J. Fractal Geom. 3 (2016), 33–74
DOI 10.4171/JFG/29

Journal of Fractal Geometry

On the Hausdorff and packing measures of slices of dynamically defined sets

Ariel Rapaport[1]

Abstract. Let $1 \leq m < n$ be integers, and let $K \subset \mathbb{R}^n$ be a self-similar set satisfying the strong separation condition, and with $\dim K = s > m$. We study the a.s. values of the $s - m$-dimensional Hausdorff and packing measures of $K \cap V$, where V is a typical $n - m$-dimensional affine subspace.

For $0 < \rho < \frac{1}{2}$ let $C_\rho \subset [0,1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$, where $f_{\rho,1}(t) = \rho \cdot t$ and $f_{\rho,2}(t) = \rho \cdot t + 1 - \rho$ for each $t \in \mathbb{R}$. We show that for certain numbers $0 < a, b < \frac{1}{2}$, for instance $a = \frac{1}{4}$ and $b = \frac{1}{3}$, if $K = C_a \times C_b$, then typically we have $\mathcal{H}^{s-m}(K \cap V) = 0$.

Mathematics Subject Classification (2010). 28A80; 28A78.

Keywords. Self-similar set, self-affine set, Hausdorff measures, packing measures.

Contents

1. Introduction

Let $1 \leq m < n$ be integers, and given $0 \leq t \leq n$ let $\mathcal{H}^t$ and $\mathcal{P}^t$ be the t-dimensional Hausdorff and packing measures in $\mathbb{R}^n$ respectively. Let $s \in (m, n)$ be a real number, and let $K \subset \mathbb{R}^n$ be compact with $0 < \mathcal{H}^s(K) < \infty$. Denote by μ the

[1] Supported by ERC grant 306494.

restriction of $\mathcal{H}^s$ to K, by G the set of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$, and by ξ_G the natural measure on G. For each $V \in G$ and $x \in \mathbb{R}^n$ set

$$K_{V,x} = K \cap (x + V).$$

It is well known that $\dim_H(K_{V,x}) = s - m$ and $\mathcal{H}^{s-m}(K_{V,x}) < \infty$, for $\mu \times \xi_G$-a.e. $(x, V) \in K \times G$ (see Theorem 10.11 in [10]). It is also known that if $s = \dim_P K$, then $\dim_P(K_{V,x}) \leq s - m$ for every $V \in G$ and $\mathcal{H}^m$-a.e. $x \in V^\perp$ (see Lemma 5 in [3]), where $\dim_P$ stands for the packing dimension. In this paper K will denote certain self-similar or self-affine sets, in which cases it will be shown that more can be said about the $\mu \times \xi_G$-typical values of $\mathcal{H}^{s-m}(K_{V,x})$ and $\mathcal{P}^{s-m}(K_{V,x})$.

Assume that K is a self-similar set which satisfies the strong separation condition (SSC), then since $s > m$ we have $P_{V^\perp}\mu \ll \mathcal{H}^m$ for ξ_G-a.e. $V \in G$ (see the proof of Lemma 3.2 below). Here $P_{V^\perp}$ is the orthogonal projection onto $V^\perp$. Firstly we are interested in the validity of the condition

$$\mathcal{H}^{s-m}(K_{V,x}) > 0 \quad \text{for } \mu \times \xi_G\text{-a.e. } (x, V) \in K \times G. \tag{1.1}$$

If $m = 1$ and K is rotation-free, then by a result by Kempton (Theorem 6.1 in [9]) it follows that (1.1) holds if and only if the density $\frac{dP_{V^\perp}\mu}{d\mathcal{H}^m}$ is bounded for ξ_G-a.e. $V \in G$. In Theorem 2.1 below the case of a general $1 \leq m < n$ and a general self-similar set K, satisfying the SSC, will be considered. Extending Kempton's result, a necessary and sufficient condition for (1.1) will be given. In order to state this condition, let H be the closed group generated by the orthogonal parts of the contracting similarities defining K (see Section 2), and let ξ_H be the Haar measure corresponding to H. We show that (1.1) holds if and only if for ξ_G-a.e. V the densities $\frac{dP_{(hV)^\perp}\mu}{d\mathcal{H}^m}$ are ξ_H-essentially bounded, i.e. there exists a constant $M_V > 0$ with $\frac{dP_{(hV)^\perp}\mu}{d\mathcal{H}^m} \leq M_V$ for ξ_H-a.e. $h \in H$. We also prove the analogous statement, which says that $\mathcal{H}^{s-m}(K_{V,x}) = 0$ for $\mu \times \xi_G$-a.e. (x, V) if and only if for ξ_G-a.e. V the densities $\frac{dP_{(hV)^\perp}\mu}{d\mathcal{H}^m}$ are not ξ_H-essentially bounded.

It is proven in [4] and independently in [16] that if ν is a compactly supported Radon measure on $\mathbb{R}^n$ with finite $2m$-energy, then $P_{V^\perp}\nu \ll \mathcal{H}^m$ with continuous density for ξ_G-a.e. $V \in G$. By combining this fact with Theorem 2.1, we prove in Corollary 2.2 that (1.1) holds whenever $s > 2m$ and H is finite. Unfortunately, in the general case, it seems out of reach to establish the ξ_H-essential boundedness of the densities by current methods. Hence whether or not (1.1) generally holds remains an open problem, which is probably quite hard. This is demonstrated by Corollary 2.3, where it is shown that if (1.1) holds and H equals the entire orthogonal group, then $P_{V^\perp}\mu \ll \mathcal{H}^m$ for every $V \in G$. Note that the validity of the last statement is a major open problem.

Next we describe our results regarding the $s-m$-dimensional packing measure of typical slices. We continue to assume that K is a self-similar set with the SSC, and observe that $\dim_P(K_{V,x}) = s-m$ for $\mu \times \xi_G$-a.e. (x, V). This follows by Lemma 5 in [3], which was mentioned above, and since $P_{V^\perp}\mu \ll \mathcal{H}^m$ for ξ_G-a.e. V. It will be shown in Theorem 2.4 that we always have $\mathcal{P}^{s-m}(K_{V,x}) > 0$ for $\mu \times \xi_G$-a.e. (x, V). We also prove that if for ξ_G-a.e. V the densities $\frac{dP_{(hV)^\perp}\mu}{d\mathcal{H}^m}$ are not essentially bounded from 0, then

$$\mathcal{P}^{s-m}(K_{V,x}) = \infty \quad \text{for } \mu \times \xi_G\text{-a.e. } (x, V) \in K \times G. \tag{1.2}$$

By using this, it is shown in Corollary 2.5 that if $s > 2m$, then (1.2) holds true. Here we again utilize the continuity of the densities obtained in [4] and [16]. This is related to a result by Orponen (Corollary 1.2 in [14]), which says that if $n = 2$, $s > m = 1$, and K is rotation and reflection free, then (1.2) holds.

Lastly we consider the case in which $n = 2$, $m = 1$ and K is a certain self-affine set. For $0 < \rho < \frac{1}{2}$ let $C_\rho \subset [0, 1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$, where

$$f_{\rho,1}(t) = \rho \cdot t$$

and

$$f_{\rho,2}(t) = \rho \cdot t + 1 - \rho$$

for each $t \in \mathbb{R}$. It will be assumed that

$$K = C_a \times C_b,$$

where $0 < a, b < \frac{1}{2}$ are such that a^{-1} and b^{-1} are Pisot numbers, $\frac{\log b}{\log a}$ is irrational, and $s = \dim_H(K) > 1$. Under these conditions it is shown in [13] that there exists a dense G_δ set, of 1-dimensional linear subspaces $V \subset \mathbb{R}^2$, such that $P_V\mu$ and $\mathcal{H}^1$ are singular. By using this fact, it will be proven in Theorem 2.6 below that $\mathcal{H}^{s-1}(K_{V,x}) = 0$ for $\mu \times \xi_G$-a.e. (x, V). This result demonstrates some kind of smallness of the slices $K_{V,x}$, hence it may be seen as related to a conjecture made by Furstenberg (Conjecture 5 in [6]). In our setting this conjecture basically says that for ξ_G-a.e. $V \in G$ we have $\dim_H(K_{V,x}) \le s-1$ for each $x \in \mathbb{R}^2$, which demonstrates the smallness of the slices in another manner.

The rest of this article is organized as follows: In Section 2 the results are stated. In Section **??** the results regarding self-similar sets are proven. In Section 4 we prove the aforementioned theorem regarding self-affine sets.

Acknowledgment. I would like to thank my advisor Michael Hochman, for suggesting to me problems studied in this paper, and for many helpful discussions.

2. Statement of the results

2.1. Slices of self-similar sets. Let $0 < m < n$ be integers, let G be the Grassmannian manifold consisting of all $n-m$-dimensional linear subspaces of $\mathbb{R}^n$, let $O(n)$ be the orthogonal group of $\mathbb{R}^n$, and let ξ_O be the Haar measure corresponding to $O(n)$. Fix $U \in G$ and for each Borel set $E \subset G$ define

$$\xi_G(E) = \xi_O\{g \in O(n) \colon gU \in E\}, \tag{2.1}$$

then ξ_G is the unique rotation invariant Radon probability measure on G. For a linear subspace V of $\mathbb{R}^n$ let P_V be the orthogonal projection onto V, let $V^\perp$ be the orthogonal complement of V, and set

$$V_x = x + V \quad \text{for each } x \in \mathbb{R}^n.$$

Let Λ be a finite and nonempty set. Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a self-similar IFS in $\mathbb{R}^n$, with attractor $K \subset \mathbb{R}^n$ and with $\dim_H K = s > m$. For each $\lambda \in \Lambda$ there exist $0 < r_\lambda < 1$, $h_\lambda \in O(n)$ and $a_\lambda \in \mathbb{R}^n$, such that $\varphi_\lambda(x) = r_\lambda \cdot h_\lambda(x) + a_\lambda$ for each $x \in \mathbb{R}^n$. We assume that $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ satisfies the strong separation condition, i.e. that the sets $\{\varphi_\lambda(K)\}_{\lambda\in\Lambda}$ are pairwise disjoint. Let H be the smallest closed sub-group of $O(n)$ which contains $\{h_\lambda\}_{\lambda\in\Lambda}$, and let ξ_H be the Haar measure corresponding to H. For each $E \subset \mathbb{R}^n$ set

$$\mu(E) = \frac{\mathcal{H}^s(K \cap E)}{\mathcal{H}^s(K)};$$

then μ is a Radon probability measure which is supported on K.

For each $0 \le t < \infty$, ν a Radon probability measure on $\mathbb{R}^n$, and $x \in \mathbb{R}^n$ set

$$\Theta^{*t}(\nu, x) = \limsup_{\epsilon\downarrow 0} \frac{\nu(B(x,\epsilon))}{(2\epsilon)^t} \quad \text{and} \quad \Theta^t_*(\nu, x) = \liminf_{\epsilon\downarrow 0} \frac{\nu(B(x,\epsilon))}{(2\epsilon)^t}, \tag{2.2}$$

where $B(x,\epsilon)$ is the closed ball in $\mathbb{R}^n$ with center x and radius ϵ. It holds that $\Theta^{*t}(\nu,\cdot)$ and $\Theta^t_*(\nu,\cdot)$ are Borel functions (see remark 2.10 in [10]). For $V \in G$ define

$$F_V(x,h) = \Theta^m_*(P_{(hV)^\perp}\mu, P_{(hV)^\perp}(x)) \quad \text{for } (x,h) \in K \times H;$$

then F_V is a Borel function from $K \times H$ to $[0,\infty]$. In what follows the collection $\{F_V\}_{V\in G}$ will be of great importance for us.

Let $\mathcal{V}$ be the set of all $V \in G$ with

$$\xi_H(H \setminus \{h \in H : P_{(hV)^\perp}\mu \ll \mathcal{H}^m\}) = 0;$$

then in Lemma 3.2 below it will be shown that $\xi_G(G \setminus \mathcal{V}) = 0$. Note that by Theorem 2.12 in [10] it follows that for each $V \in \mathcal{V}$

$$F_V(x, h) = \frac{dP_{(hV)^\perp}\mu}{d\mathcal{H}^m}(P_{(hV)^\perp}(x)) \quad \text{for } \mu \times \xi_H\text{-a.e. } (x, h) \in K \times H.$$

First we state our results regarding the Hausdorff measure of typical slices of K.

