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Approximation using hidden variable fractal interpolation function

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Abstract. The notion of hidden variable fractal interpolation provides a method to approximate functions that are self-referential or non-self-referential, and consequently allows great flexibility and diversity for the fractal modeling problem. The current article intends to apply hidden variable fractal interpolation to associate a class of $\mathbb{R}^2$-valued continuous fractal functions with a prescribed continuous function. Suitable values of the parameters are identified so that the fractal functions retain positivity and regularity of the germ function. As an application of the developed theory, we obtain positive $\mathcal{C}^1$-cubic spline hidden variable fractal interpolation functions corresponding to a prescribed set of positive data, thus initiating a new approach to shape preserving approximation via hidden variable fractal function. Depending on the values of the parameters, these positive interpolants can reflect the self-referentiality or non-self-referentiality of the original data defining function and fractality of its derivative. Therefore, the present scheme outperforms the traditional nonrecursive positivity preserving $\mathcal{C}^1$-cubic spline interpolation scheme and its fractal extension studied recently in the literature.

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Contents

1. Introductory remarks

Fractal interpolation introduced by Barnsley [2, 3, 4] and followed in earnest by a host of researchers (see, for instance, [8, 9, 10, 12, 18, 20, 23, 29]) is a relatively recent technique of interpolation and approximation, which is more versatile than the traditional nonrecursive approximation methods. Construction of interpolation schemes that reflect the intrinsic shape (expressed mathematically in terms of positivity, monotonicity, convexity, etc.) inferred by a prescribed set of data points, which is generally referred to as shape preserving interpolation (isogeometric interpolation), has received a great interest in the last decades (see, for instance, [11, 13, 14, 24] and references quoted therein).

Ubiquity of fractal functions is claimed by the fractal researchers in various contexts, inside the scope of pure mathematics and the real world applications. One of the promising topic, at least in our opinion, in this regard is the demonstration that the fractal functions can be explored in the field of shape preserving approximation, thus unifying the two fields that are otherwise developing independently and in parallel. As a humble contribution to this goal, our group has initiated a study on shape preserving fractal interpolation and approximation using Iterated Function Systems (IFSs) composed of suitable families of polynomial and rational (see [10, 25, 27]) functions. These shape preserving fractal interpolation schemes possess the novelty that the interpolants inherit the shape property in question and at the same time the derivatives of these interpolants own irregularity in a finite or dense subset of the interpolation interval. Consequently, these schemes are relatively more amenable in the study of nonlinear and nonequilibrium phenomena, for instance, in electromechanical systems and fluid dynamics problems. However, these fractal functions are self-affine, self-similar or more generally self-referential in the sense that the graph of the function is a union of transformed copies of itself. This leads to the loss of flexibility and inadequacies in approximation of a function that shows no self-similarity or less self-similarity.

Barnsley et al. [5] conceived the idea of hidden variable Fractal Interpolation Functions (FIFs), which are more diverse, appealing and irregular than a traditional FIF. Hidden variable FIF are generally non-self-referential. For simulating curves that exhibit partly self-referential and partly non-self-referential nature, Chand and Kapoor [9] introduced the notion of coalescence hidden variable FIF. We shall supply more particulars – of a technical nature – concerning the hidden variable FIF in the next section soon after we finish discussing this general introductory remarks.

Due to a seemingly complicated structure, it is *a priori* dubious whether a useful and elegant theory of shape preserving interpolation can be built upon the hidden variable or coalescence hidden variable FIF, and the current article is an attempt in this direction. The primary intent of this article is to employ $\mathcal{C}^1$-cubic spline hidden variable FIF for positivity preserving interpolation, thereby giving a class of positive interpolants that include traditional positive $\mathcal{C}^1$-cubic spline [24] and its fractal extension [10] studied recently as special cases. However, our methods are general enough in scope to enable one to carry out the analysis with different spline structures. A cheerful potent is that the current approach provides more theoretical insights to the hidden variable fractal interpolation, making it potentially applicable in approximation theory.

Using an IFS whose maps are chosen based on a given univariate continuous function f defined on a compact real interval I, a family

$$\{f^\alpha : \alpha \in \mathbb{R}^M, |\alpha_n| < 1, n = 1, 2, \dots, M\}$$

of fractal functions associated with f can be constructed, where M depends upon the number of points at which f is sampled. Navascués [19] introduced an operator

$$\mathcal{F}^\alpha : \mathcal{C}(I) \longrightarrow \mathcal{C}(I)$$

defined through

$$f \longmapsto f^\alpha$$

and developed properties of this operator. This enriched the fractal approximation theory and facilitated the theory of fractal functions to interact with the fields such as functional analysis and operator theory (see, for instance, [20, 21]).

Following a similar rationale, we apply hidden variable FIF to associate a family of $\mathbb{R}^2$-valued continuous fractal functions with a prescribed continuous function $\boldsymbol{f} : I \to \mathbb{R}^2$, where $\mathbb{R}^2$ is endowed with the l^1-norm (taxicab norm). To be more precise, given a continuous function

$$\boldsymbol{f} : I \longrightarrow \mathbb{R}^2,$$

we obtain a family of continuous fractal functions $\boldsymbol{f}[\boldsymbol{A}]$ parameterized by a block matrix

$$\boldsymbol{A} = [A_n]_{n=1}^{M},$$

where each A_n is a suitable matrix in $M^{2\times 2}(\mathbb{R})$, the space of all 2×2 matrices having real entries. As the reader will discern, this is a natural extension of the notion of α-fractal function f^α. The advantage is that the function whose graph is the orthogonal projection of graph$(\boldsymbol{f}[\boldsymbol{A}]) \subset I \times \mathbb{R}^2$ provides a non-self-referential fractal function corresponding to a given real valued continuous function in contrast to the self-referential fractal generalizations obtained through α-fractal operator. Furthermore, by a proper choice of elements of the hidden variable IFS, the projection can be made self-referential as well, thus providing more flexibility and diversity in the process of approximation. We may refer $\boldsymbol{f}[\boldsymbol{A}]$ as the *$\boldsymbol{A}$-fractal function corresponding to $\boldsymbol{f}$* or the *fractal perturbation of $\boldsymbol{f}$*. Some properties of the map $\boldsymbol{f} \mapsto \boldsymbol{f}[\boldsymbol{A}]$ are established; see Section 3.

The presence of the block matrix parameter $\boldsymbol{A}$ in the constructed function undoubtedly provides more flexibility, which may be exploited in various approximation problems. However, it is not devoid of interest to note that this flexibility raises, quite naturally, the question of an "optimal" choice of parameter, which is an "inverse problem" of fractal approximation. To this end, in Section 4 we prove that given a continuous function $\boldsymbol{\Phi} : I \to \mathbb{R}^2$ and its Lipschitz continuous approximant $\boldsymbol{f}$, the question of obtaining a hidden variable fractal analogue $\boldsymbol{f}[\boldsymbol{A}]$ close to $\boldsymbol{\Phi}$ reduces to a constrained nonlinear convex optimization problem. Though we do not attempt to solve this optimization problem completely, it is worth to recall here that a constrained optimization problem where the objective function and the constraints are convex possesses nice theoretical properties and can be efficiently solved numerically. By providing an estimate for the error caused by a perturbation of the parameter $\boldsymbol{A}$ occurring in the fractal function $\boldsymbol{f}[\boldsymbol{A}]$, sensitivity analysis is also broached.

As we alluded earlier, our own interest in the present formalism of the hidden variable FIF has been stimulated from the practical need to construct shape preserving interpolants that are more flexible than the traditional non-recursive $\mathcal{C}^1$-cubic interpolant and its fractal extension. To achieve this goal, we shall adopt a two step procedure as follows. The first step is intended to solve a generic problem wherein we identify suitable values of the parameters so that the perturbed fractal function $\boldsymbol{f}[\boldsymbol{A}]$ preserves order of regularity and positivity of the original function $\boldsymbol{f}$, where the positivity $\boldsymbol{f}$ is understood componentwise; see Section 5. Next, we obtain $\mathcal{C}^1$-cubic hidden variable FIF as perturbation (i.e., $\boldsymbol{A}$-fractal function) of the classical cubic interpolants of the extended data. Finally, in Section 6 we apply the developed theory so that $\mathcal{C}^1$-cubic hidden variable FIF arising out of $\boldsymbol{f}[\boldsymbol{A}]$ preserves positivity of the classical cubic spline; a procedure that culminate with the shape preserving $\mathcal{C}^1$-cubic hidden variable FIF. Section 7 provides some simple numerical examples and graphs.

2. Rudiments of fractal interpolation and hidden variable FIF

To make the article fairly self-contained we shall record here an abbreviated development of the hidden variable FIF. The reader, if so inclined, may consult the well-known treatises [3, 5] for a complete and rigorous treatment.

Let (X, d_X) be a complete metric space with metric d_X. If

$$f_n \colon X \longrightarrow X, \quad n = 1, 2, \dots, N,$$

are continuous mappings, then

$$\mathcal{I} = \{X; f_n \colon n = 1, 2, \dots, N\}$$

is called an *iterated function system* (IFS for short). If, in addition, each f_n is a contraction, then $\mathcal{I}$ is referred to as a *hyperbolic IFS*. Denote by $H(X)$, the hyperspace of all nonempty compact subsets of X. There is a natural metric, called the Hausdorff metric, which completes $H(X)$. Associated with the IFS $\mathcal{I}$, there exists a set-valued Hutchinson map F on $H(X)$ defined by

$$F(B) = \bigcup_{n=1}^{N} f_n(B) \quad \text{for all } B \in H(X).$$

Given a hyperbolic IFS $\mathcal{I}$ and any set $B \in H(X)$, there exists a unique set A, called the *attractor* of the hyperbolic IFS, such that

$$A = F(A) = \lim_{n\to\infty} F^n(B),$$

where F^n denotes the n-fold autocomposition of F and the limit is taken in the Hausdorff metric. In what follows, we consider suitable IFS whose attractor is the graph of a continuous function interpolating a prescribed data set.

Given a set of interpolation points

$$\Delta = \{(x_n, y_n) \in \mathbb{R}^2 \colon n = 1, 2, \ldots, N\}$$

with strictly increasing abscissae, set

$$I = [x_1, x_N] \quad \text{and} \quad I_n = [x_n, x_{n+1}] \quad \text{for } n \in J = \{1, 2, \ldots, N-1\}.$$

Define contraction homeomorphisms $u_n \colon I \to I_n$ satisfying

$$u_n(x_1) = x_n, \quad u_n(x_N) = x_{n+1}.$$

Consider bivariate mappings $v_n \colon I \times \mathbb{R} \to \mathbb{R}$ that are continuous in the first argument, contraction in the second argument, and fulfilling

$$v_n(x_1, y_1) = y_n, \quad v_n(x_N, y_N) = y_{n+1}, n \in J.$$

Let $X := I \times \mathbb{R}$ and consider the IFS

$$\mathcal{I} = \{X; w_n = (u_n, v_n) \colon n \in J\}.$$

The *attractor* $G = G(f)$ of this IFS is the graph of a continuous map $f \colon I \to \mathbb{R}$ satisfying

$$f(x_n) = y_n, \quad \text{for } n = 1, 2, \ldots, N.$$

Further, f is the fixed point of the operator defined by

$$(Tg)(x) = v_n(u_n^{-1}(x), g \circ u_n^{-1}(x)), \quad x \in I_n, n \in J.$$

The function f is referred to as the *fractal interpolation function* (FIF) corresponding to Δ. Since $G(f)$ is the union of transformed copies of itself, specifically

$$G(f) = \bigcup_{n \in J} w_n(G(f)),$$

the map f is a *self-referential function*.

The non-self-referential functions can be approximated by using the notion of hidden variable FIF, which we shall succinctly review in the following.

