J. Fractal Geom. 3 (2016), 95–117
DOI 10.4171/JFG/31

Journal of Fractal Geometry

Attractors for iterated function systems

Emma D'Aniello and T. H. Steele[1]

Abstract. Let X be a compact metric space with $\mathcal{S} = \{S_1, \dots, S_N\}$ a finite set of contraction maps from X to itself. Call a non-empty compact subset F of X an attractor for the iterated function system (IFS) $\mathcal{S}$ if $F = \bigcup_{i=1}^{N} S_i(F)$. Working primarily on the unit interval $I = [0, 1]$, we show that

(1) the typical closed set in $[0, 1]$ is not an attractor of an IFS, and describe the closed sets that comprise the F_σ set of attractors;

(2) both the set of attractors and its complement are dense subsets of $(\mathcal{K}([0, 1]), \mathcal{H})$;

(3) the set of attractors is path-connected;

(4) every countable compact subset of $[0, 1]$ of finite Cantor–Bendixon rank is homeomorphic to an attractor, and

(5) every nowhere dense uncountable compact subset of $[0, 1]$ is homeomorphic to an attractor.

Mathematics Subject Classification (2010). Primary: 26A18; Secondary: 28A80, 28A78.

Keywords. Iterated function system, attractor, self-similar set, typical (according to Baire) element.

Contents

[1] The authors are grateful to the Istituto Nazionale di Alta Matematica (INdAM) as this research has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica.

1. Introduction

Let X be a complete metric space with $\mathcal{S} = \{S_1, \dots, S_N\}$ a finite set of contraction maps from X to itself. We call a non-empty subset F of X *invariant*, or *an attractor*, for the iterated function system ([15], [18]) (IFS) $\mathcal{S}$ if

$$F = \bigcup_{i=1}^{N} S_i(F) = \mathcal{S}(F).$$

It turns out that, for a particular finite set of contraction maps $\mathcal{S}$, there exists a unique invariant compact set $F \subseteq X$. This and other results from [18] are summarized below.

Theorem 1.1. *Let (X, d) be a complete metric space with $\mathcal{S} = \{S_1, \dots, S_N\}$ a finite set of contraction maps from X to itself.*

(a) *There exists a unique non-empty compact set $F \subseteq X$ such that $F = \mathcal{S}(F)$.*

(b) *The set F is the closure of the set of fixed points $s_{i_1 \dots i_p}$ of finite compositions $S_{i_1} \circ \dots \circ S_{i_p}$ of members of $\mathcal{S}$.*

(c) *If A is any non-empty compact set in X, then*

$$\lim_{p \to \infty} \mathcal{S}^p(A) = F$$

in the Hausdorff metric.

We, sometimes, as in [18], denote F of the previous theorem with $|\mathcal{S}|$. Now, suppose that the contraction maps are similarities, so that

$$|S_i(x) - S_i(y)| = r_i |x - y|$$

for all x, y in X, and $0 < r_i < 1$. Each S_i transforms subsets of X into geometrically similar sets, giving rise to invariant sets that are self-similar. When the images of the $S_i(F)$ do not overlap "too much" (see the *open set condition* in Section 2), the self-similar set $F = \bigcup_{i=1}^{N} S_i(F)$ has Hausdorff dimension equal to the value of s satisfying $\sum_{i=1}^{N} {r_i}^s = 1$.

Here, we dedicate our attention to the structure of attractors, as well as the structure of the set of attractors, for iterated function systems. We are motivated by related work found in [16], [4], [6], [12], [13], [10], [9] and [11]. While not necessarily linked to results or techniques presented here, [1], [2], [3], [14], [15], [17], [18], [19], [21] and [23], provide some overview of recent work concerning the structure, application and generalizations of attractors for iterated function systems.

We furnish the class $\mathcal{K}$ of non-empty compact subsets of X with the Hausdorff metric $\mathcal{H}$; this space is complete, so that good use can be made of the Baire category theorem. Let

$$\mathcal{T} = \{F \in \mathcal{K}(X) \colon F = \mathcal{S}(F); \mathcal{S} \text{ a finite collection of contraction maps}\}.$$

We show that $\mathcal{T}$ is an F_σ subset of $\mathcal{K}$ and, in the case that $X = [0, 1]$, we show that $\mathcal{T}$ is of the first category. When $X = [0, 1]$, both $\mathcal{T}$ and $\mathcal{K}(X) \setminus \mathcal{T}$ are dense in $\mathcal{K}(X)$, and $\mathcal{T}$ is path-connected. Moreover, every nowhere dense uncountable element of $(\mathcal{K}([0, 1]), \mathcal{H})$ is homeomorphic to an element of $\mathcal{T}$, as is the case for any countable element of $(\mathcal{K}([0, 1]), \mathcal{H})$ of finite Cantor–Bendixon rank.

We proceed through several sections. After presenting notation, definitions and necessary previously known results in Section 2, we present several examples of invariant sets in Section 3. In Section 4 we show that the collection of invariant sets is an F_σ set for any X, while the collection of invariant sets is also of the first category, should $X = [0, 1]$. We do this by establishing that the set $\mathcal{K}^\star \setminus \mathcal{A}$, comprised of certain nowhere dense perfect sets contained in the irrationals, is both residual in $\mathcal{K}([0, 1])$, and has the property that $(\mathcal{K}^\star \setminus \mathcal{A}) \cap \mathcal{T} = \emptyset$. Every element of $\mathcal{K}^\star \setminus \mathcal{A}$ is nowhere dense and perfect in $[0, 1]$, hence homeomorphic to the middle thirds Cantor set, Q. Since, as is well-known, Q is an attractor (see Example 3.1), each element of the residual $\mathcal{K}^\star \setminus \mathcal{A}$, while not itself an attractor, is homeomorphic to the element $Q \in \mathcal{T}$. This motivates the final section, which establishes that every nowhere dense uncountable subset of $[0, 1]$ is homeomorphic to an attractor for a contractive system.

2. Preliminaries

We will do most of our work in two metric spaces. Let (X, d) be a complete metric space. As in [18], let $\mathcal{B}(X)$ be the class of non-empty closed and bounded subsets of X. We endow $\mathcal{B}(X)$ with the Hausdorff metric $\mathcal{H}$ given by

$$\mathcal{H}(E, F) = \inf\{\delta > 0 \colon E \subset B_\delta(F), F \subset B_\delta(E)\}.$$

This space is complete. In the case that X is also compact, then $\mathcal{B}(X) = \mathcal{K}(X)$, where $\mathcal{K}(X)$ is the class of non-empty compact subsets of X, and $(\mathcal{K}(X), \mathcal{H})$ is also compact [5]. As for the second metric space, let $C(X, X)$ be the collection of continuous self-maps of X. When coupled with the supremum norm, $C(X, X)$

is complete. Let Lip denote the collection of Lipschitz maps $f : X \to X$. For a fixed $m > 0$, let Lip(m) denote the collection of Lipschitz maps $f : X \to X$ with Lipschitz constant less than or equal to m. Should $m < 1$, then $f \in \mathrm{Lip}(m)$ is a contraction map.

Let $n \in \mathbb{N}$ with $E \subseteq [0, 1]$. For $s > 0$, set

$$\mathcal{H}^s_n(E) = \inf\Big\{\sum \phi(|I_j|) \colon E \subseteq \cup I_j,\, I_j \text{ an open interval of length } |I_j| \leq \frac{1}{n}\Big\}.$$

Then $\mathcal{H}^s = \lim_{n\to\infty} \mathcal{H}^s_n$ defines a measure on the Borel sets in $[0, 1]$, generally referred to as the s-dimensional Hausdorff measure ([15], [20]).

A *portion* P of a closed set $E \subseteq [0, 1]$ is a nonempty set of the form

$$P = E \cap J,$$

where J is an open interval, and take $\overline{\mathrm{conv}}(F)$ to be the convex closure of $F \subseteq [0, 1]$.

Let $\mathcal{S} = \{S_1, \dots, S_N\}$ be a finite set of contraction maps. Then, $\mathcal{S}$ is said to satisfy the *open set condition* (*OSC*) if there is a non-empty open set V such that

$$\bigcup_{i=1}^{N} S_i(V) \subseteq V \quad \text{and} \quad S_i(V) \cap S_j(V) = \emptyset,$$

whenever $i \neq j$ (see [18] and [23]). Now, suppose that, for each i, one has $S_i(x) = r_i x + b_i$, where $b_i \in \mathbb{R}$ with $0 < |r_i| < 1$, and take s such that one gets $\sum_{i=1}^{N} |r_i|^s = 1$. If $F = \bigcup_{i=1}^{N} S_i(F)$, then $0 < \mathcal{H}^s(F) < \infty$, and $\mathcal{H}^s(S_i(F) \cap S_j(F)) = 0$, whenever $i \neq j$.

