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Kneading with weights

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Abstract. We generalize Milnor–Thurston's kneading theory to the setting of piecewise continuous and monotone interval maps with a weight associated to each branch. We define a weighted kneading determinant $\mathcal{D}(t)$ and establish combinatorially two kneading identities, one with the cutting invariant and one with the dynamical zeta function. For the pressure $\log \rho_1$ of the weighted system, playing the role of entropy, we prove that $\mathcal{D}(t)$ is non-zero when $|t| < 1/\rho_1$ and has a zero at $1/\rho_1$. Furthermore, our map is semi-conjugate to every map in an analytic family $\mathcal{S}_t, 0 < t < 1/\rho_1$ of piecewise linear maps with slopes proportional to the prescribed weights and defined on a Cantor set. When the original map extends to a continuous map f, the family $\mathcal{S}_t$ converges as $t \to 1/\rho_1$ to a continuous piecewise linear interval map $\tilde{f}$. Furthermore, f is semi-conjugate to $\tilde{f}$ and the two maps have the same pressure.

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1. Introduction

Let $I = [a, b]$. Let $a = c_0 < c_1 < \cdots < c_{\ell+1} = b$. Set $S = \{0, 1, \cdots, \ell\}$. For each $i \in S$, set $I_i =]c_i, c_{i+1}[$ and let $f_i : I_i \to I$ be a strictly monotone continuous map extending continuously to the closure, and finally assign a constant weight $g_i \in \mathbb{C}$.

We say that $(I_i, f_i, g_i)_{i \in S}$ is a *weighted system*. In the particular case that each g_i equals 1, we say also that the system is *unweighted*.

Milnor and Thurston [6] developed a widely used kneading theory on unweighted systems so that the maps f_i glue together to a single continuous map f. Let us recall a list of their results (see also Hall, [5] for an enlightening introduction to the subject).

Milnor and Thurston introduced a power series matrix $\mathcal{N}(t)$, called the *kneading matrix*, which records combinatorially the forward orbits of the cutting points. They establish two identities:

1. the *main kneading identity,* relating $\mathcal{N}(t)$ to the growth of the cutting points of f^n on any subinterval J, and taking the form

$$\gamma_J(t) \cdot \mathcal{N}(t) = \text{terms involving boundaries of } J;$$

2. the *zeta-function identity,* relating $\mathcal{N}(t)$ to a dynamical Artin-Mazur zeta function that counts the global growth of the f^n-fixed points, taking the form

$$\zeta(t) \cdot \det \mathcal{N}(t) = 1.$$

Using these identities, Milnor–Thurston derive the following important consequences:

3. for $\log s$ the topological entropy of the map, the matrix $\mathcal{N}(t)$ is invertible when $|t| < 1/s$. If $s > 1$ the matrix $\mathcal{N}(t)$ is singular at $t = 1/s$ and the growth rate of the periodic points is precisely s;

4. if $s > 1$, the map is semi-conjugate to a simple model dynamical system which is a continuous PL (i.e. piecewise-linear) map of slope s.

Most of this theory has been extended by Preston [8] to the general unweighted setting without the assumption of global continuity. An advantage to allow discontinuity at the cutting points is that one can treat tree and graph maps as interval unweighted systems after edge concatenation. See for example Tiozzo [9]. There exist also works that treat tree maps as they are. See for example Alves and Sousa-Ramos [1], Baillif [2] and Baillif and de Carvalho [3].

An essential difference in Preston's approach as compared to Milnor–Thurston's lies in the proof of the zeta-function identity. Preston's method is purely combinatorial whereas the original proof tests on a concrete example and then studies behaviors under perturbations.

In this work we will generalize all four results above to weighted systems, where the pressure $\log \rho_1$ will play the role of entropy. Points 1–4 will become Theorems 2.1, 2.2, 2.3, and 2.5 below.

Our setting is identical to that of Baladi and Ruelle [4]. In their work they define a weighted kneading matrix $\mathcal{B}$ and a weighted zeta function, and establish a version the zeta-function identity using a perturbative method similar to that of Milnor–Thurston. For our purpose we will define a somewhat different kneading matrix $\mathcal{R}$.

We will not rely on previous established results but instead provide self-contained proofs. In a way our results recover partially results in [4, 6, 8].

Our proofs will be fairly elementary, with, as the only background, some basic knowledge of complex analysis. The rest is to play carefully with the combinatorics of iterations, following mostly Milnor and Thurston.

There is however a notable exception, concerning the proof of the zeta-function identity. For this we choose to follow the combinatorial method of Preston, along with several significant differences. Preston cuts off the graph above the diagonal in order to count the intersections, instead we keep the graph intact but change signs across the diagonal. Preston's kneading matrix is similar to that of Milnor–Thurston, by recording the sequence of visited intervals of a critical orbit. Instead we take the kneading matrix $\mathcal{B}$ of Baladi-Ruelle, which records the orbit's position relative to every given critical point. We then add one more dimension to $\mathcal{B}$ to obtain our kneading matrix $\mathcal{R}$, by incorporating the influence of the boundary points (with a somewhat different choice of sign). These modifications are designed to simplify, even in the unweighted case, Preston's proof of the zeta-function identity. Preston's idea is to express $-(\log \zeta(t))'_t$ as the trace of a certain matrix $\mathcal{F}$, and then use repeatedly the main kneading identity to connect $\mathcal{F}$ with the derivative of the kneading matrix. Here, many choices are possible but most give rise to additional correcting terms. Having tested various possibilities we came up with the current choice of the kneading matrix $\mathcal{R}$ and a matrix $\mathcal{F}$ for which we have the simplest relation possible, i.e. $\mathcal{F}\mathcal{R} = \mathcal{R}'$ (see Theorem 4.1). Once this relation established, the zeta-function identity will follow from a one-line computation (cf. Section 4.1),

$$-\frac{d}{dt}\log\zeta(t) = \operatorname{Tr}\mathcal{F} = \operatorname{Tr}\mathcal{R}'\mathcal{R}^{-1} = \frac{d}{dt}\log\det\mathcal{R},$$

and the fact that $\zeta(0) = \det\mathcal{R}(0) = 1$.

The kneading matrix and its smallest positive zero cost relatively little to evaluate. This enables a fast and accurate computation of the pressure/entropy as well as the semi-conjugacy and the PL model map. While experimenting these ideas we noticed that the system is also semi-conjugate to a PL map for every $0 < t < \rho_1$, although the conjugated system acts on a Cantor set instead of an interval. This numerical observation can easily be proved and has now become our Theorem 2.4. To the best of our knowledge this statement is new, also in the unweighted setting, even though its proof does not require any new ideas.

A further justification of our choice of the kneading determinant $\det\mathcal{R}$ as compared to $\det\mathcal{B}$, is that the latter may have a spurious small zero unrelated to the pressure (in Appendix C we give an example).

Another originality of this work is the systematic treatment of point-germs relative to points. Each point x in the interior of the interval generates two point-germs: x^+ and x^-. They have often distinct dynamical behavior and it is convenient to treat the two germs independently. The idea is certainly present to all the papers

in the theory. But highlighting the notion transforms our computations in more concise forms.

Why adding weights to piecewise continuous and monotone maps? One motivation is that one can prescribe slope ratios for the PL model maps, the other is that one can choose to ignore some parts of a dynamical system by assigning zero weights, so to reveal deeper entropies hidden for example in renormalisation pieces.

A further application, not pursued in the current work, is to construct various invariant measures by playing with weights and following Preston's construction of measures maximizing the entropy.

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2. Notation and results

Let $I = [a, b]$. Let $a = c_0 < c_1 < \cdots < c_{\ell+1} = b$. Set $I_i =]c_i, c_{i+1}[$ and let $f_i : I_i \to I$ be strictly monotone continuous maps for $i = 0, \cdots, \ell$. We write

$$f = (f_0|_{I_0}, \cdots, f_\ell|_{I_\ell})$$

and let

$$s_i = +1 \quad \text{(respectively } s_i = -1)$$

when f_i is increasing (respectively decreasing). We consider f as undefined at the cutting points. On the other hand, each f_i extends to a continuous map from the closed interval $[c_i, c_{i+1}]$ to $[a, b]$.

We call

$$\mathcal{C}(f) = \{c_i : 1 \le i \le \ell\}$$

the *interior cutting points* of the interval. We write $\mathcal{C}_*(f)$ for the set of cutting points including c_0 and $c_{\ell+1}$.

In order to treat monotonicity and discontinuities in a consistent manner it is convenient to extend our base interval I to its unit-tangent bundle, also denoted the space of point-germs $\widehat{I}$: each interior point $x \in I \smallsetminus \{a, b\}$ generates two point-germs denoted

$$x^+ = (x, +1) \quad \text{and} \quad x^- = (x, -1)$$

while the boundary points a, b each has only one point-germ associated, a^+ and b^-. $\widehat{I}$ is the union of these point germs. We write

$$\varepsilon(x^+) := 1 \quad \text{and} \quad \varepsilon(x^-) := -1$$

for the direction of a point germ. In order to make some formulae in Section 4 more concise, we set (artificially) $c_0^- := b^-$, so that $\{c_0^+, c_0^-\} = \{a^+, b^-\}$. For $x \in I$ we denote by $\hat{x} = (x, \sigma)$ the point-germ based at x and in the direction $\sigma \in \{\pm 1\}$.

It is notationally convenient to define an order $<$ on the collection of point-germs together with base points, $I \cup \widehat{I}$, by declaring that for two base points $x < y$ we have $x < x^+ < y^- < y < y^+$. Given two point-germs $\hat{u}, \hat{v} \in \widehat{I}$ with $\hat{u} < \hat{v}$, we define

$$\langle \hat{u}, \hat{v} \rangle := \{x \in I \mid \hat{u} < x < \hat{v}\}$$

as a sub interval of I. It is then consistent to write e.g. $[u, v[= \langle u^-, v^- \rangle$ and $]u, v[= \langle u^+, v^- \rangle$. Note that the boundary points a, b never belong to an interval of the form $\langle \hat{u}, \hat{v} \rangle$. When $J =]u, v[$ is an open interval we set

$$\widehat{J} = \{\hat{x} : u < x < v\} \cup \{u^+\} \cup \{v^-\}.$$

In particular,

$$\widehat{I}_i = \{\hat{x} : c_i < x < c_{i+1}\} \cup \{c_i^+\} \cup \{c_{i+1}^-\}, \quad 0 \le i \le \ell.$$

We observe that the $\widehat{I}_i$'s are disjoint and their union is $\widehat{I}$.

Our original map f induces a well-defined map

$$\hat{f} : \widehat{I} \longrightarrow \widehat{I}.$$

When $\hat{x} = (x, \sigma) \in \widehat{I}_i$ then $f(\hat{x}) = (y, \sigma')$ is simply the germ based at $y = \lim_{t \to 0^+} f_i(x + \sigma t)$ whose direction is $\sigma' = s_i \sigma$. Note that on each $\widehat{I}_i$, $\hat{f}$ is monotone because f is *strictly* monotone. We will usually write f also for the extended map $\hat{f}$.