For the data set Δ, by introducing a set of real parameters $\{z_n \colon n = 1, 2, \ldots, N\}$ whose selection is highly subjective, we define a generalized set of data to be

$$\widehat{\Delta} = \{(x_n, y_n, z_n) \in I \times \mathbb{R}^2 \colon n = 1, 2, \ldots, N\}.$$

The idea is to construct a fractal interpolation function for $\widehat{\Delta}$, and project its graph into $I \times \mathbb{R}$ in such a way that the projection is the graph of a function that interpolates Δ. To this end, one proceeds as follows. For $n \in J$, let the affine functions $L_n \colon I \to I_n$ be defined so as to satisfy

$$L_n(x) = a_n x + b_n, \quad L_n(x_1) = x_n, \quad L_n(x_N) = x_{n+1}. \tag{1}$$

Let $\mathbb{R}^2$ be endowed with the Manhattan metric

$$d_M((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

which is induced by the l^1-norm. Here we note that an element in $\mathbb{R}^2$ may be regarded as an ordered pair (a_1, a_2) or as a column matrix $(a_1, a_2)^T$ which will be clear from the context. Let $F_n : I \times \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{aligned} F_n(x, \boldsymbol{y}) &= F_n(x, y, z) \\ &= (F_n^1(x, y, z), F_n^2(x, z))^T \\ &:= A_n(y, z)^T + (p_n(x), q_n(x))^T, \end{aligned} \tag{2}$$

where T denotes the transpose, A_n are upper-triangular matrices $\begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}$, and p_n, q_n are suitable real valued Lipschitz continuous functions so that the following conditions are satisfied for all $n \in J$:

(1) $d_M(F_n(x, y, z), F_n(x^*, y, z)) \leq c|x - x^*|$, for some constant $c > 0$;

(2) $d_M(F_n(x, y, z), F_n(x, y^*, z^*)) \leq s\, d_M((y, z), (y^*, z^*))$, for $0 \leq s < 1$;

(3) join-up conditions:

$$F_n(x_1, y_1, z_1) = (y_n, z_n) \quad \text{and} \quad F_n(x_N, y_N, z_N) = (y_{n+1}, z_{n+1}).$$

Here the variables α_n, γ_n, and β_n are selected such that $\|A_n\|_1 < 1$ for all $n \in J$. That the functions

$$\begin{aligned} w_n : \ I \times \mathbb{R}^2 &\longrightarrow I \times \mathbb{R}^2, \\ (x, y, z) &\longmapsto (L_n(x), F_n(x, y, z)), \end{aligned}$$

are contraction maps with respect to the metric d_M^* defined by

$$d_M^*((x, y, z), (x^*, y^*, z^*)) = |x - x^*| + \theta\, d_M((y, z), (y^*, z^*))$$

for a suitable value of θ follows from the conditions on maps L_n and F_n. Consequently, the IFS $\{I \times \mathbb{R}^2; w_n : n \in J\}$ admits an attractor $A \in H(I \times \mathbb{R}^2)$. Further, the attractor A is the graph of a continuous function $\boldsymbol{g} : I \to \mathbb{R}^2$ such that $\boldsymbol{g}(x_n) = (y_n, z_n)$ for all $n \in J$. Letting $\boldsymbol{g} = (g_1, g_2)$ it follows that $g_1 : I \to \mathbb{R}$ is a continuous function interpolating Δ. The aforementioned function $g_1 : I \to \mathbb{R}$ is called (coalescence) *hidden variable fractal interpolation function* associated with the set of data Δ (see, for instance, [9]).

Let

$$\mathcal{G} := \{\boldsymbol{h}\colon I \to \mathbb{R}^2\colon \boldsymbol{h} \text{ is continuous on } I,\ \boldsymbol{h}(x_1) = (y_1, z_1), \boldsymbol{h}(x_N) = (y_N, z_N)\}$$

be endowed with the metric

$$d(\boldsymbol{h}, \boldsymbol{h}^*) = \max\{d_M(\boldsymbol{h}(x), \boldsymbol{h}^*(x))\colon x \in I\}.$$

To obtain a functional equation for $\boldsymbol{g}$ we recall that $\boldsymbol{g}$ is the fixed point of the operator

$$\mathrm{T}\colon \mathcal{G} \longrightarrow \mathcal{G}, \quad (\mathrm{T}\boldsymbol{h})(x) = F_n(L_n^{-1}(x), \boldsymbol{h}(L_n^{-1}(x))), \quad \text{for } x \in I_n, n \in J.$$

Whence, the vector-valued function $\boldsymbol{g}$ enjoys the functional equation

$$\boldsymbol{g}(L_n(x)) = A_n \boldsymbol{g}(x) + (p_n(x), q_n(x))^T, \quad x \in I.$$

Consequently, the component functions g_1 and g_2 obey the following coupled functional equations.

$$\left.\begin{aligned} g_1(L_n(x)) &= \alpha_n g_1(x) + \beta_n g_2(x) + p_n(x), \\ g_2(L_n(x)) &= \gamma_n g_2(x) + q_n(x), \quad x \in I. \end{aligned}\right\} \tag{3}$$

Remark 2.1. The projection $G(g_1)$ of the attractor $G(\boldsymbol{g})$ is not always the union of transformed copies of itself. Hence, g_1 is, in general, non-self-referential. It can be observed that

$$G(g_2) = \bigcup_{n \in J} w_n^2(G(g_2)),$$

where

$$w_n^2(x, z) := (L_n(x), F_n^2(x, z)) = (a_n x + b_n, \gamma_n z + q_n(x)) \quad \text{for all } n \in J.$$

Thus, g_2 is a self-referential fractal function interpolating

$$\{(x_n, z_n) : n = 1, 2, \dots, N\}.$$

If the elements of the hidden variable IFS are chosen such that $z_n = y_n$ for all $n = 1, 2, \dots, N$, $\alpha_n + \beta_n = \gamma_n$, and $p_n = q_n$ for all $n \in J$, then g_1 coincides with g_2, and hence in this case one obtains a self-referential hidden variable FIF. Similarly, if $\beta_n = 0$ for all $n \in J$, then

$$G(g_1) = \bigcup_{n \in J} w_n^1(G(g_1)),$$

where

$$w_n^1(x, y, z) := (L_n(x), F_n^1(x, y, z)) = (a_n x + b_n, \alpha_n y + p_n(x))\ \forall\ n \in J.$$

Ergo, we infer that g_1 is self-referential in this case as well.

Let us close this section with a word on notation. Throughout the rest of the paper, we use the block matrix $\boldsymbol{A} = [A_1\ A_2\ \dots\ A_{N-1}] = [A_n]_{n\in J}$ to collectively represent the parameters involved in the definition of the hidden variable FIF.

3. Continuous $\mathbb{R}^2$-valued functions as special cases of hidden variable FIFs

Having dispensed with an overall flavor of the hidden variable FIF, in this section, we enunciate that a continuous function $\boldsymbol{f}: I \to \mathbb{R}^2$ provides a family of fractal functions $\boldsymbol{f}[\boldsymbol{A}]$ parameterized by a certain block matrix $\boldsymbol{A} = [A_n]_{n\in J}$ with

$$A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix},$$

where $\boldsymbol{f}[\mathbf{0}] = \boldsymbol{f}$. We shall also consider certain properties of the corresponding map $\boldsymbol{f} \mapsto \boldsymbol{f}[\boldsymbol{A}]$ for a fixed $\boldsymbol{A}$. The researches of Navascués and collaborators (see, for instance, [20, 21]) influenced our work in this section. While the results herein share a natural kinship with the corresponding results for a real valued fractal function, the reader will also discern a considerable degree of disparity due to the vector valuedness considered here.

Let $\mathbb{R}^2$ be endowed with the l^1-norm that induces the Manhattan metric and $\boldsymbol{f} = (f_1, f_2) \in \mathrm{Lip}(I, \mathbb{R}^2)$, the space of all Lipschitz $\mathbb{R}^2$-valued functions on $I = [x_1, x_N]$. Choose a partition $\{x_1, x_2, \dots, x_N\}$ of I with increasing abscissae, and consider the data set $\Delta = \{(x_n, f_1(x_n), f_2(x_n)): n = 1, 2, \dots, N\}$.

In the IFS $\{I \times \mathbb{R}^2; (L_n, F_n): n \in J\}$ defined through (1) and (2), we consider the following special choice

$$p_n(x) = f_1 \circ L_n(x) - \alpha_n b_1(x) - \beta_n b_2(x),$$

$$q_n(x) = f_2 \circ L_n(x) - \gamma_n b_2(x),$$

where $\boldsymbol{b} = (b_1, b_2) \in \mathrm{Lip}(I, \mathbb{R}^2)$ satisfies $\boldsymbol{b}(x_1) = \boldsymbol{f}(x_1)$ and $\boldsymbol{b}(x_N) = \boldsymbol{f}(x_N)$. In this case, the IFS provides an attractor that is the graph of a continuous function denoted here as $\boldsymbol{f}[\boldsymbol{A}] = (f_1[\boldsymbol{A}], f_2[\boldsymbol{A}])$. The fixed point $\boldsymbol{f}[\boldsymbol{A}]$ also depends on the choice of $\boldsymbol{b} \in \mathrm{Lip}(I, \mathbb{R}^2)$, although we suppress this dependence in our notation. We christen this function $\boldsymbol{f}[\boldsymbol{A}]$ as $\boldsymbol{A}$-fractal function of $\boldsymbol{f}$ with respect to the partition Δ and the function $\boldsymbol{b}$. Let $\boldsymbol{f}$ be a Lipschitz continuous classical interpolant for the data set $\Delta = \{(x_n, y_n, z_n): n = 1, 2, \dots, N\}$. Since $\boldsymbol{f}[\boldsymbol{A}](x_n) = \boldsymbol{f}(x_n)$ for all $n = 1, 2, \dots, N$, for any choice of $\boldsymbol{A} = [A_n]_{n\in J}$ and any choice of $\boldsymbol{b}$ satisfying the conditions specified earlier, the fractal function $\boldsymbol{f}[\boldsymbol{A}]$ can be regarded as "fractal generalization" of the classical interpolant $\boldsymbol{f}$. Following (3), we stipulate that $\boldsymbol{f}[\boldsymbol{A}]$ satisfies

$$\boldsymbol{f}[\boldsymbol{A}](x) = \boldsymbol{f}(x) + A_n(\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{b})(L_n^{-1}(x)), \quad x \in I_n, n \in J. \tag{4}$$

We may also assume that $\boldsymbol{b}$ depends linearly on $\boldsymbol{f}$, that is to say,

$$\boldsymbol{b}_{\lambda \boldsymbol{f}+\boldsymbol{g}} = \lambda \boldsymbol{b}_{\boldsymbol{f}} + \boldsymbol{b}_{\boldsymbol{g}}$$

or

$$\boldsymbol{b} = L\boldsymbol{f}, \text{ where } L\colon \operatorname{Lip}(I, \mathbb{R}^2) \to \operatorname{Lip}(I, \mathbb{R}^2)$$

is a linear operator which is bounded with respect to the norm

$$\|\boldsymbol{f}\|_\infty := \sup\{\|\boldsymbol{f}(x)\|_{l^1} : x \in I\} = \sup\{|f_1(x)| + |f_2(x)|\colon x \in I\}.$$

For a fixed partition Δ, parameter matrix $\boldsymbol{A}$, and Lipschitz function $\boldsymbol{b}$, let us consider the (fractal) operator

$$\mathcal{F}[\boldsymbol{A}]\colon \operatorname{Lip}(I, \mathbb{R}^2) \subset \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2); \quad (\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}) = \boldsymbol{f}[\boldsymbol{A}].$$

It is worth pointing out that for a prescribed Lipschitz function $f : I \to \mathbb{R}$, we can select $\boldsymbol{f} = (f, f)$, $\boldsymbol{b} = (b, b)$ satisfying $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$, and $\alpha_n + \beta_n = \gamma_n$ to construct $\boldsymbol{A}$-fractal function for $\boldsymbol{f}$. In this case, we obtain $\boldsymbol{f}[\boldsymbol{A}] = (f[\boldsymbol{A}], f[\boldsymbol{A}])$, where $f[\boldsymbol{A}]$ coincides with the standard γ-fractal function f^γ corresponding to f with $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{N-1})$.

Let us recall that a linear operator $T\colon X_0 \subseteq X \to Y$ is a *closed operator* if for every sequence $\{x_n\}$ in X_0 that satisfies $x_n \to x$ and $Tx_n \to y$ for some $x \in X$ and $y \in Y$, we have

$$x \in X_0 \quad \text{and} \quad Tx = y.$$

Note that, in general, a closed linear operator need not be a bounded operator and vice versa. If Y is a Banach space, then a bounded linear operator $T\colon X_0 \subseteq X \to Y$ is closed if and only if X_0 is a closed subspace of X. The book [1] is a good reference on closed operators and their properties.