Since $(\mathcal{K}(X), \mathcal{H})$ is complete, we will be able to make good use of the Baire category theorem. A set is of the first category in (X, d) if it can be written as a countable union of nowhere dense sets; otherwise, the set is of the second category. A set is residual if it is the complement of a first category set; an element of a residual subset of (X, d) is called a typical, or generic, element of X. With these definitions in mind, we recall Baire's theorem on category [22].

Theorem 2.1 (Baire category theorem). *Let (X, d) be a complete metric space with B a first category subset of X. Then $X \setminus B$ is dense in X.*

Let $\mathcal{K}^\star$ be the collection of all Cantor sets contained in the irrationals, and let

$$\mathcal{A} = \{E \in \mathcal{K}([0, 1]) \colon \text{there are disjoint portions } P \text{ and } Q \text{ of } E \text{ and } f \in \mathrm{Lip} \text{ such that } f(P) \supseteq Q\}.$$

In [7] it is proved that $\mathcal{K}^\star$ is a residual subset of $\mathcal{K}([0,1])$ and that $\mathcal{A}$ is a first category F_σ in $\mathcal{K}([0,1])$, so that $\mathcal{K}^\star \setminus \mathcal{A}$ is also residual in $\mathcal{K}([0,1])$. Consequently, we have the following result.

Theorem 2.2. *Let $E \in \mathcal{K}^\star \setminus \mathcal{A}$. Then the only Lipschitz map that maps E onto E is the identity map.*

In Section 4 we use Theorem 2.2.

To facilitate our work on Q, the middle thirds Cantor set in $[0,1]$, we use the following notation suggested in [8]. Let

$$\mathcal{N} = \{0,1\}^{\mathbb{N}},$$

and if $\boldsymbol{n} \in \mathcal{N}$ with $\boldsymbol{n} = \{n_j\}_{j=1}^{\infty}$, we set

$$\boldsymbol{n}|k = (n_1, n_2, \ldots, n_k).$$

By $\mathbf{0}$ (respectively $\mathbf{1}$), we mean that $\boldsymbol{n} \in \mathcal{N}$ such that $n_i = 0$ (respectively $n_i = 1$) for all i. We let $\{J_{\boldsymbol{n}|k} : \boldsymbol{n} \in \mathcal{N}\}$ be the 2^k closed intervals of length $\frac{1}{3^k}$ found in the k^{th} step of the canonical construction of Q, with $J_{\boldsymbol{n}|k,0}$ lying to the left of $J_{\boldsymbol{n}|k,1}$, for all $k \geq 0$. Let

$$E_k = \bigcup_{\boldsymbol{n} \in \mathcal{N}} J_{\boldsymbol{n}|k}.$$

It follows, then, that

$$Q = \bigcup_{\boldsymbol{n} \in \mathcal{N}} \bigcap_{k=1}^{\infty} J_{\boldsymbol{n}|k} = \bigcap_{k=1}^{\infty} E_k,$$

and

$$J_{\boldsymbol{n}} = \bigcap_{k=1}^{\infty} J_{\boldsymbol{n}|k}$$

is a singleton for every $\boldsymbol{n} \in \mathcal{N}$. Let

$$G = \left(\frac{1}{3}, \frac{2}{3}\right)$$

be the complementary interval of Q between J_0 and J_1. In general, let $G_{\boldsymbol{n}|k}$ be that component of $[0,1] \setminus Q$ between $J_{\boldsymbol{n}|k,0}$ and $J_{\boldsymbol{n}|k,1}$. In Section 5 we make extensive use of clopen portions of Q of the form $J_{\boldsymbol{n}|k} \cap Q$, and frequently refer to them as *canonical portions of Q*.

We also need the Ascoli–Arzelà theorem in Section 4.

Theorem 2.3 (Ascoli–Arzelà theorem). *Let (X, d) be a compact metric space, and let K be a closed subset of $C(X, X)$. Then K is compact if and only if K is equicontinuous.*

Finally, let A be a countable compact subset of $[0, 1]$. Define a transfinite subsequence $\{A_\alpha\}_{\alpha \in \Omega}$ as follows:

$$A_0 = A,$$

$$A_\gamma = \begin{cases} \bigcap_{\alpha < \gamma} A_\alpha & \text{if } \gamma \text{ is a limit ordinal,} \\ A_\gamma & \text{is the set of the limit points of } A_{\gamma-1}. \end{cases}$$

Hence, by [24, Theorem 37, Section 26], for every such set there is an ordinal number $\beta < \Omega$ such that A_β is non-empty and finite, and $A_{\beta+1} = \emptyset$. Then we call β the *rank of A* and A_β its *set of highest order limit points*. We denote β by $\beta(A)$. (We recall that $\beta + 1$ is the *Cantor–Bendixon rank of A*.)

3. Examples of attractors

In Sections 4 and 5 we are concerned with the topological structure of attractors. In this brief section we present a few examples in order to give a flavor of the myriad of possibilities.

Example 3.1 (Cantor set). Let

$$S_1(x) = \frac{1}{3}x$$

and

$$S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then, as it is well-known,

$$Q = S_1(Q) \cup S_2(Q),$$

where Q is the middle thirds Cantor set. Since $\{S_1, S_2\}$ satisfies the OSC with $V = (0, 1)$, it follows that the Hausdorff dimension of Q is

$$s = \frac{\log 2}{\log 3},$$

as $2\left(\frac{1}{3}\right)^s = 1$.

Example 3.2 (interval attractor). Let

$$S_1(x) = \frac{1}{3}x,$$

$$S_2(x) = \frac{1}{3}x + \frac{1}{3},$$

and

$$S_3(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then,

$$[0,1] = \bigcup_{i=1}^{3} S_i([0,1]).$$

Moreover, $\{S_1, S_2, S_3\}$ satisfies the OSC with $V = (0,1)$, and as we expect, the Hausdorff dimension of $[0,1]$ is 1 as $3\left(\frac{1}{3}\right)^1 = 1$.

Example 3.3 (countable attractor). Let

$$S_1(x) = \frac{1}{3}x$$

and

$$S_2(x) = 1.$$

Then

$$F = \left\{\bigcup_{j=0}^{\infty} \frac{1}{3^j}\right\} \cup \{0\} = \mathcal{S}(F),$$

where

$$\mathcal{S} = \{S_1, S_2\},$$

as

$$S_1(F) = \{0\} \cup \bigcup_{j=0}^{\infty} \frac{1}{3^{j+1}}$$

and

$$S_2(F) = 1 = \frac{1}{3^0}.$$

The IFS $\mathcal{S}$ satisfies the OSC with $V = (0,1)$. Then, the Hausdorff dimension s of F is given by the equation,

$$\left(\frac{1}{3}\right)^s + 0^s = 1,$$

so that, as we expect as F is countable, $s = 0$.

The following example verifies that one can frequently combine types of attractors in a predictable way.

Example 3.4 (attractor of the form $Q \cup C$ with $Q \cap C = \emptyset$, where Q is a Cantor set, C is countable and $Q \subset \bar{C}$). Let $X = [0,1]$ and $F = Q \cup C$, where Q is the middle thirds Cantor set, and C is the collection of the mid-points of each open interval complementary to Q. Let

$$S_1(x) = \frac{1}{3}x,$$

$$S_2(x) = \frac{1}{2},$$

and

$$S_3(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then

$$S_1(F) = F \cap \left[0, \frac{1}{3}\right],$$

$$S_2(F) = \frac{1}{2},$$

and

$$S_3(F) = F \cap \left[\frac{2}{3}, 1\right].$$

The Hausdorff dimension s of F is given by the equation,

$$\left(\frac{1}{3}\right)^s + 0^s + \left(\frac{1}{3}\right)^s = 2\left(\frac{1}{3}\right)^s = 1,$$

so that, as we expect and is well-known, $s = \frac{\log 2}{\log 3}$.