For each $0 \le i \le \ell$ we let $g_i \in \mathbb{C}$ be a weight associated with the interval I_i. Both g_i and s_i gives rise to functions on $\widehat{I}_i$ by declaring $s(\hat{x}) = s_i$ and $g(\hat{x}) = g_i$ whenever $\hat{x} \in \widehat{I}_i$. We may define products along orbits, $s^n, g^n, [sg]^n$ by setting

$$s^0 = g^0 = 1,$$

and, for all $n \geq 1$,

$$s^n(\hat{x}) := \prod_{k=0}^{n-1} s(f^k(\hat{x})), \quad g^n(\hat{x}) := \prod_{k=0}^{n-1} g(f^k(\hat{x})), \quad [sg]^n := s^n g^n.$$

Note that $s^n(\hat{x})$ is the sense of monotonicity of f^n at $\hat{x}$.

We define a half-sign function

$$\sigma(\hat{x}, y) := \frac{1}{2}\operatorname{sgn}(\hat{x} - y) = \begin{cases} +1/2 & \text{if } \hat{x} > y, \\ -1/2 & \text{if } \hat{x} < y, \end{cases}$$

for all $\hat{x} \in \hat{I},\ y \in I$.

Concerning forward orbits of point-germs we set, for $j, k = 0, \cdots, \ell$,

$$\theta(\hat{x}, t; c_k) = \sum_{m \geq 0} t^m [sg]^m(\hat{x}) \cdot \sigma(f^m \hat{x}, c_k),$$

so, in particular,

$$\theta(\hat{x}, t; c_0) = \sum_{m \geq 0} t^m [sg]^m(\hat{x}) \cdot \sigma(f^m \hat{x}, c_0) = \frac{1}{2} \sum_{m \geq 0} t^m [sg]^m(\hat{x})$$

$$\varepsilon^*(\hat{c}) = \begin{cases} \varepsilon(\hat{c}) & \text{if } \hat{c} \neq c_0^{\pm}, \\ +1 & \text{if } \hat{c} = c_0^{\pm}, \end{cases}$$

$$R_{jk}(t) = \sum_{\hat{c}_j = c_j^+, c_j^-} \varepsilon^*(\hat{c}_j) \cdot \theta(\hat{c}_j, t; c_k),$$

$$\mathcal{R}(t) = (R_{jk}(t))_{0 \leq j,k \leq \ell} \quad \text{(the } \textit{kneading matrix}\text{)}$$

$$\mathcal{B}(t) = (R_{jk}(t))_{1 \leq j,k \leq \ell} \quad \text{(the } \textit{reduced kneading matrix}\text{)}$$

In particular, one has (note the signs)

$$R_{jk}(t) = \theta(c_j^+, t; c_k) - \theta(c_j^-, t; c_k) =: \Delta_{c_j} \theta(\cdot, t; c_k) \quad (j > 0),$$

while

$$R_{0k}(t) = \theta(a^+, t; c_k) + \theta(b^-, t; c_k)$$

(this choice of signs is designed to absorb boundary correcting terms in later calculations).

Regarding 'backward'-orbits we define Z_1 as the set of level-1 cylinders

$$(j) := I_j =]c_j, c_{j+1}[, \quad j = 0, 1, \cdots, \ell.$$

Define then recursively Z_n as the set of non-empty level-n cylinders of the form

$$(i_0 i_1 \cdots i_{n-1}) := I_{i_0} \cap f_{i_0}^{-1}(i_1 \cdots i_{n-1}).$$

Each $\alpha = (i_0 i_1 \cdots i_{n-1})$ is an open interval $]u, v[= \langle u^+, v^- \rangle$. We set

$$\widehat{\partial}\alpha = \{u^+, v^-\}.$$

For $0 \le j < n$, $f^j(\alpha) \subset I_{i_j}$. So f^n maps α homeomorphically onto its image, in particular each of the functions s^j and g^j, $0 \le j < n$, is constant on α.

Definition 2.1. We call $(I_i, f_i)_{0 \le i \le \ell}$ *expansive* if

$$\lim_{n \to \infty} \sup_{\alpha \in Z_n} \operatorname{diam}(\alpha) = 0.$$

For any $y \in I$, set $\Gamma_{0,y} = \{y\}$, and for $p > 0$,

$$\Gamma_{p,y} = \Big\{ x \in \bigcup_{\alpha \in Z_p} \alpha \mid f^p(x) = y \Big\}.$$

Note that $x \in \Gamma_{p,y}$ implies that $g^p(x^-) = g^p(x^+)$, for which we simply write $g^p(x)$. This is because $g^0(x) \equiv 1$ and every j-iterate $(0 \le j < p)$ of a p-cylinder $\alpha \in Z_p$ belongs to some level-1 cylinder. Define

$$\gamma_y(t) = \sum_{p \ge 0} \sum_{x \in \Gamma_{p,y}} t^p g^p(x)$$

and

$$\gamma_{y,J}(t) = \sum_{p \ge 0} \sum_{x \in \Gamma_{p,y}} t^p g^p(x) \chi_J(x), \qquad \text{for } J \subset]a, b[,$$

where χ_J is the characteristic function of the set J. These functions count the (weighted) number[1] of preimages of y.

Clearly when J and J' are disjoint subsets we have

$$\gamma_{y,J}(t) + \gamma_{y,J'}(t) = \gamma_{y,J \cup J'}(t).$$

[1] In the case $g_i \equiv 1$, we have $\gamma_{y,J}(t) = \sum_{p \ge 0} \#(\Gamma_{p,y} \cap J)\, t^p$.

Theorem 2.1 (main kneading identity, or MKI in short). *For any interval* $J = \langle \hat{u}, \hat{v} \rangle$ *in* I, *for all* $k \in \{0, \cdots, \ell\}$,

$$\sum_{j=1}^{\ell} \gamma_{c_j,J}(t) R_{jk}(t) = \theta(\hat{v}, t; c_k) - \theta(\hat{u}, t; c_k) =: \Delta_J \theta(\cdot, t; c_k)$$

(*the term* $j = 0$ *is not included in the sum, but we do allow* $k = 0$).

We also need a particular way to count the fixed points of f^n. Fix $n \geq 1$ and an n-cylinder α. The value of $g^n(x)$ is a constant on α, denoted by $g^n_{|\alpha}$. We define a (fixed point counting) weight by

$$\omega(\alpha) = -g^n_{|\alpha} \sum_{\hat{x} \in \hat{\partial}\alpha} \sigma(f^n \hat{x}, x) \cdot \varepsilon(f^n \hat{x}).$$

We refer to Appendix A for an account of the geometric meaning of this weight, notably its relation to fixed points of f^n. Introduce then

$$N_n = \sum_{\alpha \in Z_n} \omega(\alpha) \tag{1}$$

as well as the corresponding zeta-function

$$\zeta(t) = \exp\Big(\sum_{n \geq 1} \frac{1}{n} N_n t^n \Big). \tag{2}$$

Theorem 2.2. *We have the identity* (*as formal power-series*) *between the zeta-function and the* (*Milnor–Thurston*) *determinant*

$$\zeta(t) \cdot \det \mathcal{R}(t) = 1.$$

Definition 2.2. For every $n \geq 0$ we write

$$\|g^n\|_\infty = \sup_{\alpha \in Z_n} |g^n_{|\alpha}| \quad \text{and} \quad \|g^n\|_1 = \sum_{\alpha \in Z_n} |g^n_{|\alpha}|.$$

We then set

$$\rho_\infty := \limsup_{n \to \infty} \|g^n\|_\infty^{1/n} \leq \rho_1 := \limsup_{n \to \infty} \|g^n\|_1^{1/n}.$$

We also call $\log \rho_1$ the *pressure*[2] of the weighted system $(I_i, f_i, g_i)_{i \in S}$. This is consistent with usual "thermodynamic formalism" for dynamical systems.

[2] In the case $g_i \equiv 1$, we have $\rho_\infty = 1$ and ρ_1 is the growth rate of the n-cylinders. By Misiurewicz-Szlenk ([7]) $\log \rho_1$ is equal to the topological entropy of the unweighted system.

Theorem 2.3. (1) *The power series for* $\theta(\hat{x}, t; c_k)$, $R_{jk}(t)$ *define analytic functions of* t *on the disc* $\{|t| < 1/\rho_\infty\}$.

(2) *The kneading matrix* $\mathcal{R}(t)$ *is invertible when* $|t| < 1/\rho_1$.

(3) *Suppose that* $\rho_1 > \rho_\infty$ *and all* $g_i \geq 0$. *Then* $\mathcal{R}(t)$ *is non-invertible at* $t = 1/\rho_1$ *and* $1/\rho_1$ *coincides with the radius of convergence of* $\zeta(t)$.

Theorem 2.4. *Assume* $\rho_1 > \rho_\infty$ *and all* $g_i > 0$. *For each* $0 < t < 1/\rho_1$ *there is a monotone* (*non-continuous*) *map*

$$\phi_t : \widehat{I} \longrightarrow [0, 1]$$

with the following properties.

A. *For* $0 \leq i \leq \ell$, *let*

$$\widetilde{I}_{t,i} = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$$

(*which is an interval or a point*). *The collection* $\widetilde{I}_{t,i}$, $0 \leq i \leq \ell$ *is pairwise disjoint.*

B. *For each* i *there is an affine map*

$$\tilde{f}_{t,i} : \widetilde{I}_{t,i} \longrightarrow [0, 1]$$

of slope $s_i/(tg_i)$ *such that*

$$\tilde{f}_{t,i}(\phi_t(x)) = \phi_t(f(x)), \quad x \in \widehat{I}_i.$$

C. *Let*

$$\mathcal{S}_t : \bigcup_i \widetilde{I}_{t,i} \longrightarrow [0, 1]$$

be given by

$$\mathcal{S}_t(\xi) = \tilde{f}_{t,i}(\xi)$$

when $\xi \in \widetilde{I}_{t,i}$. *Then* $\mathcal{S}_t$ *is uniformly expanding* (*cf. Lemma* 5.4) *and its maximal invariant domain is*

$$\Omega_t = \phi_t(\widehat{I}).$$

Remark. Thus, ϕ_t is semi-conjugating the dynamical system $(\widehat{I}, f)$ to the uniformly expanding map $\mathcal{S}_t$ restricted to Ω_t. Often, Ω_t is a Cantor set but in some cases it is not, as it may contain isolated points. In particular, a subset $\phi_t(\widehat{I}_i)$ is trivial, i.e. reduced to an isolated point, precisely when the forward orbit of I_i

never encounters a cutting point. This does not happen if the original system is expansive.

The proof will show that the semi-conjugacy ϕ_t can be explicitly expressed as

$$\frac{h(\hat{x}) - h(a^+)}{h(b^-) - h(a^+)} \quad \text{with } h(\hat{x}) = (\theta(\hat{x}, t; c_k))_{k=0,\cdots\ell} \cdot \mathcal{R}^{-1} \cdot \begin{pmatrix} 0 \\ G(c_1, t) \\ \vdots \\ G(c_\ell, t) \end{pmatrix},$$

where $G(x, t)$ is the average of the generating functions for $g^n(x^-)$ and $g^n(x^+)$ (see (6), (10) and (13)). If $\ell = 1$, one can replace $h(\hat{x})$ by $\theta(\hat{x}, t; c_1)$, which is particularly simple to implement numerically.

When taking the limit as $t \nearrow 1/\rho_1$ we obtain a different type of semi-conjugacy.

Theorem 2.5. *Assume $\rho_1 > \rho_\infty$ and all $g_i \geq 0$. There is a monotone continuous surjective map*

$$\phi \colon I \longrightarrow [0, 1]$$

with the following properties. Denote by

$$\widetilde{S} \subset S := \{0, \cdots, \ell\}$$

the subset of i's for which $\widetilde{I}_i = \operatorname{Int} \phi(I_i)$ is non-empty.