In what follows, we shall establish certain properties of the function $\boldsymbol{f}[\boldsymbol{A}]$ and the operator $\mathcal{F}[\boldsymbol{A}]$. Before embarking on this project let us note that the block matrix $\boldsymbol{A}$ can be viewed as an element in $M^{2\times 2(N-1)}(\mathbb{R})$ and

$$\|\boldsymbol{A}\|_1 = \max_{n\in J}\{|\alpha_n|, |\beta_n| + |\gamma_n|\} < 1.$$

Theorem 3.1. *The following holds*:

(1) $\boldsymbol{f}[\mathbf{0}] = \boldsymbol{f}$. *Consequently, if* $\boldsymbol{A} = \mathbf{0}$, *then the fractal operator* $\mathcal{F}[\boldsymbol{A}]$ *is the identity operator on* $\mathrm{Lip}(I, \mathbb{R}^2)$;

(2) *if* $\boldsymbol{b} = \boldsymbol{f}$, *then* $\boldsymbol{f}[\boldsymbol{A}] = \boldsymbol{f}$;

(3) *the fractal function* $\boldsymbol{f}[\boldsymbol{A}]$ *corresponding to* $\boldsymbol{f}$ *satisfies the inequality*

$$\|\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{f}\|_\infty \leq \frac{\|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1} \|\boldsymbol{f} - \boldsymbol{b}\|_\infty;$$

(4) *for suitable choices of parameters, the fractal function* $\boldsymbol{f}[\boldsymbol{A}]$ *simultaneously interpolates and approximates* $\boldsymbol{f}$;

(5) *if the vector-valued function* $\boldsymbol{b}$ *depends linearly on* $\boldsymbol{f}$, *then the operator*

$$\begin{aligned} \mathcal{F}[\boldsymbol{A}] \colon \mathrm{Lip}(I, \mathbb{R}^2) \subset \mathcal{C}(I, \mathbb{R}^2) &\longrightarrow \mathcal{C}(I, \mathbb{R}^2), \\ \boldsymbol{f} &\longmapsto \boldsymbol{f}[\boldsymbol{A}], \end{aligned}$$

is linear;

(6) *if* $\boldsymbol{b} = L\boldsymbol{f}$, *where* $L \colon \mathrm{Lip}(I, \mathbb{R}^2) \to \mathrm{Lip}(I, \mathbb{R}^2)$ *is a bounded linear map and the parameters are chosen so that* $\|\boldsymbol{A}\|_1 < \|L\|^{-1}$, *then the fractal operator*

$$\begin{aligned} \mathcal{F}[\boldsymbol{A}] \colon \mathrm{Lip}(I, \mathbb{R}^2) \subset \mathcal{C}(I, \mathbb{R}^2) &\longrightarrow \mathcal{C}(I, \mathbb{R}^2), \\ \boldsymbol{f} &\longmapsto \boldsymbol{f}[\boldsymbol{A}], \end{aligned}$$

is bounded, but not closed;

(7) *with assumptions as in the previous item, the fractal operator* $\mathcal{F}[\boldsymbol{A}]$ *is injective. Denoting by* $\mathrm{Rg}(\mathcal{F}[\boldsymbol{A}])$, *the range of* $\mathcal{F}[\boldsymbol{A}]$, *the inverse*

$$\mathcal{F}[\boldsymbol{A}]^{-1} \colon \mathrm{Rg}(\mathcal{F}[\boldsymbol{A}]) \longrightarrow \mathrm{Lip}(I, \mathbb{R}^2)$$

is a bounded and closed operator.

Proof. (1) Follows directly from the functional equation for $\boldsymbol{f}[\boldsymbol{A}]$ (cf. (4)).

(2) Let $\boldsymbol{b} = \boldsymbol{f}$. In this case, the functional equation (4) for $\boldsymbol{f}[\boldsymbol{A}]$ reads

$$\boldsymbol{f}[\boldsymbol{A}](x) = \boldsymbol{f}(x) + A_n(\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{f})(L_n^{-1}(x)), \; x \in I_n, \; n \in J,$$

which is obviously satisfied by $\boldsymbol{f}[\boldsymbol{A}] = \boldsymbol{f}$. Since $\boldsymbol{f}[\boldsymbol{A}]$ is obtained as a fixed point of the map T, from the uniqueness of the fixed point it follows that $\boldsymbol{f}[\boldsymbol{A}] = \boldsymbol{f}$.

(3) By definition

$$\begin{aligned}\|\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{f}\|_\infty &= \sup\{\|\boldsymbol{f}[\boldsymbol{A}](x)-\boldsymbol{f}(x)\|_{l^1} : x \in I\},\\ &= \max_{n\in J} \sup\{\|(\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{f})(x)\|_{l^1} : x \in I_n\},\\ &= \max_{n\in J} \sup\{\|A_n(\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{b})(L_n^{-1}(x))\|_{l^1} : x \in I_n\}.\end{aligned}$$

Letting $\boldsymbol{f}[\boldsymbol{A}] = (f_1[\boldsymbol{A}], f_2[\boldsymbol{A}])$ and performing the matrix multiplication, through a series of self-explanatory steps we obtain

$$\begin{aligned}&\|\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{f}\|_\infty\\ &= \max_{n\in J} \sup\{|\alpha_n(f_1[\boldsymbol{A}]-b_1)(L_n^{-1}(x)) + \beta_n(f_2[\boldsymbol{A}]-b_2)(L_n^{-1}(x))|\\ &\qquad + |\gamma_n(f_2[\boldsymbol{A}]-b_2)\circ L_n^{-1}(x)| : x \in I_n\},\\ &\le \max_{n\in J} \sup\{|\alpha_n||(f_1[\boldsymbol{A}]-b_1)(L_n^{-1}(x))|\\ &\qquad + (|\beta_n|+|\gamma_n|)|(f_2[\boldsymbol{A}]-b_2)(L_n^{-1}(x))| : x \in I_n\},\\ &\le \max_{n\in J} \|A_n\|_1 \sup\{|(f_1[\boldsymbol{A}]-b_1)(L_n^{-1}(x))|\\ &+ |(f_2[\boldsymbol{A}]-b_2)(L_n^{-1}(x))| : x \in I_n\},\\ &= \|\boldsymbol{A}\|_1 \|\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{b}\|_\infty,\\ &\le \|\boldsymbol{A}\|_1(\|\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{f}\|_\infty + \|\boldsymbol{f}-\boldsymbol{b}\|_\infty),\end{aligned}$$

from which the desired estimate can be deduced.

(4) For an arbitrary selection of the partition, parameters, and function $\boldsymbol{b}$, the interpolation property of $\boldsymbol{f}[\boldsymbol{A}]$, i.e., $\boldsymbol{f}[\boldsymbol{A}](x_n) = \boldsymbol{f}(x_n)$ is evident and it is in fact a content of the construction.

Let $\epsilon > 0$. To show $\|\boldsymbol{f}[\boldsymbol{A}]-\boldsymbol{f}\|_\infty < \epsilon$, it suffices to show, thanks to part (3) of this theorem, that

$$\frac{\|\boldsymbol{A}\|_1}{1-\|\boldsymbol{A}\|_1}\|\boldsymbol{f}-\boldsymbol{b}\|_\infty < \epsilon.$$

Choose the parameters α_n, β_n, and γ_n such that

$$\|\boldsymbol{A}\|_1 < \frac{\epsilon}{\epsilon + \|\boldsymbol{f}-\boldsymbol{b}\|_\infty} < 1.$$

With this selection, it is a matter of direct verification that

$$\frac{\|\boldsymbol{A}\|_1}{1-\|\boldsymbol{A}\|_1}\|\boldsymbol{f}-\boldsymbol{b}\|_\infty < \epsilon,$$

whence the stated result follows.

(5) Let $\boldsymbol{f}, \boldsymbol{g}$ be in $\mathrm{Lip}(I, \mathbb{R}^2)$ and $\lambda, \mu \in \mathbb{R}$. We have

$$(\mathcal{F}[A])(\boldsymbol{f}) = \boldsymbol{f}[\boldsymbol{A}] \quad \text{and} \quad (\mathcal{F}[A])(\boldsymbol{g}) = \boldsymbol{g}[\boldsymbol{A}].$$

We have to prove $(\mathcal{F}[A])(\lambda \boldsymbol{f} + \mu \boldsymbol{g}) = \lambda \boldsymbol{f}[\boldsymbol{A}] + \mu \boldsymbol{g}[\boldsymbol{A}]$. Recall that, for all $x \in I_n$,

$$\boldsymbol{f}[\boldsymbol{A}](x) = \boldsymbol{f}(x) + A_n(\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{b}_{\boldsymbol{f}})(L_n^{-1}(x)),$$

$$\boldsymbol{g}[\boldsymbol{A}](x) = \boldsymbol{g}(x) + A_n(\boldsymbol{g}[\boldsymbol{A}] - \boldsymbol{b}_{\boldsymbol{g}})(L_n^{-1}(x)),$$

and then

$$\begin{aligned}&(\lambda \boldsymbol{f}[\boldsymbol{A}] + \mu \boldsymbol{g}[\boldsymbol{A}])(x)\\ &\quad = (\lambda \boldsymbol{f} + \mu \boldsymbol{g})(x) + A_n(\lambda \boldsymbol{f}[\boldsymbol{A}] + \mu \boldsymbol{g}[\boldsymbol{A}] - \boldsymbol{b}_{\lambda \boldsymbol{f} + \mu \boldsymbol{g}})(L_n^{-1}(x)).\end{aligned}$$

Therefore, $\lambda \boldsymbol{f}[\boldsymbol{A}] + \mu \boldsymbol{g}[\boldsymbol{A}]$ is the fixed point of the operator

$$\begin{aligned}(T\boldsymbol{h})(x) &= F_n(L_n^{-1}(x), \boldsymbol{h}(L_n^{-1}(x)))\\ &= (\lambda \boldsymbol{f} + \mu \boldsymbol{g})(x) + A_n(\boldsymbol{h} - \boldsymbol{b}_{\lambda \boldsymbol{f} + \mu \boldsymbol{g}})(L_n^{-1}(x)).\end{aligned}$$

From the uniqueness of the fixed point we gather that

$$(\lambda \boldsymbol{f} + \mu \boldsymbol{g})[\boldsymbol{A}] = \lambda \boldsymbol{f}[\boldsymbol{A}] + \mu \boldsymbol{g}[\boldsymbol{A}],$$

demonstrating the linearity of $\mathcal{F}[\boldsymbol{A}]$.

(6) From part (3) of this theorem, we have

$$\begin{aligned}\|(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f})\|_\infty &= \|(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}) - \boldsymbol{f} + \boldsymbol{f}\|_\infty,\\ &\le \|\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{f}\|_\infty + \|\boldsymbol{f}\|_\infty,\\ &\le \Big(\frac{\|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1}\|\boldsymbol{f} - \boldsymbol{b}\|_\infty + \|\boldsymbol{f}\|_\infty\Big),\\ &\le \Big(\frac{\|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1}\|I_d - L\| + 1\Big)\|\boldsymbol{f}\|_\infty,\end{aligned}$$

where I_d is the identity operator on $\mathrm{Lip}(I, \mathbb{R}^2)$. This shows that the linear map $\mathcal{F}[\boldsymbol{A}]$ is bounded, and the operator norm satisfies

$$\|\mathcal{F}[\boldsymbol{A}]\| \le \frac{\|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1}\|I_d - L\| + 1.$$

Note that $\mathrm{Lip}(I, \mathbb{R}^2)$ is a proper dense subspace of $\mathcal{C}(I, \mathbb{R}^2)$, with details supplied in due course. Therefore, the bounded linear operator $\mathcal{F}[\boldsymbol{A}]$ is not closed follow at once from the result quoted elsewhere. However, we shall provide the details in the following. We have proved already that $\mathcal{F}[\boldsymbol{A}]$ is bounded. Let us assume, contrariwise, that $\mathcal{F}[\boldsymbol{A}]$ is closed as well. Let $\boldsymbol{f} \in \mathcal{C}(I, \mathbb{R}^2) \setminus \mathrm{Lip}(I, \mathbb{R}^2)$. By the denseness, there exists a sequence $\{\boldsymbol{f}_n\}$ in $\mathrm{Lip}(I, \mathbb{R}^2)$ such that $\boldsymbol{f}_n \to \boldsymbol{f}$. Since $\mathcal{F}[\boldsymbol{A}]$ is bounded, $\{(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}_n)\}$ is a Cauchy sequence in $\mathcal{C}(I, \mathbb{R}^2)$. Consequently, by the completeness of $\mathcal{C}(I, \mathbb{R}^2)$, $(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}_n) \to \boldsymbol{g}$ for some $\boldsymbol{g} \in \mathcal{C}(I, \mathbb{R}^2)$. Using the assumption that $\mathcal{F}[\boldsymbol{A}]$ is a closed operator, we obtain $\boldsymbol{f} \in \mathrm{Lip}(I, \mathbb{R}^2)$ and $(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}) = \boldsymbol{g}$, contradicting the choice of $\boldsymbol{f}$. Hence the assertion.