Example 3.5 (attractors with countably many non-degenerate closed intervals which converge to a unique point $\{0\}$). Let $X = [0,1]$ with

$$\begin{aligned} F &= \{0\} \cup \bigcup_{j=0}^{\infty} \left[\frac{2}{3^{j+1}}, \frac{1}{3^j}\right] \\ &= \{0\} \cup \cdots \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]. \end{aligned}$$

Let

$$S_1(x) = \frac{1}{3}x,$$

$$S_2(x) = \frac{2}{3} + \frac{2}{3}\lambda([0,x] \cap F),$$

and λ be Lebesgue measure. Then, S_2 is linear with slope $\frac{2}{3}$ on each component of F, and constant on the intervals complementary to F. We conclude that $S_2 \in \operatorname{Lip}\left(\frac{2}{3}\right)$. Moreover,

$$S_1(F) = \{0\} \cup \bigcup_{j=1}^{\infty} \left[\frac{2}{3^{j+1}}, \frac{1}{3^j}\right]$$

and

$$S_2(F) = \left[\frac{2}{3}, 1\right].$$

Note that $\mathcal{S} = \{S_1, S_2\}$ satisfies the OSC with $V = (0, 1)$.

Example 3.6. We construct new examples of attractors based on the existence of some $F = \bigcup_{i=1}^{n} S_i(F) = \mathcal{S}(F) \subseteq [0, 1]$, and $S_i : I \to I$.

(1) Let

$$h \colon [a, b] \longrightarrow [p, q] = \overline{\operatorname{conv}} F$$

be a linear homeomorphism, and $\mathcal{S}' = \{S'_1, \dots, S'_n\}$, where $S'_i \colon [0, 1] \to [0, 1]$ and

(i) $S'_i = h^{-1} \circ S_i \circ h$ on $[a, b]$,

(ii) $S'_i(x) = S'_i(a)$ for $x \in [0, a]$, and

(iii) $S'_i(x) = S'_i(b)$ for $x \in [b, 1]$.

Since S_i is a contraction and $S_i(F) \subseteq F$, it follows that $S_i([p, q]) \subseteq [p, q]$, and $S'_i([0, 1]) \subseteq [a, b]$. Now,

$$S'_i(h^{-1}(F)) = (h^{-1} \circ S_i \circ h)(h^{-1}(F)) = (h^{-1} \circ S_i)(F).$$

Thus,

$$\begin{aligned} \bigcup_{i=1}^{n} S'_i(h^{-1}(F)) &= \bigcup_{i=1}^{n} (h^{-1} \circ S_i)(F) \\ &= h^{-1}\Big(\bigcup_{i=1}^{n} S_i(F)\Big) \\ &= h^{-1}(F), \end{aligned}$$

and $h^{-1}(F)$ is the attractor of $\mathcal{S}' = \{S'_1, \dots, S'_n\}$.

(2) Set

$$[a,b] = \left[\frac{3}{4}, 1\right]$$

and let

$$\mathcal{S}'' = \left\{\mathcal{S}', S_{n+1} = \frac{1}{3}x\right\}.$$

Let

$$E = \{0\} \cup \bigcup_{j=0}^{\infty} (\frac{1}{3})^{j} h^{-1}(F)$$

and

$$\left(\frac{1}{3}\right)^{j} h^{-1}(F) = \left\{\frac{x}{3^j} : x \in h^{-1}(F)\right\}.$$

We show that $E = \mathcal{S}''(E)$. This follows from the observation that we have (a) $\mathcal{S}'(E) = h^{-1}(F)$, and (b) $S_{n+1}(E) = \{0\} \cup \bigcup_{j=1}^{\infty} \left(\frac{1}{3}\right)^{j} h^{-1}(F)$.

(a) Let $S_i{}' \in \mathcal{S}'$. Then

$$\begin{aligned} S_i{}'(E) &= S_i{}'\Big(\{0\} \cup \bigcup_{j=0}^{\infty} \left(\frac{1}{3}\right)^{j} h^{-1}(F)\Big) \\ &= S_i{}'\Big(\{0\} \cup \bigcup_{j=1}^{\infty} \left(\frac{1}{3}\right)^{j} h^{-1}(F)\Big) \cup S_i'(h^{-1}(F)) \\ &= S_i{}'(a) \cup h^{-1}(S_i(F)), \end{aligned}$$

and, since $S_i{}'(a) = (h^{-1} \circ S_i \circ h)(a) = h^{-1}(S_i(h(a)))$ is an element of $h^{-1}((S_i(F))$, we have

$$S_i{}'(E) = h^{-1}(S_i(F)).$$

(b) This, of course, follows from $S_{n+1}(x) = \frac{1}{3}x$ and the definition of E.

We conclude that

$$\begin{aligned} \mathcal{S}''(E) &= \mathcal{S}'(E) \cup S_{n+1}(E) \\ &= h^{-1}(F) \cup \Big(\{0\} \cup \bigcup_{j=1}^{\infty} \left(\frac{1}{3}\right)^{j} h^{-1}(F)\Big) \\ &= \{0\} \cup \bigcup_{j=0}^{\infty} \left(\frac{1}{3}\right)^{j} h^{-1}(F) \\ &= E. \end{aligned}$$

Suppose that A and B are countable compact subsets of $[0,1]$. From [25, Proposition 2] we know that if A and B have the same rank and the same number of highest order limit points, then they are homeomorphic. This observation, coupled with Example 3.3 and Example 3.6, gives the following result.

Theorem 3.7. *If $F \in \mathcal{K}([0,1])$ is a countable set of finite rank, then there exist F' homeomorphic to F and $\mathcal{S} = \{S_1, \dots, S_n\}$ such that $F' = \mathcal{S}(F')$.*

Proof. Let $t \in \mathbb{N}$. We first show that there exists a countable set F in $\mathcal{K}([0,1])$ of rank t, and possessing a single limit point of that rank. We show this using induction. Consider $\mathcal{S} = \{\frac{1}{3}x, 1\}$ so that $S_2(x)$ is the constant function 1. Then $|\mathcal{S}| = \{0\} \cup \bigcup_{j=0}^{\infty} (\frac{1}{3})^j$, and $|\mathcal{S}|$ is of rank 1, with $\{0\}$ as its unique limit point of that rank. Now, suppose $F = \bigcup_{i=1}^{n} S_i(F) = \mathcal{S}(F) \subseteq [0,1]$, $S_i\colon I \to I$ and F is of rank t. Using Example 3.6, we construct $\mathcal{S}'' = \{\mathcal{S}', \mathcal{S}_{n+1} = \frac{1}{3}x\}$ and $E = \{0\} \cup \bigcup_{j=1}^{\infty} (\frac{1}{3})^j h^{-1}(F)$ so that $\mathcal{S}''(E) = E$. Since $(\frac{1}{3})^j h^{-1}(F) \to 0$ as $j \to \infty$, it follows that E is of rank $t+1$, and $\{0\}$ is the unique limit point of that rank.

Now, suppose $F = \bigcup_{i=1}^{n} S_i(F) = \mathcal{S}(F) \subseteq [0,1]$, $S_i\colon I \to I$ and F is of rank t. Let $k \in \mathbb{N}$. For $1 \le j \le k$, we construct $\mathcal{S}_j{}' = \{S_{1,j}{}', \dots, S_{n,j}{}'\}$ so that ${h_j}^{-1}(F)$ is a homeomorphic copy of F contained in $[\frac{2j-1}{2k}, \frac{2j}{2k}]$. Let $E = \bigcup_{j=1}^{k} {h_j}^{-1}(F)$, and $\widetilde{\mathcal{S}} = \{\mathcal{S}_1{}', \dots, \mathcal{S}_k{}'\}$. Then, $\widetilde{\mathcal{S}}(E) = E$ and E is of rank t with k limit points of that order. We conclude by recalling that if A and B are countable compact subsets having the same rank and the same number of highest order limit points, then they are homeomorphic [25, Proposition 2]. $\square$

We conclude this section by showing that any finite union of non-degenerated closed intervals in $[0,1]$ is an attractor.