A. *For every $i \in \widetilde{S}$, there is an affine map $\tilde{f}_i$ of slope $s_i \rho_1 / g_i$ such that*

$$\tilde{f}_i(\phi(x)) = \phi(f_i(x)), x \in I_i.$$

B. *The two weighted systems $(I_i, f_i, g_i)_{i \in S}$ and $(\widetilde{I}_i, \tilde{f}_i, g_i)_{i \in \widetilde{S}}$ have equal pressures.*

C. *If the sytem $f = (I_i, f_i)_{i \in S}$ extends to a continuous map on $[a, b]$ then so does $\tilde{f} = (\widetilde{I}_i, \tilde{f}_i)_{i \in \widetilde{S}}$ on $[0, 1]$ and ϕ gives a genuine topological semi-conjugacy. We have in this case, for every $x \in [a, b]$,*

$$\tilde{f}(\phi(x)) = \phi(f(x)).$$

Furthermore, the map $\tilde{f}$ is uniformly expanding.

For the last theorem, some intervals may disappear under the semi-conjugacy, i.e. the set $\widetilde{S}$ becomes a strict subset of S. This happens in particular, when the original system is not transitive and contains sub-systems of a smaller pressure. The set $\widetilde{S}$ may even depend on the choice of the weights g_i. In particular, intervals for which $g_i = 0$ disappear under the conjugacy.

3. The main kneading identity

Lemma 3.1. *We have* $\mathcal{R}(0) = \mathrm{id}$.

Proof. Note that

$$\begin{aligned} R_{jk}(t) &= \sum_{\hat{c}_j = c_j^+, c_j^-} \varepsilon^*(\hat{c}_j) \cdot \theta(\hat{c}_j, t; c_k) \\ &= \sum_{n \geq 0} t^n \sum_{\hat{c}_j = c_j^+, c_j^-} \varepsilon^*(\hat{c}_j)[sg]^n(\hat{c}_j) \cdot \sigma(f^n \hat{c}_j, c_k) \end{aligned}$$

By convention $f^0 = \mathrm{id}$. Recall that $\varepsilon^*(\hat{c}_j) = \varepsilon(\hat{c}_j)$ if $j \neq 0$ and $\varepsilon^*(\hat{c}_0) = 1$. Assume first $j > 0$. Then, for all $k = 0, \cdots, \ell$,

$$\begin{aligned} R_{jk}(0) &= \sum_{\hat{c}_j = c_j^+, c_j^-} \varepsilon^*(\hat{c}_j) \cdot [sg]^0(\hat{c}_j) \cdot \sigma(f^0 \hat{c}_j, c_k) \\ &= \sum_{\hat{c}_j = c_j^+, c_j^-} \varepsilon(\hat{c}_j) \cdot \sigma(\hat{c}_j, c_k) \\ &= \delta_{jk}. \end{aligned}$$

Also,

$$\begin{aligned} R_{0k}(0) &= \theta(a^+, 0; c_k) + \theta(b^-, 0; c_k) \\ &= \sigma(a^+, c_k) + \sigma(b^-, c_k) \\ &= \delta_{0k}. \end{aligned}$$

□

Proof of Theorem 2.1. Consider first an open interval $J =]u, v[\subset]a, b[$, and a c_k for some $k \in \{0, \cdots, \ell\}$. For each $n \geq 0$ and each $(n+1)$-cylinder $\alpha \in Z_{n+1}$, the functions

$$[sg]^n(\hat{x}) = \prod_{j=0}^{n-1} s(f^j \hat{x}) g(f^j \hat{x})$$

and $\sigma(f^n \hat{x}, c_k)$, $\hat{x} \in \hat{\alpha}$ are constants. When $\alpha \in Z_{n+1}$ and $\alpha \cap J \neq \emptyset$, then obviously

$$\sum_{\hat{x} \in \hat{\partial}(J \cap \alpha)} \varepsilon(\hat{x}) = 1 + (-1) = 0.$$

So the following power series vanishes identically:

$$\sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1},\, \hat{x}\in\widehat{\partial}(J\cap\alpha)} \varepsilon(\hat{x})\cdot[sg]^n(\hat{x})\cdot\sigma(f^n\hat{x},c_k)=0.$$

In this sum, $\hat{x}=u^+, v^-$ appears for every $n\geq 0$. Extracting their contributions we write

$$\begin{aligned}&\sum_{\hat{x}\in\widehat{\partial}J}\theta(\hat{x},t;c_k)\cdot\varepsilon(\hat{x})\\ &\quad+\sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1},\, \hat{x}\in\widehat{\partial}\alpha}\chi_J(x)\cdot\varepsilon(\hat{x})\cdot[sg]^n(\hat{x})\cdot\sigma(f^n\hat{x},c_k)=0.\end{aligned}\tag{3}$$

Now when $\alpha\in Z_{n+1}, \hat{x}\in\widehat{\partial}(J\cap\alpha)$, there is a unique minimal integer $0\leq p\leq n$ for which $f^p(\hat{x})=\hat{c}$ for some $c\in\{c_1,\cdots,c_\ell\}=:\mathcal{C}(f)$ and $\hat{c}=c^+$ or c^- (note that the boundary points a,b are excluded here, since for an interior point to be mapped to them, it has to pass an interior cutting point just before). Recall that $\Gamma_{p,c}=\{x\in\bigcup_{\alpha\in Z_p}\alpha \mid f^p x=c\}$ and $\Gamma_{0,c}=\{c\}$. When $x\in\Gamma_{p,c}$ and $f^p\hat{x}=\hat{c}$, then

$$\begin{aligned}g^p(\hat{x})&=g^p(x),\\ \sigma(f^n\hat{x},c_k)&=\sigma(f^{n-p}\hat{c},c_k),\end{aligned}$$

and also (the essential point here is that the sign $s^p(\hat{x})$ is absorbed in $\varepsilon(\hat{c})$)

$$\begin{aligned}\varepsilon(\hat{x})\cdot[sg]^n(\hat{x})&=g^p(x)(\varepsilon(\hat{x})s^p(\hat{x}))[sg]^{n-p}(\hat{c})\\ &=g^p(x)\cdot\varepsilon(\hat{c})\cdot[sg]^{n-p}(\hat{c}).\end{aligned}$$

So we obtain, for the second term in (3) (writing $t^n=t^pt^q$),

$$\begin{aligned}&\sum_{c\in\mathcal{C}(f)}\Big[\Big(\sum_{p\geq 0}t^p\sum_{x\in\Gamma_{p,c}}g^p(x)\chi_J(x)\Big)\cdot\sum_{\hat{c}=c^\pm,\,q\geq 0}t^q\cdot\varepsilon(\hat{c})\cdot[sg]^q(\hat{c})\sigma(f^q\hat{c},c_k)\Big]\\ &\quad=\sum_{c\in\mathcal{C}(f)}\gamma_{J,c}(t)\cdot\Delta_c\theta(\cdot,t;c_k).\end{aligned}$$

Combining with (3) we get the Main kneading Identity when J is an open interval.

It remains to prove the case that J is half closed or closed. Consider for example $J=\langle u^-,v^-\rangle$ with $a<u<v\leq b$. We have $\langle a^+,v^-\rangle=\langle a^+,u^-\rangle\sqcup J$ and the additivity $\gamma_{c,\langle a^+,v^-\rangle}=\gamma_{c,\langle a^+,u^-\rangle}+\gamma_{c,J}$. The result then follows by applying the identity to the two intervals $\langle a^+,u^-\rangle$ and $\langle a^+,v^-\rangle$ and subtracting. □

4. Zeta functions and kneading determinants

In this section we prove Theorems 2.2 and 2.3.

Set

$$\widehat{\mathcal{C}}(f) := \{a^+ = c_0^+, c_1^-, c_1^+, \cdots, c_\ell^-, c_\ell^+, b^- = c_0^-\},$$

$$\Gamma_{0,\hat{c}} = \{\hat{c}\}, \quad \text{for all } \hat{c} \in \widehat{\mathcal{C}}(f),$$

and

$$\Gamma_{p,\hat{c}} = \{\hat{x} \in \widehat{I} \mid f^p\hat{x} = \hat{c},\ f^j\hat{x} \notin \widehat{\mathcal{C}}(f) \text{ for } 0 \le j < p\}, \quad \text{for } p \ge 1.$$

If $\hat{c} \neq c_0^\pm$, then for any $\hat{x} \in \Gamma_{p,\hat{c}}$ we have $x \in \Gamma_{p,c}$. Conversely for any $x \in \Gamma_{p,c}$ exactly one of $x^\pm$ belongs to $\Gamma_{p,\hat{c}}$.

Notice that if $\hat{c} = c_0^\pm$ then $\Gamma_{p,\hat{c}} = \emptyset$ when $p \ge 1$: due to the forward invariance of I we have $f^{-1}(\{a,b\}) \subset \{a, b, c_1, \cdots, c_\ell\}$, so every orbit passing though $\{a,b\}$ must pass through $\{c_1, \cdots, c_\ell\}$ just before.

Fix $n \ge 1$ and an n-cylinder α. Note that for each $\hat{x} \in \widehat{\partial}\alpha$, we have

$$g^n_{|\alpha} \cdot \varepsilon(f^n\hat{x}) = [sg]^n(\hat{x}) \cdot \varepsilon(\hat{x}),$$

so

$$\begin{aligned} -\omega(\alpha) &:= g^n_{|\alpha} \sum_{\hat{x}\in\widehat{\partial}\alpha} \sigma(f^n\hat{x}, x) \cdot \varepsilon(f^n\hat{x}) \\ &= \sum_{\hat{x}\in\widehat{\partial}\alpha} \sigma(f^n\hat{x}, x)[sg]^n(\hat{x}) \cdot \varepsilon(\hat{x}). \end{aligned}$$

To each $\hat{x} \in \widehat{\partial}\alpha$, there is a unique $\hat{c} \in \widehat{\mathcal{C}}(f)$ and $0 \le p < n$ such that $x \in \Gamma_{p,\hat{c}}$. Schematically,

$$\hat{x} \xrightarrow[p \text{ minimal}]{f^p} \hat{c} \xrightarrow{f^{q+1}} f^n\hat{x} = f^{q+1}\hat{c}.$$

Setting q such that $p + q = n - 1$, we have the "co-cycle" properties:

$$s^n(\hat{x})\varepsilon(\hat{x}) = s^{q+1}(\hat{c})\varepsilon(\hat{c}),$$

$$g^0(\hat{x}) = 1,$$

$$g^n(\hat{x}) = g^{q+1}(\hat{c})g^p(\hat{x}).$$

Recalling the definitions (1) and (2) we then for the zeta-function we get (as formal power-series):

$$
\begin{aligned}
&\zeta'(t)/\zeta(t)\\
&\quad= \sum_{n\geq 1} t^{n-1} N_n\\
&\quad= \sum_{n\geq 1} t^{n-1} \sum_{\alpha\in Z_n} \omega(\alpha)\\
&\quad= -\sum_{n\geq 1} t^{n-1} \sum_{\alpha\in Z_n, \hat{x}\in\hat{\partial}\alpha} \sigma(f^n\hat{x}, x)[sg]^n(\hat{x})\cdot\varepsilon(\hat{x})\\
&\quad= -\sum_{\hat{c}\in\hat{\mathcal{C}}(f)} \sum_{q\geq 0} t^q [sg]^{q+1}(\hat{c})\cdot\varepsilon(\hat{c}) \sum_{p\geq 0,\, \hat{x}\in\Gamma_{p,\hat{c}}} t^p g^p(\hat{x})\sigma(f^{q+1}\hat{c}, x)\\
&\quad= -\sum_{\hat{c}\in\hat{\mathcal{C}}(f)\setminus\{c_0^{\pm}\}} \sum_{q\geq 0} t^q [sg]^{q+1}(\hat{c})\cdot\varepsilon(\hat{c})\Big(\sum_{p\geq 0,\, \hat{x}\in\Gamma_{p,\hat{c}}} t^p g^p(\hat{x})\sigma(f^{q+1}\hat{c}, x)\Big)\\
&\qquad - \sum_{\hat{c}=c_0^{\pm}} \sum_{q\geq 0} t^q [sg]^{q+1}(\hat{c})\cdot\Big(\varepsilon(\hat{c}) \sum_{p\geq 0,\, \hat{x}\in\Gamma_{p,\hat{c}}} t^p g^p(\hat{x})\sigma(f^{q+1}\hat{c}, x)\Big).
\end{aligned}
$$

