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Embedding topological fractals in universal spaces

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Abstract. A compact metric space X is called a Rakotch (Banach) fractal if $X = \bigcup_{f\in\mathcal{F}} f(X)$ for some finite system $\mathcal{F}$ of Rakotch (Banach) contracting self-maps of X. A Hausdorff topological space X is called a topological fractal if $X = \bigcup_{f\in\mathcal{F}} f(X)$ for some finite system $\mathcal{F}$ of continuous self-maps, which is topologically contracting in the sense that for any sequence $(f_n)_{n\in\omega} \in \mathcal{F}^\omega$ the intersection $\bigcap_{n\in\omega} f_0 \circ \cdots \circ f_n(X)$ is a singleton. It is known that each topological fractal is homeomorphic to a Rakotch fractal. We prove that each Rakotch (Banach) fractal is isometric to the attractor of a Rakotch (Banach) contracting function system on the universal Urysohn space $\mathbb{U}$. Also we prove that each topological fractal is homemorphic to the attractor $A_{\mathcal{F}}$ of a topologically contracting function system $\mathcal{F}$ on an arbitrary Tychonoff space U, which contains a topological copy of the Hilbert cube. If the space U is metrizable, then its topology can be generated by a bounded metric making all maps $f \in \mathcal{F}$ Rakotch contracting.

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1. Introduction

Let X be a topological space. By a *function system* on X we shall understand any finite family $\mathcal{F}$ of continuous self-maps of X. Every function system $\mathcal{F}$ generates the mapping

$$\mathcal{F}\colon \mathbb{K}(X) \longrightarrow \mathbb{K}(X), \quad K \longmapsto \bigcup_{f\in\mathcal{F}} f(K),$$

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on the hyperspace $\mathbb{K}(X)$ of non-empty compact subsets of X, endowed with the Vietoris topology. If the topology of X is generated by a metric d, then the Vietoris topology on $\mathbb{K}(X)$ is generated by the Hausdorff metric

$$d_H(A, B) = \max\{\max_{a\in A} d(a, B), \max_{b\in B} d(b, A)\}.$$

We shall say that a compact set $A \in \mathbb{K}(X)$ is an *attractor* of a function system $\mathcal{F}$ if $\mathcal{F}(A) = A$ and for every compact set $K \in \mathbb{K}(X)$ the sequence of iterations $\mathcal{F}^n(K) = \mathcal{F}\circ\cdots\circ\mathcal{F}(K)$ converges to A in the hyperspace $\mathbb{K}(X)$. A function system $\mathcal{F}$ on a Hausdorff topological space X can have at most one attractor, which will be denoted by $A_{\mathcal{F}}$. The following classical result of the theory of deterministic fractals [11] detects function systems possessing attractors.

Theorem 1.1. *Each function system $\mathcal{F}$ consisting of Banach contractions of a complete metric space X has a unique attractor $A_{\mathcal{F}}$.*

The proof of Theorem 1.1 given in [11] (cf. also [3]) uses the observation that a Banach contracting function system $\mathcal{F}$ (i.e., a function system consisting of Banach contractions) on a complete metric space X induces a Banach contracting map $\mathcal{F}\colon \mathbb{K}(X) \to \mathbb{K}(X)$ of the hyperspace, which makes possible to apply the Banach Contracting Principle to show that $\mathcal{F}$ has an attractor $A_{\mathcal{F}}$.

It turns out that the Banach contractivity of $\mathcal{F}$ in Theorem 1.1 can be weakened to the φ-contractivity, which is defined as follows.

A map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called *φ-contracting* for a function $\varphi\colon [0,\infty) \to [0,\infty)$ if $d_Y(f(x), f(x')) \le \varphi(d_X(x, x'))$ for every points $x, x' \in X$. It follows that $f : X \to Y$ is Banach contracting (i.e., it is a Lipschitz mapping with the Lipschitz constant less than 1) if and only if it is φ-contracting for some function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\sup_{0<t<\infty} \varphi(t)/t < 1$.