Proposition 3.8. *Let $\{J_1, J_2, \dots, J_n\}$ be a finite collection of disjoint nondegenerate closed intervals in $I = [0,1]$. Then there exists $\mathcal{S} = \{S_1, \dots, S_n\}$ a finite set of contraction maps on I such that $F = \bigcup_{i=1}^{n} S_i(F)$, where $F = \bigcup_{i=1}^{n} J_i$. Moreover, the set of contractions $\mathcal{S} = \{S_1, \dots, S_n\}$ satisfies the open set condition.*

Proof. Suppose $i = 1$, so we have just one interval J_1. That J_1 is an attractor follows from Examples 3.2 and 3.6. Now, let $\{J_1, J_2, \dots, J_n\}$ be a finite collection of disjoint nondegenerate closed intervals in $I = [0,1]$, with $n \ge 2$. Let $J_i = [a_i, b_i]$, with $b_i < a_j$ whenever $i < j$. Let $Q = \sum_{i=1}^{n} (b_i - a_i)$ be the sum of the lengths of the intervals J_i. Now, fix $1 \le i \le n$, and set $P = b_i - a_i$, the length of J_i. We define the contraction map $S_i\colon I \to I$ so that $S_i(F) = J_i$, where $F = \bigcup_{k=1}^{n} J_k$. As one sees from the construction, $S_i([0,1]) = J_i$, so that the resulting set of contraction maps on I satisfies the open set condition, with $V = (0,1)$.

On J_1, $S_i(a_1) = a_i$, $S_i(b_1) = a_i + \frac{P}{Q}(b_1 - a_1)$ and S_i is linearly extended to all of $[a_1, b_1]$. Now, let $S_i|[b_1, a_2]$ be the constant map $a_i + \frac{P}{Q}(b_1 - a_1)$, and set $S_i(b_2) = a_i + \frac{P}{Q}(b_1 - a_1) + \frac{P}{Q}(b_2 - a_2) = a_i + \frac{P}{Q}\sum_{k=1}^{2}(b_k - a_k)$, with S_i linear on $[a_2, b_2]$. Continuing in this manner, we have

$$S_i(b_n) = a_i + \frac{P}{Q}\sum_{k=1}^{n}(b_k - a_k) = a_i + \frac{P}{Q}Q = a_i + P = b_i,$$

S_i linear with slope $m = \frac{P}{Q}$ on each J_k and constant on the complementary intervals $[b_{k-1}, a_k]$, and $S_i(F) = S_i([0,1]) = J_i$. □

4. The set of attractors

Let (X,d) be a compact metric space. The first main result of the section is Theorem 4.4, which establishes that the set

$$\mathcal{T} = \{F \in \mathcal{K}(X) \colon F = \mathcal{S}(F) \text{ with } \mathcal{S} \text{ a finite collection of contraction maps}\}$$

is an F_σ subset of $(\mathcal{K}(X), \mathcal{H})$. Lemma 4.3 describes the structure of the closed sets that comprise the F_σ set $\mathcal{T}$.

Lemma 4.1. *Let (X,d) be a compact metric space. Let $\{F_i\}_{i\in\mathbb{N}}$ be a sequence converging to F in $(\mathcal{K}(X), \mathcal{H})$. Suppose there exists $N \in \mathbb{N}$ such that, for each i, $F_i = K_i^1 \cup \cdots \cup K_i^N$ with $K_i^j \in \mathcal{K}(X)$ for any $1 \le j \le N$. Then, there exists $\{F_{i_j}\}_{j\in\mathbb{N}} \subseteq \{F_i\}_{i\in\mathbb{N}}$ so that*

(1) *for each i_j, $F_{i_j} = K_{i_j}^{\rho(1)} \cup \cdots \cup K_{i_j}^{\rho(N)}$, where $\rho\colon \{1,\dots,N\} \to \{1,\dots N\}$ is a permutation,*

(2) *$F = K^{\rho(1)} \cup \cdots \cup K^{\rho(L)}$, $L \le N$ and, for each $1 \le i \le L$,*

$$\lim_{j\to\infty} K_{i_j}^{\rho(i)} = K^{\rho(i)}.$$

Proof. For each i, we have $F_i = K_i^1 \cup \cdots \cup K_i^N$, and $\lim_{i\to\infty} \mathcal{H}(F_i, F) = 0$. Consider $\{K_i^1\}_{i\in\mathbb{N}}$. Since $\mathcal{K}(X)$ is compact, there exists $\{K_{i_j}^1\}_{j\in\mathbb{N}} \subseteq \{K_i^1\}_{i\in\mathbb{N}}$ such that $\lim_{j\to\infty} K_{i_j}^1 = K^{\rho(1)}$. Now, if $K^{\rho(1)} = F$, then we are done. Otherwise, $F \setminus K^{\rho(1)} \neq \emptyset$.

Let $x \in F \setminus K^{\rho(1)}$. Since $\lim_{j\to\infty} F_{i_j} = F$, and $x \in F$, for each i_j there is $K_{i_j}^{\rho(2)} \in \{K_{i_j}^1, \dots, K_{i_j}^N\} \setminus K_{i_j}^{\rho(1)}$ and a subsequence $\{K_{i_{j_l}}^{\rho(2)}\}_{l\in\mathbb{N}} \subseteq \{K_{i_j}^{\rho(2)}\}_{j\in\mathbb{N}}$ such that $\lim_{l\to\infty} K_{i_{j_l}}^{\rho(2)} = K^{\rho(2)}$ and $x \in K^{\rho(2)}$. To simplify the notation, we rename the subsequence i_{j_l} as i_j. Now, if $K^{\rho(1)} \cup K^{\rho(2)} = F$, then we are done. Otherwise, let $x \in F \setminus K^{\rho(1)} \cup K^{\rho(2)}$, and as before, there exists for each i_j, $K_{i_j}^{\rho(3)} \in \{K_{i_j}^1, \dots, K_{i_j}^N\} \setminus \{K^{\rho(1)}, K^{\rho(2)}\}$ and a subsequence $\{K_{i_{j_l}}^{\rho(3)}\}_{l\in\mathbb{N}} \subseteq \{K_{i_j}^{\rho(3)}\}_{j\in\mathbb{N}}$ such that $\lim_{l\to\infty} K_{i_{j_l}}^{\rho(3)} = K^{\rho(3)}$ and $x \in K^{\rho(3)}$. We show that there exists $L \le N$ such that $F = K^{\rho(1)} \cup \dots \cup K^{\rho(L)}$. Suppose this is not the case. Then $F \setminus K^{\rho(1)} \cup \dots \cup K^{\rho(N)} \neq \emptyset$. Moreover, there is $\{i_j\}_{j\in\mathbb{N}}$ such that $F_{i_j} = K_{i_j}^{\rho(1)} \cup \dots \cup K_{i_j}^{\rho(N)}$ and $\lim_{j\to\infty} K_{i_j}^{\rho(i)} = K^{\rho(i)}$ for each $1 \le i \le N$. But now we have a contradiction, as

$$\bigcup_{l=1}^{N} K_{i_j}^{\rho(l)} = F_{i_j}$$

and

$$\lim_{j\to\infty} F_{i_j} = F,$$

yet $\lim_{j\to\infty} \bigcup_{l=1}^{N} K_{i_j}^{\rho(l)} = \bigcup_{l=1}^{N} K^{\rho(l)}$ is also a proper subset of F. □

Lemma 4.2. *Let (X, d) be a compact metric space. If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{K}(X)$ such that* $\lim_{k\to\infty} \mathcal{H}(E_k, E) = 0$ *and $\{S_k\}_{k\in\mathbb{N}}$ is a sequence in $C(X, X)$ converging uniformly to S, then* $\lim_{k\to\infty} \mathcal{H}(S_k(E_k), S(E)) = 0$.

Proof. Since

$$\mathcal{H}(S_k(E_k), S(E)) \le \mathcal{H}(S_k(E_k), S_k(E)) + \mathcal{H}(S_k(E), S(E))$$

and both, $\mathcal{H}(S_k(E_k), S_k(E))$ and $\mathcal{H}(S_k(E), S(E))$ converge to zero as k goes to ∞, the conclusion follows. □

Lemma 4.3. *Let (X, d) be a compact metric space. Let $N \in \mathbb{N}$ and $0 < m \le 1$. The set*

$$\mathcal{L}_{N,m} = \{F \in \mathcal{K}(X) \colon F = \mathcal{S}(F) \textit{ with } \mathcal{S} = \{S_1, \dots, S_L\}, \\ \textit{for } L \le N \textit{ and } \mathrm{Lip}(S_i) \le m\}$$

is closed.