Note that the $\varepsilon(\hat{c})$ factor in the last expression is treated differently for $\hat{c} = c_1^{\pm}, \ldots, c_\ell^{\pm}$ and $\hat{c} = c_0^{\pm}$. The reason for this is that we want the two expressions in the parenthesis to be independent of the direction of $\hat{c}$. Indeed, for any $\hat{u} \in \hat{I}$,

$$
\begin{aligned}
m_{\hat{c}}(\hat{u}, t) &:= \sum_{p\geq 0,\, \hat{x}\in\Gamma_{p,\hat{c}}} t^p g^p(\hat{x})\sigma(\hat{u}, x)\\
&= \sum_{p\geq 0,\, x\in\Gamma_{p,c}} t^p g^p(x)\sigma(\hat{u}, x), \quad \text{for } \hat{c} = c_1^{\pm}, \cdots, c_\ell^{\pm},
\end{aligned}
$$

and

$$
\begin{aligned}
m_{\hat{c}}(\hat{u}, t) &:= \varepsilon(\hat{c}) \sum_{p\geq 0,\, \hat{x}\in\Gamma_{p,\hat{c}}} t^p g^p(\hat{x})\sigma(\hat{u}, x)\\
&= \varepsilon(\hat{c})\sigma(\hat{u}, c)\\
&\equiv \frac{1}{2}, \quad \text{for } \hat{c} = c_0^{\pm},
\end{aligned}
$$

where we have used the facts that

$$\Gamma_{p,c_0^{\pm}} = \emptyset \quad \text{for } p > 0,$$

$$g^0(\hat{x}) \equiv 1,$$

and

$$g^p(x^+) = g^p(x^-) =: g^p(x) \quad \text{for } x \in \Gamma_{p,c_j},\ j > 0.$$

In both cases $m_{\hat{c}}(\hat{u}, t)$ is independent of $\varepsilon(\hat{c}) = +$ or $-$, so we may safely write $m_c(\hat{u}, t)$ for this quantity. To compactify the two cases we set $\mathcal{C}^*(f) = \{c_0, c_1, \cdots, c_\ell\}$. Recall that $c_0^+ = a^+$, $c_0^- = b^-$ and $\varepsilon^*(\hat{c}) := \varepsilon(\hat{c})$ if $\hat{c} \neq c_0^{\pm}$ and $\varepsilon^*(\hat{c}) = 1$ otherwise. We then have

$$\zeta'(t)/\zeta(t) = -\sum_{c \in \mathcal{C}^*(f)} \sum_{\hat{c}=c^{\pm}, q \geq 0} t^q [sg]^{q+1}(\hat{c}) \cdot \varepsilon^*(\hat{c}) \cdot m_c(f^{q+1}\hat{c}, t). \tag{4}$$

A central idea (due to Preston) is to consider the right hand side as the trace of an $(\ell+1) \times (\ell+1)$ matrix $\mathcal{F} = (F_{ij})$, and to define $\mathcal{F}$ in a way so that $\mathcal{F}\mathcal{R}$ becomes related to $\mathcal{R}'$. There are many choices suitable for this purpose with most choices giving rise to additional correcting terms. There is, however, a choice for which the relationship becomes particularly simple (note the $*$ in the epsilon factor).

For $i, j \in \{0, 1, \cdots, \ell\}$, define

$$F_{ij}(t) = \sum_{q \geq 0, \hat{c}_i = c_i^{\pm}} t^q [sg]^{q+1}(\hat{c}_i) \cdot \varepsilon^*(\hat{c}_i) \cdot m_{c_j}(f^{q+1}\hat{c}_i, t). \tag{5}$$

Theorem 4.1. *We have*

$$\zeta'(t)/\zeta(t) + \operatorname{Tr}\mathcal{F} = 0 \quad \textit{and} \quad \mathcal{F}\mathcal{R} = \mathcal{R}'.$$

Proof. The first follows from expressions (4) and (5):

$$\operatorname{Tr}\mathcal{F} = \sum_{i=0}^{\ell} F_{ii} = -\zeta'(t)/\zeta(t).$$

For the second we first establish a consequence of the main kneading identity.

Claim. *For every $\hat{w} \in \hat{I}$, $k = 0, 1, \cdots, \ell$,*

$$\sum_{j=0}^{\ell} m_{c_j}(\hat{w}, t) R_{jk}(t) = \theta(\hat{w}, t; c_k).$$

Proof. By the main kneading identity, we sum first over interior cutting points:

$$\begin{aligned}
&\sum_{j=1}^{\ell} m_{c_j}(\hat{w},t)R_{jk}(t)\\
&\quad=\sum_{j=1}^{\ell}\sum_{p\geq 0,\, x\in\Gamma_{p,c_j}} t^p g^p(x)\sigma(\hat{w},x)\cdot R_{jk}(t)\\
&\quad=\sum_{j=1}^{\ell}\sum_{p\geq 0,\, x\in\Gamma_{p,c_j}} t^p g^p(x)\frac{1}{2}(\chi_{(a^+,\hat{w})}(x)-\chi_{(\hat{w},b^-)}(x))R_{jk}(t)\\
&\quad=\frac{1}{2}(2\theta(\hat{w},t;c_k)-\theta(a^+,t;c_k)-\theta(b^-,t;c_k)).
\end{aligned}$$

Adding the boundary term

$$m_{c_0}(\hat{w},t)R_{0k}(t)=\frac{1}{2}(\theta(a^+,t;c_k)+\theta(b^-,t;c_k))$$

we get the desired result and end the proof of the claim.

Now, for $i,k\in\{0,\cdots,\ell\}$,

$$\begin{aligned}
&\sum_{j=0}^{\ell} F_{ij}R_{jk}\\
&\quad=\sum_{q\geq 0,\,\hat{c}_i=c_i^{\pm}} t^q[sg]^{q+1}(\hat{c}_i)\cdot\varepsilon^*(\hat{c}_i)\cdot\sum_{j=0}^{\ell}(m_{c_j}(f^{q+1}\hat{c}_i,t)R_{jk})\\
&\quad=\sum_{q\geq 0,\,\hat{c}_i=c_i^{\pm}} t^q[sg]^{q+1}(\hat{c}_i)\cdot\varepsilon^*(\hat{c}_i)\cdot\theta(f^{q+1}(\hat{c}_i),t;c_k)\\
&\quad=\sum_{q\geq 0,\,\hat{c}_i=c_i^{\pm}} t^q[sg]^{q+1}(\hat{c}_i)\cdot\varepsilon^*(\hat{c}_i)\sum_{p\geq 0} t^p[sg]^p(f^{q+1}\hat{c}_i)\sigma(f^p(f^{q+1}\hat{c}_i),c_k)\\
&\quad=\sum_{\hat{c}_i=c_i^{\pm}}\sum_{p,q\geq 0} t^{p+q}[sg]^{p+q+1}(\hat{c}_i)\cdot\sigma(f^{p+q+1}\hat{c}_i,c_k)\cdot\varepsilon^*(\hat{c}_i)\\
&\quad=\sum_{\hat{c}_i=c_i^{\pm}}\Big(\sum_{n\geq 1} n\cdot t^{n-1}[sg]^n(\hat{c}_i)\cdot\sigma(f^n\hat{c}_i,c_k)\Big)\varepsilon^*(\hat{c}_i)
\end{aligned}$$

$$= \sum_{\hat{c}_i = c_i^{\pm}} \Big(\frac{d}{dt} \theta(\hat{c}_j, t; c_k) \Big) \varepsilon^*(\hat{c}_i)$$

$$= \frac{d}{dt} R_{ik}(t)$$

in which we recall that

$$R_{jk}(t) = \sum_{\hat{c}_j = c_j^{\pm}} \theta(\hat{c}_j, t; c_k) \cdot \varepsilon^*(\hat{c}_j). \qquad \square$$

4.1. Proof of Theorem 2.2. Our version of the kneading matrix, $\mathcal{R}(t)$, is a matrix valued formal power series in t starting with the identity matrix. Then $D(t) = \det \mathcal{R}(t)$ is a formal power series starting with $D(0) = 1$. As formal power series one has the relation $\frac{d}{dt} \log D(t) = \operatorname{Tr} \mathcal{R}'(t)\mathcal{R}(t)^{-1}$. This is certainly true for a truncated power series (e.g. by using standard analytic methods valid for $|t|$ small enough) and then holds in general by a degree argument which allows the identification of terms with the same degree.

Theorem 4.1 yields as identities between formal power-series

$$-\frac{d}{dt} \log \zeta(t) = -\zeta'(t)/\zeta(t) = Tr\mathcal{F}(t) = Tr\mathcal{R}'(t)\mathcal{R}(t)^{-1}.$$

If one truncates to a finite order N in t then $\mathcal{R}(t)$ becomes analytic in t and when $\mathcal{R}(t)$ is invertible a standard calculation gives

$$\frac{d}{dt} \log \det(\mathcal{R}(t)) = \lim_{h \to 0} \frac{1}{h} \log \det(\mathcal{R}(t+h)\mathcal{R}(t)^{-1}) = \operatorname{Tr} \mathcal{R}'(t)\mathcal{R}(t)^{-1}.$$

When the constant term $\mathcal{R}(0)$ is invertible, this identity is naturally graded in the degree of t so is valid also in the case of a formal power-series. Here we have by Lemma 3.1 $\det \mathcal{R}(0) = 1$ and since $\zeta(0) = 1$ we get $\log [\zeta(t) \det (\mathcal{R}(t))] = 0$, or equivalently,

$$\zeta(t) \det (\mathcal{R}(t)) = 1.$$

as claimed. $\square$

4.2. Weighted lap function and proof of Theorem 2.3. Let us consider the generating function of $g^n(\hat{x})$,

$$G(\hat{x},t) = \sum_{n\geq 0} t^n g^n(\hat{x}), \qquad \text{for } \hat{x} \in \widehat{I}, \tag{6a}$$

and then

$$G(x,t) = \frac{1}{2}(G(x^-,t) + G(x^+,t)), \quad \text{when } a < x < b. \tag{6b}$$

Let $J = \langle \hat{u}, \hat{v} \rangle \subset]a,b[$ be an (open, closed or half-closed) interval or a point. We define the *weighted lap function*[3]

$$L(J,t) := \frac{1}{2}\sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1}} \sum_{\hat{x}\in\widehat{\partial}\alpha} g^n(\hat{x})\chi_J(x).$$

Repeating the calculation in our proof of the main kneading identity without the sign factors s, ε and σ, it follows easily that

$$\begin{aligned} L(J,t) &= \sum_{j=1}^{\ell}\Big(\sum_{p\geq 0,\, x\in\Gamma_{p,c_j}} t^p g^p(x)\chi_J(x)\Big)\Big(\sum_{\hat{c}=c_j^{\pm}} \frac{1}{2}\sum_{q\geq 0} t^q \cdot g^q(\hat{c})\Big) \\ &= \sum_{j=1}^{\ell} \gamma_{c_j,J}(t)\cdot G(c_j,t) \end{aligned} \tag{7}$$

In particular, for a one-point set $J = \{x\}$ we have simply

$$L(\{x\},t) = \begin{cases} t^p g^p(x)\cdot G(c_i,t) & \text{for } x \in \Gamma_{p,c_i},\ p\geq 0,\ 1\leq i\leq \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 4.2. *Fix any subinterval $J = \langle \hat{u}, \hat{v}\rangle$. The functions G, θ, $\Delta_J\theta$, $\mathcal{R}_{jk}$ are all analytic functions of t on the disc $\{|t| < 1/\rho_\infty\}$. The kneading matrix is invertible when $|t| < 1/\rho_1$. The function $L(J,t)$ is meromorphic on $\{|t| < 1/\rho_\infty\}$ and analytic on $\{|t| < 1/\rho_1\}$.*

[3] If $g_i \equiv 1$ the G-functions are $\dfrac{1}{1-t}$ and the function $L(J,t)$ is the generating function for the numbers of $(n+1)$-cylinders in J, and $L(]a,b[,t)$ has radius of convergence equal to $1/\rho_1$.