A function $f : X \to Y$ is called

- *Rakotch contracting* if f is φ-contracting for a function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\sup_{a<t<\infty} \varphi(t)/t < 1$ for every $a > 0$;
- *Edelstein contracting* if $d_Y(f(x), f(x')) < d_X(x, x')$ for any distinct points $x, x' \in X$.

It is known ([12]) that each Rakotch contracting map is Edelstein contracting, and each Edelstein contracting map $f : X \to Y$ with compact domain X is Rakotch contracting. The notions of Rakotch and Edelstein contracting maps are connected with certain generalizations of the Banach Contracting Principle, cf. [8], [12], and [23].

A topological version of Theorem 1.1 was recently proved by Mihail [20] who introduced the following notion (cf. also [4] for a particular version of it). A function system $\mathcal{F}$ on a Hausdorff topological space X is called *topologically contracting* if

- for every $K \in \mathbb{K}(X)$, there is $D \in \mathbb{K}(X)$ such that $K \subset D$ and $\mathcal{F}(D) \subset D$;
- for every $D \in \mathbb{K}(X)$ with $\mathcal{F}(D) \subset D$, and a function sequence

$$\vec{f} = (f_n)_{n\in\omega} \in \mathcal{F}^\omega,$$

  the set $\bigcap_{n\in\omega} f_0 \circ \cdots \circ f_n(D)$ is a singleton.

It can be seen that in this case the singleton $\{\pi(\vec{f})\} = \bigcap_{n\in\omega} f_0 \circ \cdots \circ f_n(D)$ does not depend on the choice of the compact set D and the map $\pi : \mathcal{F}^\omega \to X$, $\vec{f} \mapsto \pi(\vec{f})$, is continuous (here the countable power $\mathcal{F}^\omega$ of $\mathcal{F}$ carries the compact metrizable topology of Tychonoff product of countably many copies of the finite space $\mathcal{F}$ endowed with the discrete topology). Moreover, the compact metrizable space $A_{\mathcal{F}} = \pi(\mathcal{F}^\omega)$ is the attractor of the function system $\mathcal{F}$. This fact was proved by Mihail [20].

Theorem 1.2. *Every topologically contracting function system $\mathcal{F}$ on a Hausdorff topological space X has an attractor $A_{\mathcal{F}}$, which is a compact metrizable space.*

A compact Hausdorff topological space X is called a *topological fractal* if $X = \bigcup_{f\in\mathcal{F}} f(X)$ for some topologically contracting function system $\mathcal{F}$ on X. It follows that for every topologically contracting function system $\mathcal{F}$ on a Hausdorff topological space X its attractor $A_{\mathcal{F}}$ is a topological fractal. Mihail's Theorem 1.2 implies that each topological fractal is a compact metrizable space. Moreover, the topology of X is generated by a metric d making all maps $f \in \mathcal{F}$ Rakotch contracting (see [2] or [21]). Topological fractals were introduced and investigated by Kameyama [13] and Kigami [17] (who called them *topologically self similar sets*), and considered also in [1], [7], [14], [15], and [22].

In this paper we shall search for copies of (Banach) topological fractals in universal (metric) spaces. A topological space X will be called *topologically universal* if every compact metrizable space K admits a topological embedding into X. By a *Tychonoff space* we understand a completely regular T_1-space. Equivalently, Tychonoff spaces can be defined as spaces homeomorphic to subspaces of powers of the real line. The following realization theorem will be proved in Section 2.