Proof. Let $\{F_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathcal{L}_{N,m}$ such that $\lim_{i\to\infty} F_i = F$. It suffices to find $\mathcal{S} = \{S_1, \ldots, S_L\} \subseteq \mathrm{Lip}(m)$, where $L \le N$, such that $F = \mathcal{S}(F)$. From Lemma 4.1, there exists $\{F_{i_j}\}_{j\in\mathbb{N}} \subseteq \{F_i\}_{i\in\mathbb{N}}$ such that each

$$F_{i_j} = S_{i_j}^{\rho(1)}(F_{i_j}) \cup \cdots \cup S_{i_j}^{\rho(N)}(F_{i_j}) = K_{i_j}^{\rho(1)} \cup \cdots \cup K_{i_j}^{\rho(N)} \quad (S_{i_j}^{\rho(l)}(F_{i_j}) := K_{i_j}^{\rho(l)}),$$

we have

$$\lim_{j\to\infty} F_{i_j} = F,$$

and

$$F = K^{\rho(1)} \cup \cdots \cup K^{\rho(L)}, \quad \text{with } L \le N,$$

and, for each $1 \le l \le L$, we have

$$\lim_{j\to\infty} K_{i_j}^{\rho(l)} = K^{\rho(l)}.$$

Recall that, for each l, $S_{i_j}^{\rho(l)}(F) = K_{i_j}^{\rho(l)}$, and $\{S_{i_j}^{\rho(l)}\}_{j\in\mathbb{N}}$ is uniformly bounded and equicontinuous. Thus, by restricting our attention to a subsequence of $\{i_j\}_{j\in\mathbb{N}}$ if necessary, we may assume that for each l, $S_{i_j}^{\rho(l)}(F_{i_j})$ converges to $S^{\rho(l)}$ uniformly. This follows from the Ascoli–Arzelà theorem. Thus, by Lemma 4.2, as F_{i_j} converges to F in $(\mathcal{K}(X), \mathcal{H})$ and $S_{i_j}^{\rho(l)}$ converges to $S^{\rho(l)}$ uniformly,

$$\lim_{j\to\infty} S_{i_j}^{\rho(l)}(F_{i_j}) = \lim_{j\to\infty} K_{i_j}{}^{\rho(l)} = K^{\rho(l)} = S^{\rho(l)}(F).$$

We conclude that $F = \mathcal{S}(F)$, where $\mathcal{S} = \{S^{\rho(1)}, \ldots, S^{\rho(L)}\}$. □

Theorem 4.4. *Let (X, d) be a compact metric space. The set*

$$\mathcal{T} = \{F \in \mathcal{K}(X) \colon F = \mathcal{S}(F) \textit{ with } \mathcal{S} \textit{ a finite collection of contraction maps}\}$$

is an F_σ subset of $(\mathcal{K}(X), \mathcal{H})$.

Proof. Let N and l be in $\mathbb{N}$. By Lemma 4.3 the set

$$\mathcal{L}_{N,l} = \Big\{F \in \mathcal{K}(X) \colon F = \mathcal{S}(F) \text{ with } \mathcal{S} = \{S_1, \ldots, S_L\}, \quad \text{for } L \le N \text{ and } \mathrm{Lip}(S_i) \le \frac{l-1}{l}\Big\}$$

is closed for any N and l in $\mathbb{N}$. Now,

$$\mathcal{T} = \bigcup_{N=1}^{\infty} \bigcup_{l=1}^{\infty} \mathcal{L}_{N,l}.$$ □

We now turn our attention to the case $X = [0,1]$. Here we see that $\mathcal{T}$ is quite "small" in $\mathcal{K}([0,1])$, as the typical element of $\mathcal{K}([0,1])$ is not an attractor for any contractive system defined on the interval.

Theorem 4.5. *Let $\mathcal{S} = \{S_1, \ldots, S_N\}$ be a finite set of contraction maps on* $[0,1]$, *with $F = \bigcup_{i=1}^{N} S_i(F)$. Then $F \notin \mathcal{K}^\star \setminus \mathcal{A}$.*

Proof. Let $E \in \mathcal{K}^\star \setminus \mathcal{A}$. Each $S_i : I \to I$ is a contraction map, so that each S_i is Lipschitz. Since $E \in \mathcal{K}^\star \setminus \mathcal{A}$, it follows from Theorem 2.2 that $S_i(E)$ is nowhere dense in E for each i. One concludes that $\bigcup_{i=1}^{n} S_i(E)$ is also nowhere dense in E and $E \neq \bigcup_{i=1}^{n} S_i(E)$. □

Corollary 4.6. *The collection*

$$\mathcal{T} = \{F \in \mathcal{K}([0,1]) : F = \mathcal{S}(F) \textit{ with } \mathcal{S} \textit{ a finite collection of contraction maps}\}$$

is a first category F_σ subset of $(\mathcal{K}([0,1]), \mathcal{H})$.

Proof. This follows immediately from Theorems 2.2, 4.4, and 4.5. □

Remark 4.7. This result is the best possible, since $\{E \in \mathcal{K}([0,1]) : E \text{ is finite}\}$ is dense in $\mathcal{K}([0,1])$ and contained in $\mathcal{T}$, and $\mathcal{K}^\star \setminus \mathcal{A}$ is dense in $\mathcal{K}([0,1])$ and contained in $\mathcal{K}([0,1]) \setminus \mathcal{T}$. Moreover, $\mathcal{T}$ is path-connected in $\mathcal{K}([0,1])$. In particular, let $F \in \mathcal{T}$, with h_ϵ the linear homeomorphism taking $[0,1]$ onto $[0,\epsilon]$, as in Example 3.6. Then, $h_\epsilon(F) \subseteq [0,\epsilon]$, and $h_\epsilon(F) \in \mathcal{T}$. Since $\lim_{\epsilon\to 0} h_\epsilon(F) = \{0\}$, we have a path-connected subset of $\mathcal{T}$ connecting $h_1(F) = F$ to $\lim_{\epsilon\to 0} h_\epsilon(F) = \{0\}$.

5. Attractors and homeomorphisms

In Section 4 we saw that there exists a residual subset $\mathcal{K}^\star \setminus \mathcal{A}$ of $\mathcal{K}([0,1])$ contained in the complement of

$$\mathcal{T} = \{F \in \mathcal{K}(X) : F = \mathcal{S}(F) \text{ with } \mathcal{S} \text{ a finite collection of contraction maps}\}.$$

Every element of $\mathcal{K}^\star \setminus \mathcal{A}$ is a Cantor set, and hence homeomorphic to Q, the middle thirds Cantor set. From Theorem 4.5 it follows, then, that no element of $\mathcal{K}^\star \setminus \mathcal{A}$ is an attractor, yet, by Example 3.1, every element of this residual subset of $\mathcal{K}([0,1])$ is homeomorphic to an attractor, namely Q. This observation is the starting point of this section, where we show that every nowhere dense uncountable element of $\mathcal{K}([0,1])$ is homeomorphic to an element of $\mathcal{T}$.

Lemma 5.1. *Let Q be the middle thirds Cantor set, $C \in \mathcal{K}([0,1])$ be countable, and $\epsilon > 0$. Then there exist a homeomorphic copy F_0 of C so that $F_0 \subset Q$ and $f\colon Q \to F_0$ with $f \in \mathrm{Lip}(\epsilon)$.*

Proof. We first show that there exists F a homeomorphic copy of C contained in Q.

Let $C = \{x_i\}_{i\in\mathbb{N}}$. If there exists $\delta > 0$ such that $B_\delta(x_i) \cap C = \{x_i\}$, then x_i is isolated in C. Let $\{x_{i_j}\}_{j\in\mathbb{N}} \subseteq \{x_i\}_{i\in\mathbb{N}}$ be the set of isolated points of C. Take $F_k = J_{\boldsymbol{n}|k} \cap Q$, for some $\boldsymbol{n} \in \mathbb{N}$, so that a translation F_{i_j} of F_k contains x_{i_j} and has the property that $F_{i_j} \cap C \setminus \{x_{i_j}\} = \emptyset$. It follows that $C \subset \overline{\bigcup_{j=1}^{\infty} F_{i_j}}$ and $\overline{\bigcup_{j=1}^{\infty} F_{i_j}}$ is nowhere dense and perfect. Take h a homeomorphism such that $h(\overline{\bigcup_{j=1}^{\infty} F_{i_j}}) = Q$. Then, $F = h(C)$ is a homeomorphic copy of C contained in Q.

We now show that there exists $f\colon Q \to F$ such that $f \in \mathrm{Lip}(3)$.

We define $f\colon I \to I$ such that $f \in \mathrm{Lip}(3)$ and $f(Q) = F$.