(7) It is patent from a moment's reflection on the proof of part (3) of this theorem that

$$\|\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{f}\|_\infty \leq \|\boldsymbol{A}\|_1 \|\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{b}\|_\infty = \|\boldsymbol{A}\|_1 \|\boldsymbol{f}[\boldsymbol{A}] - L\boldsymbol{f}\|_\infty.$$

Assume $(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}) = \boldsymbol{f}[\boldsymbol{A}] = \mathbf{0}$. Then, we have

$$\|\boldsymbol{f}\|_\infty \leq \|\boldsymbol{A}\|_1 \|L\| \|\boldsymbol{f}\|_\infty.$$

Since $\|\boldsymbol{A}\|_1 < \|L\|^{-1}$, it is manifest that $\|\boldsymbol{f}\|_\infty = 0$. That is $\boldsymbol{f} = \mathbf{0}$, yielding injectivity of the operator $\mathcal{F}[\boldsymbol{A}]$. Again using part (3)

$$\|\boldsymbol{f}\|_\infty \leq \|\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{f}\|_\infty + \|\boldsymbol{f}[\boldsymbol{A}]\|_\infty \leq \|\boldsymbol{A}\|_1 \|\boldsymbol{f}[\boldsymbol{A}] - L\boldsymbol{f}\|_\infty + \|\boldsymbol{f}[\boldsymbol{A}]\|_\infty.$$

By some simple manipulations on the above inequality we obtain

$$\|\boldsymbol{f}\|_\infty \leq \frac{1 + \|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1 \|L\|} \|\boldsymbol{f}[\boldsymbol{A}]\|_\infty,$$

from which we see that the inverse operator $\mathcal{F}[\boldsymbol{A}]^{-1}$ is bounded.

Let $\{\boldsymbol{g}_n\}$ be a sequence in $\mathrm{Rg}(\mathcal{F}[\boldsymbol{A}])$ be such that $\boldsymbol{g}_n \to \boldsymbol{g}$ in $\mathcal{C}(I, \mathbb{R}^2)$ and $(\mathcal{F}[\boldsymbol{A}]^{-1})(\boldsymbol{g}_n) \to \boldsymbol{f}$ in $\mathrm{Lip}(I, \mathbb{R}^2)$. Letting

$$\boldsymbol{f}_n = (\mathcal{F}[\boldsymbol{A}]^{-1})(\boldsymbol{g}_n),$$

we get

$$\boldsymbol{f}_n \longrightarrow \boldsymbol{f} \quad \text{and} \quad (\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}_n) \longrightarrow \boldsymbol{g}.$$

By the boundedness of the map $\mathcal{F}[\boldsymbol{A}]$ and uniqueness of limit, we infer that $\boldsymbol{g} = (\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}) \in \mathrm{Rg}(\mathcal{F}[\boldsymbol{A}])$ and $\boldsymbol{f} = (\mathcal{F}[\boldsymbol{A}]^{-1})(\boldsymbol{g})$. In particular, it follows that $\mathcal{F}[\boldsymbol{A}]$ has a closed range. This concludes the proof. $\square$

For the remaining part of this section, let us assume that the function $\boldsymbol{b}$ used to generate $\boldsymbol{f}[\boldsymbol{A}]$ from $\boldsymbol{f}$ satisfies $\boldsymbol{b} = L\boldsymbol{f}$, where L is a bounded linear operator on $\mathrm{Lip}(I, \mathbb{R}^2)$. As indicated at the beginning of this section, we now turn to the task of obtaining fractal analogue of a vector-valued function $\boldsymbol{f} \in \mathcal{C}(I, \mathbb{R}^2)$. Since we have already accomplished the fractal analogue of a vector-valued function $\boldsymbol{f} \in \mathrm{Lip}(I, \mathbb{R}^2)$, what remains is just to adapt the following elementary theorem on the extension of a bounded linear map.

Theorem 3.2 ([6]). *Let $V_0 \subset V$ be a dense subspace of a normed linear space V and W be a Banach space. Let $T: V_0 \to W$ be a bounded linear operator. Then there is a unique bounded linear operator*

$$\bar{T}: V \longrightarrow W$$

such that

$$\bar{T}v = Tv, \quad \text{for all } v \in V_0,$$

and

$$\|\bar{T}\| = \|T\|.$$

The following remark sheds some light on this extension.

Remark 3.3. Given $v \in V$, by density of V_0 in V there exists a sequence $\{v_n\}$ in V_0 such that $\lim v_n = v$. We define $\bar{T}(v) = \lim T(v_n)$.

To apply the aforementioned extension theorem in the present setting we need denseness of $\mathrm{Lip}(I, \mathbb{R}^2)$ in $\mathcal{C}(I, \mathbb{R}^2)$, which can be deduced from the Weierstrass theorem for uniform approximation as follows. Let $\boldsymbol{f} = (f_1, f_2) \in \mathcal{C}(I, \mathbb{R}^2)$ and $\epsilon > 0$. Since $f_i: I \to \mathbb{R}$, $i = 1, 2$, are continuous, by the Weierstrass theorem, there exist polynomials p_i, $i = 1, 2$, such that $|f_i(x) - p_i(x)| < \frac{\epsilon}{2}$ for all $x \in I$. Consider $\boldsymbol{p} = (p_1, p_2) \in \mathrm{Lip}(I, \mathbb{R}^2)$. We have

$$\begin{aligned}
\|\boldsymbol{f} - \boldsymbol{p}\|_\infty &= \sup\{\|(\boldsymbol{f} - \boldsymbol{p})(x)\|_{l^1} : x \in I\} \\
&= \sup\{\sum_{i=1}^{2} |f_i(x) - p_i(x)| : x \in I\} \\
&< \epsilon,
\end{aligned}$$

demonstrating the denseness of $\mathrm{Lip}(I, \mathbb{R}^2)$ in $\mathcal{C}(I, \mathbb{R}^2)$.

Consider $\boldsymbol{f} \in \mathcal{C}(I, \mathbb{R}^2)$. For the bounded linear map

$$\mathcal{F}[\boldsymbol{A}]\colon \operatorname{Lip}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2),$$

consider the extension

$$\overline{\mathcal{F}}[\boldsymbol{A}]\colon \mathcal{C}(I, \mathbb{R}^2) \to \mathcal{C}(I, \mathbb{R}^2)$$

and let

$$\bar{\boldsymbol{f}}[\boldsymbol{A}] = (\overline{\mathcal{F}}[\boldsymbol{A}])(\boldsymbol{f}).$$

The function $\bar{\boldsymbol{f}}[\boldsymbol{A}]$ is defined as the $\boldsymbol{A}$-fractal function associated with $\boldsymbol{f}$. The following diagram epitomizes the extension procedure.

$$\begin{array}{ccc} \{\boldsymbol{f}_n\} \subset \operatorname{Lip}(I, \mathbb{R}^2) & \xrightarrow{\|.\|_\infty} & \boldsymbol{f} \in \mathcal{C}(I, \mathbb{R}^2) \\ \Big\downarrow \mathcal{F}[\boldsymbol{A}] & & \Big\downarrow \bar{\mathcal{F}}[\boldsymbol{A}] \\ \{\boldsymbol{f}_n[\boldsymbol{A}]\} \subset \mathcal{C}(I, \mathbb{R}^2) & \xrightarrow{\|.\|_\infty} & \bar{\boldsymbol{f}}[\boldsymbol{A}] \in \mathcal{C}(I, \mathbb{R}^2) \end{array}$$

Thus, for a given $\mathbb{R}^2$-valued continuous function $\boldsymbol{f}$, we produce a class of fractal functions parameterized by a block matrix $\boldsymbol{A}$, the germ being $\boldsymbol{f}$ and the "base function" being $\boldsymbol{b}$, as promised earlier.

On lines similar to Theorem 3.1, some elementary properties of the operator $\overline{\mathcal{F}}[\boldsymbol{A}]$ can be established, for instance, we have the following.

Theorem 3.4. *For the variables α_n, β_n, and γ_n, $n \in J$ selected so that $\|\boldsymbol{A}\|_1 < (1 + \|I_d - L\|)^{-1}$, the corresponding fractal operator $\overline{\mathcal{F}}[\boldsymbol{A}]$ is a topological automorphism on $\mathcal{C}(I, \mathbb{R}^2)$.*

Proof. With stated assumptions, $\overline{\mathcal{F}}[\boldsymbol{A}]$ is a bounded linear map. Let $\overline{L}$ be the norm preserving extension of L to $\mathcal{C}(I, \mathbb{R}^2)$. Consider $\boldsymbol{f} \in \mathcal{C}(I, \mathbb{R}^2)$ and let $\{\boldsymbol{f}_n\}$ be a sequence in $\operatorname{Lip}(I, \mathbb{R}^2)$ such that $\boldsymbol{f}_n \to \boldsymbol{f}$. Then

$$\begin{aligned} \|(\overline{\mathcal{F}}[\boldsymbol{A}])(\boldsymbol{f}) - \boldsymbol{f}\|_\infty &= \lim \|(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}_n) - \boldsymbol{f}_n\|_\infty, \\ &\leq \|\boldsymbol{A}\|_1 \lim \|(\mathcal{F}[\boldsymbol{A}])(\boldsymbol{f}_n) - L\boldsymbol{f}_n\|_\infty, \\ &= \|\boldsymbol{A}\|_1 \lim \|(\overline{\mathcal{F}}[\boldsymbol{A}])(\boldsymbol{f}_n) - \overline{L}\boldsymbol{f}_n\|_\infty, \\ &= \|\boldsymbol{A}\|_1 \|(\overline{\mathcal{F}}[\boldsymbol{A}])\boldsymbol{f} - \overline{L}\boldsymbol{f}\|_\infty, \\ &\leq \|\boldsymbol{A}\|_1 (\|(\overline{\mathcal{F}}[\boldsymbol{A}])\boldsymbol{f} - \boldsymbol{f}\|_\infty + \|\boldsymbol{f} - \overline{L}\boldsymbol{f}\|_\infty). \end{aligned}$$

Hence,

$$\|(\overline{\mathcal{F}}[\boldsymbol{A}])(\boldsymbol{f}) - \boldsymbol{f}\|_\infty \leq \frac{\|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1} \|\boldsymbol{f} - \overline{L}\boldsymbol{f}\|_\infty.$$

Consequently,

$$\|I_d - \overline{\mathfrak{F}}[\boldsymbol{A}]\| \le \frac{\|\boldsymbol{A}\|_1}{1 - \|\boldsymbol{A}\|_1} \|I_d - \overline{L}\|.$$

It is straight forward to see that extension of $I_d - L$ is $I_d - \overline{L}$ and

$$\|I_d - L\| = \|I_d - \overline{L}\|,$$

where I_d stands for identity operator on appropriate spaces (see, for instance, [20, 21]). The hypothesis $\|\boldsymbol{A}\|_1 < (1 + \|I_d - L\|)^{-1}$ now yields $\|I_d - \overline{\mathfrak{F}}[\boldsymbol{A}]\| < 1$. That the operator $\overline{\mathfrak{F}}[\boldsymbol{A}] = I_d - (I_d - \overline{\mathfrak{F}}[\boldsymbol{A}])$ has bounded inverse follows from the standard theorem which reads: if T is a bounded linear operator from a Banach space into itself such that $\|T\| < 1$, then $I - T$ has bounded inverse and the Neumann series $\sum_{k=0}^{\infty} T^k$ converges in operator norm to $(I - T)^{-1}$. This completes the proof. □

4. On perturbation in parameter matrix and an optimal choice

The process of associating a parameterized family of fractal functions $\boldsymbol{f}[\boldsymbol{A}]$ with a function $\boldsymbol{f}$ clearly offers flexibility, and the parameters may be selected so as to cater to the situations one encounters in practice. The dependence of fractal function on the parameter matrix $\boldsymbol{A}$ and the vast diversity in the choice of parameters raise the question of quantification of the error caused in the fractal function $\boldsymbol{f}[\boldsymbol{A}]$ due to a perturbation in $\boldsymbol{A}$ and also an "optimal" choice of $\boldsymbol{A}$. In this section, we make an earnest attempt to address these questions. To avoid technicalities, we shall assume that the germ $\boldsymbol{f}$ of the family of fractal functions $\{\boldsymbol{f}[\boldsymbol{A}]\}$ belongs to the function class $\mathrm{Lip}(I, \mathbb{R}^2)$.