Theorem 1.3. *If a Tychonoff space X is topologically universal, then every topological fractal is homeomorphic to the attractor $A_{\mathcal{F}}$ of a topologically contracting function system $\mathcal{F}$ on X. If the space X is metrizable, then we can additionally assume that all maps $f \in \mathcal{F}$ are Rakotch contracting with respect to some bounded metric d generating the topology of X.*

For the universal Urysohn space we can prove a bit more. Let us recall that the *universal Urysohn space* is a separable complete metric space $\mathbb{U}$ such that each isometric embedding $f: B \to \mathbb{U}$ of a subspace B of a finite metric space A extends to an isometric embeddig $\bar{f}: A \to \mathbb{U}$. By [24], a universal Urysohn space exists and is unique up to a bijective isometry.

A compact metric space X will be called a (*Banach*) *Rakotch fractal* if $X = \bigcup_{f \in \mathcal{F}} f(X)$ for some function system $\mathcal{F}$ consisting of (Banach) Rakotch contractions of X. By [2] and [21], each topological fractal is homeomorphic to a Rakotch fractal. On the other hand, there are examples of Rakotch fractals which are not homeomorphic to Banach fractals (see [1], [14], [22]). A compact metric space is a Rakotch fractal if and only if it is *self-similar* in the sense of Hata [10, 2.2].

The following realization theorem will be proved in Section 3.

Theorem 1.4. *Each* (*Banach*) *Rakotch fractal X is isometric to the attractor $A_{\mathcal{F}}$ of a function system $\mathcal{F}$ consisting of Banach* (*Rakotch*) *contractions of the universal Urysohn space* $\mathbb{U}$.

Theorems 1.3 and 1.4 show that the problem of studying the inner topological or metric structure of attractors of topologically (Banach, Rakotch) contracting function systems in arbitrary spaces can be reduced to studying such systems in some nice universal spaces (such as the universal Urysohn space $\mathbb{U}$).

2. Copies of topological fractals in universal spaces

In this section we shall prove Theorem 1.3. At first we need to recall some information on spaces of probability measures.

For a compact metric space (X, d_X) by PX we shall denote the space of Borel probability measures on X endowed with the metric

$$d_{PX}(\mu, \eta) := \inf \Big\{ \int_{X \times X} d_X(x, y)\, d\lambda : \lambda \in \mathcal{B}(\mu, \eta) \Big\},$$

where $\mathcal{B}(\mu, \eta)$ is the space of all Borel probability measures on $X \times X$ such that $\pi_1(\lambda) = \mu$ and $\pi_2(\lambda) = \eta$ (here π_1 and π_2 stand for the projections onto the first and the second coordinate, respectively). It is known that (PX, d_{PX}) is a compact metric space and for any measures $\mu, \eta \in PX$, there is $\lambda \in \mathcal{B}(\mu, \eta)$ such that $d_{PX}(\mu, \eta) = \int_{X\times X} d(x, y)\, d\lambda$ (cf. [5, Chapter 8, especially Theorem 8.9.3, Theorem 8.10.45, p. 234]; here we use the fact that every Borel probability measure on a compact metrizable spaces is Radon [5, p. 70]). The metric d_{PX} was introduced by Kantorovich who studied the mass transportation problem (see the survey [25] for more detail historical information). According to Kantorovich–Rubinshtein Theorem [16], this metric can be equivalently defined as

$$d_{PX}(\mu, \eta) := \sup\{|\mu(f) - \eta(f)| : f \in \mathrm{Lip}_1(X)\}$$

where the supremum is taken over all non-expanding functions $f : X \to \mathbb{R}$ on the compact metric space X.

It follows that the map $X \ni x \to \delta_x \in PX$ assigning to each point $x \in X$ the Dirac measure δ_x supported at x is an isometric embedding of X into PX. Every continuous map $f : X \to Y$ between compact metric spaces (X, d_X) and (Y, d_Y) induces a continuous map $Pf : PX \to PY$ between their spaces of probability measures. The map Pf assigns to each measure $\mu \in PX$ the measure $Pf(\mu) \in PY$ defined by $Pf(\mu)(B) = \mu(f^{-1}(B))$ for a Borel subset $B \subset Y$.