Proof. The first claim follows from the definition of ρ_∞ and the following estimates:

$$|G(\hat{x},t)| \le \sum_{n\ge 0} |t|^n \|g^n\|_\infty < \infty \quad \text{for } |t| < 1/\rho_\infty, \text{ for all } \hat{x} \in \hat{I}.$$

Similarly, for all k,

$$|\Delta_J \theta(\cdot, t; c_k)| \le \sum_{n\ge 0} |t|^n \|g^n\|_\infty < \infty \quad \text{for } |t| < 1/\rho_\infty.$$

To see that the kneading matrix is invertible when $|t| < 1/\rho_1$ we use the relationship to the zeta function. By Theorem 2.2 we have $\zeta(t) \cdot \det \mathcal{R}(t) = 1$, where

$$\zeta(t) = \exp\Big(\sum_{n\ge 1} \frac{N_n}{n} t^n\Big)$$

and each $|N_n| \le \|g^n\|_1$. So $\zeta(t)$ is analytic and non-zero for $|t| < 1/\rho_1$ whence $\mathcal{R}(t)$ is invertible for $|t| < 1/\rho_1$.

We have

$$\begin{aligned} |L(J,t)| &\le \sum_{n\ge 0} |t|^n \sum_{\alpha\in Z_{n+1}} |g^n_{|\alpha}| \\ &\le \sum_{n\ge 0} |t|^n \sum_{\alpha\in Z_n} |g^n_{|\alpha}|(\ell+1) \\ &= (\ell+1) \sum_{n\ge 0} |t|^n \|g^n\|_1, \end{aligned} \tag{9}$$

which shows that $L(J,t)$ has radius of convergence at least $1/\rho_1$.

Using the MKI itself for the γ factor in (7) we get

$$L(J,t) = \sum_{k=0}^{\ell} \Delta_J \theta(\cdot, t; c_k)\Big(\sum_{j=1}^{\ell} \mathcal{R}^{-1}(t)_{kj} \cdot G(c_j, t)\Big). \tag{10}$$

The above identities are valid as formal power series but also when the functions involved are analytic and $\mathcal{R}(t)$ is invertible. As $1/\rho_1 \le 1/\rho_\infty$, so when $|t| < 1/\rho_1$, the identity (10) is valid. □

Proof of Theorem 2.3. Both (1) and (2) have already been proved in Lemma 4.2.

We proceed to prove (3). When all g_i's are positive and $t \geq 0$ we have

$$\begin{aligned} L(]a,b[,t) + G(a^+,t) + G(b^-,t) &= \sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1}} g^n_{|\alpha} \\ &\geq \sum_{n\geq 0} t^n \|g^n\|_1. \end{aligned}$$

By definition the RHS has radius of convergence equal to $1/\rho_1$. Being a power-series with positive coefficients it follows that the RHS diverges as $t \nearrow 1/\rho_1$.

Under the further assumption $1/\rho_1 < 1/\rho_\infty$, the functions $t \mapsto G(\hat{x},t)$, in particular $G(a^+,t)$ and $G(b^-,t)$, remain bounded at $t = 1/\rho_1$. So $L(]a,b[,t)$ must diverge as $t \nearrow 1/\rho_1$. Combining with (9) we know that the radius of convergence of $L(]a,b[,t)$ is equal to $1/\rho_1$. Now, the functions $\Delta_J\theta$ and G involved in (10) remain bounded on $|t| \leq 1/\rho_1$. Letting $t \nearrow 1/\rho_1$ in (10) we conclude that $\mathcal{R}(t)$ must be non-invertible at $t = 1/\rho_1$. □

5. Semi-conjugacies to piecewise linear models

In this section we prove Theorems 2.4 and 2.5.

Lemma 5.1. *Fix $J = \langle \hat{u}, \hat{v}\rangle \subset I_j =]c_j^+, c_{j+1}^-[$. For $k = 0, \cdots, \ell$ and $|t| < 1/\rho_\infty$,*

$$\theta(\hat{v},t;c_k) - \theta(\hat{u},t;c_k) = t \cdot s_j g_j (\theta(f\hat{v},t;c_k) - \theta(f\hat{u},t;c_k)) \tag{11}$$

When also $|t| < 1/\rho_1$ we have for the weighted lap function

$$L(J,t) = t\, g_j \cdot L(f_j J, t) \tag{12}$$

Proof. Let us fix $k \in \{0, \cdots, \ell\}$. By definition, we have the following relation for $\theta(\cdot,t;c_k)$ when applied to $\hat{x}$ and $f\hat{x}$:

$$\begin{aligned} \theta(\hat{x},t;c_k) &= \sum_{m\geq 0} t^m [sg]^m(\hat{x}) \cdot \sigma(f^m\hat{x}, c_k) \\ &= \sigma(\hat{x},c_k) + t \cdot [sg](\hat{x}) \cdot \theta(f\hat{x},t;c_k), \quad \text{for all } \hat{x} \in \hat{I}. \end{aligned}$$

This implies (11) when restricting to $\widehat{I}_j$. Now,

$$\Delta_J \theta(\cdot, t; c_k) = \theta(\hat{v}, t; c_k) - \theta(\hat{u}, t; c_k)$$

and (as f may reverse the orientation)

$$\Delta_{f_j J} \theta(\cdot, t; c_k), = s_j(\theta(f\hat{v}, t; c_k) - \theta(f\hat{u}, t; c_k)),$$

so

$$\Delta_J \theta(\cdot, t; c_k) = t g_j \Delta_{f_j J} \theta(\cdot, t; c_k).$$

The result for $L(J, t)$ now follows by linearity in equation (10) which is valid when $|t| < 1/\rho_1$. □

5.1. Proof of Theorem 2.4. We assume here that all $g_i > 0$ and that $\rho_1 > \rho_\infty$. Fix $0 < t < 1/\rho_1 < 1/\rho_\infty$. Noting that $0 < L(]a, b[, t) < +\infty$ we define our conjugating map

$$\phi_t : \widehat{I} \longrightarrow \mathbb{R}$$

by setting

$$\phi_t(\hat{x}) = \frac{L(\langle a^+, \hat{x}\rangle, t)}{L(]a, b[, t)}, \quad \hat{x} \in \widehat{I}. \tag{13}$$

Notice that ϕ_t maps point-germs to genuine real numbers.

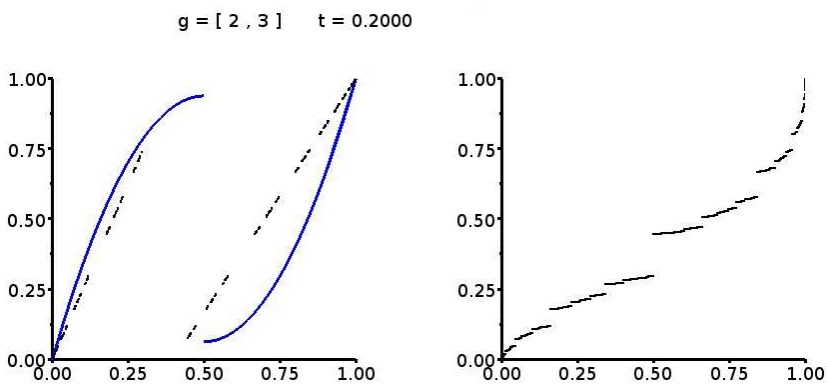

Figure 1. Left: Example of a piecewise continuous map f and the graph restricted to Ω_t of its conjugated map $\tilde{f}_t$. Right: The graph of ϕ_t. Here, $t = 0.2 < 1/\rho_1 = 0.2684$. The ratio of slopes of the two branches is 3 : 2 (coming from the choice of weights).

Part A. Using (12) we get for any $\hat{x}_1, \hat{x}_2 \in \widehat{I}_j$ (the sign enters again)

$$\phi_t(\hat{x}_2) - \phi_t(\hat{x}_1) = t s_j g_j (\phi_t(f\hat{x}_2) - \phi_t(f\hat{x}_1)). \tag{14}$$

Similarly, we get by iterating this argument for $\hat{x}_1, \hat{x}_2 \in \hat{\alpha}$ with $\alpha \in Z_n$

$$\phi_t(\hat{x}_2) - \phi_t(\hat{x}_1) = t^n s^n_{|\alpha}\, g^n_{|\alpha} (\phi_t(f^n\hat{x}_2) - \phi_t(f^n\hat{x}_1)). \tag{15}$$

When the g_i's are non-negative, we clearly have $L(J,t) \geq 0$ for any interval J so by set-additivity with respect to J it follows that ϕ_t is monotone increasing and takes values in $[0,1]$. Let $\Omega_t = \phi_t(\widehat{I}) \subset [0,1]$ and set $\Omega_{t,i} = \phi_t(\widehat{I}_i)$. By monotonicity of ϕ_t the convex hull of $\Omega_{t,i}$ is the closed interval (or a one-point set) $\widetilde{I}_{t,i} = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$

Let now $a < x < b$. As all $g_i > 0$, by (8)

$$\phi_t(x^+) - \phi_t(x^-) = \frac{L(\{x\},t)}{L(]a,b[,t)} > 0 \tag{16}$$

precisely when x is a cutting point or a pre-image of such (also called an eventual cutting point). We have in particular $L(\{c_i\},t) \geq 1$ so that $\sup \Omega_{t,i} < \inf \Omega_{t,i+1}$ and also $\sup \widetilde{I}_{t,i} < \inf \widetilde{I}_{t,i+1}$, so the intervals are pairwise disjoint, proving Part A.

