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The Hausdorff dimension of sets of numbers defined by their Q-Cantor series expansions

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Abstract. Following in the footsteps of P. Erdős, A. Rényi, and T. Šalát we compute the Hausdorff dimension of sets of numbers whose digits with respect to their Q-Cantor series expansions satisfy various statistical properties. In particular, we consider difference sets associated with various notions of normality and sets of numbers with a prescribed range of digits.

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1. Introduction

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first done by P. Erdős and A. Rényi in [7] and [8] and by A. Rényi in [19], [20], and [21] and by P. Turán in [24].

Denote by $N_n^b(B,x)$ the number of times a block B occurs with its starting position no greater than n in the b-ary expansion of x.

Definition 1.1. A real number x is *normal in base b* if for all k and blocks B in base b of length k, one has

$$\lim_{n\to\infty} \frac{N_n^b(B,x)}{n} = b^{-k}. \tag{1.1}$$

A number x is *simply normal in base b* if (1.1) holds for $k = 1$.

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Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers are normal in all bases. The best known example of a number that is normal in base 10 is due to Champernowne [5]. The number

$$H_{10} = 0.1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ldots,$$

formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. Any H_b, formed similarly to H_{10} but in base b, is known to be normal in base b. Since then, many examples have been given of numbers that are normal in at least one base. One can find a more thorough literature review in [12, 6, 3].

We wish to mention one of the most fundamental and important results relating to normal numbers in base b. The following is due to D. D. Wall in his Ph.D. dissertation [29].

Theorem 1.2 (D. D. Wall). *A real number x is normal in base b if and only if the sequence $(b^n x)$ is uniformly distributed mod* 1.

The Q-Cantor series expansions, first studied by G. Cantor in [4], are a natural generalization of the b-ary expansions. G. Cantor's motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [11]. Let $\mathbb{N}_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in \mathbb{N}_2^{\mathbb{N}}$, then we say that Q is a *basic sequence*. Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the *Q-Cantor series expansion* of a real number x is the (unique)[1] expansion of the form

$$x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \tag{1.2}$$

where $E_0 = \lfloor x \rfloor$ and E_n is in $\{0, 1, \ldots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (1.2) with the notation $x = E_0.E_1E_2E_3\ldots$ with respect to Q.

A *block* is an ordered tuple of non-negative integers, a *block of length k* is an ordered k-tuple of integers, and *block of length k in base b* is an ordered k-tuple of integers in $\{0, 1, \ldots, b-1\}$. Let $N_n^Q(B, x)$ denote the number of occurrences of the block B in the digits of the Q-Cantor series expansion of x up to position n.

[1] Uniqueness can be proven in the same way as for the b-ary expansions.

Let

$$Q_n^{(k)} := \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}$$

and

$$T_{Q,n}(x) := \left(\prod_{j=1}^{n} q_j\right) x \pmod{1}.$$

A. Rényi [20] defined a real number x to be *normal* with respect to Q if for all blocks B of length 1,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(1)}} = 1. \tag{1.3}$$

If $q_n = b$ for all n and we restrict B to consist of only digits less than b, then (1.3) is equivalent to *simple normality in base* b, but not equivalent to *normality in base* b. A basic sequence Q is k*-divergent* if

$$\lim_{n\to\infty} Q_n^{(k)} = \infty,$$

fully divergent if Q is k-divergent for all k, and k*-convergent* if it is not k-divergent. A basic sequence Q is *infinite in limit* if $q_n \to \infty$.

Motivated by Theorem 1.2, we make the following definitions of normality for Cantor series expansions.

Definition 1.3. A real number x is Q*-normal of order* k if for all blocks B of length k,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1.$$

We let $\mathcal{N}_k(Q)$ be the set of numbers that are Q-normal of order k. The real number x is Q*-normal* if

$$x \in \mathcal{N}(Q) := \bigcap_{k=1}^{\infty} \mathcal{N}_k(Q).$$

Definition 1.4. A real number x is Q*-ratio normal of order* k (here we write $x \in \mathcal{RN}_k(Q)$) if for all blocks B_1 and B_2 of length k

$$\lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = 1.$$

We say that x is Q*-ratio normal* if

$$x \in \mathcal{RN}(Q) := \bigcap_{k=1}^{\infty} \mathcal{RN}_k(Q).$$

Definition 1.5. A real number x is called *Q-distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Let $\mathcal{DN}(Q)$ be the set of Q-distribution normal numbers.