Theorem 2.1. (i) *Given $V \in \mathcal{V}$, if $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$, then*

$$\mathcal{H}^{s-m}(K \cap (x + hV)) > 0 \quad \textit{for } \mu \times \xi_H\textit{-a.e. } (x, h) \in K \times H.$$

(ii) *Given $V \in \mathcal{V}$, if $\|F_V\|_{L^\infty(\mu\times\xi_H)} = \infty$, then*

$$\mathcal{H}^{s-m}(K \cap (x + hV)) = 0 \quad \textit{for } \mu \times \xi_H\textit{-a.e. } (x, h) \in K \times H.$$

(iii) *$\mathcal{H}^{s-m}(K \cap V_x) > 0$ for $\mu \times \xi_G$-a.e. $(x, V) \in K \times G$ if and only if*

$$\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty \quad \textit{for } \xi_G\textit{-a.e. } V \in G.$$

(iv) *$\mathcal{H}^{s-m}(K \cap V_x) = 0$ for $\mu \times \xi_G$-a.e. $(x, V) \in K \times G$ if and only if*

$$\|F_V\|_{L^\infty(\mu\times\xi_H)} = \infty \quad \textit{for } \xi_G\textit{-a.e. } V \in G\ .$$

By Theorem 2.1 we can derive the following corollaries.

Corollary 2.2. *Assume that $s > 2m$ and H is finite; then*

$$\mathcal{H}^{s-m}(K \cap V_x) > 0 \quad \textit{for } \mu \times \xi_G\textit{-a.e. } (x, V) \in K \times G.$$

Corollary 2.3. *Assume that $H = O(n)$ and*

$$\mu \times \xi_G\{(x, V) \in K \times G : \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0;$$

then there exists $0 < M < \infty$ such that for each $V \in G$ we have

$$P_{V^\perp}\mu \ll \mathcal{H}^m$$

with

$$\left\| \frac{dP_{V^\perp}\mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} \le M.$$

Remark. It is known that under the assumptions of Corollary 2.3 we have

$$\dim(P_{V^\perp}\mu) = m \quad \text{for each } V \in G$$

(see Theorem 1.6 in [7]). It is not known however if $P_{V^\perp}\mu \ll \mathcal{H}^m$ for each $V \in G$, which is in fact a major open problem. Hence Corollary 2.3 implies that determining whether

$$\mu \times \xi_G\{(x, V) \in K \times G : \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0$$

is probably quite hard.

Next we state our results regarding the packing measure of typical slices.

Theorem 2.4. (i) *For $\mu \times \xi_G$-a.e. $(x, V) \in K \times G$,*

$$\mathcal{P}^{s-m}(K \cap V_x) > 0.$$

(ii) *Given $V \in \mathcal{V}$, if $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$, then*

$$\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty \quad \textit{for } \mu \times \xi_H\textit{-a.e. } (x, h) \in K \times H.$$

(iii) *If $\left\|\frac{1}{F_V}\right\|_{L^\infty(\mu\times\xi_H)} = \infty$ for ξ_G-a.e. $V \in G$, then*

$$\mathcal{P}^{s-m}(K \cap V_x) = \infty \quad \textit{for } \mu \times \xi_G\textit{-a.e. } (x, V) \in K \times G.$$

By Theorem 2.4 the following corollary can be derived.

Corollary 2.5. *Assume $s > 2m$; then*

$$\mathcal{P}^{s-m}(K \cap V_x) = \infty \quad \textit{for } \mu \times \xi_G\textit{-a.e. } (x, V) \in K \times G.$$

Remark. In the proofs of Corollaries 2.2 and 2.5, we use the fact that if $s > 2m$, then $\frac{dP_{V^\perp}\mu}{d\mathcal{H}^m}$ is a continuous function for ξ_G-a.e. V (see Lemma 3.8 below). It is not known whether this is still true if $m < s \le 2m$, hence we need the assumption $s > 2m$. Note also that the densities $\frac{dP_{V^\perp}\mu}{d\mathcal{H}^m}$ are in $L^2(\mathcal{H}^m)$ for ξ_G-a.e. V (see Theorem 9.7 in [10]), but it seems difficult to make any use of this.

2.2. Slices of self-affine sets. Assume $n = 2$ and $m = 1$. Given $0 < \rho < \frac{1}{2}$ define

$$f_{\rho,1}, f_{\rho,2} \colon \mathbb{R} \to \mathbb{R}$$

by

$$f_{\rho,1}(x) = \rho \cdot x$$

and

$$f_{\rho,2}(x) = \rho \cdot x + 1 - \rho$$

for all $x \in \mathbb{R}$. Let $C_\rho \subset [0,1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$. Set

$$d_\rho = \dim_H C_\rho,$$

so that

$$d_\rho = \frac{\log 2}{\log \rho^{-1}},$$

and, for each $E \subset \mathbb{R}$, set

$$\mu_\rho(E) = \frac{\mathcal{H}^{d_\rho}(C_\rho \cap E)}{\mathcal{H}^{d_\rho}(C_\rho)}.$$

Theorem 2.6. *Let $0 < a < b < \frac{1}{2}$ be such that $\frac{1}{a}$ and $\frac{1}{b}$ are Pisot numbers, $\frac{\log b}{\log a}$ is irrational and $d_a + d_b > 1$; then*

$$\mathcal{H}^{d_a+d_b-1}((C_a \times C_b) \cap V_{(x,y)}) = 0$$

for $\mu_a \times \mu_b \times \xi_G$-a.e. $(x, y, V) \in C_a \times C_b \times G$.

Remark. Recall that every integer greater than 1 is a Pisot number, hence Theorem 2.6 applies for instance in the case $a = \frac{1}{4}$ and $b = \frac{1}{3}$.

Remark. Note that

$$0 < \mathcal{H}^{d_a+d_b}(C_a \times C_b) < \infty$$

and

$$\mu_a \times \mu_b(E) = \frac{\mathcal{H}^{d_a+d_b}(E)}{\mathcal{H}^{d_a+d_b}(C_a \times C_b)}$$

for each Borel set $E \subset C_a \times C_b$ (see Lemma 4.4 below). Hence by Theorem 10.11 in [10] we get

$$\dim_H((C_a \times C_b) \cap V_{(x,y)}) = d_a + d_b - 1$$

for $\mu_a \times \mu_b \times \xi_G$-a.e. $(x, y, V) \in C_a \times C_b \times G$.

3. Proof of the results regarding self-similar sets

3.1. Preliminaries. The following notation will be used in the proofs of Theorems 2.1 and 2.4. For each $\lambda \in \Lambda$ set

$$p_\lambda = r_\lambda^s.$$

Then μ is the unique self-similar probability measure corresponding to the IFS $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ and the probability vector $(p_\lambda)_{\lambda\in\Lambda}$, i.e. μ satisfies the relation

$$\mu = \sum_{\lambda\in\Lambda} p_\lambda \cdot \mu \circ \varphi_\lambda^{-1}.$$

Let Λ^* be the set of finite words over Λ; then given $\lambda_1 \cdot \dots \cdot \lambda_l = w \in \Lambda^*$ we write

$$\begin{aligned} p_w &= p_{\lambda_1} \cdot \dots \cdot p_{\lambda_l}, \\ r_w &= r_{\lambda_1} \cdot \dots \cdot r_{\lambda_l}, \\ h_w &= h_{\lambda_1} \cdot \dots \cdot h_{\lambda_l}, \\ \varphi_w &= \varphi_{\lambda_1} \circ \dots \circ \varphi_{\lambda_l} \end{aligned}$$

and

$$K_w = \varphi_w(K).$$

For each $l \geq 1$ and $x \in K$, let $w_l(x) \in \Lambda^l$ be the unique word of length l which satisfies $x \in K_{w_l(x)}$. Set also

$$\rho = \min\{d(\varphi_{\lambda_1}(K), \varphi_{\lambda_2}(K)) \colon \lambda_1, \lambda_2 \in \Lambda \text{ and } \lambda_1 \neq \lambda_2\}; \tag{3.1}$$

then $\rho > 0$ since $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ satisfies the strong separation condition. Given $V_1, V_2 \in G$ set

$$d_G(V_1, V_2) = \|P_{V_1} - P_{V_2}\|$$

(where $\|\cdot\|$ stands for operator norm); then d_G is a metric on G. Recall that for a Radon measure ν on $\mathbb{R}^n$ and $t > 0$ the t-energy of ν is defined to be

$$I_t(\nu) = \iint |x - y|^{-t} \, d\nu(x)\, d\nu(y).$$

The following dynamical system will be used in the proofs of Theorems 2.1 and 2.4. Set

$$X = K \times H$$

and for each $(x,h) \in X$ let

$$T(x,h) = (\varphi_{w_1(x)}^{-1} x, h_{w_1(x)}^{-1} \cdot h).$$

It is easy to check that the system $(X, \mu \times \xi_H, T)$ is measure preserving, and by Corollary 4.5 in [15] it follows that it is ergodic. We also have

$$T^k(x,h) = (\varphi_{w_k(x)}^{-1} x, h_{w_k(x)}^{-1} \cdot h) \quad \text{for each } k \geq 1 \text{ and } (x,h) \in X.$$

Let $\mathcal{R}$ be the Borel σ-algebra of $\mathbb{R}^n$. For each $V \in G$ set

$$\mathcal{R}_V = P_{V^\perp}^{-1}(\mathcal{R}),$$

and let $\{\mu_{V,x}\}_{x\in\mathbb{R}^n}$ be the disintegration of μ with respect to $\mathcal{R}_V$ (see Section 3 of [5]). For μ-a.e. $x \in \mathbb{R}^n$ the probability measure $\mu_{V,x}$ is defined and supported on $K \cap V_x$. Also, for each $f \in L^1(\mu)$ the map that takes $x \in \mathbb{R}^n$ to $\int f\, d\mu_{V,x}$ is $\mathcal{R}_V$-measurable, the formula

$$\int f\, d\mu = \iint f(y)\, d\mu_{V,x}(y)\, d\mu(x)$$

is satisfied, and for $P_{V^\perp}\mu$-a.e. $x \in V^\perp$ we have

$$\int f\, d\mu_{V,x} = \lim_{\epsilon \downarrow 0} \frac{1}{P_{V^\perp}\mu(B(x,\epsilon))} \cdot \int_{P_{V^\perp}^{-1}(B(x,\epsilon))} f\, d\mu\,.$$

For more details on the measures $\{\mu_{V,x}\}_{x\in\mathbb{R}^n}$ see Section 3 of [5] and the references therein.

3.2. Auxiliary lemmas. We shall now prove some lemmas that will be needed later on. The following lemma will be used with ξ_H in place of η, where ξ_H is considered as a measure on $O(n)$ (which is supported on H).

Lemma 3.1. *Let Q be a compact metric group, and let ν be its normalized Haar measure. Let η be a Borel probability measure on Q; then for each Borel set $E \subset Q$*

$$\nu(E) = \int_Q \eta(E \cdot q^{-1})\, d\nu(q).$$

Proof. For each Borel set $E \subset Q$ define

$$\zeta(E) = \int_Q \eta(E \cdot q^{-1})\, d\nu(q).$$

Since ν is invariant it follows that for each $g \in Q$

$$\begin{aligned}\zeta(Eg) &= \int_Q \eta(E \cdot g \cdot q^{-1})\, d\nu(q) \\ &= \int_Q \eta(E \cdot g \cdot (q \cdot g)^{-1})\, d\nu(q) \\ &= \zeta(E).\end{aligned}$$

This shows that ζ is a right-invariant Borel probability measure on Q, hence $\nu = \zeta$ by the uniqueness of the Haar measure, and the lemma follows. □

Lemma 3.2. *Let $\mathcal{V}$ be the set of all $V \in G$ with*

$$\xi_H(H \setminus \{h \in H : P_{(hV)^\perp}\mu \ll \mathcal{H}^m\}) = 0,$$

then $\xi_G(G \setminus \mathcal{V}) = 0$.