Let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be an enumeration of the complementary intervals of F. If $x \in I$, then either $x \in F$ or there exists $j \in \mathbb{N}$ such that $x \in (a_j, b_j)$. If $x \in F$, define $f(x) = x$. If there exists $j \in \mathbb{N}$ such that $x \in (a_j, b_j)$, let

$$G_{\boldsymbol{n}|k} = (c_j, d_j) \subset (a_j, b_j)$$

such that

$$|b_j - a_j| \le 3|G_{\boldsymbol{n}|k}|.$$

Define $f\colon [a_j, b_j] \to I$ continuous such that

$$f([a_j, c_j]) = a_j,$$

$$f([d_j, b_j]) = b_j,$$

and $f|[c_j, d_j]$ is linear. By construction, $f(I) = \overline{\mathrm{conv}}(F)$ and $f \in \mathrm{Lip}(3)$.

Now, take k such that $(\frac{1}{3})^{k-1} < \epsilon$. Let $F_k = J_{\boldsymbol{n}|k} \cap Q$ be a canonical portion of Q. Then there exists a linear homeomorphism $h\colon Q \to F_k$ such that $h \in \mathrm{Lip}\left((\frac{1}{3})^k\right)$ and let $F_0 = h(F)$. Since there exists $f\colon F_k \to F_0$ so that $f \in \mathrm{Lip}(3)$, it follows that $f \circ h\colon Q \to F_0$ is $\mathrm{Lip}\left((3)(\frac{1}{3})^k\right)$, that is $\mathrm{Lip}\left((\frac{1}{3})^{k-1}\right)$. $\square$

Our goal is to show that every nowhere dense uncountable element of $\mathcal{K}([0,1])$ is homeomorphic to an element of $\mathcal{T}$. Here is an approach.

Let $F \in \mathcal{K}([0,1])$ be nowhere dense and uncountable. Say, $F = P \cup C$, where $P \cap C = \emptyset$, P is the maximal perfect subset of F, and C is countable. Let

$$C_l = \{x \in F : x \le \min P\}$$

and

$$C_r = \{x \in F : x \ge \max P\}.$$

Take

$$P \cup C' = F \cap \overline{\operatorname{conv}}(P),$$

where $P \cap C' = \emptyset$. We take $Q \cup S$ to be a homeomorphic copy of $P \cup C'$ such that P is homeomorphic to Q, the middle thirds Cantor set, and S is contained in a particular set K which we describe below. We construct K so that $Q \cup K$ can be covered with a finite number of similarities of Q. It makes sense, then, that K be a collection of canonical portions of Q, arranged so that $Q \mapsto Q \cup K$ is relatively convenient to verify. In the construction of K that follows, we make extensive use of the notations found in Section 2.

Consider some $G_{\boldsymbol{n}|k}$. Let

$$F_{\boldsymbol{n}|k} = F^l_{\boldsymbol{n}|k} \cup F^r_{\boldsymbol{n}|k} \subseteq G_{\boldsymbol{n}|k},$$

where $F^l_{\boldsymbol{n}|k}$ and $F^r_{\boldsymbol{n}|k}$ each is a copy of the canonical portion $J_{\boldsymbol{n}|2(k+1)} \cap Q$, with $\min F^l_{\boldsymbol{n}|k} = \min \overline{G_{\boldsymbol{n}|k}}$ and $\max F^r_{\boldsymbol{n}|k} = \max \overline{G_{\boldsymbol{n}|k}}$. Similarly, take

$$F^0 = F^{0,l} \cup F^{0,r} \subset G = \left(\frac{1}{3}, \frac{2}{3}\right),$$

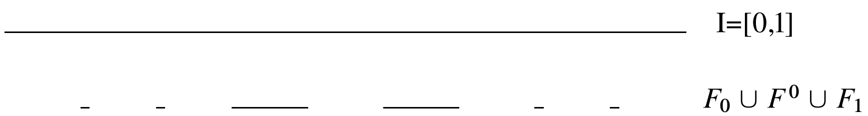
where $F^{0,l}$ and $F^{0,r}$ each is a copy of $J_{00} \cap Q$, with

$$\min F^{0,l} = \frac{1}{3} \quad \text{and} \quad \max F^{0,r} = \frac{2}{3}.$$

Let

$$K = F^0 \cup \Big(\bigcup_{\boldsymbol{n} \in \mathcal{N}} \bigcup_{k=1}^{\infty} F_{\boldsymbol{n}|k} \Big).$$

See the picture below for the first two steps of the construction of K:



Lemma 5.2. *There exists* $f \in \mathrm{Lip}(9)$ *so that* $f(Q) = Q \cup K$.

Proof. We proceed in three parts.

(1) We wish to show that there exists $f \in \mathrm{Lip}(9)$ so that $f(Q) = Q \cup K$. Let $\{K_i\}_{i\in\mathbb{N}} \subseteq \mathcal{K}$ such that $\lim_{i\to\infty} K_i = Q \cup K$ in $(\mathcal{K}, \mathcal{H})$ and for each i, $f_i(Q) = K_i$ for some $f_i \in \mathrm{Lip}(9)$. Since $\{f_i\}_{i=1}^{\infty}$ is equicontinuous, there exists $\{f_{i_j}\}_{j\in\mathbb{N}} \subseteq \{f_i\}_{i\in\mathbb{N}}$ and f such that $\| f_{i_j} - f \| \to 0$ uniformly. This follows from the Ascoli–Arzelà theorem (Theorem 2.3). Using Lemma 4.2, it follows that

$$\begin{aligned} 0 &= \lim_{j\to\infty} \mathcal{H}(f(Q), f_{i_j}(Q)) \\ &= \lim_{j\to\infty} \mathcal{H}(f(Q), K_{i_j}) \\ &= \lim_{j\to\infty} \mathcal{H}(f(Q), Q \cup K). \end{aligned}$$

Thus, it suffices to take $\{K_i\}_{i\in\mathbb{N}}$ in $\mathcal{K}$ such that $\lim_{i\to\infty} K_i = Q \cup K$ in $(\mathcal{K}, \mathcal{H})$, and determine the existence, for each i, of some $f_i \in \mathrm{Lip}(9)$ so that $f_i(Q) = K_i$.

(2) In our construction of $Q \cup K$, in each interval $G_{\boldsymbol{n}|k}$ complementary to Q, we insert $F_{\boldsymbol{n}|k}$, comprised of two canonical portions $J_{\boldsymbol{n}|2(k+1)} \cap Q$ of Q. Let

$$K_i = F^0 \cup \Big(\bigcup_{k=1}^{i} \bigcup_{\boldsymbol{n}\in\mathcal{N}} F_{\boldsymbol{n}|k} \Big),$$

so that K_i is comprised of $2\sum_{k=0}^{i} 2^k = 2(2^{i+1} - 1) = 2^{i+2} - 2$ portions, and K_i converges to $Q \cup K$ in the Hausdorff metric. Let $s = \frac{\log 2}{\log 3}$, and note that $9^s = 4$. By construction,

$$\begin{aligned} \mathcal{H}^s(K_i) &= 2\sum_{k=0}^{i} 2^k \Big(\frac{1}{3^{2(k+1)}} \Big)^s \mathcal{H}^s(Q) \\ &= \frac{2}{9^s} \Big[\sum_{k=0}^{i} \Big(\frac{2}{9^s} \Big)^k \Big] \mathcal{H}^s(Q) \\ &= \frac{1}{2} \Big[\sum_{k=0}^{i} \Big(\frac{1}{2} \Big)^k \Big] \mathcal{H}^s(Q) \\ &= \Big[1 - \Big(\frac{1}{2} \Big)^{i+1} \Big] \mathcal{H}^s(Q) \\ &< \mathcal{H}^s(Q). \end{aligned}$$

Should f_i be linear with slope 9, we have

$$\mathcal{H}^s(f_i(J_{\boldsymbol{n}|k} \cap Q)) = 4\mathcal{H}^s(J_{\boldsymbol{n}|k} \cap Q),$$

for $\boldsymbol{n} \in \mathcal{N}$ and $k \in \mathbb{N}$. In particular, we may have

$$f_i(J_{\boldsymbol{m}|2k+4} \cap Q) = F^l_{\boldsymbol{n}|k} \text{ (or, } F^r_{\boldsymbol{n}|k}).$$