We note that by Theorem 1.2, the analogous versions of the above definitions are equivalent for the b-ary expansions.

It was proven in [17] that the directed graph in Figure 1 gives the complete containment relationships between these notions when Q is infinite in limit and fully divergent. The vertices are labeled with all possible intersections of one, two, or three choices of the sets $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$, where we know that $\mathcal{N}(Q) = \mathcal{N}(Q) \cap \mathcal{RN}(Q)$ and $\mathcal{N}(Q) \cap \mathcal{DN}(Q) = \mathcal{N}(Q) \cap \mathcal{DN}(Q) \cap \mathcal{RN}(Q)$. The set labeled on vertex A is a subset of the set labeled on vertex B if and only if there is a directed path from A to B. For example, $\mathcal{N}(Q) \cap \mathcal{DN}(Q) \subseteq \mathcal{RN}(Q)$, so all numbers that are Q-normal and Q-distribution normal are also Q-ratio normal.

We remark that all inclusions suggested from Figure 1 are either easily proven ($\mathcal{N}(Q) \subseteq \mathcal{RN}(Q)$) or are trivial. The difficulty comes in showing a lack of inclusion. The most challenging of these is to prove that there is a basic sequence Q where $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q) \neq \emptyset$.

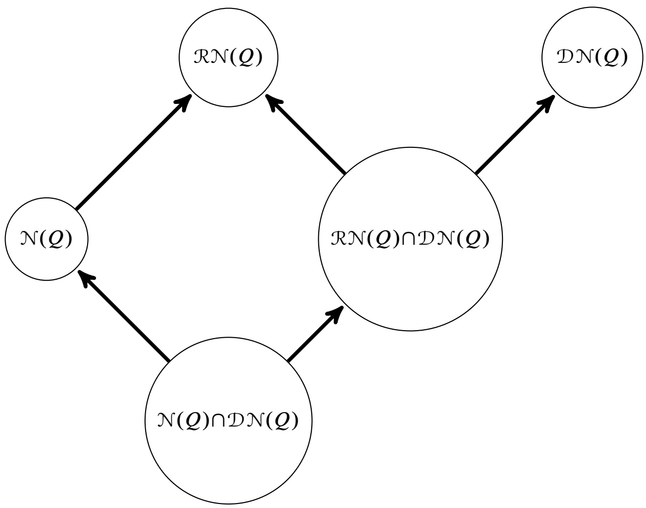

Figure 1

It follows from a well known result of H. Weyl [31, 32] that $\mathcal{DN}(Q)$ is a set of full Lebesgue measure for every basic sequence Q. We will need the following result of the second author [16] later in this paper.

Theorem 1.6. *Suppose*[2] *that* Q *is infinite in limit. Then* $\mathcal{N}_k(Q)$ *and* $\mathcal{RN}_k(Q)$ *are of full measure if and only if* Q *is* k*-divergent. The sets* $\mathcal{N}(Q)$ *and* $\mathcal{RN}(Q)$ *are of full measure if and only if* Q *is fully divergent.*

Based on Figure 1 and Theorem 1.6 it is natural to ask for the Hausdorff dimension of the difference sets. It was proven in [18] that for every basic sequence Q that is infinite in limit

$$\dim_{\mathrm{H}}(\mathcal{DN}(Q)\backslash\mathcal{N}(Q)) = \dim_{\mathrm{H}}(\mathcal{DN}(Q)\backslash\mathcal{RN}(Q)) = 1.$$

Using different methods we will prove the following theorem.

Theorem 1.7. *Every non-empty set formed by taking the difference of two sets listed in Figure 1 has full Hausdorff dimension for every* Q *that is infinite in limit, except possibly the set* $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$.

It will be shown that the set $\mathcal{N}(Q)\backslash\mathcal{DN}(Q)$ has full Hausdorff dimension for a more restricted class of basic sequences in Theorem 3.5. We should note that we can not hope to establish $\dim_{\mathrm{H}}(\mathcal{N}(Q)\backslash\mathcal{DN}(Q)) = 1$ for all Q that are infinite in limit. This follows from the result in [16] that $\mathcal{N}(Q) = \emptyset$ when Q is infinite in limit and not fully divergent.