Proof. Set

$$L = G \setminus \{V \in G : P_{V^\perp}\mu \ll \mathcal{H}^m\}.$$

It is easy to see that there exists a constant $b \in (0, \infty)$ with $\mu(B(x, r)) \le b \cdot r^s$ for each $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ (see Theorem 4.14 in [10]). By the discussion found at the beginning of Chapter 8 of [10], and since $s > m$, it follows that $I_m(\mu) < \infty$. Hence, by Theorem 9.7 and equality (3.10) in [10] we get $\xi_G(L) = 0$.

Let $U \in G$ be as in (2.1) and set

$$L' = \{g \in O(n) : gU \in L\};$$

then

$$\xi_O(L') = \xi_G(L) = 0.$$

Let $B \subset O(n)$ be a Borel set with $L' \subset B$ and $\xi_O(B) = 0$; then by Lemma 3.1 it follows that

$$0 = \xi_O(B) = \int \xi_H(B \cdot g^{-1})\, d\xi_O(g)\,.$$

We get that for ξ_O-a.e. $g \in O(n)$

$$\begin{aligned}0 = \xi_H(B \cdot g^{-1}) &\ge \xi_H(L' \cdot g^{-1}) \\ &= \xi_H\{h \in H : hg \in L'\} \\ &= \xi_H(H \setminus \{h \in H : P_{(hgU)^\perp}\mu \ll \mathcal{H}^m\}),\end{aligned}$$

and so

$$\xi_H(H \setminus \{h \in H \colon P_{(hV)^\perp}\mu \ll \mathcal{H}^m\}) = 0 \quad \text{for } \xi_G\text{-a.e. } V \in G,$$

which proves the lemma. □

Lemma 3.3. *Let $\mathcal{Z}$ be the set of all $(x, V) \in K \times G$ such that $\mu_{V,x}$* is defined and

$$\mu_{V,x}(K_w) = \lim_{\epsilon \downarrow 0} \frac{\mu(K_w \cap P_{V^\perp}^{-1}(B(P_{V^\perp}x, \epsilon)))}{P_{V^\perp}\mu(B(P_{V^\perp}x, \epsilon))} \quad \text{for each } w \in \Lambda^*;$$

then for each $V \in G$ we have

$$\mu \times \xi_H\{(x, h) \in X \colon (x, hV) \notin \mathcal{Z}\} = 0.$$

Proof. Fix $V \in G$. It holds that $\mathcal{Z}$ is a Borel set, see Section 3 of [11] for a related argument. It follows that the set

$$\mathcal{Z}_V = \{(x, h) \in X \colon (x, hV) \in \mathcal{Z}\}$$

is also a Borel set. By the properties stated in Section 3.1 we get that

$$\mu\{x \in K \colon (x, h) \notin \mathcal{Z}_V\} = 0 \quad \text{for each } h \in H,$$

and so $\mu \times \xi_H(X \setminus \mathcal{Z}_V) = 0$ by Fubini's theorem. This proves the lemma. □

Lemma 3.4. *Given a compact set $\widetilde{K} \subset \mathbb{R}^n$ and $0 < t \le n$, the map that takes $(x, V) \in \widetilde{K} \times G$ to $\mathcal{H}^t(\widetilde{K} \cap V_x)$ is Borel measurable.*

Proof. For $\delta > 0$ let $\mathcal{H}^t_\delta$ be as defined in Section 4.3 of [10]. Let $(x, V) \in \widetilde{K} \times G$, $\epsilon > 0$ and $\{(x_k, V^k)\}_{k=1}^\infty \subset \widetilde{K} \times G$, be such that

$$(x_k, V^k) \xrightarrow{k} (x, V). \tag{$*$}$$

Let $W_1, W_2, \ldots \subset \mathbb{R}^n$ be open sets with $\widetilde{K} \cap V_x \subset \bigcup_{j=1}^\infty W_j$,

$$\sum_{j=1}^\infty (\mathrm{diam}(W_j))^t \le \mathcal{H}^t_\delta(\widetilde{K} \cap V_x) + \epsilon$$

and $\mathrm{diam}(W_j) \le \delta$ for each $j \ge 1$. Since $\widetilde{K}$ is compact and ($*$), it follows that $\widetilde{K} \cap V^k_{x_k} \subset \bigcup_{j=1}^\infty W_j$ for each $k \ge 1$ which is large enough, and so for each such k

$$\mathcal{H}^t_\delta(\widetilde{K} \cap V^k_{x_k}) \le \sum_{j=1}^\infty (\mathrm{diam}(W_j))^t < \mathcal{H}^t_\delta(\widetilde{K} \cap V_x) + \epsilon.$$

It follows that the function that maps (x, V) to $\mathcal{H}^t_\delta(\widetilde{K} \cap V_x)$ is upper semi-continuous, and so Borel measurable. Now since $\mathcal{H}^s = \lim_{k\to\infty} \mathcal{H}^s_{1/k}$ the lemma follows. □

Lemma 3.5. *Given $0 < t \le n$ and a Radon probability measure ν on $K \times G$, the map that takes $(x, V) \in K \times G$ to $\mathcal{P}^t(K \cap V_x)$ is ν-measurable (i.e. this map is universally measurable).*

Proof. Let $a \ge 0$ and set

$$E = \{(x, V) \in K \times G : \mathcal{P}^t(K \cap V_x) < a\};$$

then in order to prove the lemma it suffice to show that E is ν-measurable.

Set

$$Y = \{C \subset K : C \text{ is compact}\},$$

endow Y with the Hausdorff metric, and let $\mathcal{G}$ be the σ-algebra of Y which is generated by its analytic subsets. Set

$$\mathcal{E} = \{C \in Y : \mathcal{P}^t(C) < a\};$$

then by Theorem 4.2 in [12] it follows that $\mathcal{E} \in \mathcal{G}$, and so by Theorem 21.10 in [8] we get that $\mathcal{E}$ is universally measurable.

For each $(x, V) \in K \times G$ set

$$\psi(x, V) = K \cap V_x.$$

It will now be shown that

$$\psi : K \times G \longrightarrow Y$$

is a Borel function. For each $y \in K$ the function that maps $(x, V) \in K \times G$ to $d(K \cap V_x, y)$ is lower semi-continuous, and hence a Borel function. For each $l \ge 1$ let $S_l \subset K$ be finite with $K \subset \bigcup_{y\in S_l} B(y, l^{-1})$, and set

$$\psi_l(x, V) = \{y \in S_l : d(K \cap V_x, y) \le l^{-1}\}$$

for each $(x, V) \in K \times G$. It holds that

$$\psi_l : K \times G \longrightarrow Y$$

is a Borel function and

$$\psi_l \xrightarrow{l\to\infty} \psi$$

pointwise, hence ψ is a Borel function. Note also that $E = \psi^{-1}(\mathcal{E})$.

Since $\mathcal{E}$ is universally measurable it is $\nu \circ \psi^{-1}$-measurable, and so there exist $\mathcal{A}$ and $\mathcal{C}$, Borel subsets of Y, with $\mathcal{A} \subset \mathcal{E} \subset \mathcal{C}$ and $\nu \circ \psi^{-1}(\mathcal{C} \setminus \mathcal{A}) = 0$. It holds that $\psi^{-1}(\mathcal{A})$ and $\psi^{-1}(\mathcal{C})$ are Borel subsets of $K \times G$, $\psi^{-1}(\mathcal{A}) \subset E \subset \psi^{-1}(\mathcal{C})$ and $\nu(\psi^{-1}(\mathcal{C}) \setminus \psi^{-1}(\mathcal{A})) = 0$. This shows that E is ν-measurable, and the lemma is proved. □

Lemma 3.6. *For* $(x, h, V) \in K \times H \times G$ *set*

$$\psi(x, h, V) = (x, hV)$$

and let $B \subset K \times G$ *be universally measurable. Assume that for* ξ_G*-a.e.* $V \in G$ *it holds for* ξ_H*-a.e.* $h \in H$ *that*

$$\mu\{x \in K : \psi(x, h, V) \in B\} = 0;$$

then $\mu \times \xi_G(B) = 0$.

Proof. Since B is universally measurable there exist Borel sets $A, C \subset K \times G$ with $A \subset B \subset C$ and

$$\mu \times \xi_H \times \xi_G(\psi^{-1}(C \setminus A)) = 0.$$

By the assumption on B and Fubini's theorem it follows that

$$\begin{aligned} \mu \times \xi_H \times \xi_G(\psi^{-1}(C)) &= \mu \times \xi_H \times \xi_G(\psi^{-1}(A)) \\ &= \iint \mu\{x : (x, h, V) \in \psi^{-1}(A)\}\, d\xi_H(h)\, d\xi_G(V) \\ &\le \iint \mu\{x : (x, h, V) \in \psi^{-1}(B)\}\, d\xi_H(h)\, d\xi_G(V) \\ &= 0. \end{aligned}$$

Now by Fubini's theorem, by the definition of ξ_G given in (2.1), and by Lemma 3.1, it follows that

$$\begin{aligned} 0 &= \mu \times \xi_H \times \xi_G(\psi^{-1}(C)) \\ &= \iint \xi_H\{h : (x, h, V) \in \psi^{-1}(C)\}\, d\xi_G(V)\, d\mu(x) \\ &= \iint \xi_H\{h : (x, h, gU) \in \psi^{-1}(C)\}\, d\xi_O(g)\, d\mu(x) \\ &= \iint \xi_H\{h : (x, hgU) \in C\}\, d\xi_O(g)\, d\mu(x) \end{aligned}$$

$$= \iint \xi_H(\{h\colon (x, hU) \in C\} \cdot g^{-1})\, d\xi_O(g)\, d\mu(x)$$
$$= \int \xi_O\{g\colon (x, gU) \in C\}\, d\mu(x)$$
$$= \int \xi_G\{V\colon (x, V) \in C\}\, d\mu(x)$$
$$= \mu \times \xi_G(C)$$
$$\geq \mu \times \xi_G(B),$$

which completes the proof of the lemma. □

3.3. Proofs of Theorems 2.1 and 2.4. Fix $V \in \mathcal{V}$ for the remainder of this section, set

$$F = F_V,$$

and for each $h \in H$ set

$$V^h = hV \quad \text{and} \quad P_h = P_{(V^h)^\perp}.$$

Set

$$Q = \{(x, h) \in X\colon F(x, h) \neq \Theta^{*m}(P_h\mu, P_h(x)) \text{ or } F(x, h) = \infty \text{ or } F(x, h) = 0\}$$

where Θ^{*m} is as defined in (2.2); then Q is a Borel set. By Theorem 2.12 in [10] it follows that

$$\mu\{x \in K\colon (x, h) \in Q\} = 0 \quad \text{for each } h \in H \text{ with } P_h\mu \ll \mathcal{H}^m,$$

hence since $V \in \mathcal{V}$ we have

$$\mu \times \xi_H(Q) = \int_H \mu\{x\colon (x, h) \in Q\}\, d\xi_H(h) = 0. \tag{3.2}$$

Let D be the set of all $(x, h) \in X$ such that $P_h\mu \ll \mathcal{H}^m$, $\mu_{V^h,x}$ is defined,

$$\mu_{V^h,x}(K_w) = \lim_{\epsilon \downarrow 0} \frac{\mu(K_w \cap P_h^{-1}(B(P_h x, \epsilon)))}{P_h\mu(B(P_h x, \epsilon))} \quad \text{for each } w \in \Lambda^*,$$

and

$$0 < F(x, h) = \lim_{\epsilon \downarrow 0} \frac{P_h\mu(B(P_h(x), \epsilon))}{(2\epsilon)^m} < \infty.$$

By the choice of V, by Lemma 3.3 and by equation (3.2), it follows that

$$\mu \times \xi_H(X \setminus D) = 0.$$

Set

$$D_0 = \bigcap_{j=0}^{\infty} T^{-j} D;$$

then

$$\mu \times \xi_H(X \setminus D_0) = 0$$

since T is measure preserving. The following key lemma will be used several times below.

