvex set $\mathcal{C}$ is equivalent to some measure in $\mathcal{C}$.

Lemma 22. *For any $s \in \mathbf{R}$, the set $\mathcal{C}'_s$ is countably quasiconvex.*

Proof. If $\nu_1, \nu_2, \ldots \in \mathcal{C}'_s$, then there are constants $C_1, C_2, \ldots \geq 1$ such that for each k

$$|\hat{\nu}_k(\xi)| \leq C_k |\xi|^{-s/2} \quad \text{for all } \xi.$$

Now set

$$a_k = \frac{1}{2^k C_k} \quad \text{and} \quad p_k = \frac{a_k}{\sum_{i=1}^{\infty} a_i}.$$

Then the probability measure

$$\nu = \sum_{k=1}^{\infty} p_k \nu_k$$

satisfies

$$|\hat{\nu}(\xi)| \leq \sum_{k=1}^{\infty} p_k |\hat{\nu}_k(\xi)| \leq \sum_{k=1}^{\infty} p_k C_k |\xi|^{-s/2} \lesssim |\xi|^{-s/2},$$

so $\nu \in \mathcal{C}'_s$. □

Lemma 23. *Let $\mathcal{C} \subset \mathcal{P}(\mathbf{R}^d)$ be countably quasiconvex and let $\mu \in \mathcal{P}(\mathbf{R}^d)$. Then there is a set $E \in \mathcal{B}(\mathbf{R}^d)$ such that*

$$\mu\big|_E \ll \nu \quad \textit{for some } \nu \in \mathcal{C},$$

and

$$\mu\big|_{E^c} \perp \nu \quad \textit{for all } \nu \in \mathcal{C}.$$

Proof. Let

$$r = \sup\{\mu(F);\ F \in \mathcal{B}(\mathbf{R}^d) \text{ and } \mu\big|_F \ll \nu \text{ for some } \nu \in \mathcal{C}\},$$

and for $k = 1, 2, \ldots$ let $\nu_k \in \mathcal{C}$ and $F_k \in \mathcal{B}(\mathbf{R}^d)$ be such that $\mu\big|_{F_k} \ll \nu_k$ and $\mu(F_k) \geq r - 1/k$. Set

$$E = \bigcup_{k=1}^{\infty} F_k.$$

By assumption, there is a sequence (p_k) of positive numbers such that

$$\sum_{k=1}^{\infty} p_k \nu_k \in \mathcal{C},$$

and since all p_k are positive, $\mu\big|_E$ is absolutely continuous with respect to this measure.

Suppose towards a contradiction that there is some $\nu \in \mathcal{C}$ such that $\mu\big|_{E^c} \not\perp \nu$. Then by Lebesgue decomposition of $\mu\big|_{E^c}$ with respect to ν there is a Borel set $S \subset E^c$ such that

$$\mu\big|_S \ll \nu \quad \text{and} \quad \mu(S) > 0.$$

For each k, there is a $\lambda_k \in (0,1)$ such that

$$(1 - \lambda_k)\nu + \lambda_k \nu_k \in \mathcal{C},$$

and $\mu\big|_{S \cup F_k}$ is absolutely continuous with respect to this measure. Moreover,

$$\mu(S \cup F_k) = \mu(S) + \mu(F_k) \geq \mu(S) + r - \frac{1}{k}$$

which is greater than r for large enough k – this is a contradiction. □

5.2. Proof of Theorem 20. The following theorem by Goullet de Rugy [6] is used in the proof.

Theorem 24 (Goullet de Rugy). *Let T be a compact Hausdorff space and let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{P}(T)$ of the form*

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{A}_k \quad \text{and} \quad \mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k,$$

where the $\mathcal{A}_k$ and $\mathcal{B}_k$ are weak-$*$ compact and convex, such that $\mu \perp \nu$ for all $\mu \in \mathcal{A}$, $\nu \in \mathcal{B}$. Then there exist disjoint $F_{\sigma\delta}$-sets $T_1, T_2 \subset T$ such that $\mu(T_1) = 1$ for all $\mu \in \mathcal{A}$ and $\nu(T_2) = 1$ for all $\nu \in \mathcal{B}$.

Proof of Theorem 20. As remarked in the beginning of the section, it suffices to show that $\mathcal{C}_s'^{\perp\perp} = \mathcal{C}_s$. For any $\mu \in \mathcal{C}_s$ there is some $\nu \in \mathcal{C}_s'$ such that $\mu \ll \nu$. Then any $E \in \mathcal{C}_s'^{\perp}$ is a null set for ν and hence also for μ, which means that $\mu \in \mathcal{C}_s'^{\perp\perp}$. Thus $\mathcal{C}_s \subset \mathcal{C}_s'^{\perp\perp}$.

To see the other inclusion, take any $\mu \in \mathcal{C}_s'^{\perp\perp}$. By Lemma 22 and Lemma 23, it is possible to write $\mu = \mu_1 + \mu_2$ such that μ_1 is absolutely continuous to some measure in $\mathcal{C}_s'$ (hence $\mu_1 \in \mathcal{C}_s$) and μ_2 is singular to all measures in $\mathcal{C}_s'$. It will be shown that $\mu_2 = 0$, from which it follows that $\mu = \mu_1 \in \mathcal{C}_s$.

For $N, R \in \{1, 2, \dots\}$ let

$$\mathcal{B}_R^N = \{\nu \in \mathcal{P}([-R, R]^d);\ \hat{\nu}(\xi) \le N|\xi|^{-s/2} \text{ for all } \xi\}.$$

These sets are weak-$*$ compact and convex, so Theorem 24 applied to

$$\mathcal{A}_R = \{\mu_2|_{[-R,R]^d}\} \quad \text{and} \quad \mathcal{B}_R = \bigcup_{N=1}^{\infty} \mathcal{B}_R^N$$

gives for each R a Borel set E_R such that $\mu_2(E_R) = 0$ and

$$\nu(\mathbf{R}^d \setminus E_R) = \nu([-R, R]^d \setminus E_R) = 0 \quad \text{for all } \nu \in \mathcal{B}_R.$$

Let

$$E = \bigcup_{R=1}^{\infty} E_R.$$

Then $\mu_2(E) = 0$ and $\nu(E^c) = 0$ for all $\nu \in \mathcal{C}_s'$ that have compact support.

If ν is *any* measure in $\mathcal{C}_s'$, define the measure ν_R by $\mathrm{d}\nu_R = \varphi_R \,\mathrm{d}\nu$, where φ_R for each R is a smooth function such that $\chi_{[-R,R]^d} \le \varphi_R \le \chi_{[-2R,2R]^d}$. Then each ν_R lies in $\mathcal{C}_s'$ by Lemma 1 and ν_R has compact support, so

$$\nu(E^c) = \lim_{R\to\infty} \nu(E^c \cap [-R, R]^d) \le \lim_{R\to\infty} \nu_R(E^c) = 0.$$

This shows that $E^c \in \mathcal{C}_s'^{\perp}$ and hence $\mu(E^c) = 0$. Since also $\mu_2(E) = 0$, this implies that $\mu_2 = 0$. □

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