Part B. Given $y \in \Omega_t$ suppose that $y = \phi_t(\hat{x}_1) = \phi_t(\hat{x}_2)$ with $\hat{x}_1 < \hat{x}_2$. By the previous paragraph $\hat{x}_1$ and $\hat{x}_2$ must belong to the same $\widehat{I}_j$. So by the identity (14) we must have $\phi_t(f\hat{x}_2) - \phi_t(f\hat{x}_1) = 0$. We may then define a map

$$\tilde{f}_t : \Omega_t \longrightarrow \Omega_t$$

by

$$\tilde{f}_t(y) := \phi_t(f\hat{x}), \quad y = \phi_t(\hat{x}) \in \Omega_t$$

since the value is independent of the choice of $\hat{x}$ in the pre-image of y.

Now, either $\Omega_{t,j} = \phi_t(\widehat{I}_j)$ is reduced to a point (when I_j contains no pre-image of a critical point) or, by equation (14), the conjugated map $(\tilde{f}_t)_{|\Omega_{t,j}}$ has slope $(t s_j g_j)^{-1} = s_j / t g_j$. This map then extends to a unique affine map

$$\tilde{f}_{t,j} : \widetilde{I}_{t,j} \longrightarrow [0,1]$$

which coincides with $\tilde{f}_t$ on $\Omega_{t,j} = \phi_t(\widehat{I}_j) = \Omega_t \cap \widetilde{I}_{t,j}$, thus proving Part B.

Part C. The collection of closed disjoint intervals and associated affine maps, $(\widetilde{I}_{t,i}, \tilde{f}_{t,i})_{0\le i\le \ell}$, defines a partially dynamical system (as there are "holes" in the domain of definition for the iterated map)

$$\mathcal{S}_t : \mathcal{D}_t := \bigcup_i \widetilde{I}_{t,i} \longrightarrow [0,1], \quad \mathcal{S}_{t|\widetilde{I}_{t,i}} = \tilde{f}_{t,i}.$$

Its maximal invariant set is given by $\bigcap_{n\ge 0} (\mathcal{S}_t)^{-k}\, \mathcal{D}_t$. The claim is that this invariant set is precisely $\Omega_t = \phi_t(\widehat{I})$. The proof is hampered by the fact that ϕ_t is neither continuous nor injective.

For an open interval $J =]u, v[\subset]a, b[$ we will use the short-hand notation

$$\Xi_t(J) := [\phi_t(u^+), \phi_t(v^-)].$$

Definition 5.1. We say that $J =]u_1, u_2[\subset]a, b[$ is a *cutting interval* iff each of u_1 and u_2 is an eventual cutting point, i.e. is either a cutting point or a pre-image of such.

Note that if $J =]u_1, u_2[$ is a cutting interval then because of (16) and monotonicity of ϕ_t, $\widetilde{J} = \Xi_t(J) = [\phi_t(u_1^+), \phi_t(u_2^-)]$ is disjoint from $\mathrm{Cl}(\Omega_t \setminus \widetilde{J})$.

Lemma 5.2. *Given $0 \le i, j \le \ell$, let $J \subset I_j$ be a cutting interval and set*

$$K = I_i \cap f_i^{-1} J \quad \textit{and} \quad \widetilde{K} = \Xi_t(I_i) \cap \mathcal{S}_t^{-1} \Xi_t(J).$$

Then, either K and $\widetilde{K}$ are both empty, or K is a cutting interval and $\widetilde{K} = \Xi_t(K)$.

Proof. We consider the case $s_i = +1$ (the case $s_i = -1$ being treated in a similar way). We write $J =]u_1, u_2[$ and set $\widetilde{J} = \Xi_t(J)$. If K were empty, then e.g. $f_i(c_{i+1}) \le u_1$. As u_1 is assumed to be an eventual cutting point, we have

$$\mathcal{S}_t \phi_t(c_{i+1}^-) = \phi_t(f_i c_{i+1}^-) < \phi_t(u_1^+).$$

Then $\mathcal{S}_t \Xi_t(I_i) \cap \Xi_t(J)$ is empty and so is $\widetilde{K}$.

Suppose then $K =]w_1, w_2[$ non-empty. There are two options. Either

$$u_1 \le f_i c_i < u_2$$

or

$$f_i c_i < u_1 < f_i c_{i+1}.$$

In the first case, $w_1 = \inf K = c_i$ (so is a cutting point) and we have $\phi_t(w_1^+) = \min \Xi_t(I_i)$. As also $\mathcal{S}_t\phi_t(w_1^+) = \phi_t(f_i w_1^+) = \phi_t(f_i c_i^+) \in \Xi_t(J)$ we conclude that $\widetilde{K}$ is non-empty and that $\min \widetilde{K} = \phi_t(w_1^+)$.

In the second case, $w_1 \in I_i$ with $f_i w_1 = u_1$ so w_1 is an eventual cutting point, since u_1 is. We have $\mathcal{S}_t\phi_t(w_1^+) = \phi_t(f_i w_1^+) = \phi_t(u_1^+)$ and as $\phi_t(w_1^+) \in \Xi(I_i)$ we have again $\min \widetilde{K} = \phi_t(w_1^+)$. In either case, $\min \widetilde{K} = \phi_t(w_1^+)$ and w_1 is an eventual cutting point. The same is true for the right end-point so we conclude that K is a cutting interval and that $\Xi_t(K) = \widetilde{K}$. □

For $0 \le i \le \ell$, set $\widetilde{I}_t(i) = \widetilde{I}_{t,i} = \Xi_t(I_i)$ and define recursively

$$\widetilde{I}_t(i_0, \dots, i_{n-1}) = \widetilde{I}_{t,i_0} \cap \mathcal{S}_t^{-1}\widetilde{I}_t(i_1, \dots, i_{n-1})$$

which is either empty, a point or a closed interval. We write $\widetilde{Z}_{t,n}$ for the collection of non-empty sets of this form. They constitute the n-cylinders for the system $(\mathcal{D}_t, \mathcal{S}_t)$. The n-cylinders are pairwise disjoint (shown by induction) and form a partition for the domain of definition of $(\mathcal{S}_t)^n$. They are closely related to the cylinders of the original map since

Lemma 5.3. $\widetilde{Z}_{t,n} = \{\Xi_t(\alpha) \colon \alpha \in Z_n\}$.

Proof. For $n = 1$ this is the very definition: Z_1 consists of the intervals $\{I_i : 0 \le i \le \ell\}$ and $\widetilde{Z}_{t,1}$ is the collection of $\widetilde{I}_t(i) = \Xi_t(I_i) = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$, $0 \le i \le \ell$.

Suppose the claim is true for a given n and pick $\beta \in Z_n$, $\tilde{\beta} = \Xi_t(\beta)$ (using the induction hypothesis) and $I_i \in Z_1$. As β is bounded by eventual cutting points the previous lemma shows that either $\alpha = I_i \cap f_i^{-1}\beta$ and $\tilde{\alpha} = \Xi_t(I_i) \cap \mathcal{S}_t^{-1}\Xi_t(\beta)$ are both empty or they are both non-trivial $(n+1)$-cylinders with $\tilde{\alpha} = \Xi_t(\alpha)$. The claim follows. □

Returning to the proof of Part C. By part B, the future $\mathcal{S}_t$-orbit of every $\xi = \phi_t(\hat{x})$ exists so the point ξ belongs to the maximal invariant domain of $\mathcal{S}_t$. To see that there are no more points, consider $\xi \in \bigcap_n \bigcup \widetilde{Z}_{t,n}$ which verifies $\tilde{f}_t^k(\xi) \in \widetilde{I}_{t,i_k}$ for $k \ge 0$. Then by the previous lemma, $\xi \in \tilde{\alpha}_k = \widetilde{I}_t(i_0, \dots, i_{k-1}) = \Xi_t(\alpha_k)$ for all k (a nested sequence of intervals). If ξ is a boundary point of such an interval for some $k = k_0$ then it is in fact a boundary point for all $k \ge k_0$ and clearly in the image of ϕ_t. So assume that ξ is in the interior of $\tilde{\alpha}_k = \Xi_t(\alpha_k)$ for all $k \ge 0$.

Let $\alpha_k =]u_k, v_k[$. Then $u_k \nearrow u_*$ and $v_k \searrow v_*$ with $u_* \leq v_*$. By hypothesis, $\phi_t(u_k^+) < \xi < \phi_t(v_k^-)$ and

$$0 < \phi_t(v_k^-) - \phi_t(u_k^+) \leq t^k g^k_{|\alpha_k} / L(]a, b[, t) \longrightarrow 0 \quad \text{as } k \to \infty.$$

Given any $x \in [u^*, v^*]$ we conclude by monotonicity of ϕ_t that

$$\phi_t(x^+) = \phi_t(x^-) = \xi.$$

So $\phi_t(\widehat{I}) = \Omega_t$ is indeed the maximal invariant domain for $\mathcal{S}_t$.

For the uniform expansion let us state more precisely what we mean.

Lemma 5.4. *The system $(\mathcal{S}_t, \mathcal{D}_t)$, is uniformly expanding in the following sense. There is $n_0 \geq 1$ and $\lambda > 1$ so that for all $n \geq n_0$, every n-cylinder $\alpha \in \widetilde{Z}_{t,n}$ and all $\xi_1, \xi_2 \in \alpha$,*

$$|\mathcal{S}_t^n(\xi_1) - \mathcal{S}_t^n(\xi_2)| \geq \lambda |\xi_1 - \xi_2|.$$

Proof. Either α is reduced to a point (so the inequality is trivially true) or it is an interval for which the identity (15) implies

$$\mathcal{S}_t^n(\xi_2) - \mathcal{S}_t^n(\xi_1) = \frac{s^n_{|\alpha}}{t^n g^n_{|\alpha}}(\xi_2 - \xi_1).$$

Given $\lambda > 1$ it then suffices to find $n_0 > 1$ so that, for $n \geq n_0$, $\sup_{\alpha \in Z_n} g^n_{|\alpha} \leq \rho_1^n / \lambda$ and this is possible since $t < 1/\rho_1 < 1/\rho_\infty$. □

In order to prove Theorem 2.5 we consider the limit $t \nearrow 1/\rho_1$. As the function $L(]a, b[, t)$ diverges the situation is a bit different. By Lemma 4.2 the lap-function $L(]a, b[, t)$ is meromorphic in the disc $\{|t| < 1/\rho_\infty\}$ and has a pole of some order $m \geq 1$ at $t = 1/\rho_1$. By positivity of $L(]a, b[, t)$ for $t > 0$ there is $c > 0$ so that

$$L(]a, b[, t) = \frac{c}{(1 - \rho_1 t)^m} + (\text{lower order terms}).$$

For any interval $J \subset]a, b[$ we have $0 \leq L(J, t) \leq L(]a, b[, t)$. An eventual pole of $L(J, t)$ at $1/\rho_1$ is therefore of order at most m so $L(J, t)/L(]a, b[, t)$ extends analytically to $t = 1/\rho_1$ (the singularity is removable here). We denote the limit

$$\Lambda(J) := \lim_{t \nearrow 1/\rho_1} \frac{L(J, t)}{L(]a, b[, t)} \in [0, 1].$$

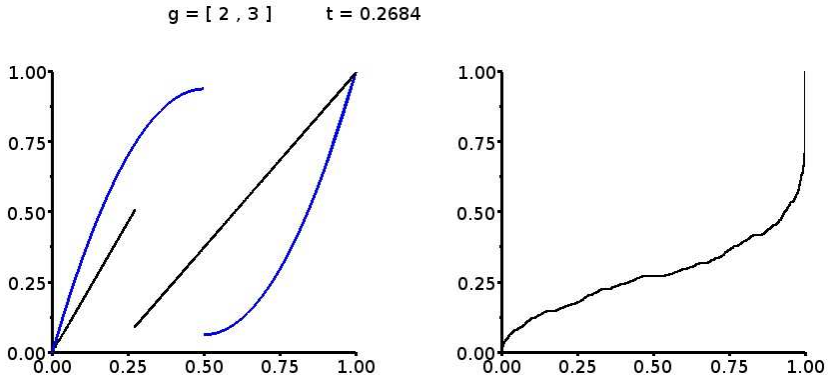

Figure 2. The same example as before but at the critical value $t = 1/\rho_1 = 0.2684$. Note that Ω_t is no longer a Cantor set and that ϕ_t is continuous.