Lemma 3.7. *Given $k \geq 1$ and $(x,h) \in D_0$, we have*

$$\mu_{V^h,x}(K_{w_k(x)}) = (F(x,h))^{-1} \cdot r_{w_k(x)}^{s-m} \cdot F(T^k(x,h)).$$

Proof. Set $u = w_k(x)$. Then

$$\begin{aligned}
\mu_{V^h,x}(K_u) &= \lim_{\epsilon \downarrow 0} \frac{\mu(K_u \cap P_h^{-1}(B(P_h x, \epsilon)))}{P_h\mu(B(P_h x, \epsilon))} \\
&= \lim_{\epsilon \downarrow 0} \frac{(2\epsilon)^m}{P_h\mu(B(P_h x, \epsilon))} \cdot \frac{\mu(K_u \cap P_h^{-1}(B(P_h x, \epsilon)))}{(2\epsilon)^m} \\
&= (F(x,h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(K_u \cap P_h^{-1}(B(P_h x, \epsilon)))}{(2\epsilon)^m}.
\end{aligned}$$

For each $\epsilon > 0$ set

$$E_\epsilon = P_{h_u^{-1}h}^{-1}(B(P_{h_u^{-1}h}(\varphi_u^{-1}(x)), \epsilon \cdot r_u^{-1}));$$

then since

$$\begin{aligned}
P_h^{-1}(B(P_h x, \epsilon)) &= x + V^h + B(0, \epsilon) \\
&= \varphi_u \circ \varphi_u^{-1}(x + V^h + B(0, \epsilon)) \\
&= \varphi_u(\varphi_u^{-1}(x) + V^{h_u^{-1}h} + B(0, \epsilon \cdot r_u^{-1})) \\
&= \varphi_u(E_\epsilon),
\end{aligned}$$

it follows that

$$\begin{aligned}
\mu_{V^h,x}(K_u) &= (F(x,h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(\varphi_u(K \cap E_\epsilon))}{(2\epsilon)^m} \\
&= (F(x,h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{1}{(2\epsilon)^m} \sum_{w \in \Lambda^k} p_w \cdot \mu(\varphi_w^{-1}(\varphi_u(K \cap E_\epsilon))).
\end{aligned}$$

Given $w \in \Lambda^k \setminus \{u\}$ we have

$$\varphi_u(K) \cap \varphi_w(K) = \emptyset,$$

so

$$\varphi_w^{-1}(\varphi_u(K)) \cap K = \emptyset,$$

and so

$$\begin{aligned} \mu_{V^h,x}(K_u) &= (F(x,h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{p_u \cdot \mu(K \cap E_\epsilon)}{(2\epsilon)^m} \\ &= (F(x,h))^{-1} \cdot r_u^{s-m} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(E_\epsilon)}{(2\epsilon \cdot r_u^{-1})^m} \\ &= (F(x,h))^{-1} \cdot r_u^{s-m} \cdot F(\varphi_u^{-1}(x), h_u^{-1}h) \\ &= (F(x,h))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x,h)), \end{aligned}$$

which proves the lemma. □

Proof of Theorem 2.1 (i). Assume that V is such that $\|F\|_{L^\infty(\mu\times\xi_H)} < \infty$. Set

$$M = \|F\|_{L^\infty(\mu\times\xi_H)},$$

$$E = \{(x,h) : F(x,h) \le M\}$$

and

$$E_1 = D_0 \cap \Big(\bigcap_{j=0}^{\infty} T^{-j}(E)\Big);$$

then

$$\mu \times \xi_H(X \setminus E_1) = 0.$$

For ξ_H-a.e. $h \in H$ we have

$$\mu\{x \in K : (x,h) \notin E_1\} = 0,$$

fix such $h_0 \in H$. For each $l \ge 1$ set

$$A_l = \{x \in K : (x,h_0) \in E_1 \text{ and } F(x,h_0) \ge l^{-1}\},$$

and fix $l_0 \ge 1$. Setting $\kappa = \min\{r_\lambda : \lambda \in \Lambda\}$, it will now be shown that

$$\Theta^{*s-m}(\mu_{V^{h_0},x}, x) \le (2\rho\kappa)^{m-s} l_0 M \quad \text{for each } x \in A_{l_0}, \tag{3.3}$$

where ρ is as defined in (3.1). Let $x \in A_{l_0}$ and let $\kappa\rho > \delta > 0$. Let $k \geq 1$ be such that

$$r_{w_k(x)} \geq \frac{\delta}{\rho} > r_{w_{k+1}(x)},$$

and set

$$u = w_k(x).$$

By Lemma 3.7 and by $T^k(x, h_0) \in E$ we get that

$$\mu_{V^{h_0},x}(K_u) = (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \leq l_0 \cdot r_u^{s-m} \cdot M,$$

and so

$$\begin{aligned}\frac{\mu_{V^{h_0},x}(B(x,\delta))}{(2\delta)^{s-m}} &\leq \frac{\mu_{V^{h_0},x}(B(x,\rho \cdot r_{w_k(x)}))}{(2\rho \cdot r_{w_{k+1}(x)})^{s-m}} \\ &\leq \frac{\mu_{V^{h_0},x}(K_u)}{(2\rho\kappa \cdot r_u)^{s-m}} \\ &\leq \frac{l_0 r_u^{s-m} M}{(2\rho\kappa \cdot r_u)^{s-m}} \\ &= (2\rho\kappa)^{m-s} l_0 M,\end{aligned}$$

which proves (3.3).

It holds that

$$\{x \in K \colon (x, h_0) \in E_1\} = \bigcup_{l=1}^{\infty} A_l,$$

hence

$$0 = \mu(K \setminus \bigcup_{l=1}^{\infty} A_l) = \int \mu_{V^{h_0},x}\Big(K \setminus \bigcup_{l=1}^{\infty} A_l\Big)\, d\mu(x),$$

and so for μ-a.e. $x \in K$ there exist $l_x \geq 1$ with

$$\mu_{V^{h_0},x}(A_{l_x} \cap V_x^{h_0}) = \mu_{V^{h_0},x}(A_{l_x}) > 0.$$

Fix such $x_0 \in K$ and let $y \in A_{l_{x_0}} \cap V_{x_0}^{h_0}$; then by (3.3) we get that

$$\Theta^{*s-m}(\mu_{V^{h_0},x_0}, y) = \Theta^{*s-m}(\mu_{V^{h_0},y}, y) \leq (2\rho\kappa)^{m-s} l_{x_0} M,$$

and so by Theorem 6.9 in [10] it follows that

$$\begin{aligned}\mathcal{H}^{s-m}(K \cap V_{x_0}^{h_0}) &\geq \mathcal{H}^{s-m}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) \\ &\geq 2^{-(s-m)}(2\rho\kappa)^{s-m} l_{x_0}^{-1} M^{-1} \cdot \mu_{V^{h_0},x_0}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) \\ &> 0.\end{aligned}$$

This proves that if $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$, then for ξ_H-a.e. $h \in H$ we have

$$\mathcal{H}^{s-m}(K \cap (x + hV)) > 0 \quad \text{for } \mu\text{-a.e. } x \in K,$$

and so (i) follows by Lemma 3.4 and Fubini's theorem. □

Proof of Theorem 2.1 (ii). Assume that V is such that

$$\|F\|_{L^\infty(\mu\times\xi_H)} = \infty;$$

then

$$\mu \times \xi_H\{(x,h)\colon F(x,h) > M\} > 0 \quad \text{for each } 0 < M < \infty.$$

For each integer $M \geq 1$ set

$$E_M = \{(x,h) \in X\colon F(x,h) > M\}$$

and

$$E_{0,M} = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} T^{-j}(E_M);$$

then

$$\mu \times \xi_H(E_M) > 0,$$

and so

$$\mu \times \xi_H(X \setminus E_{0,M}) = 0$$

since $\mu \times \xi_H$ is ergodic (see Theorem 1.5 in [17]). Set

$$\widetilde{E} = D_0 \cap \Big(\bigcap_{M=1}^{\infty} E_{0,M}\Big);$$

then

$$\mu \times \xi_H(X \setminus \widetilde{E}) = 0.$$

For ξ_H-a.e. $h \in H$ it holds that

$$\mu\{x \in K\colon (x,h) \notin \widetilde{E}\} = 0,$$

fix such $h_0 \in H$ and set

$$A = \{x \in K\colon (x,h_0) \in \widetilde{E}\}.$$

Note that since $(x, h_0) \in D_0$ for some $x \in K$, it follows that $P_{h_0}\mu \ll \mathcal{H}^m$. It will now be shown that

$$\Theta^{*s-m}(\mu_{V^{h_0},x}, x) = \infty \quad \text{for each } x \in A. \tag{3.4}$$

Let $x \in A$, $M \geq 1$ and $N \geq 1$ be given; then there exists $k \geq N$ with $T^k(x, h_0) \in D_0 \cap E_M$, and so $F(T^k(x, h_0)) > M$. Set

$$u = w_k(x)$$

and

$$\beta = (F(x, h_0))^{-1};$$

then by Lemma 3.7

$$\begin{aligned} \mu_{V^{h_0},x}(K_u) &= \beta \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \\ &\geq \beta \cdot r_u^{s-m} \cdot M. \end{aligned}$$

Set

$$d = \sup\{|y_1 - y_2| : y_1, y_2 \in K\}.$$

Then

$$\begin{aligned} \frac{\mu_{V^{h_0},x}(B(x, d \cdot r_{w_k(x)}))}{(2d \cdot r_{w_k(x)})^{s-m}} &\geq \frac{\mu_{V^{h_0},x}(K_u)}{(2d \cdot r_u)^{s-m}} \\ &\geq \frac{\beta \cdot r_u^{s-m} \cdot M}{(2d \cdot r_u)^{s-m}} \\ &= \frac{M\beta}{(2d)^{s-m}}. \end{aligned}$$

Since $\lim_{k\to\infty} r_{w_k(x)} = 0$ we get that

$$\Theta^{*s-m}(\mu_{V^{h_0},x}, x) \geq \frac{M\beta}{(2d)^{s-m}},$$

and so (3.4) follows since M can be chosen arbitrarily large.

Let $x \in A$ and $y \in A \cap V_x^{h_0}$; then by (3.4) we get

$$\Theta^{*s-m}(\mu_{V^{h_0},x}, y) = \Theta^{*s-m}(\mu_{V^{h_0},y}, y) = \infty.$$

Now by Theorem 6.9 in [10] it follows that for each $M \geq 1$

$$\mathcal{H}^{s-m}(A \cap V_x^{h_0}) \leq M^{-1} \cdot \mu_{V^{h_0},x}(A \cap V_x^{h_0}) \leq M^{-1},$$

and so

$$\mathcal{H}^{s-m}(A \cap V_x^{h_0}) = 0$$

since M can be chosen arbitrarily large. Also, since $\mu(K \setminus A) = 0$, by Theorem 7.7 in [10] we get that

$$\begin{aligned} \int_{(V^{h_0})^\perp} & \mathcal{H}^{s-m}((K \setminus A) \cap V_y^{h_0}) \, d\mathcal{H}^m(y) \\ & \leq \text{const} \cdot \mathcal{H}^s(K \setminus A) \\ & = \text{const} \cdot \mu(K \setminus A) \\ & = 0. \end{aligned}$$

This shows that

$$\mathcal{H}^{s-m}((K \setminus A) \cap V_y^{h_0}) = 0 \quad \text{for } \mathcal{H}^m\text{-a.e. } y \in (V^{h_0})^\perp,$$

and so

$$\mathcal{H}^{s-m}((K \setminus A) \cap V_x^{h_0}) = 0 \quad \text{for } \mu\text{-a.e. } x \in K \text{ since } P_{h_0}\mu \ll \mathcal{H}^m.$$

It follows that for μ-a.e. $x \in A$ (and so for μ-a.e. $x \in K$) we have

$$\mathcal{H}^{s-m}(K \cap V_x^{h_0}) = \mathcal{H}^{s-m}(A \cap V_x^{h_0}) + \mathcal{H}^{s-m}((K \setminus A) \cap V_x^{h_0}) = 0\,.$$

From Lemma (3.4) and Fubini's theorem, it follows that

$$\mathcal{H}^{s-m}(K \cap V_x^h) = 0 \quad \text{for } \mu \times \xi_H\text{-a.e. } (x, h) \in K \times H,$$

which proves (ii). □

Proof of Theorem 2.1 (iii). Assume that $\|F_V\|_\infty < \infty$ for ξ_G-a.e. $V \in G$. By Lemma 3.2 and part (i), it follows that for ξ_G-a.e. $V \in G$ it holds for ξ_H-a.e. $h \in H$ that

$$\mathcal{H}^{s-m}(K \cap (x + hV)) > 0 \quad \text{for } \mu\text{-a.e. } x \in K\,.$$

Set

$$B = \{(x, V) \in K \times G : \mathcal{H}^{s-m}(K \cap V_x) = 0\};$$

then by Lemma 3.4 we get that B is a Borel set (hence universally measurable), and so $\mu \times \xi_G(B) = 0$ by Lemma 3.6.