Lemma 2.1. *If a map $f : X \to Y$ between compact metric spaces X, Y is Rakotch contracting, then the induced map $Pf : P(X) \to P(Y)$ is Rakotch contracting too.*

Proof. Being Rakotch contracting, the map f is φ-contracting for some function $\varphi : [0, \infty) \to [0, \infty)$ such that $c_\delta = \sup_{\delta \le t < \infty} \varphi(t)/t < 1$ for every $\delta > 0$. By the compactness of PX, the Rakotch contractivity of Pf is equivalent to its Edelstein contractivity. So, it suffices to prove that $d_{PY}(Pf(\mu), Pf(\eta)) < d_{PX}(\mu, \eta)$ for any distinct measures $\mu, \eta \in PX$. Take any distinct measures $\mu, \eta \in PX$. As stated earlier, there is a measure $\lambda \in \mathcal{B}(\mu, \eta)$ such that $d_{PX}(\mu, \eta) = \int_{X\times X} d_X(x, y) d\lambda$. Since $d_{PX}(\mu, \eta) > 0$, for some $\delta > 0$ the compact set

$$X_\delta = \{(x, x') \in X \times X : d_X(x, x') \ge \delta\}$$

has positive measure $\lambda(X_\delta) > 0$.

Let $\tilde{\lambda} \in P(Y \times Y)$ be the image of the measure λ under the map

$$f \times f : X \times X \longrightarrow Y \times Y, \quad (x, y) \longmapsto (f(x), f(y)).$$

Observe that for every Borel subset $B \subset Y$ we get

$$\begin{aligned}\tilde{\lambda}(B \times Y) &= \lambda((f \times f)^{-1}(B \times Y)) \\ &= \lambda(f^{-1}(B) \times f^{-1}(Y)) \\ &= \lambda(f^{-1}(B) \times X) \\ &= \mu(f^{-1}(B)) \\ &= Pf(\mu)(B)\end{aligned}$$

and similarly $\tilde{\lambda}(Y \times B) = Pf(\eta)(B)$, which means that $\tilde{\lambda} \in \mathcal{B}(Pf(\mu), Pf(\eta))$. Moreover,

$$\begin{aligned}d_{PY}(Pf(\mu), Pf(\eta)) &\le \int_{Y \times Y} d_Y(y, y')\, d\tilde{\lambda} \\ &= \int_{X \times X} d_Y(f(x), f(x'))\, d\lambda \\ &= \int_{X_\delta} d_Y(f(x), f(x'))\, d\lambda + \int_{X^2 \setminus X_\delta} d_Y(f(x), f(x'))\, d\lambda \\ &\le \int_{X_\delta} c_\delta \cdot d_X(x, x')\, d\lambda + \int_{X^2 \setminus X_\delta} d_X(x, x')\, d\lambda \\ &< \int_{X_\delta} d_X(x, x')\, d\lambda + \int_{X^2 \setminus X_\delta} d_X(x, x')\, d\lambda \\ &= \int_{X^2} d_X(x, x')\, d\lambda \\ &= d_{PX}(\mu, \eta).\end{aligned}$$

□

We shall also need a metrization theorem for globally contracting function systems, proved in [2]. A function system $\mathcal{F}$ on a Hausdorff topological space X is called *globally contracting* ([2, Definition 2.1]) if there exists a non-empty compact set $K \subset X$ such that $\mathcal{F}(K) \subset K$ and for every open cover $\mathcal{U}$ of X there is $n \in \mathbb{N}$ such that for every map $f \in \mathcal{F}^n = \{f_1 \circ \cdots \circ f_n : f_1, \dots, f_n \in \mathcal{F}\}$ the set $f(X)$ is contained in some set $U \in \mathcal{U}$. The following result was proved in [2, Theorem 6.7].