Lemma 5.5. *We have the following properties for* Λ.

(1) *For any* $x \in I$, $\Lambda(\{x\}) = 0$.

(2) *For all* $\alpha \in Z_n$, $\Lambda(\alpha) = \dfrac{1}{\rho_1^n}\, g^n_{|\alpha}\, \Lambda(f^n\alpha)$.

(3) $\delta_n = \sup\limits_{\alpha \in Z_n} \Lambda(\alpha) \xrightarrow[n\to\infty]{} 0$.

Proof. The expression (8) shows that the function $L(\{x\}, t)$ is analytic on $\{|t| < 1/\rho_\infty\}$ in particular remains bounded on $\{|t| \le 1/\rho_1\}$. As $t \nearrow 1/\rho_1$, the denominator $L(]a, b[, t)$ diverges, the first claim follows.

For $J \subset I_j$ for some j we divide (12) by $L(]a, b[, t)$ and take the limit $t \nearrow 1/\rho_1$ to obtain

$$\Lambda(J) = \frac{1}{\rho_1} g_j \Lambda(f_j J).$$

In particular, for $\alpha = (i_0 i_1 \cdots i_{n-1}) \in Z_n$ we have $\alpha \in I_{i_0}$ so that

$$\Lambda(\alpha) = \frac{1}{\rho_1} g_{i_0} \cdot \Lambda(f\alpha).$$

Iterating this we get the formula.

The last claim follows from

$$\Lambda(\alpha) = \frac{1}{\rho_1^n}\, g^n_{|\alpha}\, \Lambda(f^n\alpha) \le \frac{\rho_\infty^n}{\rho_1^n} \xrightarrow[n\to\infty]{\rho_1 > \rho_\infty} 0. \qquad \square$$

Lemma 5.6. *The map* $\phi\colon [a,b] \to [0,1]$ *defined by*

$$\phi(x) = \Lambda(]a,x[), \quad x \in [a,b]$$

is non-decreasing, continuous and surjective. For $x \in]a,b[$,

$$\phi(x) = \lim_{t \nearrow 1/\rho_1} \phi_t(x^-) = \lim_{t \nearrow 1/\rho_1} \phi_t(x^+) \tag{17}$$

Proof. Monotonicity follows from positivity and additivity of $\Lambda(J)$, $J \subset]a,b[$. Let $x \in [a,b]$ and $\varepsilon > 0$. Choose n so that $\delta_n < \varepsilon/2$ (δ_n from the previous lemma). Either x is inside some n-cylinder or on the boundary of two such cylinders. In any case, we may find at most two n-cylinders α_1, α_2 with $\overline{\alpha_1} \cap \overline{\alpha_2} = \{x'\}$ so that $J = \alpha_1 \cup \{x'\} \cup \alpha_2$ is an open neighborhood of x and $\Lambda(J) < \varepsilon$. For $h > 0$ small enough $\phi(x+h) - \phi(x-h) \le \Lambda(J) < \varepsilon$. As $\phi(a) = \Lambda(\emptyset) = 0$ and $\phi(b) = 1$ the map is surjective. The first equality in (17) is essentially the definition of ϕ and the second follows from the continuity just shown. □

We write

$$\tilde{c}_i = \phi(c_i), \quad i = 0, \cdots, \ell+1$$

and let

$$\widetilde{S} \subset S := \{0, \cdots, \ell\}$$

denote the (possibly strict) subset of indices i for which

$$0 < \tilde{c}_{i+1} - \tilde{c}_i = \Lambda(]c_i, c_{i+1}[).$$

For $i \in \widetilde{S}$ we set

$$\widetilde{I}_i =]\tilde{c}_i, \tilde{c}_{i+1}[.$$

5.2. Proof of Theorem 2.5

Part A. For $\hat{x}_1, \hat{x}_2 \in I_j$, taking the limit $t \nearrow 1/\rho_1$ in the identity (14) yields

$$\phi(\hat{x}_2) - \phi(\hat{x}_1) = t s_j g_j (\phi(f_j \hat{x}_2) - \phi(f_j \hat{x}_1)). \tag{18}$$

Continuity of ϕ and f_j shows that this identity is independent of the direction of the point-germs. If $g_j = 0$ then this identity shows that $\tilde{c}_i = \tilde{c}_{i+1}$ so $j \notin \widetilde{S}$ and otherwise the affine map

$$\tilde{f}_j(y) = \hat{c}_j + \frac{s_j}{t g_j}(y - \hat{c}_j)$$

will satisfy the required identity.

Part B. Recall that Z_n consists of the non-empty n-cylinder for $(I_i, f_i)_{i\in S}$. Let $\widetilde{Z}_n$ be the collection of non-empty open intervals of the form $\tilde{\alpha} = \text{Int } \phi(\alpha)$ where $\alpha = (i_0 \cdots i_{n-1}) \in Z_n$. Here each $i_k \in \widetilde{S}$, $0 \le k < n$ (or else $\tilde{\alpha}$ is a fortiori empty) and $\tilde{f}^k\tilde{\alpha} \subset \widetilde{I}_{i_k}$. Therefore $\tilde{\alpha}$ is contained in an n-cylinder for the dynamical system $(\widetilde{I}_i, \tilde{f}_i)_{i\in\widetilde{S}}$. We claim that $\tilde{\alpha}$ is actually equal to an n-cylinder for that system and $\widetilde{Z}_n$ is precisely the set of non-empty n-cylinders for the same system. To see this note that

$$1 = \sum_{\alpha\in Z_n} \Lambda(\alpha) = \sum_{\tilde{\alpha}\in\widetilde{Z}_n} |\tilde{\alpha}|, \tag{19}$$

where $|\cdot|$ denotes the length of intervals. There is no room for any other or any larger open cylinder.

Now, by Lemma 5.5 we have

$$|\tilde{\alpha}| = \Lambda(\alpha) = \frac{g^n_{|\alpha}}{\rho_1^n}\Lambda(f^n\alpha) \le \frac{g^n_{|\alpha}}{\rho_1^n}.$$

So using (19) we get

$$\begin{aligned}
\rho_1^n &= \sum_{\tilde{\alpha}\in\widetilde{Z}_n} |\tilde{\alpha}|\rho_1^n \\
&= \sum_{\tilde{\alpha}\in\widetilde{Z}_n} \Lambda(\alpha)\rho_1^n \\
&= \sum_{\tilde{\alpha}\in\widetilde{Z}_n} g^n_{|\alpha}\Lambda(f^n\alpha) \\
&\le \sum_{\tilde{\alpha}\in\widetilde{Z}_n} g^n_{|\alpha} \\
&\le \sum_{\alpha\in Z_n} g^n_{|\alpha} \\
&= \|g^n\|_1.
\end{aligned}$$

So

$$\rho_1 = \limsup_{n\to\infty} \Big(\sum_{\tilde{\alpha}\in\widetilde{Z}_n} g^n_{|\alpha} \Big)^{1/n}.$$

The pressures of $(I_i, f_i, g_i)_{i\in S}$ and $(\widetilde{I}_i, \tilde{f}_i, g_i)_{i\in\widetilde{S}}$ are therefore the same.

Part C. We assume here that f extends to a continuous map of $[a, b]$. When $J \subset [a, b]$ is an interval then $f(J) \setminus \bigcup_i f(J \cap I_i)$ consists of a finite number of points. By Lemma 5.5 this set difference has zero mass. By the same lemma we get

$$\Lambda(J) = \sum_{i=0}^{\ell} \frac{1}{\rho_1} g_i \Lambda(f_i(J \cap I_i)).$$

Thus,

$$\Big(\min_i g_i \frac{1}{\rho_1}\Big)\Lambda(fJ) \le \Lambda(J) \le \Big(\sum_i g_i \frac{1}{\rho_1}\Big)\Lambda(fJ).$$

In particular

$$\Lambda(J) = 0 \iff \Lambda(fJ) = 0.$$

(Note, however, that $\Lambda(J) > 0$ does not imply $\Lambda(f^{-1}J) > 0$ as the latter set might be empty).

Let us write $x \sim x'$ if $\phi(x) = \phi(x')$,

When $x, x' \in I$ and $x \sim x'$ then $\Lambda([x, x']) = 0$ so also $\Lambda(f[x, x']) = 0$. As we have assumed f continuous, $f([x, x'])$ is connected and contains $f(x)$, $f(x')$. Therefore, $\phi(f(x)) = \phi(f(x'))$, i.e. $f(x) \sim f(x')$. For $y \in [0, 1]$, we may thus define $\tilde{f}(y) = \phi(f(x))$ with $x \in \phi^{-1}(y)$ (independent of the choice of x). Then for every $x \in [a, b]$,

$$\tilde{f}(\phi(x)) = \phi(f(x)).$$

The same argument also shows that for any two $x, x' \in I$ we have

$$|\tilde{f}(\phi(x)) - \tilde{f}(\phi(x'))| \le \max_{i \in \widetilde{S}} \frac{\rho_1}{g_i} |\phi(x) - \phi(x')|$$

so $\tilde{f}$ is a continuous endomorphism of $[0, 1]$. Uniform expansion should be understood as eventual uniform expansion on cylinders as in Lemma 5.4 and is proved in the same way.

Remark 1. The set $\widetilde{S}$ may depend upon the weights g_i. If, however, f is transitive then $\widetilde{S} = S$ for any choice of non-zero weights and $\widetilde{Z}_n = Z_n$ for all n. We leave the exercise to the reader.

A. Geometry of the weight function $\omega(\alpha)$

Fix $n \geq 1$ and an n-cylinder $\alpha \in Z_n$. Recall that we have associated a weight

$$\omega(\alpha) = -\, g^n_{|\alpha} \sum_{\hat{x} \in \widehat{\partial}\alpha} \sigma(f^n \hat{x}, x) \cdot \varepsilon(f^n \hat{x}).$$

Set

$$\pi(\alpha) := -\sum_{\hat{x} \in \widehat{\partial}\alpha} \sigma(f^n \hat{x}, x) \cdot \varepsilon(f^n \hat{x}).$$

This quantity depends only on the boundary values and their positions relative to the diagonal. Let h be an affine map on α coinciding with f^n on the boundary.

Lemma A.1. *We have*

$$\pi(\alpha) = -\sum_{\hat{x} \in \widehat{\partial}\alpha} \sigma(h(\hat{x}), x) \cdot \varepsilon(h(\hat{x}))$$

and

- $\pi(\alpha) = -1$ *if* $0 < \mathrm{slope}(h) \leq 1$ *and* $\overline{h(\alpha)}$ *touches the diagonal;*
- $\pi(\alpha) = 1$ *if* $h(\alpha)$ *transverses the diagonal with slope either* > 1 *or* < 0*;*
- $\pi(\alpha) = 0$ *in all other cases, namely*
  - *either* $\overline{h(\alpha)}$ *does not touch the diagonal*
  - *or* $\overline{h(\alpha)}$ *touches the diagonal at one end only, with slope* > 1 *or* < 0.