Similarly, if f_i is linear with slope 9 on some $G_{\boldsymbol{n}|k} = (a, b)$, then

$$|f_i(b) - f_i(a)| = 9(b - a).$$

(3) Since $K_i \cap \left[0, \frac{1}{2}\right]$ is a reflection of $K_i \cap \left[\frac{1}{2}, 1\right]$, it suffices to determine the map $f_i \in \text{Lip}(9)$ from $Q \cap \left[0, \frac{1}{3}\right]$ to $K_i \cap \left[0, \frac{1}{3} + \frac{1}{9}\right]$. As we develop f_i, for each maximal portion $F^l_{\boldsymbol{n}|k}$ (or $F^r_{\boldsymbol{n}|k}$) of K_i, we take a canonical portion $J_{\boldsymbol{m}|2k+4} \cap Q$ of Q, so that $f_i(J_{\boldsymbol{m}|2k+4} \cap Q) = F^l_{\boldsymbol{n}|k}$ (or $F^r_{\boldsymbol{n}|k}$) is Lip(9) there. One takes an appropriate complementary interval $G_{\boldsymbol{j}|q}$ to bridge the gap between successive portions of K_i, so that $f_i|G_{\boldsymbol{j}|q} \in \text{Lip}(m)$, $m \leq 9$. Fix i; we develop f_i from right to left. Let $\boldsymbol{n} = 0111\ldots$ so that $J_{\boldsymbol{n}|4}$ is the right most component of $E_4 \cap \left[0, \frac{1}{3}\right]$. Let $f_i\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{1}{9}$, and take f_i so that $f_i(J_{\boldsymbol{n}|4} \cap Q) = F^{0,l}$. Then $f_i|J_{\boldsymbol{n}|4} \cap Q \in \text{Lip}(9)$ since $F^{0,l}$ is congruent to $J_{\boldsymbol{n}|2} \cap Q$. Now, consider (a, b), the first interval from the right complementary to portions of K_i, so that $b = \frac{1}{3}$ and $a = \max\left\{x \in K_i : x < \frac{1}{3}\right\}$. Let $G_{\boldsymbol{\sigma}|k}$ be the first complementary interval from the right of $Q \cap [0, \min J_{\boldsymbol{n}|4}]$ such that $\frac{b-a}{|G_{\boldsymbol{\sigma}|k}|} \leq 9$. If $G_{\boldsymbol{\sigma}|k} = (c, d)$, then $f_i(c) = a$, $f_i(d) = b$ and $f_i([d, \min J_{\boldsymbol{n}|4}]) = f_i(\min J_{\boldsymbol{n}|4}) = b = \frac{1}{3}$.

Now, let $F^r_{\boldsymbol{m}|s}$ be the next maximal portion from the right of K_i. By construction, $F^r_{\boldsymbol{m}|s}$ is a canonical portion $J_{\boldsymbol{m}|2(s+1)} \cap Q$. We take $J_{\boldsymbol{p}|2s+4} \cap Q$ to be the right most canonical portion of $Q \cap [0, c]$, and take f_i so that $f_i(J_{\boldsymbol{p}|2s+4} \cap Q) = F^r_{\boldsymbol{m}|s}$. Then $f_i|J_{\boldsymbol{p}|2s+4} \cap Q \in \text{Lip}(9)$.

Set $f_i([\max J_{\boldsymbol{p}|2s+4}, c]) = f_i(c) = a$. We continue this construction by choosing the first element $G_{\boldsymbol{\sigma}'|k'} = (a', b')$ from the right contained in $Q \cap [0, \min J_{\boldsymbol{p}|2s+4}]$ so that

$$\frac{\min F^r_{\boldsymbol{m}|s} - \max F^l_{\boldsymbol{m}|s}}{b' - a'} \leq 9,$$

and set $f_i(a') = \max F^l_{\boldsymbol{m}|s}$, $f_i(b') = \min F^r_{\boldsymbol{m}|s}$, with $f([b', \min J_{\boldsymbol{p}|2s+4}]) = \min F^r_{\boldsymbol{m}|s}$.

From our considerations in (2), this construction terminates after considering the $2^{i+2} - 2$ portions of K_i, and the intervals complementary to its portions. Finally, set $f_i([0, x]) = \min K_i$, where $f_i(x) = \min K_i$ from our construction. □

Corollary 5.3. *Let $k \geq 3$. Let F be a canonical component of E_k.There exists $f \in \mathrm{Lip}\left(\left(\frac{1}{3}\right)^{k-2}\right)$ such that $f(Q) = F \cap (Q \cup K)$.*

Proof. By Lemma 5.2 there exists $f_1 \in \mathrm{Lip}(9)$ so that $f_1(Q) = Q \cup K$. Let $f_2 \in \mathrm{Lip}\left(\left(\frac{1}{3}\right)^{k}\right)$ such that $f_2(Q \cup K) = F \cap (Q \cup K)$. Let $f = f_2 \circ f_1$. Then, clearly, $f \in \mathrm{Lip}\left(\left(\frac{1}{3}\right)^{k-2}\right)$ and $f(Q) = F \cap (Q \cup K)$. $\square$

Theorem 5.4. *If $F \in \mathcal{K}([0,1])$ is nowhere dense and uncountable, then there exist F', a homeomorphic copy of F, and an IFS, $\mathcal{S} = \{S_1, \ldots, S_n\}$ such that $F' = \mathcal{S}(F')$.*

Proof. Let $F = P \cup C$, where $P \cap C = \emptyset$, P is the maximal perfect subset of F, and C is countable. Set $C_l = \{x \in F : x \leq \min P\}$, $C_r = \{x \in F : x \geq \max P\}$, and $P \cup C' = F \cap \overline{\mathrm{conv}}(P)$, where $P \cap C' = \emptyset$.

Now, take $Q \cup S$ a homeomorphic copy of $P \cup C'$ such that P is homeomorphic to Q and C' is homeomorphic to S, with $S \subset K$.

Our plan is to cover $Q \cup S$ with a finite number of images of Q. First, we cover portions of the form $J_{\boldsymbol{n}|k} \cap (Q \cup S)$ with Q using Corollary 5.3. This gives rise to 2^k $\mathrm{Lip}\left(\left(\frac{1}{3}\right)^{k-2}\right)$ maps, say $g_1, g_2, \ldots, g_{2^k}$, with $g_i(Q) = J_{\boldsymbol{n}|k} \cap (Q \cup S)$ and each g_i a distinct contraction map for each element of $\{\boldsymbol{n}|k : \boldsymbol{n} \in \mathcal{N}\}$. Now, consider

$$S \cap \left[\left(\bigcup_{j=1}^{k-1} \bigcup_{\boldsymbol{n} \in \mathcal{N}} G_{\boldsymbol{n}|j}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right)\right],$$

or that part of S not contained in $\bigcup_{\boldsymbol{n} \in \mathcal{N}} J_{\boldsymbol{n}|k}$. Consider the set

$$S \cap G_{\boldsymbol{n}|j} \subset F_{\boldsymbol{n}|j} = {F_{\boldsymbol{n}|j}}^{l} \cup F^{r}_{\boldsymbol{n}|j}.$$

As one sees from the proof of Lemma 5.2, there exists a $\mathrm{Lip}\left(\left(\frac{1}{3}\right)^{2j+1}\right)$ map ${g_i}^{\star}$ so that ${g_i}^{\star}(Q) = S \cap {F_{\boldsymbol{n}|j}}^{\star}$, where $\star$ is either l or r. Similarly, there exists a $\mathrm{Lip}\left(\frac{1}{3}\right)$ map $g^{\star}$ so that $g^{\star}(Q) = S \cap (F^{0,\star})$, where $\star$ is either l or r.

This gives rise to $2^{k+1} - 2$ contraction maps, each having a Lipschitz constant less than $\frac{1}{3}$, which collectively cover that part of S contained in

$$F^0 \cup \left(\bigcup_{\boldsymbol{n} \in \mathcal{N}} \bigcup_{j=1}^{k-1} F_{\boldsymbol{n}|k}\right).$$

Again, using Lemma 5.1, there exist in Q homeomorphic copies $\widetilde{C}_l$ and $\widetilde{C}_r$ of C_l and C_r, respectively, so that $g_{C_l}(Q) = \widetilde{C}_l$, $g_{C_r}(Q) = \widetilde{C}_r$ and g_{C_l} and g_{C_r} are each Lip $\left(\left(\frac{1}{3}\right)^{k-1}\right)$. We now have a set of $2^k + (2^{k+1} - 2) + 2 = 3 \cdot 2^k$ contraction maps, that we list as $\mathcal{S}' = \{S_i'\}_{i=1}^{3\cdot 2^k}$, each defined on Q, so that

$$\bigcup_{i=1}^{3\cdot 2^k} S_i'(Q) = (Q \cup S) \cup (\widetilde{C}_l \cup \widetilde{C}_r) = Q \cup D,$$

and $Q \cup D$ is a homeomorphic copy of F.