Theorem 2.2. *A function system $\mathcal{F}$ on a metrizable space X is globally contracting if and only if the topology of X is generated by a bounded metric d making all maps $f \in \mathcal{F}$ Rakotch contractive.*

Now we are able to present:

Proof of Theorem 1.3. Assuming that K is a topological fractal, find a topologically contracting function system $\mathcal{F}$ on K such that $K = \bigcup_{f \in \mathcal{F}} f(K)$. By [2, Theorem 6.8(8)], the topology of K is generated by a metric d_K making all maps $f \in \mathcal{F}$ Rakotch contracting. By Lemma 2.1, the function system $P\mathcal{F} = \{Pf : f \in \mathcal{F}\}$ consists of Rakotch contracting self-maps of the metric space (PK, d_{PK}).

Given any topologically universal Tychonoff space X, identify the compact metrizable space PK with a (closed) subspace of X. The space PK, being a metrizable compact convex subset of a locally convex space, is an absolute retract in the class of Tychonoff spaces (this follows from [6] and Tietze-Urysohn Theorem [9, 2.1.8]). This implies that each map $Pf : PK \to PK$, $f \in \mathcal{F}$, can be extended to a continuous map $\overline{Pf} : X \to PK$. The Rakotch contractivity of $P\mathcal{F}$ and Theorem 2.2 implies that the function system $P\mathcal{F}$ is globally contractive and so is the function system $\overline{P\mathcal{F}} = \{\overline{Pf} : Pf \in P\mathcal{F}\}$ (as $\overline{P\mathcal{F}}(X) \subset PK$).

The global contractivity of $\overline{P\mathcal{F}}$ implies the topological contractivity of $\overline{P\mathcal{F}}$ ([2, Theorem 2.2]). Then the function system $\overline{P\mathcal{F}}$ has a unique attractor, which coincides with K by the uniqueness of the fixed point of the map

$$\overline{P\mathcal{F}} : \mathbb{K}(X) \longrightarrow \mathbb{K}(X).$$

Therefore, the topological fractal K is homeomorphic to the attractor of the topologically contracting function system $\overline{P\mathcal{F}}$ on X.

If the space X is metrizable, then by Theorem 2.2, the topology of X is generated by a bounded metric d making all maps $f \in \overline{P\mathcal{F}}$ Rakotch contracting. □

3. Embedding fractals into the Urysohn universal space

Recall that the Urysohn space $\mathbb{U}$ is the unique (up to isometry) complete separable metric space $\mathbb{U}$ such that every isometric embedding $f : B \to \mathbb{U}$ of a finite subset $B \subset \mathbb{U}$ extends to an isometric embedding $\bar{f} : \mathbb{U} \to \mathbb{U}$, and any separable metric space is isometric to a subspace of $\mathbb{U}$. According to [19, Theorem 4.1], the universal Urysohn space has a stronger universality property: every isometric embedding $f : B \to \mathbb{U}$ of a compact subspace $B \subset \mathbb{U}$ extends to an isometric embedding $\bar{f} : \mathbb{U} \to \mathbb{U}$. In fact, isometric embeddings in this result can be replaced by maps with given oscillation.

For a map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) its *oscillation* is the function $\omega_f : [0, \infty] \to [0, \infty]$ assigning to each $\delta \in [0, \infty]$ the number

$$\omega_f(\delta) = \sup\{d_Y(f(x), f(x')) : x, x' \in X,\ d_X(x, x') \le \delta\} \in [0, \infty].$$

It follows that $d_Y(f(x), f(x')) \le \omega_f(d_X(x, x'))$ for every points $x, x' \in X$. It is clear that the function $f: X \to Y$ is uniformly continuous if and only if $\lim_{\delta\to 0} \omega_f(\delta) = 0$.