For the other direction, set

$$\mathcal{W} = \{V \in G \colon \|F_V\|_\infty = \infty\}$$

and assume that

$$\xi_G(\mathcal{W}) > 0.$$

By part (ii) it follows that for ξ_G-a.e. $V \in \mathcal{W}$ we have

$$\mathcal{H}^{s-m}(K \cap (x + hV)) = 0 \quad \text{for } \mu \times \xi_H\text{-a.e. } (x,h) \in X,$$

and so by Lemma 3.1

$$\begin{aligned}
0 &< \xi_G(\mathcal{W}) \\
&\le \int \mu \times \xi_H\{(x,h) \colon \mathcal{H}^{s-m}(K \cap (x + hV)) = 0\}\, d\xi_G(V) \\
&= \iint \xi_H\{h \colon \mathcal{H}^{s-m}(K \cap (x + hgU)) = 0\}\, d\xi_O(g)\, d\mu(x) \\
&= \iint \xi_H(\{h \colon \mathcal{H}^{s-m}(K \cap (x + hU)) = 0\} \cdot g^{-1})\, d\xi_O(g)\, d\mu(x) \\
&= \int \xi_O\{g \colon \mathcal{H}^{s-m}(K \cap (x + gU)) = 0\}\, d\mu(x) \\
&= \int \xi_G\{V \colon \mathcal{H}^{s-m}(K \cap V_x) = 0\}\, d\mu(x) \\
&= \mu \times \xi_G\{(x,V) \colon \mathcal{H}^{s-m}(K \cap V_x) = 0\},
\end{aligned}$$

which completes the proof of (iii). □

Part (iv) can be proven in a similar manner, so the proof of Theorem 2.1 is complete.

Proof of Theorem 2.4 (i). Let $M > 0$ be so large such that for

$$E = \{(x,h) \in X \colon F(x,h) \le M\}$$

we have

$$\mu \times \xi_H(E) > 0.$$

Set

$$E_0 = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} T^{-j}(E);$$

then

$$\mu \times \xi_H(X \setminus E_0) = 0$$

since $\mu \times \xi_H$ is ergodic. Set

$$E_1 = E_0 \cap D_0;$$

then

$$\mu \times \xi_H(X \setminus E_1) = 0.$$

For ξ_H-a.e. $h \in H$ it holds that

$$\mu\{x \in K : (x,h) \notin E_1\} = 0.$$

Fix such $h_0 \in H$. For each $l \geq 1$ set

$$A_l = \{x \in K : (x,h_0) \in E_1 \text{ and } F(x,h_0) \geq l^{-1}\},$$

and fix $l_0 \geq 1$. It will now be shown that

$$\Theta_*^{s-m}(\mu_{V^{h_0},x}, x) \leq (2\rho)^{m-s} l_0 M \qquad \text{for each } x \in A_{l_0}. \tag{3.5}$$

Let $x \in A_{l_0}$ and let $N \geq 1$ be given. Then since $(x,h_0) \in E_1$ it follows that there exist $k \geq N$ with $T^k(x,h_0) \in E \cap D_0$, and so $F(T^k(x,h_0)) \leq M$. Set

$$u = w_k(x);$$

then by Lemma 3.7 we have

$$\begin{aligned} \mu_{V^{h_0},x}(K_u) &= (F(x,h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x,h_0)) \\ &\leq l_0 r_u^{s-m} M. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\mu_{V^{h_0},x}(B(x,\rho \cdot r_{w_k(x)}))}{(2\rho \cdot r_{w_k(x)})^{s-m}} &\leq \frac{\mu_{V^{h_0},x}(K_u)}{(2\rho \cdot r_u)^{s-m}} \\ &\leq \frac{l_0 r_u^{s-m} M}{(2\rho \cdot r_u)^{s-m}} \\ &= (2\rho)^{m-s} l_0 M. \end{aligned}$$

This proves (3.5) since $r_{w_k(x)}$ tends to 0 as k tends to ∞.

As in the proof of part (i) of Theorem 2.1, since

$$\mu\Big(K \setminus \bigcup_{l=1}^{\infty} A_l\Big) = 0,$$

it follows that for μ-a.e. $x \in K$ there exists $l_x \geq 1$ with

$$\mu_{V^{h_0},x}(A_{l_x} \cap V_x^{h_0}) > 0.$$

Fix such an x_0 and let $y \in A_{l_{x_0}} \cap V_{x_0}^{h_0}$. Then by (3.5) we get

$$\Theta_*^{s-m}(\mu_{V^{h_0},x_0}, y) = \Theta_*^{s-m}(\mu_{V^{h_0},y}, y) \leq (2\rho)^{m-s} l_{x_0} M,$$

and so by Theorem 6.11 in [10] it follows that

$$\begin{aligned}\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) &\geq \mathcal{P}^{s-m}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) \\ &\geq (2\rho)^{s-m} l_{x_0}^{-1} M^{-1} \cdot \mu_{V^{h_0},x_0}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) \\ &> 0.\end{aligned}$$

Since $\xi_G(G \setminus \mathcal{V}) = 0$, this shows that for ξ_G-a.e. $V \in G$ it holds for ξ_H-a.e. $h \in H$ that

$$\mathcal{P}^{s-m}(K \cap (x + hV)) > 0 \quad \text{for } \mu\text{-a.e. } x \in K.$$

Set

$$B = \{(x, V) \in K \times G : \mathcal{P}^{s-m}(K \cap V_x) = 0\};$$

then by Lemma 3.5 we get that B is universally measurable, and so the claim stated in (i) follows by Lemma 3.6. □

Proof of Theorem 2.4 (ii). Assume V is such that

$$\Big\| \frac{1}{F} \Big\|_{L^\infty(\mu \times \xi_H)} = \infty.$$

Then

$$\mu \times \xi_H\{(x, h) : F(x, h) < M^{-1}\} > 0 \quad \text{for each } 0 < M < \infty.$$

For each integer $M \geq 1$ set

$$E_M = \{(x, h) : F(x, h) < M^{-1}\}$$

and

$$E_{0,M} = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} T^{-j}(E_M);$$

then since $\mu \times \xi_H$ is ergodic and $\mu \times \xi_H(E_M) > 0$ it follows that

$$\mu \times \xi_H(X \setminus E_{0,M}) = 0.$$

Set

$$\widetilde{E} = D_0 \cap \Big(\bigcap_{M=1}^{\infty} E_{0,M} \Big);$$

then

$$\mu \times \xi_H(X \setminus \widetilde{E}) = 0.$$

For ξ_H-a.e. $h \in H$ it holds that

$$\mu\{x \in K : (x, h) \notin \widetilde{E}\} = 0,$$

fix such $h_0 \in H$ and set

$$A = \{x \in K : (x, h_0) \in \widetilde{E})\}.$$

It will now be shown that

$$\Theta_*^{s-m}(\mu_{V^{h_0},x}, x) = 0 \text{ for each } x \in A\,. \tag{3.6}$$

Let $x \in A$, $M \geq 1$ and $N \geq 1$ be given. Then there exists $k \geq N$ with $T^k(x, h_0) \in D_0 \cap E_M$, and so $F(T^k(x, h_0)) < M^{-1}$. Set

$$u = w_k(x).$$

Then by Lemma 3.7

$$\begin{aligned} \mu_{V^{h_0},x}(K_u) &= (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \\ &\leq (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot M^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{\mu_{V^{h_0},x}(B(x, \rho \cdot r_{w_k(x)}))}{(2\rho \cdot r_{w_k(x)})^{s-m}} \\ &\leq \frac{\mu_{V^{h_0},x}(K_u)}{(2\rho \cdot r_u)^{s-m}} \\ &\leq \frac{(F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot M^{-1}}{(2\rho \cdot r_u)^{s-m}} \\ &= (2\rho)^{m-s} \cdot (F(x, h_0))^{-1} \cdot M^{-1}. \end{aligned}$$

This shows that

$$\Theta_*^{s-m}(\mu_{V^{h_0},x}, x) \leq (2\rho)^{m-s} \cdot (F(x,h_0))^{-1} \cdot M^{-1},$$

and so (3.6) holds since M can be chosen arbitrarily large.

We have

$$0 = \mu(K \setminus A) = \int \mu_{V^{h_0},x}(K \setminus A)\, d\mu(x),$$

hence $\mu_{V^{h_0},x}(A \cap V_x^{h_0}) > 0$ for μ-a.e. $x \in K$. Fix such $x_0 \in K$ and let $y \in A \cap V_{x_0}^{h_0}$. Then by (3.6) we get

$$\Theta_*^{s-m}(\mu_{V^{h_0},x_0}, y) = \Theta_*^{s-m}(\mu_{V^{h_0},y}, y) = 0\,.$$

Now by Theorem 6.11 in [10] it follows that for each $\epsilon > 0$

$$\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) \geq \mathcal{P}^{s-m}(A \cap V_{x_0}^{h_0}) \geq \epsilon^{-1} \cdot \mu_{V^{h_0},x_0}(A \cap V_{x_0}^{h_0}),$$

which shows that

$$\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) = \infty$$

since ϵ can be chosen arbitrarily small and $\mu_{V^{h_0},x_0}(A \cap V_{x_0}^{h_0}) > 0$.

This proves that if $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$, then for ξ_H-a.e. $h \in H$ we have

$$\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty \quad \text{for } \mu\text{-a.e. } x \in K,$$

and so (ii) follows by Lemma 3.5 and Fubini's theorem. □

Proof of Theorem 2.4 (iii). Assume that

$$\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty \quad \text{for } \xi_G\text{-a.e. } V \in G;$$

then by Lemma 3.2 and part (ii) it follows that for ξ_G-a.e. $V \in G$ it holds for ξ_H-a.e. $h \in H$ that

$$\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty \quad \text{for } \mu\text{-a.e. } x \in K.$$

Set

$$B = \{(x, V) \in K \times G : \mathcal{P}^{s-m}(K \cap V_x) < \infty\};$$

then by Lemma 3.5 we get that B is universally measurable, and so the claim stated in (iii) follows by Lemma 3.6. □

This completes the proof of Theorem 2.4.