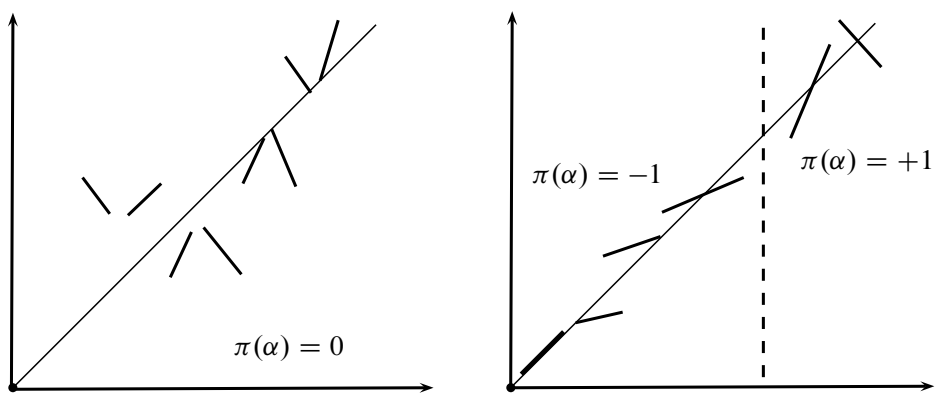

Figure 3. fixed points counting

Proof. Since $f^n|_\alpha$ is a continuous strictly monotone map, we have

$$\sigma(f^n\hat{x}, x) = \sigma(h(\hat{x}), x)$$

and

$$\varepsilon(f^n\hat{x}) = \varepsilon(h(\hat{x}))$$

at the two ends of α. So we can replace f^n by h in $\pi(\alpha)$.

Extend h continuously to the boundary points. Let $\hat{x}$ be a boundary germ. We check case by case the value of

$$-\sigma(h(\hat{x}), x)\cdot\varepsilon(h(\hat{x})) = \begin{cases} \frac{1}{2} & \text{if } h(x) < x \text{ and } h(\hat{x}) > h(x), \\ & \text{or if } h(x) > x \text{ and } h(\hat{x}) < h(x), \\ -\frac{1}{2} & \text{if } h(x) = x, \\ & \text{or } h(x) > x \text{ and } h(\hat{x}) > h(x), \\ & \text{or if } h(x) < x \text{ and } h(\hat{x}) < h(x). \end{cases}$$

Adding the values at the two ends, we get the lemma. □

The quantity $\pi(\alpha)$ counts the number of 'effective' fixed points of $f^n_{|\alpha}$. If e.g. f^n has 3 fixed points in α then they only count as one provided the middle is given the opposite sign of the two others.

We also notice that if f is Lipschitz-expanding then $\pi(\alpha)$ is either 0 or $+1$ for all n and $\alpha \in Z_n$. So in that case $N_n \geq 0$ for all n.

B. Relation between det $\mathcal{R}$, det $\mathcal{B}$, and Milnor–Thurston's kneading determinant

We relate here our definition of the kneading determinant to that of Milnor and Thurston (modified by adding weights) and that of Baladi and Ruelle [4]. We write

$$I_k =]c_k, c_{k+1}[, \quad 0 \leq k \leq \ell$$

and define

$$\eta_k(\hat{x},t) = \begin{cases} \theta(\hat{x},t;c_k) - \theta(\hat{x},t;c_{k+1}), & 0 \le k < \ell, \\ \theta(\hat{x},t;c_\ell) + \theta(\hat{x},t;c_0), & k = \ell, \end{cases}$$

or

$$\eta_k(\hat{x},t) = \sum_{j=0}^{\ell} \theta(\hat{x},t;c_j) Q_{jk} \tag{20}$$

with

$$Q = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

For further use we note that by adding the first line in Q to the second, the resulting second line to the third, etc., one obtains a triangular matrix with 1 in the diagonal except for the bottom right element which becomes 2. Therefore, $\det Q = 2$.

As

$$\sigma(\hat{x},c_k) - \sigma(\hat{x},c_{k+1}) = \chi_{I_k}(\hat{x}), \quad 0 \le k \le \ell,$$

we may also write

$$\eta_k(\hat{x},t) = \sum_{m \ge 0} t^m [sg]^m(\hat{x}) \chi_{I_k}(f^m \hat{x}).$$

We have the following identity.

Lemma B.1. *We have*

$$\sum_{k=0}^{\ell} \eta_k(\hat{x},t)(1 - t s_k g_k) \equiv 1 \quad \textit{for all } a^+ \le \hat{x} \le b^-.$$

Proof. Using

$$\chi_{I_k}(f^m \hat{x})\, s_k g_k = [sg](f^m \hat{x}) \chi_{I_k}(\hat{x}),$$

we get

$$\begin{aligned}
\sum_{k=0}^{\ell} \eta_k(\hat{x},t) &= \sum_{m\geq 0} t^m [sg]^m(\hat{x}) \\
&= 1 + \sum_{m\geq 0} t^{m+1}[sg]^{m+1}(\hat{x}) \\
&= 1 + \sum_{k=0}^{\ell}\sum_{m\geq 0} t^{m+1}[sg]^{m+1}(\hat{x})\chi_{I_k}(f^m\hat{x}) \\
&= 1 + \sum_{k=0}^{\ell}\sum_{m\geq 0} t^{m}[sg]^{m}(\hat{x})\chi_{I_k}(f^m\hat{x})\; ts_k g_k \\
&= 1 + \sum_{k=0}^{\ell} \eta_k(\hat{x},t)\; ts_k g_k.
\end{aligned}$$

□

Consider now the cutting point increments. For every $0 \leq i, k \leq \ell$, let

$$\Delta_{c_i}\eta_k(t) = \begin{cases} \eta_k(a^+,t) + \eta_k(b^-,t), & i = 0, \\ \eta_k(c_i^+,t) - \eta_k(c_i^-,t), & 0 < i \leq \ell, \end{cases} \tag{21}$$

and define the augmented Milnor–Thurston kneading matrix

$$\mathcal{N}(t) = (\Delta_{c_i}\eta_k(t))_{0\leq i,k\leq \ell}\ .$$

Now, Milnor and Thurston originally defined their kneading matrix as the $\ell\times(\ell+1)$ sub-matrix consisting of $\mathcal{N}(t)$ without the first line:

$$\widehat{\mathcal{N}}(t) = (\Delta_{c_i}\eta_k(t))_{i=1,\cdots,\ell,\ k=0,1,\cdots,\ell}\ .$$

Denote by $D_j(t)$ the determinant of $\widehat{\mathcal{N}}(t)$ after deleting the j-th column. Baladi and Ruelle [4] used another matrix $\mathcal{B}(t)$ obtained from

$$\mathcal{R}(t) = (\Delta_{c_i}\theta(\cdot,t;c_k))_{0\leq i,k\leq \ell}$$

by deleting the first line and the first column.

Proposition B.2. *The quantity*

$$D_{\mathrm{MT}}(t) := \frac{(-1)^j D_j(t)}{1 - s_j g_j t}$$

is independent of j. *Setting*

$$H(t) = 1 - t(s_0 g_0 + s_\ell g_\ell)/2,$$

we have

$$\frac{1}{2}\det \mathcal{N}(t) = D_{\mathrm{MT}}(t) = \det \mathcal{R}(t) = \frac{\det \mathcal{B}(t)}{H(t)}. \tag{22}$$

Proof. Let

$$\mathbf{v} = \begin{pmatrix} 1 - s_0 g_0 t \\ \vdots \\ 1 - s_\ell g_\ell t \end{pmatrix}$$

and let $(\mathbf{e}_0)_i = \delta_{i,0}$ be the canonical first base vector in $\mathbb{C}^{\ell+1}$. By Lemma B.1, $\sum_{k=0}^{\ell} \eta_k(\hat{x}, t) v_k = 1$. In view of the signs in (21) with the first line having a plus sign and the rest a minus sign, we get $\mathcal{N}(t)\mathbf{v} = 2\mathbf{e}_0$. By Cramer's solution for a linear system, $v_j = 1 - s_j g_j t = 2(-1)^j D_j(t)/\det \mathcal{N}(t)$ which implies the first statement as well as the first equality in (22). From definitions (20) and (21) we see that $\mathcal{N}(t) = \mathcal{R}(t)Q$. As $\det Q = 2$ we obtain the second equality. For the last, note that the vector $Q\mathbf{v}$ has $v_0 + v_\ell = 2H(t)$ as its first row. Since $\mathcal{R}(Q\mathbf{v}) = \mathcal{R}Q\mathbf{v} = \mathcal{N}\mathbf{v} = 2\mathbf{e}_0$ we obtain again by Cramer's formulae (eliminating the first row and the first line in $\mathcal{R}$): $2H(t) = v_0 + v_\ell = 2\det \mathcal{B}(t)/\det \mathcal{R}(t)$. □

Corollary B.3. *If all the weights* g_i *are equal to* 1, *all three determinants* D_{MT}, $\det \mathcal{R}$, $\det \mathcal{B}$ *have the same zeros in* $\mathbb{D}$.

Proof. In this case $H(t) = 1 - t(s_0 + s_\ell)/2 = 1$ or $1 - t$ so $H(t)$ has no zeros in $\{|t| < 1/\rho_\infty\} = \mathbb{D}$. □

C. The first zero of $\det \mathcal{B}$ may not correspond to the pressure

We have shown in Theorem 2.3, Part 3, that the first zero of $\det \mathcal{R}$ corresponds to the pressure. And in case all the weights g_i are 1, one can also use the first zero of $\det \mathcal{B}$ (Corollary B.3). This need not, however, be true with more general weights. Here is a counter example.

Let

$$I = [a, b] = [0, 3], \quad I_0 =]0, 1[, \quad I_1 =]1, 2[, \quad I_2 =]2, 3[.$$

We have

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 2 - 2(x-1), & 1 \le x \le 2, \\ 2(x-2), & 2 \le x \le 3. \end{cases}$$

Let us assign weights $g_0 = g_1 = 1$ and $g_2 = M$.

Note that $f(I_2) = [0, 2]$ and that

$$f : [0, 2] \longrightarrow [0, 2]$$

is the full tent map. There is no periodic points in I_2. Using Lemma A.1 and the definition one obtains

$$\zeta(t) = \exp\Big(\sum_{n \ge 1} \frac{t^n}{n} 2^n\Big) = (1 - 2t)^{-1}.$$

So by (22) and Theorem 2.2 we have

$$D_{\mathrm{MT}}(t) = \det \mathcal{R}(t) = \frac{1}{\zeta(t)} = 1 - 2t.$$

The first zero being $1/2$ one obtains that the pressure is $\log 2$ (this pressure can also be computed directly). It is easily seen that the topological entropy is also $\log 2$.

On the other hand,

$$H(t) = 1 - \frac{t}{2}(s_0 g_0 + s_2 g_2) = 1 - \frac{t}{2}(1 + M).$$

So by (22) again

$$\det \mathcal{B}(t) = H(t) \det \mathcal{R}(t) = \Big(1 - \frac{t}{2}(1 + M)\Big)(1 - 2t).$$

If $M > 3$, then $\det \mathcal{B}(t)$ has a 'spurious' zero at $2/(1 + M)$ smaller than $1/2$.

So the first positive zero of $\mathcal{B}(t)$ does not correspond to the pressure in this case. By increasing M, one can make this first zero arbitrarily small without changing the pressure.

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