If the metric space (X, d_X) is *geodesic* (in the sense that for every points $x, x' \in X$ there is an isometric embedding $\gamma: [0, d_X(x, x')] \to X$ such that $\gamma(0) = x$ and $\gamma(d_X(x, x')) = x'$), then for any map $f: X \to Y$ its oscillation ω_f is *subadditive* in the sense that $\omega_f(s + t) \le \omega_f(s) + \omega_f(t)$ for any $s, t \in [0, \infty)$. This motivates the following definition.

By a *continuity modulus* we shall understand any non-decreasing continuous subadditive function $\varphi: [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$. It is easy to see that each continuous, concave function $\varphi: [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ is a continuity modulus. The following lemma uses some ideas from [18].

Lemma 3.1. *Let φ be a continuity modulus. Every map $f: B \to \mathbb{U}$ with $\omega_f \le \varphi$ defined on a compact subset B of a separable metric space (A, d_A) extends to a map $\bar{f}: A \to \mathbb{U}$ with continuity modulus $\omega_{\bar{f}} \le \varphi$.*

Proof. The statement of the lemma is trivial if the map f is constant. So, we assume that f is not constant, which imples that $\omega_f \ne 0$ and hence $\varphi(t) > 0$ for every $t > 0$ by the subadditivity of φ.

Fix a countable dense subset $\{a_n\}_{n\in\mathbb{N}}$ in A and put $B_0 = B$ and $B_{n+1} = B_n \cup \{a_{n+1}\}$ for $n \in \omega$. Let $f_0 = f$. By induction we shall construct a sequence of maps $(f_n: B_n \to \mathbb{U})_{n\in\omega}$ such that $f_{n+1}|B_n = f_n$ and $\omega_{f_n} \le \varphi$ for all $n \in \omega$. Assume that for some $n \in \omega$ the map $f_n: B_n \to \mathbb{U}$ has been constructed. Consider the compact space $B_{n+1} = B_n \cup \{a_{n+1}\}$. If $B_{n+1} = B_n$, then put $f_{n+1} = f_n$. If $B_{n+1} \ne B_n$, then $a_{n+1} \notin B_n$. Consider the subspace $f_n(B_n) \subset \mathbb{U}$ of the Urysohn space and fix any point $y \notin f_n(B_n)$. On the union $Y = f_n(B_n) \cup \{y\}$ consider the metric d_Y which coincides on $f_n(B_n)$ with the metric $d_{\mathbb{U}}$ of the Urysohn space $\mathbb{U}$ and

$$d_Y(z, y) = \min\{d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, a_{n+1})) : b \in B_n\}$$

for $z \in f_n(B_n)$. It follows from the compactness of B_n and $a_{n+1} \notin B_n$ that $d_Y(z, y) > 0$ for every $z \in f_n(B_n)$. Let us show that the metric d_Y satisfies the triangle inequality and hence is well-defined. Indeed, for any points $z, z' \in f_n(B_n)$, we can find points $b, b' \in B_n$ such that $d_Y(z, y) = d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b, a_{n+1}))$ and $d_Y(z', y) = d_{\mathbb{U}}(z', f_n(b')) + \varphi(d_A(b', a_{n+1}))$. Then

$$\begin{aligned} d_Y(z,z') = d_{\mathbb{U}}(z,z') &\le d_{\mathbb{U}}(z, f_n(b)) + d_{\mathbb{U}}(f_n(b), f_n(b')) + d_{\mathbb{U}}(f_n(b'), z') \\ &\le d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b,b')) + d_{\mathbb{U}}(f_n(b'), z') \\ &\le d_{\mathbb{U}}(z, f_n(b)) + \varphi\big(d_A(b,a_{n+1}) + d_A(a_{n+1},b')\big) + d_{\mathbb{U}}(f_n(b'), z') \\ &\le d_{\mathbb{U}}(z, f_n(b)) + \varphi(d_A(b,a_{n+1})) + \varphi(d_A(a_{n+1},b')) + d_{\mathbb{U}}(f_n(b'), z') \\ &= d_Y(z,y) + d_Y(y,z'). \end{aligned}$$

On the other hand,

$$\begin{aligned} d_Y(z,y) &\le d_{\mathbb{U}}(z, f_n(b')) + \varphi(d_A(b',a_{n+1})) \\ &\le d_{\mathbb{U}}(z,z') + d(z', f_n(b')) + \varphi(d_A(b',a_{n+1})) \\ &= d_Y(z,z') + d_Y(z',y). \end{aligned}$$

So, the metric d_Y satisfies the triangle inequality and hence is well-defined.