3.4. Proofs of Corollaries 2.2, 2.3 and 2.5. The following lemma will be used in the proofs of Corollaries 2.2 and 2.5.

Lemma 3.8. *Assume $s > 2m$. Then $P_{V^\perp}\mu \ll \mathcal{H}^m$ and $\frac{dP_{V^\perp}\mu}{d\mathcal{H}^m}$ has a continuous version for ξ_G-a.e. $V \in G$.*

Proof. It is proven in [4] and independently in [16] that if ν is a compactly supported Radon measure on $\mathbb{R}^n$ with $I_{2m}(\nu) < \infty$, then $P_{V^\perp}\nu \ll \mathcal{H}^m$ and $\frac{dP_{V^\perp}\nu}{d\mathcal{H}^m}$ has a continuous version for ξ_G-a.e. $V \in G$. By the assumption $s > 2m$ we get $I_{2m}(\mu) < \infty$ (see the proof of Lemma 3.2), hence the lemma follows. □

Proof of corollary 2.2. Assuming $s > 2m$ and $|H| < \infty$, it will be shown that $\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty$ for ξ_G-a.e. $V \in G$. By Theorem 2.1 (iii) the corollary will follow. Set

$$E = \left\{V \in G : P_{V^\perp}\mu \ll \mathcal{H}^m \text{ and } \frac{dP_{V^\perp}\mu}{d\mathcal{H}^m} \text{ is continuous}\right\};$$

then by Lemma 3.8 we get $\xi_G(G \setminus E) = 0$. By Lemma 3.1 it now follows that

$$\begin{aligned} 0 &= \xi_G(G \setminus E) \\ &= \xi_O\{g \in O(n) : gU \notin E\} \\ &= \int \xi_H\{h : hgU \notin E\}\, d\xi_O(g) \\ &= \int \xi_H\{h : hV \notin E\}\, d\xi_G(V), \end{aligned}$$

and so

$$\xi_H\{h : hV \notin E\} = 0 \quad \text{for } \xi_G\text{-a.e. } V.$$

Fix such a $V \in G$. Then since $|H| < \infty$ we have

$$\xi_H\{h\} > 0 \quad \text{for each } h \in H,$$

and so

$$hV \in E \qquad \text{for every } h \in H.$$

For each $h \in H$ and $y \in (hV)^{\perp}$ set

$$Q_h(y) = \Theta_*^m(P_{(hV)^{\perp}}\mu, y),$$

fix $h_0 \in H$, and set

$$W = (h_0 V)^{\perp}.$$

Since $\mathcal{H}^m(B(y,\epsilon) \cap W) = (2\epsilon)^m$ for each $y \in W$ and $0 < \epsilon < \infty$, it follows by Theorem 2.12 in [10] that

$$Q_{h_0}(y) = \frac{dP_W\mu}{d\mathcal{H}^m}(y) \quad \text{for } \mathcal{H}^m\text{-a.e. } y \in W,$$

i.e. the function Q_{h_0} equals a continuous function as members of $L^1(W, \mathcal{H}^m)$. Also, since μ is supported on a compact set it follows that the set

$$\{y \in W : Q_{h_0}(y) \neq 0\}$$

is bounded, so Q_{h_0} equals a continuous function with compact support in $L^1(W, \mathcal{H}^m)$, which shows that

$$\|Q_{h_0}\|_{L^\infty(W,\mathcal{H}^m)} < \infty.$$

Since $P_W\mu \ll \mathcal{H}^m$ it follows that

$$\|Q_{h_0}\|_{L^\infty(P_W\mu)} < \infty.$$

Now set

$$M = \max\{\|Q_h\|_{L^\infty(P_{(hV)^{\perp}}\mu)} : h \in H\}.$$

Then $M < \infty$ since $|H| < \infty$. Also we have

$$\begin{aligned}
0 &= \frac{1}{|H|}\sum_{h\in H} P_{(hV)^{\perp}}\mu\{y \in (hV)^{\perp} : |Q_h(y)| > M\} \\
&= \frac{1}{|H|}\sum_{h\in H} \mu\{x \in K : |Q_h(P_{(hV)^{\perp}}(x))| > M\} \\
&= \frac{1}{|H|}\sum_{h\in H} \mu\{x \in K : |F_V(x,h)| > M\} \\
&= \int \mu\{x \in K : |F_V(x,h)| > M\}\, d\xi_H(h) \\
&= \mu \times \xi_H\{(x,h) \in K \times H : |F_V(x,h)| > M\},
\end{aligned}$$

which shows that

$$\|F_V\|_{L^\infty(\mu\times\xi_H)} \le M < \infty.$$

This completes the proof of corollary 2.2. □

Proof of corollary 2.3. Assume that $H = O(n)$ and

$$\mu \times \xi_G\{(x, V) : \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0 .$$

Let $V \in \mathcal{V}$. Then since $\xi_H = \xi_O$ we have

$$\mu \times \xi_H\{(x, h) : \mathcal{H}^{s-m}(K \cap (x + hV)) > 0\} > 0 ,$$

and so by Theorem 2.1 (ii) it follows that

$$\|F_V\|_{L^\infty(\mu\times\xi_H)} < \infty.$$

Set

$$M = \|F_V\|_{L^\infty(\mu\times\xi_H)}$$

and

$$E = \left\{W \in G : P_{W^\perp}\mu \ll \mathcal{H}^m \text{ and } \left\|\frac{dP_{W^\perp}\mu}{d\mathcal{H}^m}\right\|_{L^\infty(\mathcal{H}^m)} \le M\right\},$$

and for each $h \in H$ set

$$P_h = P_{(hV)^\perp}.$$

We shall first show that

$$\xi_G(G \setminus E) = 0.$$

Since $P_{W^\perp}\mu \ll \mathcal{H}^m$ for ξ_G-a.e. $W \in G$ (see the proof of Lemma 3.2), and since $\xi_H = \xi_O$, we have

$$\begin{aligned}\xi_G(G \setminus E) &= \xi_G(G \setminus \{W \in G : P_{W^\perp}\mu \ll \mathcal{H}^m\}) \\ &\quad + \xi_G\left\{W \in G : P_{W^\perp}\mu \ll \mathcal{H}^m \text{ and } \left\|\frac{dP_{W^\perp}\mu}{d\mathcal{H}^m}\right\|_{L^\infty(\mathcal{H}^m)} > M\right\} \\ &= \xi_H\left\{h : P_h\mu \ll \mathcal{H}^m \text{ and } \left\|\frac{dP_h\mu}{d\mathcal{H}^m}\right\|_{L^\infty(\mathcal{H}^m)} > M\right\}.\end{aligned} \tag{3.7}$$

Let $h \in H$ be such that
$$P_h\mu \ll \mathcal{H}^m$$
and
$$\left\| \frac{dP_h\mu}{d\mathcal{H}^m} \right\|_{L^\infty(P_h\mu)} \le M.$$
Then
$$\begin{aligned} 0 &= P_h\mu\Big\{y \in (hV)^\perp : \frac{dP_h\mu}{d\mathcal{H}^m}(y) > M\Big\} \\ &= \int_{(hV)^\perp} 1_{\left\{\frac{dP_h\mu}{d\mathcal{H}^m} > M\right\}} \cdot \frac{dP_h\mu}{d\mathcal{H}^m}\, d\mathcal{H}^m \\ &\ge M \cdot \mathcal{H}^m\Big\{y \in (hV)^\perp : \frac{dP_h\mu}{d\mathcal{H}^m}(y) > M\Big\}, \end{aligned}$$
which shows that
$$\left\| \frac{dP_h\mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} \le M.$$
By (3.7) it follows that
$$\xi_G(G \setminus E) = \xi_H \left\{ h : P_h\mu \ll \mathcal{H}^m \text{ and } \left\| \frac{dP_h\mu}{d\mathcal{H}^m} \right\|_{L^\infty(P_h\mu)} > M \right\}. \tag{3.8}$$
By Theorem 2.12 in [10] we get that for each $h \in H$ with $P_h\mu \ll \mathcal{H}^m$
$$F_V(x,h) = \frac{dP_h\mu}{d\mathcal{H}^m}(P_h(x)) \quad \text{for } \mu\text{-a.e. } x \in K,$$
and so by (3.8)
$$\begin{aligned} \xi_G(G \setminus E) &\le \xi_H\{h : \|F_V(\cdot,h)\|_{L^\infty(\mu)} > M\} \\ &= \xi_H\{h : \mu\{x : F_V(x,h) > \|F_V\|_{L^\infty(\mu\times\xi_H)}\} > 0\} \\ &= 0. \end{aligned}$$

Since $\xi_G(\mathcal{W}) > 0$ for every non-empty open set $\mathcal{W} \subset G$ and since $\xi_G(G \setminus E) = 0$, it follows that E is dense in G, and so in order to prove the corollary it suffices to show that E is a closed subset of G. Let $W_0 \in \overline{E}$, let $y \in W_0^\perp$ and let $r \in (0,\infty)$. Given $\epsilon > 0$ there exists $W \in E$ so close to W_0 in G (with respect to the metric d_G defined in Section 3.1) such that
$$P_{W_0^\perp}^{-1}(B(y,r)) \cap K \subset P_{W^\perp}^{-1}(B(P_{W^\perp}y, r+\epsilon)).$$

Since $W \in E$, it follows that

$$\begin{aligned} P_{W_0^\perp}\mu(B(y,r)) &= \mu(P_{W_0^\perp}^{-1}(B(y,r)) \cap K) \\ &\le \mu(P_{W^\perp}^{-1}(B(P_{W^\perp}y, r+\epsilon))) \\ &= P_{W^\perp}\mu((B(P_{W^\perp}y, r+\epsilon))) \\ &= \int_{B(P_{W^\perp}y,r+\epsilon)\cap W^\perp} \frac{dP_{W^\perp}\mu}{d\mathcal{H}^m}\, d\mathcal{H}^m \\ &\le M \cdot \mathcal{H}^m(B(P_{W^\perp}y, r+\epsilon) \cap W^\perp) \\ &= M \cdot (2 \cdot (r+\epsilon))^m, \end{aligned}$$

and since this holds for each $\epsilon > 0$ we have

$$P_{W_0^\perp}\mu(B(y,r) \cap W_0^\perp) \le M \cdot (2r)^m = M \cdot \mathcal{H}^m(B(y,r) \cap W_0^\perp)\,.$$

This holds for every $y \in W_0^\perp$ and $r \in (0,\infty)$, hence $W_0 \in E$ by Theorem 2.12 in [10], which shows that E is closed in G and completes the proof of the corollary. □

Proof of corollary 2.5. Assuming $s > 2m$ it will be shown that

$$\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu\times\xi_H)} = \infty \quad \text{for } \xi_G\text{-a.e. } V \in G.$$

By Theorem 2.4 (iii) the corollary will follow. Set

$$E = \left\{ V \in G : \; P_{V^\perp}\mu \ll \mathcal{H}^m \text{ and } \frac{dP_{V^\perp}\mu}{d\mathcal{H}^m} \text{ is continuous} \right\};$$

then as in the proof of corollary 2.2 it follows by Lemma 3.8 and Lemma 3.1 that

$$0 = \xi_G(G \setminus E) = \int \xi_H\{h : hV \notin E\}\, d\xi_G(V),$$

and so

$$\xi_H\{h : hV \notin E\} = 0 \quad \text{for } \xi_G\text{-a.e. } V.$$

Fix such $V \in G$, let $M > 0$, set

$$A = \{h \in H : hV \in E\},$$

and for each $h \in H$ and $y \in (hV)^\perp$ set

$$Q_h(y) = \Theta_*^m(P_{(hV)^\perp}\mu, y)$$

and

$$L_h = \{y \in (hV)^\perp \colon 0 < Q_h(y) \le M^{-1}\}.$$

Fix $h_0 \in A$ and set

$$W = (h_0 V)^\perp.$$

By Theorem 2.12 in [10] it follows that

$$Q_{h_0}(y) = \frac{dP_W\mu}{d\mathcal{H}^m}(y) \quad \text{for } \mathcal{H}^m\text{-a.e. } y \in W,$$

hence the function Q_{h_0} equals a continuous function in $L^1(W, \mathcal{H}^m)$. Also, since μ is supported on a compact set, it follows that the set

$$\{y \in W \colon Q_{h_0}(y) \neq 0\}$$

is bounded. By these two facts it easily follows that

$$\mathcal{H}^m(L_{h_0}) > 0,$$

and so

$$P_W\mu(L_{h_0}) > 0$$

since $Q_{h_0} = \frac{dP_W\mu}{d\mathcal{H}^m}$ and $Q_{h_0} > 0$ on L_{h_0}. We get that

$$\begin{aligned} 0 &< \mu\{x \in K \colon Q_{h_0}(P_W(x)) \le M^{-1}\} \\ &= \mu\{x \in K \colon F_V(x, h_0) \le M^{-1}\}, \end{aligned}$$

and so by Fubini's theorem

$$\begin{aligned} \mu \times \xi_H\left\{(x,h) \colon \frac{1}{F_V(x,h)} \ge M\right\} &= \int_A \mu\{x \in K \colon F_V(x,h) \le M^{-1}\}\, d\xi_H(h) \\ &> 0. \end{aligned}$$

It follows that $\|\frac{1}{F_V}\|_{L^\infty(\mu\times\xi_H)} \ge M$, and so $\|\frac{1}{F_V}\|_{L^\infty(\mu\times\xi_H)} = \infty$ since we can choose M as large as we want. This completes the proof of the corollary. $\square$