Let the fractal function corresponding to $\boldsymbol{f} \in \mathrm{Lip}(I, \mathbb{R}^2)$ with respect to an arbitrary partition Δ of I, base function $\boldsymbol{b} \in \mathrm{Lip}(I, \mathbb{R}^2)$, and the parameters α_n, β_n, γ_n, $n \in J$ satisfying $|\alpha_n| < 1$, $|\beta_n| + |\gamma_n| < 1$ be denoted by $\boldsymbol{f}[\boldsymbol{A}]$, where $\boldsymbol{A} = [A_n]_{n \in J}$,

$$A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}.$$

Suppose that the parameters are perturbed slightly, say to $\tilde{\alpha}_n = \alpha_n + \epsilon_n$, $\tilde{\beta}_n = \beta_n + \delta_n$, and $\tilde{\gamma}_n = \gamma_n + \eta_n$ so that

$$|\tilde{\alpha}_n| < 1 \quad \text{and} \quad |\tilde{\beta}_n| + |\tilde{\gamma}_n| < 1, \quad \text{for } n \in J,$$

keeping other elements in the IFS unaltered. We denote the corresponding matrix by $\widetilde{\boldsymbol{A}} = [\widetilde{A}_n]_{n\in J}$, where

$$\widetilde{A}_n = \begin{bmatrix} \tilde{\alpha}_n & \tilde{\beta}_n \\ 0 & \tilde{\gamma}_n \end{bmatrix}.$$

Let us denote the fractal function associated with $\boldsymbol{f}$ generated via this perturbed IFS by $\boldsymbol{f}[\widetilde{\boldsymbol{A}}]$. Both $\boldsymbol{f}[\boldsymbol{A}]$ and $\boldsymbol{f}[\widetilde{\boldsymbol{A}}]$ interpolate $\boldsymbol{f}$ at the partition points in Δ. With these notation we have the following theorem that analyzes the perturbation error.

Theorem 4.1. *Let $\boldsymbol{f} \in \mathrm{Lip}(I, \mathbb{R}^2)$ be the original function for which $\boldsymbol{f}[\boldsymbol{A}]$ and $\boldsymbol{f}[\widetilde{\boldsymbol{A}}]$ are the fractal functions corresponding to two different sets of parameters as specified above. Then*

$$\|\boldsymbol{f}[\widetilde{\boldsymbol{A}}] - \boldsymbol{f}[\boldsymbol{A}]\|_\infty \le \frac{\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\|_1}{(1-\|\boldsymbol{A}\|_1)(1-\|\widetilde{\boldsymbol{A}}\|_1)}\|\boldsymbol{f} - \boldsymbol{b}\|_\infty.$$

Proof. We begin by noting that for $x \in I_n$, $n \in J$, the following functional equations hold:

$$\boldsymbol{f}[\boldsymbol{A}](x) = \boldsymbol{f}(x) + A_n(\boldsymbol{f}[\boldsymbol{A}] - \boldsymbol{b})(L_n^{-1}(x)), \tag{5a}$$

$$\boldsymbol{f}[\widetilde{\boldsymbol{A}}](x) = \boldsymbol{f}(x) + \widetilde{A}_n(\boldsymbol{f}[\widetilde{\boldsymbol{A}}] - \boldsymbol{b})(L_n^{-1}(x)). \tag{5b}$$

Denoting $\boldsymbol{f}[\boldsymbol{A}]$ componentwise by $(f_1[\boldsymbol{A}], f_2[\boldsymbol{A}])$ we obtain

$$\begin{aligned}&\|\boldsymbol{f}[\widetilde{\boldsymbol{A}}] - \boldsymbol{f}[\boldsymbol{A}]\|_\infty \\ &\quad = \sup\{|f_1[\boldsymbol{A}](x) - f_1[\widetilde{\boldsymbol{A}}](x)| + |f_2[\boldsymbol{A}](x) - f_2[\widetilde{\boldsymbol{A}}](x)| \colon x \in I\}.\end{aligned}$$

Writing the functional equations expressed in (5) componentwise and performing calculations similar to that in part (3) of Theorem 3.1 we infer that

$$\begin{aligned}\|\boldsymbol{f}[\widetilde{\boldsymbol{A}}] - \boldsymbol{f}[\boldsymbol{A}]\|_\infty &\le \frac{\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\|_1}{1-\|\boldsymbol{A}\|_1}\|\boldsymbol{f}[\widetilde{\boldsymbol{A}}] - \boldsymbol{b}\|_\infty, \\ &\le \frac{\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\|_1}{1-\|\boldsymbol{A}\|_1}(\|\boldsymbol{f}[\widetilde{\boldsymbol{A}}] - \boldsymbol{f}\|_\infty + \|\boldsymbol{f} - \boldsymbol{b}\|_\infty), \\ &\le \frac{\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\|_1}{1-\|\boldsymbol{A}\|_1}\Big(\frac{\|\widetilde{\boldsymbol{A}}\|_1}{1-\|\widetilde{\boldsymbol{A}}\|_1}\|\boldsymbol{f} - \boldsymbol{b}\|_\infty + \|\boldsymbol{f} - \boldsymbol{b}\|_\infty\Big), \\ &= \frac{\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\|_1}{(1-\|\boldsymbol{A}\|_1)(1-\|\widetilde{\boldsymbol{A}}\|_1)}\|\boldsymbol{f} - \boldsymbol{b}\|_\infty.\end{aligned}$$

Second step in the preceding analysis uses the triangle inequality, penultimate step employs Theorem 3.1, part (3), and the last step is a matter of direct verification, finishing the proof. □

Remark 4.2. The foregoing theorem illustrates that a small perturbation in variables $\alpha_n, \beta_n, \gamma_n, n \in J$ leads to small change in the corresponding members of the parameterized family $\{\boldsymbol{f}[\boldsymbol{A}]\}$, as expected.

The parameters involved in the defining IFS possess a decisive influence on the closeness of fit of the corresponding FIF. Different approaches have been suggested to find optimum values of these parameters (see, for instance, [15, 16, 17]) in the case of a FIF. However, our search for a method of selecting optimum parameters in the case of hidden variable FIF came up empty-handed. In what follows, we set out some facts on the optimal choice of parameters (i.e., the partition matrix $\boldsymbol{A}$) from an approximation point of view. To be precise, given a $\mathbb{R}^2$-valued Lipschitz continuous function $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$ and a Lipschitz continuous approximant $\boldsymbol{f}$, we prove that finding a block matrix $\boldsymbol{A}$ of parameters for which $\boldsymbol{f}[\boldsymbol{A}]$ is close to $\boldsymbol{\Phi}$ is a nonlinear constrained convex optimization problem with a solution.

Let

$$\mathcal{B}_\tau := \left\{ \boldsymbol{A} = [A_n]_{n\in J} : A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix} : \|\boldsymbol{A}\|_1 \le \tau < 1 \right\}.$$

The problem is to find $\boldsymbol{A} \in \mathcal{B}_\tau$ for which $\|\boldsymbol{\Phi} - \boldsymbol{f}[\boldsymbol{A}]\|_\infty$ is minimum. Recalling that $\boldsymbol{f}[\boldsymbol{A}]$ is the fixed point of the contraction operator $\mathrm{T}_{\boldsymbol{A}}$ (to emphasize the dependence of T on $\boldsymbol{A}$), in view of the collage theorem (see, for instance, [2]), the problem reduces to that of finding an $\boldsymbol{A}^* \in \mathcal{B}_\tau$ for which

$$F(\boldsymbol{A}^*) = \min_{\boldsymbol{A}\in\mathcal{B}_\tau} \|\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty.$$

We do not provide a complete implementation of this convex optimization problem, however, the philosophy is that there are great advantages, both theoretical and practical, to recognize or formulate a problem as a convex optimization problem. The reference [22] provided us with an insight to the problem and an array of basic tools which we have modified and adapted. Let us note also that the set $\mathcal{B}_\tau$ is convex and parameter (block) matrix $\boldsymbol{A}$ may be treated as an element in $M^{2\times 2(N-1)}(\mathbb{R})$ equipped with the matrix norm corresponding to the 1-norm for vectors.

Theorem 4.3. *There exists an optimal selection of parameters for which the corresponding matrix $\boldsymbol{A}^*$ is such that*

$$F(\boldsymbol{A}^*) := \min_{\boldsymbol{A}\in\mathcal{B}_\tau} \|\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty.$$

Finding such an optimal $\boldsymbol{A}^*$ is a convex optimization problem.

Proof. Consider

$$F : \mathcal{B}_\tau \longrightarrow \mathbb{R}^+ \cup \{0\}$$

defined by

$$F(\boldsymbol{A}) = \|\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty.$$

Firstly, let us prove that F is a convex function on

$$\mathcal{B} = \left\{\boldsymbol{A} = [A_n]_{n\in J} : A_n = \begin{bmatrix}\alpha_n & \beta_n\\ 0 & \gamma_n\end{bmatrix} : \|\boldsymbol{A}\|_1 < 1\right\}.$$

Choose elements $\boldsymbol{A} = [A_n]_{n\in J}$ and $\boldsymbol{B} = [B_n]_{n\in J}$ in $\mathcal{B}$, where $A_n = \left[\begin{smallmatrix}\alpha_n & \beta_n\\ 0 & \gamma_n\end{smallmatrix}\right]$ and $B_n = \left[\begin{smallmatrix}\alpha'_n & \beta'_n\\ 0 & \gamma'_n\end{smallmatrix}\right]$. Let $0 < \lambda < 1$, and let us denote a generic $\mathbb{R}^2$-valued function $\boldsymbol{g}$ by $(\boldsymbol{g}_1, \boldsymbol{g}_2)$. Recall that

$$(\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi})(x) = \boldsymbol{f}(x) + A_n(\boldsymbol{\Phi} - \boldsymbol{b})(L_n^{-1}(x)), \quad x \in I_n,\ n \in J.$$

We have

$$\begin{aligned}
&\|\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty\\
&\quad = \sup\{|(\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi})_1(x) - \Phi_1(x)| + |(\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi})_2(x) - \Phi_2(x)| : x \in I\},\\
&\quad = \max_{n\in J} \sup\{|f_1(x) + \alpha_n(\Phi_1 - b_1)(L_n^{-1}(x)) + \beta_n(\Phi_2 - b_2)(L_n^{-1}(x)) - \Phi_1(x)|\\
&\qquad\qquad + |f_2(x) + \gamma_n(\Phi_2 - b_2)\circ L_n^{-1}(x) - \Phi_2(x)|\}
\end{aligned}$$

Applying the above equations for $\lambda\boldsymbol{A} + (1-\lambda)\boldsymbol{B}$ in place of $\boldsymbol{A}$ and using

$$f_i(x) = (\lambda f_i + (1-\lambda) f_i)(x)$$

and

$$\Phi_i(x) = (\lambda\Phi_i + (1-\lambda)\Phi_i)(x),$$

for $i = 1, 2$, a routine calculation yields

$$\|\mathrm{T}_{\lambda\boldsymbol{A}+(1-\lambda)\boldsymbol{B}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty \le \lambda\|\mathrm{T}_{\boldsymbol{A}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty + (1-\lambda)\|\mathrm{T}_{\boldsymbol{B}}\boldsymbol{\Phi} - \boldsymbol{\Phi}\|_\infty.$$

Thus, F is a convex map on $\mathcal{B}$. Since a real valued convex function on an open convex set in a normed linear space is continuous, F is continuous on $\mathcal{B}$ and hence on the subset $\mathcal{B}_\tau$. The finite dimensionality of $M^{2\times 2(N-1)}(\mathbb{R})$ yields compactness to the closed and bounded set $\mathcal{B}_\tau$. Now it follows from a standard theorem in analysis that $\mathcal{B}_\tau$ admits a point $\boldsymbol{A}^*$ at which the continuous function F is minimum. Further, detecting such an $\boldsymbol{A}^*$ is a convex optimization problem since F is a convex function and $\mathcal{B}_\tau$ is convex. □

5. Fractal function $\boldsymbol{f}[A]$ preserving some properties of the original function $\boldsymbol{f}$

In this section our perspective changes slightly, as we begin to perceive the hidden variable fractal perturbation process, that is the process of obtaining fractal function $\boldsymbol{f}[\boldsymbol{A}]$ from $\boldsymbol{f}$, as a key to develop a new approach to shape preserving interpolation. Towards this goal first we shall identify suitable parameters in the IFS so that $\boldsymbol{f}[\boldsymbol{A}]$ preserves the properties (for instance, positivity and regularity) inherent in $\boldsymbol{f}$. The elements of the IFS that may require appropriate selection for this "property preserving fractal perturbation" are function $\boldsymbol{b}$, matrix $\boldsymbol{A}$ containing parameters, and perhaps partition Δ. The purpose of this section is to elaborate upon this sentiment.