By [19, Theorem 4.1], the identity embedding $f_n(B_n) \to \mathbb{U}$ extends to an isometric embedding $e\colon Y \to \mathbb{U}$. Let $f_{n+1}\colon B_{n+1} \to \mathbb{U}$ be the map such that $f_{n+1}|B_n = f_n$ and $f_{n+1}(a_{n+1}) = e(y)$. It follows from $\omega_{f_n} \le \varphi$ and

$$d_{\mathbb{U}}(f_{n+1}(x), f_{n+1}(a_{n+1})) = d_Y(f_n(x), y) \le \varphi(d_A(x,a_{n+1})) \quad \text{for } x \in B_n$$

that $\omega_{f_{n+1}} \le \varphi$. This completes the inductive step.

After the completion of the inductive construction, consider the map

$$f_\omega\colon B_\omega \longrightarrow \mathbb{U}$$

defined on the set $B_\omega = \bigcup_{n\in\omega} B_n$ by

$$f_\omega|B_n = f_n \quad \text{for } n \in \omega.$$

It follows from $\omega_{f_n} \le \varphi$, $n \in \omega$, that $\omega_{f_\omega} \le \varphi$. This implies that the map f_ω is uniformly continuous and hence extends to a uniformly continuous map $\bar{f}\colon A \to \mathbb{U}$ having the oscillation $\omega_{\bar{f}} \le \varphi$. It is clear that $\bar{f}|B = f_0 = f$. □

Now we are able to present the

Proof of Theorem 1.4. Given a (Banach) Rakotch fractal X, choose a function system $\mathcal{F}$ consisting of (Banach) Rakotch contractions of X such that $X = \bigcup_{f\in\mathcal{F}} f(X)$. The maps $f \in \mathcal{F}$, being Rakotch contracting on the compact metric space X, are φ-contracting for some bounded continuity modulus φ

such that $\sup_{t\geq\delta}\varphi(t)/t < 1$ for all $\delta > 0$. (Moreover, if all maps $f \in \mathcal{F}$ are Banach contractions, then we can assume that $\sup_{t>0}\varphi(t)/t < 1$).

Since the universal Urysohn space $\mathbb{U}$ contains an isometric copy of each compact metric space, we can assume that X is a subspace of $\mathbb{U}$. By Lemma 3.1, each map $f \in \mathcal{F}$ extends to a map $\bar{f}\colon \mathbb{U} \to \mathbb{U}$ such that $\omega_{\bar{f}} \leq \varphi$, which implies that $\bar{f}$ is a (Banach) Rakotch contraction of $\mathbb{U}$. By Theorems 1.2 and 2.2, the function system $\bar{\mathcal{F}} = \{\bar{f}\colon f \in \mathcal{F}\}$ on the universal Urysohn space $\mathbb{U}$ has an attractor $A_{\bar{\mathcal{F}}}$, which is a unique fixed point of the map $\bar{\mathcal{F}}\colon \mathbb{K}(\mathbb{U}) \to \mathbb{K}(\mathbb{U})$ on the hyperspace $\mathbb{K}(\mathbb{U})$ of the Urysohn space $\mathbb{U}$. Taking into account that $\bar{\mathcal{F}}(X) = \bigcup_{f\in\mathcal{F}}\bar{f}(X) = \bigcup_{f\in\mathcal{F}}f(X) = X$, we conclude that $X = A_{\bar{\mathcal{F}}}$. □

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