4. Proof of the result regarding self-affine sets

Set

$$\Lambda = \{1, 2\}.$$

Given $0 < \rho < \frac{1}{2}$ define

$$f_{\rho,1}, f_{\rho,2} \colon \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$f_{\rho,1}(x) = \rho \cdot x$$

and

$$f_{\rho,2}(x) = \rho \cdot x + 1 - \rho$$

for each $x \in \mathbb{R}$, let $C_\rho \subset [0, 1]$ be the attractor of the IFS $\{f_{\rho,1}, f_{\rho,2}\}$, set

$$d_\rho = \dim_H C_\rho,$$

so that $d_\rho = \frac{\log 2}{\log \rho^{-1}}$, and for each $E \subset \mathbb{R}$ set

$$\mu_\rho(E) = \frac{\mathcal{H}^{d_\rho}(C_\rho \cap E)}{\mathcal{H}^{d_\rho}(C_\rho)}.$$

Let $0 < a < b < \frac{1}{2}$ be such that $\frac{1}{a}$ and $\frac{1}{b}$ are Pisot numbers, $\frac{\log b}{\log a}$ is irrational, and $d_a + d_b > 1$. Let $I = [0, 1)$ and let $\mathcal{L}$ be the Lebesgue measure on I. Fix $\tau \in (0, \infty)$, and for each $t \in I$ and $z \in \mathbb{R}^2$ define

$$W^t = \{x \cdot (1, \tau \cdot a^t) \colon x \in \mathbb{R}\},$$
$$V^t = (W^t)^\perp$$

and

$$V^t_z = z + V^t.$$

In order to prove Theorem 2.6 we shall first prove the following:

Theorem 4.1. *For $\mu_a \times \mu_b \times \mathcal{L}$-a.e. $(x, y, t) \in C_a \times C_b \times I$ it holds that*

$$\mathcal{H}^{d_a + d_b - 1}((C_a \times C_b) \cap V^t_{(x,y)}) = 0\,.$$

4.1. Preliminaries. Set

$$\alpha = \frac{\log b}{\log a}$$

(so $\alpha \in I \setminus \mathbb{Q}$) and, for each $t \in I$,

$$R(t) = t + \alpha \mod 1.$$

Given $0 < \rho < \frac{1}{2}$ and a word $\lambda_1 \cdot \cdots \cdot \lambda_l = w \in \Lambda^*$, write

$$f_{\rho,w} = f_{\rho,\lambda_1} \circ \cdots \circ f_{\rho,\lambda_l}$$

and

$$C_{\rho,w} = f_{\rho,w}(C_\rho).$$

For each $n \geq 1$ and $x \in C_\rho$ let $w_{\rho,n}(x) \in \Lambda^n$ be the unique word of length n which satisfies

$$x \in C_{\rho,w_{\rho,n}(x)},$$

and let

$$S_\rho(x) = f^{-1}_{\rho,w_{\rho,1}(x)}(x).$$

We also write

$$w_{\rho,0}(x) = \emptyset$$

and

$$C_{\rho,\emptyset} = C_\rho.$$

The following dynamical system will be used in the proof of Theorem 4.1. The idea of using this system comes from the partition operator introduced in Section 10 of [7]. Set

$$K = C_a \times C_b,$$

$$X = K \times I,$$

$$\mu = \mu_a \times \mu_b,$$

$$\nu = \mu \times \mathcal{L},$$

and for each $(x, y, t) \in X$ define

$$T(x, y, t) = \begin{cases} (x, S_b(y), R(t)) & \text{if } t \in [0, 1-\alpha), \\ ((S_a(x), S_b(y), R(t)) & \text{otherwise.} \end{cases}$$

It is easy to check that the system (X, ν, T) is measure preserving, and by Lemma 2.2 in [2] it follows that it is ergodic.

Let $\mathcal{R}$ be the Borel σ-algebra of $\mathbb{R}^2$. For each $t \in I$ let P_t be the orthogonal projection onto W^t, and let $\{\mu_{t,z}\}_{z\in\mathbb{R}^2}$ be the disintegration of μ with respect to $P_t^{-1}(\mathcal{R})$ (see Section 3.1 above). Also, for each $(z,t) \in X$ define

$$F(z,t) = \Theta_*^1(P_t\mu, P_t z).$$

4.2. Auxiliary lemmas

Lemma 4.2. *It holds that $I_1(\mu) < \infty$, where recall that $I_1(\mu)$ is the 1-energy of μ.*

Proof. Set

$$\delta = 1 - 2b.$$

Then, for each $(x,y) \in \mathbb{R}^2$, and $k \geq 1$

$$\begin{aligned}
\mu(B((x,y),\delta\cdot a^k)) &\leq \mu((x-\delta\cdot a^k, x+\delta\cdot a^k)\times(y-\delta\cdot a^k, y+\delta\cdot a^k)) \\
&\leq \mu_a(x-\delta\cdot a^k, x+\delta\cdot a^k)\cdot\mu_b(y-\delta\cdot a^k, y+\delta\cdot a^k) \\
&\leq 2^{-k}\cdot 2^{-[k\log_b a]} \\
&\leq 2^{-k}\cdot 2^{1-k\log_b a} \\
&= 2\cdot a^{k(1+\log_b a)\log_a 2^{-1}} \\
&= 2\cdot a^{k(d_a+d_b)}.
\end{aligned}$$

This shows that there exists a constant $M > 0$ with

$$\mu(B(z,r)) \leq M\cdot r^{d_a+d_b} \quad \text{for all } z\in\mathbb{R}^2 \text{ and } r > 0.$$

Since $d_a + d_b > 1$, the lemma follows by the discussion found at the beginning of Chapter 8 of [10]. □

Lemma 4.3. *Let $n_1, n_2 \geq 1$, $w_1 \in \Lambda^{n_1}$ and $w_2 \in \Lambda^{n_2}$. For each $(x,y) \in K$ set*

$$g(x,y) = (f_{a,w_1}(x), f_{b,w_2}(y)).$$

Then for each Borel set $B \subset K$

$$\mu(g(B)) = 2^{-n_1-n_2}\cdot\mu(B).$$

Proof. We prove this by using the $\pi - \lambda$ theorem (see Section 3 of Chapter 1 of [1]). Let $\mathcal{E}$ be the collection of all Borel sets $B \subset K$ which satisfy

$$\mu(g(B)) = 2^{-n_1-n_2} \cdot \mu(B);$$

then $\mathcal{E}$ is a λ-system. Set

$$\mathcal{P} = \{C_{a,u_1} \times C_{b,u_2} : u_1, u_2 \in \Lambda^*\} \cup \{\emptyset\};$$

then $\mathcal{P}$ is a π-system, $\mathcal{P} \subset \mathcal{E}$ and $\sigma(\mathcal{P})$ equals the collection of all Borel subsets of K. By the $\pi - \lambda$ theorem it follows that $\sigma(\mathcal{P}) \subset \mathcal{E}$, hence $\mathcal{E}$ equals the collection of all Borel subsets of K, and the lemma is proven. □

Lemma 4.4. *It holds that*

$$0 < \mathcal{H}^{d_a+d_b}(K) < \infty,$$

and

$$\mu(E) = \frac{\mathcal{H}^{d_a+d_b}(K \cap E)}{\mathcal{H}^{d_a+d_b}(K)}$$

for each Borel set $E \subset \mathbb{R}^2$.

Proof. By Theorem 8.10 in [10] it follows that

$$\mathcal{H}^{d_a+d_b}(K) > 0,$$

and by an elementary covering argument it can be shown that

$$\mathcal{H}^{d_a+d_b}(K) < \infty.$$

The rest of the lemma can be proven by using the $\pi - \lambda$ theorem, as in the proof of Lemma 4.3. □

Lemma 4.5. *Let* $0 < M < \infty$ *and set*

$$E_M = \{(z, t) \in X : F(z, t) > M\}.$$

Then

$$\nu(E_M) > 0.$$

Proof. Assume by contradiction that $\nu(E_M) = 0$ and set

$$L = \{t \in I : \mu\{z : (z,t) \in E_M\} = 0\}.$$

Then $\mathcal{L}(I \setminus L) = 0$, and so $\overline{L} = I$. Set

$$A = \left\{ t \in I : P_t\mu \ll \mathcal{H}^1 \text{ and } \left\| \frac{dP_t\mu}{d\mathcal{H}^1} \right\|_{L^\infty(\mathcal{H}^1)} \leq M \right\},$$

and let $t \in L$. For $P_t\mu$-a.e. $z \in W^t$ we have $\Theta^1_*(P_t\mu, z) \leq M$, hence by parts (2) and (3) of Theorem 2.12 in [10] it follows that $t \in A$. This shows that $L \subset A$, and so that $\overline{A} = I$. By an argument similar to the one given at the end of the proof of Corollary 2.3, it can be shown that A is a closed subset of I. Hence $A = I$, and in particular $P_t\mu \ll \mathcal{H}^1$ for each $t \in I$. This contradicts Theorem 4.1 in [13], which says that there exists a dense G_δ set of 1-dimensional linear subspaces $V \subset \mathbb{R}^2$ such that $P_V\mu$ and $\mathcal{H}^1$ are singular. It follows that we must have $\nu(E_M) > 0$, and the lemma is proven. □

4.3. Proofs of Theorems 4.1 and 2.6

Proof of theorem 4.1. Let D be the set of all $(z,t) \in X$ such that $P_t\mu \ll \mathcal{H}^1$, $\mu_{t,z}$ is defined,

$$\mu_{t,z}(C_{a,w_1} \times C_{b,w_2}) = \lim_{\epsilon \downarrow 0} \frac{\mu((C_{a,w_1} \times C_{b,w_2}) \cap P_t^{-1}(B(P_t z, \epsilon)))}{P_t\mu(B(P_t z, \epsilon))}$$

for each $w_1, w_2 \in \Lambda^*$, and

$$0 < F(z,t) = \lim_{\epsilon \downarrow 0} \frac{P_t\mu(B(P_t z, \epsilon))}{2\epsilon} < \infty.$$

By Lemma 4.2 and by the same arguments as the ones given at the beginning of Section 3.3, it follows that

$$\nu(X \setminus D) = 0.$$

Set

$$D_0 = \bigcap_{j=0}^{\infty} T^{-j} D.$$

Then

$$\nu(X \setminus D_0) = 0$$

since T is measure preserving.

For $0 < M < \infty$ let E_M be as in Lemma 4.5, and set

$$E_{0,M} = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} T^{-j}(E_M).$$

Since $\nu(E_M) > 0$, it follows by the ergodicity of (X, ν, T) that

$$\nu(X \setminus E_{0,M}) = 0.$$

Set

$$D_1 = D_0 \cap \Big(\bigcap_{M=1}^{\infty} E_{0,M} \Big);$$

then $\nu(X \setminus D_1) = 0$. For $\mathcal{L}$-a.e. $t \in I$ it holds that

$$\mu\{z \in K : (z,t) \notin D_1\} = 0.$$

Fix such $t_0 \in I$ and set

$$A = \{z \in K : (z, t_0) \in D_1\}.$$

Note that by $A \neq \emptyset$ it follows that $P_{t_0}\mu \ll \mathcal{H}^1$.