We now extend each contraction map S_i', and, for notational simplicity, we continue to call the extension S_i', to $Q \cup D$ so that

(1) $S_i'(\widetilde{C}_l) = S_i'(\min Q)$ and $S_i'(\widetilde{C}_r) = S_i'(\max Q)$,

(2) $S_i'(F^{0,l}) = S_i'(\frac{1}{3})$ and $S_i'(F^{0,r}) = S_i'(\frac{2}{3})$, and

(3) for each $\boldsymbol{n} \in \mathcal{N}$ and $k \in \mathbb{N}$,

$$S_i'(D \cap F^l_{\boldsymbol{n}|k}) = S_i{}'(\min \overline{G}_{\boldsymbol{n}|k})$$

and

$$S_i'(D \cap F^r_{\boldsymbol{n}|k}) = S_i{}'(\max \overline{G}_{\boldsymbol{n}|k}).$$

Thus, $S_i' \colon Q \cup D \to J_{\boldsymbol{n}|k} \cap (Q \cup S)$ is Lip $\left(\left(\frac{1}{3}\right)^{k-3}\right)$, $S_i' \colon Q \cup D \to S \cap F^\star_{\boldsymbol{n}|j}$ is Lip $\left(\left(\frac{1}{3}\right)^{2j}\right)$, $S_i' \colon Q \cup D \to S \cap F^{0,\star}$ is less than Lip(1) and $S_i' \colon Q \cup D \to \widetilde{C}_\star$ is Lip $\left(\left(\frac{1}{3}\right)^{k-2}\right)$.

Take $k = 5$, and send a homeomorphic copy of $Q \cup D$ into $[0, 1]$ with a linear homeomorphism h. Our conclusion now follows, with $h(Q \cup D) = F'$, and $\mathcal{S}$ comprised of $3 \cdot 2^k$ contraction maps $S_i = h \circ S_i' \circ h^{-1}$. □

From Theorem 5.4 we know that every nowhere dense uncountable element of $\mathcal{K}([0, 1])$ is homeomorphic to an attractor, and from Theorem 3.7 that the same may be said for countable sets of finite rank. Proposition 3.8 shows that any finite union of nondegenerate closed intervals in $[0, 1]$ is itself an attractor. We conclude with the rather natural open problem.

Characterize those elements of $\mathcal{K}([0, 1])$ *which are* (*homeomorphic to*) *attractors for some contractive system* $\mathcal{S} = \{S_1, \ldots, S_N\}$.

References

[1] T. Banakh and M. Nowak, A 1-dimensional Peano continuum which is not an IFS attractor. *Proc. Amer. Math. Soc.* **141** (2013), no. 3, 931—935. MR 3003685 Zbl 1262.28001

[2] M. F. Barnsley, *Fractals everywhere.* Second edition. Revised with the assistance of and with a foreword by Hawley Rising III. Academic Press Professional, Boston, MA, 1993. MR 1231795 Zbl 0784.58002

[3] M. F. Barnsley and A. Vince, The chaos game on a general iterated function system. *Ergodic Theory Dynam. Systems* **31** (2011), no. 4, 1073–1079. MR 2818686 Zbl 1221.37079

[4] A. Blokh, A. M. Bruckner, P. D. Humke, and J. Smítal, The space of ω-limit sets of a continuous map of the interval. *Trans. Amer. Math. Soc.* **348** (1996), no. 4, 1357–1372. MR 1348857 Zbl 0860.54036

[5] A. M. Bruckner, J. B. Bruckner, and B. S. Thomson, *Real Analysis.* Prentice-Hall Iternational, Upper Saddle River, N.J., 1997. Zbl 0872.26001

[6] A. M. Bruckner and J. Smítal, A characterization of ω-limit sets of maps of the interval with zero topological entropy. *Ergodic Theory Dynam. Systems* **13** (1993), no. 1, 7–19. MR 1213076 Zbl 0788.58021

[7] A. M. Bruckner and T. H. Steele, The Lipschitz structure of continuous self-maps of generic compact sets. *J. Math. Anal. Appl.* **188** (1994), no. 3, 798–808. MR 1305486 Zbl 0820.26001

[8] R. L. Devaney, *Chaotic dynamical systems.* The Benjamin/Cummings Publishing Co., Menlo Park, CA, 1986. MR 0811850 Zbl 0632.58005

[9] E. D'Aniello, U. B. Darji, and T. H. Steele, Ubiquity of odometers in topological dynamical systems. *Topology Appl.* **156** (2008), no. 2, 240–245. MR 2475110 Zbl 1153.37003

[10] E. D'Aniello, P. D. Humke, and T. H. Steele, The space of adding machines generated by continuous self maps of manifolds. *Topology Appl.* **157** (2010), no. 5, 954–960. MR 2593709 Zbl 1191.39022

[11] E. D'Aniello and T. H. Steele, A C^1 function for which the ω-limit points are not contained in the closure of the periodic points. *Trans. Amer. Math. Soc.* **355** (2003), no. 6, 2545–2556. MR 1974002 Zbl 1026.26003

[12] E. D'Aniello and T. H. Steele, The persistence of ω-limit sets defined on compact spaces. *J. Math. Anal. Appl.* **413** (2014), no. 2, 789–799. MR 3159805 Zbl 1317.37025

[13] E. D'Aniello and T. H. Steele, Self-similarity in the collection of ω-limit sets. *Z. Anal. Anwend.* **33** (2014), no. 1, 87–100. MR 3148625 Zbl 1296.28010

[14] P. F. Duvall, Jr., and L. S. Husch, Attractors of iterated function systems. *Proc. Amer. Math. Soc.* **116** (1992), no. 1, 279–284. MR 1132850 Zbl 0765.54031

[15] K. Falconer, *Fractal geometry.* Mathematical foundations and applications. John Wiley & Sons, Chichester, 1990. MR 1102677 Zbl 0689.28003

[16] Feng, De-Jun and Wang, Yang, On the structures of generating iterated function systems of Cantor sets. *Adv. Math.* **222** (2009), no. 6, 1964–1981. MR 2562770 Zbl 1184.28013

[17] M. Hata, On the structure of self-similar sets. *Japan J. Appl. Math.* **2** (1985), no. 2, 381–414. MR 0839336 Zbl 0608.28003

[18] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747. MR 0625600 Zbl 0598.28011

[19] M. Kulczycki and M. Nowak, A class of continua that are not attractors of any IFS. *Cent. Eur. J. Math.* **10** (2012), no. 6, 2073–2076. MR 2983147 Zbl 1259.28013

[20] P. Mattila, *Geometry of sets and measures in Euclidean spaces.* Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. MR 1333890 Zbl 0819.28004

[21] M. Nowak and T. Sharek, A “shark tooth” space is a topological IFS-attractor. *Sibirsk. Mat. Zh.* **55** (2014), no. 2, 364–369. In Russian. English translation, *Sib. Math. J.* **55** (2014), no. 2, 296–300. MR 3237340 Zbl 1317.28019

[22] J. C. Oxtoby, *Measure and category.* A survey of the analogies between topological and measure spaces. Graduate Texts in Mathematics, 2. Springer, Berlin etc., 1971. MR 0393403 Zbl 0217.09201

[23] A. Schief, Separation properties for self-similar sets. *Proc. Amer. Math. Soc.* **122** (1994), no. 1, 111–115. MR 1191872 Zbl 0807.28005

[24] W. Sierpiński, *General Topology.* Translated by C. Cecilia Krieger. Mathematical Expositions, 7. University of Toronto Press, Toronto, 1952. Reprint, Dover Books on Mathematics. Dover Publications, Mineola, N.Y., 2000. MR 0050870 Zbl 0047.16003

[25] T. H. Steele, The topological structure of attractors for differentiable functions. *Real Anal. Exchange* **21** (1995/96), no. 1, 181–193. MR 1377528 Zbl 0849.26003

Received June 9, 2014

Emma D’Aniello, Dipartimento di Matematica e Fisica,
Scuola Politecnica e delle Scienze di Base, Seconda Università degli Studi di Napoli,
Viale Lincoln n. 5, 81100 Caserta, Italia

e-mail: emma.daniello@unina2.it

T.H. Steele, Department of Mathematics, Weber State University,
Ogden, UT 84408-2517, USA

e-mail: thsteele@weber.edu