For the sake of simplicity of presentation, let us introduce the following notation. For a given $\boldsymbol{f} = (f_1, f_2) \in \mathrm{Lip}(I, \mathbb{R}^2)$ and $\boldsymbol{b} = (b_1, b_2)$ selected to construct the corresponding $\boldsymbol{A}$-fractal function $\boldsymbol{f}[\boldsymbol{A}] = (f_1[\boldsymbol{A}], f_2[\boldsymbol{A}])$, let

$$M_i = \max_{x\in I} b_i(x), \quad m_{in} = \min_{x\in I} f_i(L_n(x)), \quad \text{for } i = 1, 2,\ n \in J.$$

Note that the existence of these parameters follows from the continuity of functions involved in their definition and the compactness of the domain.

Theorem 5.1. *Let $\boldsymbol{f} = (f_1, f_2) \in \mathrm{Lip}(I, \mathbb{R}^2)$ be such that $\boldsymbol{f} \geq \boldsymbol{0}$, i.e.,*

$$f_1(x) \geq 0 \quad \textit{and} \quad f_2(x) \geq 0 \quad \textit{for all } x \in I.$$

Consider an arbitrary partition $\{x_1, x_2, \ldots, x_N\}$ of $I = [x_1, x_N]$ satisfying

$$x_1 < x_2 < \cdots < x_N$$

and a function $\boldsymbol{b} = (b_1, b_2)$ satisfying

$$\boldsymbol{b}(x_1) = \boldsymbol{f}(x_1) \quad \textit{and} \quad \boldsymbol{b}(x_N) = \boldsymbol{f}(x_N).$$

Further, let the block matrix $\boldsymbol{A} = [A_n]_{n\in J}$, with $A_n = \left[\begin{smallmatrix}\alpha_n & \beta_n\\ 0 & \gamma_n\end{smallmatrix}\right]$, be selected such that the entries $\alpha_n, \beta_n, \gamma_n$ that lie in $[0, 1)$ satisfy

$$\gamma_n \leq \frac{m_{2n}}{M_2}, \quad \beta_n + \gamma_n < 1, \quad \alpha_n M_1 + \beta_n M_2 \leq m_{1n}.$$

Then the corresponding fractal function $\boldsymbol{f}[\boldsymbol{A}]$ preserves the positivity of $\boldsymbol{f}$, that is to say,

$$f_1[\boldsymbol{A}](x) \geq 0 \quad \textit{and} \quad f_2[\boldsymbol{A}](x) \geq 0 \quad \textit{for all } x \in I.$$

Proof. Note that the fractal function $\boldsymbol{f}[\boldsymbol{A}] = (f_1[A], f_2[A])$ is constructed by iteration via the functional equations

$$\begin{aligned} f_1[\boldsymbol{A}](L_n(x)) &= F_n^1(x, f_1[\boldsymbol{A}](x), f_2[\boldsymbol{A}](x)), \\ &= \alpha_n f_1[\boldsymbol{A}](x) + \beta_n f_2[\boldsymbol{A}](x) + f_1(L_n(x)) - \alpha_n b_1(x) - \beta_n b_2(x), \end{aligned}$$

and

$$\begin{aligned} f_2[\boldsymbol{A}](L_n(x)) &= F_n^2(x, f_2[\boldsymbol{A}](x)), \\ &= \gamma_n f_2[\boldsymbol{A}](x) + f_2(L_n(x)) - \gamma_n b_2(x). \end{aligned}$$

Thus the self-referential function $f_2[\boldsymbol{A}]$ is obtained recursively by inserting new points between existing data points, iterations being initialized with the node points $\{(x_n, f_2[\boldsymbol{A}](x_n)) \colon n = 1, 2, \ldots, N\}$, where $f_2[\boldsymbol{A}](x_n) = f_2(x_n) \geq 0$. Therefore, to prove that $f_2[\boldsymbol{A}](x) \geq 0$ for all $x \in I$, it is enough (by induction) to prove that $f_2[\boldsymbol{A}]$ evaluated at points in $(k+1)$-th iteration is positive whenever $f_2[\boldsymbol{A}]$ at distinct points in k-th iteration is known to be positive. This is equivalent to prove that

$$F_n^2(x, z) = \gamma_n z + f_2(L_n(x)) - \gamma_n b_2(x) \geq 0, \quad \text{for all } n \in J,$$

whenever $(x, z) \in I \times \mathbb{R}$ and $z \geq 0$. Again for $\gamma_n \geq 0$, the conditions $F_n^2(x, z) \geq 0$ for all $(x, z) \in I \times \mathbb{R}$ and $z \geq 0$ are met if

$$f_2(L_n(x)) - \gamma_n b_2(x) \geq 0.$$

By the definition of m_{2n} and M_2, we have

$$f_2(L_n(x)) - \gamma_n b_2(x) \geq m_{2n} - \gamma_n M_2.$$

With the aforementioned points one can deduce that $f_2[A](x) \geq 0$ for all $x \in I$ is satisfied if $\gamma_n \in [0, 1)$ is selected so that $\gamma_n \leq \dfrac{m_{2n}}{M_2}$ for all $n \in J$. Note also that if $M_2 = 0$, then no additional constraint on γ_n needs to be imposed.

Having selected γ_n, $n \in J$ according to the above prescription, by similar arguments it can be seen that $f_1[A](x) \geq 0$ for all $x \in I$ is fulfilled, if

$$\alpha_n \geq 0, \beta_n \geq 0 \quad \text{and} \quad f_1(L_n(x)) - \alpha_n b_1(x) - \beta_n b_2(x) \geq 0, \quad \text{for all } x \in I, n \in J.$$

Note that

$$f_1(L_n(x)) - \alpha_n b_1(x) - \beta_n b_2(x) \geq m_{1n} - \alpha_n M_1 - \beta_n M_2.$$

Consequently, the desired condition turns out to be true if $\alpha_n M_1 + \beta_n M_2 \leq m_{1n}$. This completes the proof. □

The selection of parameters may seem to be computationally demanding due to the presence of many inequality constraints in the above theorem, but in actuality it is easy to select the parameter values satisfying all the required conditions. It is also worth to mention that our intention here is to provide (sufficient) conditions on parameters that ensure the positivity. Automatic strategy for a clever selection of the parameters, i.e., finding "optimum" values of parameters is indeed a difficult problem for which one approach in approximation point of view is via convex optimization stated previously. However, the particular nature of the problem dictates what type of optimization needs to be employed and exploring various other approaches deserves further research.

The next theorem points to the conditions on the elements of the (hidden variable) IFS so that the fractal function $\boldsymbol{f}[\boldsymbol{A}]$ retains the $\mathcal{C}^1$-continuity of $\boldsymbol{f}$.

Theorem 5.2. *Let $\boldsymbol{f} = (f_1, f_2)\colon I \to \mathbb{R}^2$ be a continuously differentiable function. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be an arbitrary partition on I satisfying*

$$x_1 < x_2 < \cdots < x_N.$$

Consider a block matrix $\boldsymbol{A} = [A_n]_{n\in J}$, $A_n = \left[\begin{smallmatrix}\alpha_n & \beta_n\\ 0 & \gamma_n\end{smallmatrix}\right]$ whose parameters satisfy

$$|\alpha_n| < a_n, \quad |\beta_n| + |\gamma_n| < a_n, \quad \text{for all } n \in J.$$

Let $\boldsymbol{b} = (b_1, b_2)$ be a continuously differentiable function satisfying

$$\boldsymbol{b}^{(j)}(x_1) = \boldsymbol{f}^{(j)}(x_1), \quad \boldsymbol{b}^{(j)}(x_N) = \boldsymbol{f}^{(j)}(x_N), \quad \text{for } j = 0, 1.$$

Then the corresponding $\mathbb{R}^2$-valued fractal function $\boldsymbol{f}[\boldsymbol{A}]$ is continuously differentiable, and

$$\boldsymbol{f}[\boldsymbol{A}]^{(j)}(x_n) = \boldsymbol{f}^{(j)}(x_n) \quad \text{for } j = 0, 1 \text{ and } n = 1, 2, \dots, N.$$

Proof. Consider

$$\begin{aligned}\mathcal{D}^1(I, \mathbb{R}^2) := \{\boldsymbol{h} \in \mathcal{C}^1(I, \mathbb{R}^2)\colon &\boldsymbol{h}^{(j)}(x_1) = \boldsymbol{f}^{(j)}(x_1),\\ &\boldsymbol{h}^{(j)}(x_N) = \boldsymbol{f}^{(j)}(x_N), j = 0, 1\}\end{aligned}$$

endowed with the metric induced by the $\mathcal{C}^1$-norm

$$\|\boldsymbol{h}\|_{\mathcal{C}^1} := \|\boldsymbol{h}\|_\infty + \|\boldsymbol{h}^{(1)}\|_\infty.$$

Note that $(\mathcal{D}^1(I,\mathbb{R}^2), \|.\|_{\mathcal{C}^1})$ is a complete metric space. Define a map

$$\mathrm{T}_{\boldsymbol{A}}: \mathcal{D}^1(I,\mathbb{R}^2) \longrightarrow \mathcal{D}^1(I,\mathbb{R}^2)$$

via

$$\begin{aligned}(\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h})(x) &= F_n(L_n^{-1}(x), \boldsymbol{h}(L_n^{-1}(x))),\\ &= \boldsymbol{f}(x) + A_n(\boldsymbol{h}-\boldsymbol{b})(L_n^{-1}(x)), \quad x \in I_n,\ n \in J.\end{aligned}$$

From the conditions on $\boldsymbol{f}$ and $\boldsymbol{b}$ it can be seen that $\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h}$ is continuously differentiable on (x_n, x_{n+1}). We prove that $\mathrm{T}_{\boldsymbol{A}}$ maps $\mathcal{D}^1(I,\mathbb{R}^2)$ into $\mathcal{D}^1(I,\mathbb{R}^2)$. With the choice of $\boldsymbol{b}$, we have

$$(\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h})(x_1) = \boldsymbol{f}(x_1) \quad \text{and} \quad (\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h})(x_N) = \boldsymbol{f}(x_N).$$

Next we verify

$$(\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h})^{(1)}(x_n^-) = (\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h})^{(1)}(x_n^+) \quad \text{for all } n \in J. \tag{6}$$

Differentiating the expression for $\mathrm{T}_{\boldsymbol{A}}\boldsymbol{h}$, and writing the conditions prescribed in (6) componentwise we obtain the equations

$$\begin{aligned}f_1^{(1)}(x_n) &+ a_{n-1}^{-1}\{\alpha_{n-1}h_1^{(1)}(x_N) + \beta_{n-1}h_2^{(1)}(x_N)\\ &\qquad - \alpha_{n-1}b_1^{(1)}(x_N) - \beta_{n-1}b_2^{(1)}(x_N)\} \qquad (7)\\ &= f_1^{(1)}(x_n) + a_n^{-1}\{\alpha_n h_1^{(1)}(x_1) + \beta_n h_2^{(1)}(x_1) - \alpha_n b_1^{(1)}(x_1) - \beta_n b_2^{(1)}(x_1)\},\end{aligned}$$

and

$$\begin{aligned}f_2^{(1)}(x_n) &+ a_{n-1}^{-1}\gamma_{n-1}\{h_2^{(1)}(x_N) - b_2^{(1)}(x_N)\}\\ &= f_2^{(1)}(x_n) + a_n^{-1}\gamma_n\{h_2^{(1)}(x_1) - b_2^{(1)}(x_1)\}.\end{aligned} \tag{8}$$

Since $\boldsymbol{h} \in \mathcal{D}^1(I,\mathbb{R}^2)$ and $\boldsymbol{b}$ satisfies the conditions

$$\boldsymbol{f}^{(1)}(x_j) = \boldsymbol{b}^{(1)}(x_j), \quad \text{for } j = 1, N,$$

both sides of (7) coincide with $f_1^{(1)}(x_n)$ and both sides of (8) reduce to $f_2^{(1)}(x_n)$. This establishes (6), and consequently $\mathrm{T}_{\boldsymbol{A}}$ is well-defined. Using the conditions on entries of matrix A_n, from calculations similar to that in Theorem 4.3, we infer that $T_{\boldsymbol{A}}$ is a contraction map. The corresponding fixed point $\boldsymbol{f}[\boldsymbol{A}]$ is $\mathcal{C}^1$-continuous. Our analysis also reveals that $\boldsymbol{f}[\boldsymbol{A}]^{(1)}$ interpolates to $\boldsymbol{f}^{(1)}$ at the knots of the partition. □

Remark 5.3. By a similar rendition of the arguments as in the foregoing theorem, it can be shown that $\boldsymbol{f}[\boldsymbol{A}]$ preserves r-smoothness of $\boldsymbol{f}$, if the parameters satisfy the following conditions: (i) $|\alpha_n| < a_n^r$, $|\beta_n| + |\gamma_n| < a_n^r$ for all $n \in J$, (ii) $\boldsymbol{b} \in \mathcal{C}^r(I,\mathbb{R}^2)$ and (iii) the derivatives up to the order r of $\boldsymbol{b}$ at the extremes of the interval coincide with the derivatives of $\boldsymbol{f}$.