Set

$$\eta = d_a + d_b - 1.$$

It will now be shown that

$$\Theta^{*\eta}(\mu_{t_0,z}, z) = \infty \quad \text{for each } z \in A. \tag{4.1}$$

Let $(x, y) = z \in A$ and set

$$\beta = (F(z, t_0))^{-1}.$$

Then $0 < \beta < \infty$ since $(z, t_0) \in D_0$. Let $M \geq 1$ and $N \geq 1$ be given; then there exists $k \geq N$ with

$$T^k(z, t_0) \in D_0 \cap E_M,$$

and so

$$F(T^k(z, t_0)) > M.$$

Set

$$l = [t_0 + k\alpha];$$

then

$$\begin{aligned}
&\mu_{t_0,z}(C_{a,w_l(x)} \times C_{b,w_k(y)}) \\
&\quad = \lim_{\epsilon \downarrow 0} \frac{\mu((C_{a,w_l(x)} \times C_{b,w_k(y)}) \cap P_{t_0}^{-1}(B(P_{t_0}z,\epsilon)))}{P_{t_0}\mu(B(P_{t_0}z,\epsilon))} \\
&\quad = \lim_{\epsilon \downarrow 0} \frac{2\epsilon}{P_{t_0}\mu(B(P_{t_0}z,\epsilon))} \cdot \frac{\mu((C_{a,w_l(x)} \times C_{b,w_k(y)}) \cap P_{t_0}^{-1}(B(P_{t_0}z,\epsilon)))}{2\epsilon} \\
&\quad = \beta \cdot \lim_{\epsilon \downarrow 0} \frac{\mu((C_{a,w_l(x)} \times C_{b,w_k(y)}) \cap P_{t_0}^{-1}(B(P_{t_0}z,\epsilon)))}{2\epsilon}\,.
\end{aligned} \tag{4.2}$$

For each $(x', y') \in \mathbb{R}^2$ set

$$g(x', y') = (f_{a,w_l(x)}(x'), f_{b,w_k(y)}(y'));$$

then

$$C_{a,w_l(x)} \times C_{b,w_k(y)} = f_{a,w_l(x)}(C_a) \times f_{b,w_k(y)}(C_b) = g(C_a \times C_b). \tag{4.3}$$

Let $\epsilon > 0$, and let

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

be a linear map with

$$L(1,0) = (a^l, 0)$$

and

$$L(0,1) = (0, b^k).$$

Since L is the linear part of the affine transformation g, we have

$$\begin{aligned}
P_{t_0}^{-1}(B(P_{t_0}z,\epsilon)) &= z + V^{t_0} + B(0,\epsilon) \\
&= g \circ g^{-1}(z) + L \circ L^{-1}(V^{t_0}) + L \circ L^{-1}(B(0,\epsilon)) \\
&= g(g^{-1}(z) + L^{-1}(V^{t_0}) + L^{-1}(B(0,\epsilon))).
\end{aligned} \tag{4.4}$$

Since $a^{-l} \geq a^{-t_0-k\alpha+1} \geq a \cdot b^{-k}$, we obtain

$$L^{-1}(B(0,\epsilon)) \supset B(0, \epsilon \cdot a \cdot b^{-k}). \tag{4.5}$$

Also we have

$$\begin{aligned} L^{-1}(V^{t_0}) = L^{-1}((W^{t_0})^{\perp}) &= L^{-1}(((1, \tau \cdot a^{t_0}) \cdot \mathbb{R})^{\perp}) \\ &= L^{-1}((\tau \cdot a^{t_0}, -1) \cdot \mathbb{R}) \\ &= (\tau \cdot a^{t_0} \cdot a^{-l}, -b^{-k}) \cdot \mathbb{R} \\ &= \left(\tau \cdot a^{t_0} \cdot \frac{b^k}{a^l}, -1\right) \cdot \mathbb{R}, \end{aligned}$$

and so since

$$\frac{b^k}{a^l} = a^{k \cdot \log_a b - l} = a^{k\alpha - [t_0 + k\alpha]},$$

it follows that

$$\begin{aligned} L^{-1}(V^{t_0}) &= (\tau \cdot a^{t_0 + k\alpha - [t_0 + k\alpha]}, -1) \cdot \mathbb{R} \\ &= ((1, \tau \cdot a^{R^k(t_0)}) \cdot \mathbb{R})^{\perp} \qquad (4.6) \\ &= V^{R^k(t_0)}. \end{aligned}$$

Set

$$Q_\epsilon = P^{-1}_{R^k(t_0)}(B(P_{R^k(t_0)}(f^{-1}_{a,w_l(x)}(x), f^{-1}_{b,w_k(y)}(y)), \epsilon a b^{-k}));$$

then by (4.4), (4.5), and (4.6) it follows that

$$\begin{aligned} P^{-1}_{t_0}(B(P_{t_0} z, \epsilon)) &= g(g^{-1}(z) + L^{-1}(V^{t_0}) + L^{-1}(B(0, \epsilon))) \\ &\supset g((f^{-1}_{a,w_l(x)}(x), f^{-1}_{b,w_k(y)}(y)) + V^{R^k(t_0)} + B(0, \epsilon a b^{-k})) \\ &= g(Q_\epsilon). \end{aligned}$$

Now by (4.2), (4.3), and Lemma 4.3 we get that

$$\begin{aligned} &\mu_{t_0,z}(C_{a,w_l(x)} \times C_{b,w_k(y)}) \\ &\quad = \beta \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(g((C_a \times C_b) \cap Q_\epsilon))}{2\epsilon} \\ &\quad = \beta \cdot 2^{-l-k} \cdot \frac{a}{b^k} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu((C_a \times C_b) \cap Q_\epsilon)}{2\epsilon a b^{-k}} \\ &\quad \geq \frac{\beta}{2} \cdot 2^{-k-k\alpha} \cdot \frac{a}{b^k} \cdot F((f^{-1}_{a,w_l(x)}(x), f^{-1}_{b,w_k(y)}(y)), R^k(t_0)) \\ &\quad = \frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot F(T^k(z, t_0)) \\ &\quad \geq \frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot M. \end{aligned}$$

Since

$$C_{a,w_l(x)} \times C_{b,w_k(y)} \subset B\Big(z, \frac{2 \cdot b^k}{a}\Big)$$

and

$$2^{-k-k\alpha} \cdot b^{-k} \cdot b^{-k\eta} = 1,$$

it follows that

$$\begin{aligned}
\frac{\mu_{t_0,z}(B(z, \frac{2 \cdot b^k}{a}))}{(4a^{-1} \cdot b^k)^\eta} &\geq \frac{\mu_{t_0,z}(C_{a,w_l(x)} \times C_{b,w_k(y)})}{(4a^{-1} \cdot b^k)^\eta} \\
&\geq \frac{\frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot M}{(4a^{-1} \cdot b^k)^\eta} \\
&\geq \frac{\beta a^2}{8} \cdot M \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot b^{-k\eta} \\
&= \frac{\beta a^2}{8} \cdot M.
\end{aligned}$$

This shows that

$$\Theta^{*\eta}(\mu_{t_0,z}, z) \geq \frac{\beta a^2}{8} \cdot M,$$

which proves (4.1) since $\beta > 0$ and M can be chosen arbitrarily large.

Let $z \in A$ and $u \in A \cap V_z^{t_0}$. Then by (4.1)

$$\Theta^{*\eta}(\mu_{t_0,z}, u) = \Theta^{*\eta}(\mu_{t_0,u}, u) = \infty,$$

and so by Theorem 6.9 in [10] we get that

$$\mathcal{H}^\eta(A \cap V_z^{t_0}) = 0.$$

Also it holds that

$$\mu(K \setminus A) = 0;$$

hence by Theorem 7.7 in [10] and by Lemma 4.4 we get that

$$\begin{aligned}
\int_{W^{t_0}} \mathcal{H}^\eta((K \setminus A) \cap V_u^{t_0})\, d\mathcal{H}^1(u) &\leq \text{const} \cdot \mathcal{H}^{\eta+1}(K \setminus A) \\
&= \text{const} \cdot \mu(K \setminus A) \\
&= 0.
\end{aligned}$$

This shows that

$$\mathcal{H}^\eta((K \setminus A) \cap V_u^{t_0}) = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } u \in W^{t_0},$$

and so

$$\mathcal{H}^\eta((K \setminus A) \cap V_z^{t_0}) = 0 \quad \text{for } \mu\text{-a.e. } z \in K$$

since $P_{t_0}\mu \ll \mathcal{H}^1$. It follows that for μ-a.e. $z \in A$, and so for μ-a.e. $z \in K$,

$$\mathcal{H}^\eta(K \cap V_z^{t_0}) = \mathcal{H}^\eta(A \cap V_z^{t_0}) + \mathcal{H}^\eta((K \setminus A) \cap V_z^{t_0}) = 0\,.$$

By Lemma 3.4, and by Fubini's theorem it follows that

$$\mathcal{H}^\eta(K \cap V_z^t) = 0 \quad \text{for } \nu\text{-a.e. } (z,t) \in X,$$

which completes the proof of Theorem 4.1. □

Proof of Theorem 2.6. Let G be the set of all 1-dimensional linear subspaces of $\mathbb{R}^2$, and set

$$E = \{(z,V) \in K \times G : \mathcal{H}^{d_a+d_b-1}(K \cap V_z) = 0\}.$$

For each $-\infty \le t_1 < t_2 \le \infty$ set

$$G_{t_1,t_2} = \{V \in G : V = (t,-1)\cdot\mathbb{R} \text{ with } t \in (t_1,t_2)\}.$$

Given $k \in \mathbb{Z}$ we can apply theorem 4.1 with $\tau = a^k$, in order to get that $(z,V) \in E$ for $\mu \times \xi_G$-a.e. $(z,V) \in K \times G_{a^{k+1},a^k}$. By doing this for each $k \in \mathbb{Z}$ we get that $(z,V) \in E$ for $\mu \times \xi_G$-a.e. $(z,V) \in K \times G_{0,\infty}$. Now Theorem 2.6 follows by the symmetry of K with respect to the map that takes $(x,y) \in K$ to $(1-x,y)$. □

References

[1] P. Billingsley, *Probability and measure*. 3rd ed. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York, 1995. MR 1324786 Zbl 0822.60002

[2] J. Brettschneider, On uniform convergence in ergodic theorems for a class of skew product transformations. *Discrete Contin. Dyn. Syst.* **29** (2011), no. 3, 873–891. MR 2773156 Zbl 1215.37012

[3] K. J. Falconer, Sets with large intersection properties. *J. London Math. Soc.* (2) **49** (1994), 267–280. MR 1260112 Zbl 0798.28004

[4] K. J. Falconer and T. O'Neil, Convolutions and the geometry of multifractal measures. *Math. Nachr.* **204** (1999), 61–82. MR 1705154 Zbl 1040.28011

[5] D.-J. Feng and H. Hu, Dimension theory of iterated function systems. *Comm. Pure Appl. Math.* **62** (2009), no. 11, 1435–1500. MR 2560042 Zbl 1230.37031

[6] H. Furstenberg, Intersections of Cantor sets and transversality of semigroups. In R. C. Gunning (ed.), *Problems in analysis.* A symposium in honor of S. Bochner, Princeton University, Princeton, N.J., 1–3 April 1969. Princeton University Press, Princeton, N.J., 1970, 41-59. MR 0354562 MR 0337450 (collection) Zbl 0208.32203

[7] M. Hochman and P. Shmerkin, Local entropy averages and projections of fractal measures. *Ann. of Math.* (2) **175** (2012), no. 3, 1001–1059. MR 2912701 Zbl 1251.28008

[8] A. Kechris, Classical descriptive set theory. Graduate Texts in Mathematics, 156. Springer, New York, 1995. MR 1321597 Zbl 0819.04002

[9] T. Kempton, Sets of β-expansions and the Hausdorff measure of slices through fractals. Preprint 2013. arXiv:1307.2091 [math.DS]

[10] P. Mattila, *Geometry of sets and measures in euclidean spaces.* Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. MR 1333890 Zbl 0819.28004

[11] P. Mattila, Integral geometric properties of capacities. *Trans. Amer. Math. Soc.* **266** (1981), 539–554. MR 0617550 Zbl 0479.31003

[12] P. Mattila and R. Mauldin, Measure and dimension functions: measurability and densities. *Math. Proc. Cambridge Philos. Soc.* **121** (1997), no. 1, 81–100. MR 1418362 Zbl 0885.28005

[13] F. Nazarov, Y. Peres, and P. Shmerkin, Convolutions of Cantor measures without resonance. *Israel J. Math.* **187** (2012), 93–116. MR 2891700 Zbl 1258.28008

[14] T. Orponen, On the packing measure of slices of self-similar sets. Preprint 2013. arXiv:1309.3896 [math.DS]

[15] W. Parry, Skew products of shifts with a compact Lie group. *J. London Math. Soc.* (2) **56** (1997), no. 2, 395–404. MR 1489145 Zbl 0944.37003

[16] Y. Peres and W. Schlag, Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions. *Duke Math. J.* **102** (2000), no. 2, 193–251. MR 1749437 Zbl 0961.42007

[17] P. Walters, *An introduction to ergodic theory.* Graduate Texts in Mathematics, 79. Springer, Berlin etc., 1982. MR 0648108 Zbl 0475.28009

Received February 19, 2015

Ariel Rapaport, Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel

e-mail: ariel.rapaport@mail.huji.ac.il