6. Positivity of $\mathcal{C}^1$-cubic spline hidden variable FIF

The purpose of the current section is twofold. First is to illustrate the fractal perturbation process and its positivity aspect enunciated in the previous sections by taking cubic spline as an example. Secondly and most importantly, to develop a positivity preserving cubic spline interpolation scheme that extends and supplements the methods described in the references [10, 24]. Let us elaborate this a little more. Schmidt and Heß [24] have established the condition on the derivative parameters so that the $\mathcal{C}^1$-cubic spline reflect the positivity property inherent in a prescribed data set. However, the derivatives of these traditional positive cubic splines are always smooth, except possibly at a finite number of points. To improve the situation and to represent a positive function Φ with derivative $\Phi^{(1)}$ being irregular in a dense subset of the interval, positivity of $\mathcal{C}^1$-cubic spline FIF is studied recently in [10]. These interpolants represent self-referential functions. The aforementioned limitations of the positive cubic spline and its fractal analogue recommended our attention to the $\mathcal{C}^1$-cubic spline hidden variable FIF for the positivity preserving interpolation. That is, the $\mathcal{C}^1$-cubic spline hidden variable FIF in this section can be employed to represent self-referential or non-self-referential positive function Φ with derivative $\Phi^{(1)}$ having irregularity in a finite or dense subset of the interpolation interval.

Consider a set of data points

$$\Delta = \{(x_n, y_n, d_n) \colon n = 1, 2, \dots, N\},$$

where y_n denotes the function value and d_n denotes the derivative value of an unknown function Φ_1 at the knot point x_n. To obtain a $\mathcal{C}^1$-cubic spline hidden variable interpolant corresponding to Δ, we extend it to a generalized data set

$$\widehat{\Delta} = \{(x_n, y_n, d_n, y_n^*, d_n^*) \colon n = 1, 2, \dots, N\},$$

where y_n^* and d_n^* are real parameters that are assumed to be the function values and the derivative values of a function Φ_2 at the knot point x_n. A natural way of constructing a $\mathcal{C}^1$-cubic spline hidden variable FIF corresponding to Δ is to employ the general theory given in Section 2 coupled with conditions of differentiability by taking p_n and q_n as cubic polynomials. However, to fit the cubic spline hidden variable FIF to the realm of positivity preserving interpolation via the theory developed in previous sections, we shall obtain it as $\boldsymbol{A}$-fractal function corresponding to $\boldsymbol{f} = (f_1, f_2) \in \mathrm{Lip}(I, \mathbb{R}^2)$. Here f_1 and f_2 are the traditional nonrecursive $\mathcal{C}^1$-cubic splines corresponding to the data sets

$$\Delta = \{(x_n, y_n, d_n) \colon n = 1, 2, \dots, N\}$$

and

$$\widetilde{\Delta} = \{(x_n, y_n^*, d_n^*) : n = 1, 2, \dots, N\},$$

respectively. With

$$h_n = x_{n+1} - x_n \quad \text{and} \quad \theta := \frac{x - x_1}{x_N - x_1},$$

the traditional nonrecursive cubic interpolant corresponding to Δ and $\widetilde{\Delta}$ can be represented as

$$\begin{aligned} f_1(L_n(x)) = {} & \{h_n(d_n + d_{n+1}) - 2(y_{n+1} - y_n)\}\theta^3 \\ & + \{-h_n(2d_n + d_{n+1}) + 3(y_{n+1} - y_n)\}\theta^2 + h_n d_n \theta + y_n, \end{aligned} \tag{9a}$$

$$\begin{aligned} f_2(L_n(x)) = {} & \{h_n(d_n^* + d_{n+1}^*) - 2(y_{n+1}^* - y_n^*)\}\theta^3 \\ & + \{-h_n(2d_n^* + d_{n+1}^*) + 3(y_{n+1}^* - y_n^*)\}\theta^2 + h_n d_n^* \theta + y_n^*. \end{aligned} \tag{9b}$$

To obtain a continuously differentiable fractal perturbation for $\boldsymbol{f} = (f_1, f_2) \in \mathcal{C}^1(I, \mathbb{R}^2)$, we have to select the parameter matrix $\boldsymbol{A}$ and function $\boldsymbol{b} = (b_1, b_2)$ according to the specifications in Theorem 5.2. A natural choice of $\boldsymbol{b} = (b_1, b_2)$ is the one in which b_1 and b_2 are the two-point Hermite interpolants (with knots at x_1 and x_N) corresponding to f_1 and f_2 respectively. That is,

$$\begin{aligned} b_1(x) = {} & [(x_N - x_1)(d_1 + d_N) - 2(y_N - y_1)]\theta^3 \\ & + [-(x_N - x_1)(2d_1 + d_N) + 3(y_N - y_1)]\theta^2 + d_1(x - x_1) + y_1, \end{aligned} \tag{10a}$$

$$\begin{aligned} b_2(x) = {} & [(x_N - x_1)(d_1^* + d_N^*) - 2(y_N^* - y_1^*)]\theta^3 \\ & + [-(x_N - x_1)(2d_1^* + d_N^*) + 3(y_N^* - y_1^*)]\theta^2 + d_1^*(x - x_1) + y_1^*. \end{aligned} \tag{10b}$$

With these choices of component functions and with $\boldsymbol{A} = [A_n]_{n \in J}$, where $A_n = \begin{bmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{bmatrix}$, $n \in J$ satisfy $|\alpha_n| < a_n$, $|\beta_n| + |\gamma_n| < a_n$, we obtain $\boldsymbol{A}$-fractal function

$$\boldsymbol{f}[\boldsymbol{A}] = (f_1[\boldsymbol{A}], f_2[\boldsymbol{A}]) \in \mathcal{C}^1(I, \mathbb{R}^2)$$

corresponding to $\boldsymbol{f} = (f_1, f_2) \in \mathcal{C}^1(I, \mathbb{R}^2)$ defined as

$$f_1[\boldsymbol{A}](L_n(x)) = f_1(L_n(x)) + \alpha_n(f_1[\boldsymbol{A}] - b_1)(x) + \beta_n(f_2[\boldsymbol{A}] - b_2)(x), \tag{11a}$$

$$f_2[\boldsymbol{A}](L_n(x)) = f_2(L_n(x)) + \gamma_n(f_2[\boldsymbol{A}] - b_2)(x), \; x \in I, \; n \in J. \tag{11b}$$

The function $f_1[\boldsymbol{A}]\colon I \to \mathbb{R}$ enjoying the Hermite interpolation conditions

$$f_1[\boldsymbol{A}](x_n) = y_n \quad \text{and} \quad f_1[\boldsymbol{A}]^{(1)}(x_n) = d_n$$

is the desired $\mathcal{C}^1$*-cubic spline hidden variable FIF* corresponding to Δ. If we choose the "hidden variables" y_n^* and d_n^* such that $y_n^* = y_n$ and $d_n^* = d_n$ for all $n \in J$, and the parameters according to the relation $\alpha_n + \beta_n = \gamma_n$ for all $n \in J$, then the cubic hidden variable FIF $f_1[\boldsymbol{A}]$ coincides with $f_2[\boldsymbol{A}]$, representing a self-referential $\mathcal{C}^1$-cubic FIF approached constructively by Chand and Viswanathan [10]. For other choices of the hidden variables and parameters, $f_1[\boldsymbol{A}]$ is, in general, non-self-referential. Thus, the method is suitable for representing both self-referential and non-self-referential function, hence referred to as cubic spline coalescence hidden variable FIF, see also [28].

The preceding discussion accomplishes the first half of what we set about to do in this section. The rest of this section is devoted to study positivity aspects of the $\mathcal{C}^1$-cubic spline hidden variable FIF. Most of our work has already been accomplished; what remains is to recast the findings in Theorem 5.1 to the present context. We quote the following result from the reference [24] as an overture.

Proposition 6.1 (Schmidt and Heß [24]). *For the data set* $\{(x_n, y_n, d_n)\colon n = 1, 2, \dots, N\}$*, consider the traditional* $\mathcal{C}^1$*-cubic spline* s *defined as*

$$\begin{aligned} s(x) = {} & \{h_n(d_n + d_{n+1}) - 2(y_{n+1} - y_n)\}t^3 \\ & + \{-h_n(2d_n + d_{n+1}) + 3(y_{n+1} - y_n)\}t^2 + h_n d_n t + y_n, \end{aligned}$$

with $t = \frac{x - x_n}{h_n}$*, where* $x \in [x_n, x_{n+1}]$*. The cubic spline* s *is nonnegative on* $I_n = [x_n, x_{n+1}]$ *if and only if* $(d_n, d_{n+1}) \in W_n$*, where*

$$\begin{aligned} W_n := {} & \{(x, y)\colon h_n x \geq -3y_n, h_n y \leq 3y_{n+1}\} \\ & \cup \{(x, y)\colon 36 y_n y_{n+1}(x^2 + xy + y^2 - 3\Delta_n(x + y) + 3\Delta_n^2) \\ & + 3(y_{n+1}x - y_n y)(2h_n xy - 3y_{n+1}x + 3y_n y) \\ & + 4h_n(y_{n+1}x^3 - y_n y^3) - h_n^2 x^2 y^2 \geq 0\}. \end{aligned}$$

Remark 6.2. The above result gives rise to some simpler conditions for the positivity of the cubic spline s on I_n. That is, s is positive on I_n if (d_n, d_{n+1}) belongs to one of the following subregions of W_n:

$$T_n := \{(x, y)\colon h_n x \geq -3y_n, h_n y \leq 3y_{n+1}\},$$

$$S_n := \Big\{(x, y)\colon x \geq \frac{-2(y_n + \sqrt{y_n y_{n+1}})}{h_n},\ y \leq \frac{2(y_n + \sqrt{y_n y_{n+1}})}{h_n}\Big\}.$$

Coupling Theorem 5.2 with Proposition 6.1, we obtain the following algorithm for constructing positive (nonnegative) $\mathcal{C}^1$-cubic hidden variable FIF corresponding to $\Delta = \{(x_n, y_n, d_n) \colon n = 1, 2, \dots, N\}$, where $y_n \geq 0$ for all $n = 1, 2, \dots, N$.

6.1. An Algorithm for positive $\mathcal{C}^1$-cubic spline hidden variable FIF.

Step 1. Given a data set $\Delta = \{(x_n, y_n) \colon n = 1, 2, \dots, N\}$ wherein $y_n \geq 0$, extend it to $\widehat{\Delta} = \{(x_n, y_n, y_n^*) \colon n = 1, 2, \dots, N\}$ by augmenting real parameters (hidden variables) y_n^* such that $y_n^* \geq 0$ for $n = 1, 2, \dots, N$.

Step 2. Choose the derivative pairs such that $(d_n, d_{n+1}) \in T_n$ and $(d_n^*, d_{n+1}^*) \in T_n^*$ for all $n \in J$, where

$$T_n = \Big\{(x, y) \colon x \geq \frac{-3y_n}{h_n}, y \leq \frac{3y_{n+1}}{h_n}\Big\},$$
$$T_n^* = \Big\{(x, y) \colon x \geq \frac{-3y_n^*}{h_n}, y \leq \frac{3y_{n+1}^*}{h_n}\Big\}.$$

Further, construct the corresponding positive cubic splines f_i, $i = 1, 2$, (cf. (9)) and the functions b_i, $i = 1, 2$ (cf. (10)).

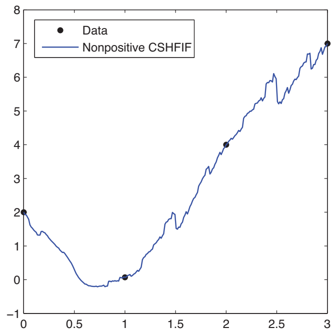
Step 3. For f_i and b_i, $i = 1, 2$, as obtained at the end of the previous step, compute the constants $M_i = \max_{x \in I} b_i(x)$, $m_{in} = \min_{x \in I} f_i(L_n(x))$ for $i = 1, 2$. Choose variables satisfying the constrains

$$0 \leq \alpha_n < a_n, \quad \beta_n \geq 0, \quad \gamma_n \in [0, \frac{m_{2n}}{M_2}], \beta_n + \gamma_n < a_n, \quad \alpha_n M_1 + \beta_n M_2 \leq m_{1n}.$$

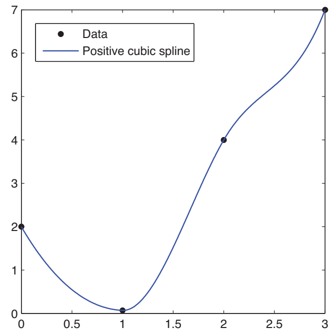
Step 4. Input the derivative values chosen in Step 2 and parameters as prescribed by Step 3 in the functional equations represented by (11) whereupon the points of the graph of $f_1[\boldsymbol{A}]$ and $f_2[\boldsymbol{A}]$ are computed.

Remark 6.3. If we take the auxiliary variables y_n^* and d_n^* such that $y_n^* = y_n \geq 0$ and $d_n^* = d_n$ for all $n = 1, 2, \dots, N$, then the fractal function $f_2[\boldsymbol{A}]$ provides a self-referential positive function corresponding to $\{(x_n, y_n) \colon n = 1, 2, \dots, N\}$. Therefore the algorithm presented above provides, in particular, an alternative to the positivity preserving $\mathcal{C}^1$-cubic FIF scheme given in the reference [10]. Advantage of the method in [10] is that it allows both positive and negative values of the scaling factors for generating positive $\mathcal{C}^1$-cubic FIF. On the other hand, the present approach provides a more general method wherein any suitable positive cubic spline scheme (see, for instance, [7]), can be employed as an interlude (although we used a very special case of Schmidt and Heß algorithm), and a family of more diverse and flexible positive fractal curves can be obtained by using the (hidden variable) fractal perturbation process.

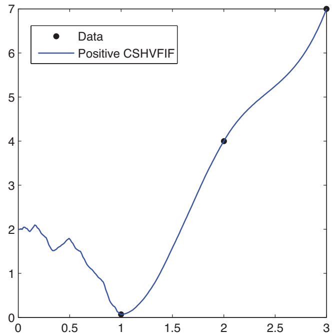
7. Numerical illustration



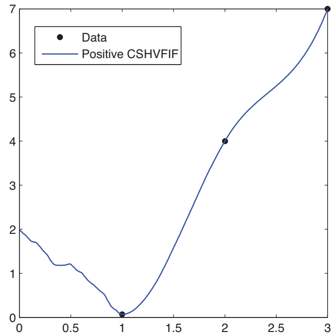
(a) Nonpositive cubic spline HVFIF.



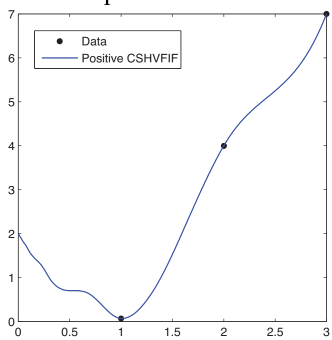
(b) Positive nonrecursive cubic spline interpolant.



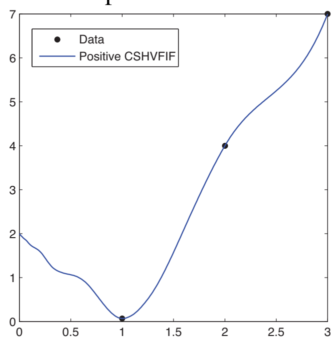
(c) Positive non-self-referential cubic spline HVFIF.



(d) Positive non-self-referential cubic spline HVFIF.



(e) Positive self-referential cubic spline HVFIF.



(f) Positive self-referential cubic spline HVFIF.

Figure 1. Cubic spline hidden variable fractal interpolation functions (HVFIFs) (the interpolating data points are given by the circles and the relevant hidden variable fractal interpolants by the solid lines).

Purpose of this section is to illustrate the positivity preserving $\mathcal{C}^1$-cubic spline hidden variable FIF scheme with some simple examples. To this end, let us take a set of positive data $\Delta = \{(0,2),(1,0.07),(2,4),(3,7)\}$ reported in [24].

We have written a simple computer program in MatLab for plotting the graphs of $\mathcal{C}^1$-cubic spline hidden variable FIFs. One inputs the data points, derivative values, hidden variables, and scaling parameters, whereupon points on the graph are recursively generated. Theoretically, to obtain the actual fractal interpolant, one needs to continue the iterations indefinitely. However, in practice, computation is very fast (note that for a data set with N points exactly $(N-1)^{r+1}+1$ points with distinct x-coordinate are obtained at the r-th iteration) and a good view of the whole function is quickly obtained and may be printed with a graphics printer.

Extend the given data set to

$$\{(x_n, y_n, y_n^*)\colon n = 1,2,3,4\} = \{(0,2,1),(1,0.07,3),(2,4,2),(3,7,5)\}.$$

Note that for the implementation of the $\mathcal{C}^1$-cubic spline hidden variable FIF scheme one requires in input the values of the derivatives at the knot points. Therefore, in the absence of other conditions/information, estimates of derivatives are necessary. Values (rounded off to two decimal places) of $d_n, d_n^*, n = 1,2,3,4$ estimated using the arithmetic mean method are $d_1 = -4.86$, $d_2 = 1$, $d_3 = 3.47$, $d_4 = 2.54$, $d_1^* = 3.5$, $d_2^* = 0.5$, $d_3^* = 1$, and $d_4^* = 5$. The $\mathcal{C}^1$-cubic spline hidden variable FIF $f_1[\boldsymbol{A}]$ displayed in Figure 1(a), which is obtained by taking parameters values as $\alpha_1 = -0.2$, $\alpha_2 = \gamma_3 = 0.2$, $\alpha_3 = 0.4$, $\beta_1 = 0.3$, $\beta_2 = \gamma_2 = 0.6$, $\beta_3 = 0.8$, $\gamma_1 = -0.4$ and iterating the functional equations given in (11) via (9) and (10) is nonpositive. This illustrates the importance of the positivity preserving $\mathcal{C}^1$-cubic spline hidden variable FIF algorithm developed in the previous section. Taking $\alpha_n = \beta_n = \gamma_n = 0$ for all $n = 1,2,3$, and the derivatives parameters as recommended by Schmidt and Heß (see Step 2 of our algorithm), we recover a positive nonrecursive $\mathcal{C}^1$-cubic spline in Figure 1(b). Selecting $\alpha_1 = 0.3$, $\beta_1 = 0.2$, $\alpha_2 = \beta_2 = 0.01$, $\alpha_3 = \beta_3 = 0$, $\gamma_1 = 0.1$, and $\gamma_2 = \gamma_3 = 0.3$ arbitrarily from the range of permissible values given in Step 3, and the derivative parameters as in Figure 1(b), we obtain Figure 1(c). Note that Figure 1(c) represents a non-self-referential cubic spline hidden variable FIF. Again, taking a different set of values for the variables, namely, $\alpha_1 = 0.2$, $\beta_1 = 0.1$, $\alpha_2 = \beta_2 = 0.01$, $\alpha_3 = \beta_3 = 0$, $\gamma_1 = 0.1$, and $\gamma_2 = \gamma_3 = 0.3$ and iterating the corresponding functional equations, we obtain a non-self-referential cubic spline hidden variable FIF $f_1[\boldsymbol{A}]$ plotted in Figure 1(d) (see also Remark 2.1). With derivative parameters d_n, d_n^*, and variables α_n, γ_n, $n = 1,2,3$ as in Figure 1(d), and $\beta_n = 0$ for all $n = 1,2,3$, we construct a positivity preserving self-referential $\mathcal{C}^1$-cubic spline hidden variable FIF of Figure 1(e). Finally, we take the "hidden variables" y_n^* and corresponding derivative parameters d_n^* such that $y_n^* = y_n$, $d_n^* = d_n$ for $n = 1, 2, 3$. Further, the parameters are selected to satisfy $\beta_n + \gamma_n = \alpha_n$ for $n = 1, 2, 3$, apart from the conditions stated in Step 3, for instance, $\alpha_1 = 0.2$, $\alpha_2 = 0.02$, $\alpha_3 = \beta_3 = \gamma_3 = 0$, $\beta_1 = \gamma_1 = 0.1$, and $\beta_2 = \gamma_2 = 0.01$. The corresponding functional equations are iterated to obtain a good approximation for the positive self-referential hidden variable FIF corresponding to the data set Δ, which we display in Figure 1(f). Note that due to the fractal nature of the corresponding derivative functions, the graphs of FIFs depicted in Figure 1 themselves have some artifacts when compared with the classical counterpart.

By plotting the derivatives of these hidden variable fractal interpolants, it can be observed that the derivatives may have irregularity in finite number of points or on dense subsets of the interpolation interval $I = [0, 3]$. Thus the flexibility in the selection of parameters can be aptly used to find an interpolant satisfying chosen properties such as self-referentiality or non-self-referentiality, regularity or fractality in the derivative, recursiveness or nonrecursiveness, locality or nonlocality, and positivity. As mentioned elsewhere, finding optimal values of the parameters may be considered for a future investigation.

8. Concluding remarks

In this paper we have presented hidden variable fractal interpolation function as a tool to associate a family of $\mathbb{R}^2$-valued continuous functions $\boldsymbol{f}[\boldsymbol{A}]$ parameterized by a suitable block matrix $\boldsymbol{A}$ with a given continuous $\mathbb{R}^2$-valued function $\boldsymbol{f}$. Depending on the choice of parameters, the members of the family may be smooth (even infinitely differentiable), piecewise smooth (differentiable except at finite number of points), irregular (even nowhere differentiable), self-referential, or non-self-referential, thus yielding more diversity in the process of approximation. This may explain why the present method performs rather well than the family of fractal functions f^α associated with a real-valued continuous function f studied earlier in the literature. We have derived estimate for the approximation of function $\boldsymbol{f}$ by their fractal analogue $\boldsymbol{f}[\boldsymbol{A}]$. Suitable values for the parameters are identified so that the function $\boldsymbol{f}[\boldsymbol{A}]$ preserves the positivity and $\mathcal{C}^1$-continuity of $\boldsymbol{f}$. Thus the paper proposes a novel approach to shape preserving approximation using hidden variable fractal interpolation functions. Our method is general enough in scope to yield hidden variable fractal function analogue of any traditional nonrecursive positivity preserving polynomial interpolation scheme. For our part, we have chosen positivity preserving $\mathcal{C}^1$-cubic spline for illustrative purpose. The positivity preserving cubic FIF scheme studied recently in [10], though conceptually quite general than the corresponding traditional counterpart, work better in certain situations, for instance for approximating a self-referential curve, than they do in some others. The positivity preserving $\mathcal{C}^1$-cubic hidden variable FIFs obtained in this paper supplements and extends that in [10]. Further, using a family of functions $\{\boldsymbol{b}_m = (b_{m1}, b_{m2})\colon m \in J\}$ instead of a single function $\boldsymbol{b}$ in the fractal perturbation process developed herein and modifying Theorem 5.2 accordingly, positivity preserving hidden variable corresponding to the traditional nonrecursive rational splines with shape parameters whose fractal extensions are studied in [26, 27] can be obtained.

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