A surprising property of Q-normality of order k is that we may not conclude that $\mathcal{N}_k(Q) \subseteq \mathcal{N}_j(Q)$ for all $j < k$ like we may for the b-ary expansions. In fact, it was shown in [14] that for every k there exists a basic sequence Q and a real number x such that $\mathcal{N}_k(Q)\backslash\bigcup_{j=1}^{k-1}\mathcal{N}_j(Q)$ is non-empty. Thus, we will have to be more careful in stating exactly what our theorems prove since lack of Q-normality of order 2 does not imply lack of Q-normality of order 338, for example. Furthermore, we will greatly expand on this result in Theorem 3.6 where for each natural number ℓ we exhibit a class of basic sequences such that

$$\dim_{\mathrm{H}}\Big(\bigcap_{j=\ell}^{\infty}\mathcal{N}_j(Q)\Big\backslash\bigcup_{j=1}^{\ell-1}\mathcal{N}_j(Q)\Big) = 1.$$

For $x = E_0.E_1E_2\cdots$ with respect to Q, define the set

$$\mathcal{S}_Q(x) = \{E_1, E_2, E_3, \dots\}.$$

P. Erdős and A. Rényi [7] proved the following theorems.

[2] Early work in this direction has been done by A. Rényi [20], T. Šalát [27], and F. Schweiger [23].

Theorem 1.8 (P. Erdős and A. Rényi). *If Q is 1-convergent, then $\mathcal{S}_Q(x)$ has density 0 for almost every real number x.*

Theorem 1.9 (P. Erdős and A. Rényi). *For $x = E_0.E_1E_2\cdots$ with respect to Q, let $d_n(x)$ denote the number of different numbers in the sequence $E_1, \ldots, E_n$. If Q is 1-convergent, then for almost every x we have*

$$\lim_{n\to\infty} \frac{d_n(x)}{n} = 1.$$

If Q is infinite in limit and 1-divergent, we have that almost every real number is simply Q-normal by Theorem 1.6. On the other hand, if Q is 1-convergent we have that almost every real number is not simply Q-normal in a particularly strong sense by Theorem 1.8.

It should be noted that T. Šalát [28] considered sets related to those mentioned in Theorem 1.8 and Theorem 1.9. We will need the following definition from [2].

Definition 1.10. For $S \subseteq \mathbb{Z}$, define the *mass dimension of S* to be the limit

$$\dim_{\mathrm{M}}(S) = \lim_{n\to\infty} \frac{\log \#(S \cap (-n/2, n/2))}{\log n},$$

if it exists.

We note that an *upper mass dimension* and a *lower mass dimension* may be defined similarly by changing the limit in Definition 1.10 to a lim sup or a lim inf.

For non-empty $S \subseteq \mathbb{N}_0$, define

$$\mathcal{W}_Q(S) = \left\{x \in \mathbb{R} : \mathcal{S}_Q(x) = S\right\}.$$

We will build on Theorem 1.8 and Theorem 1.9 by proving the following theorem.

Theorem 1.11. *If Q is infinite in limit, $\lim_{n\to\infty} \frac{\log q_n}{\sum_{i=1}^{n} \log q_i} = 0$, and $S \subseteq \mathbb{N}$ such that $\min S < \min Q$ and $\dim_M(S)$ exists, then*

$$\dim_H\left(\mathcal{W}_Q(S)\right) = \dim_M(S).$$

T. Šalát proved in [26] that under some conditions on the basic sequence Q the set of real numbers whose digits in their Q-Cantor expansion is bounded has zero Hausdorff dimension. We remark that his result may be sharpened with his conditions weakened by use of our Lemma 2.4 instead of Satz 1 from [25]. The proof of this otherwise follows identically to his original proof, so we do not record it in this paper.

We remark that some of the techniques developed in this paper and Lemma 2.4 are used to study fractals associated with normality-preserving operations in [1]. Interesting results of a slightly different flavor may be found in [15, 30, 9].

2. Lemmata

Let (n_k) be a sequence of positive integers and (c_k) be a sequence of positive numbers such that $n_k \geq 2$, $0 < c_k < 1$, $n_1 c_1 \leq \delta$, and $n_k c_k \leq 1$, where δ is a positive real number. For any k, let

$$D_k = \{(i_1, \dots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\},$$

and

$$D = \bigcup D_k,$$

where $D_0 = \emptyset$. If $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$, $\tau = (\tau_1, \dots, \tau_m) \in D_m$, put

$$\sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m).$$

Definition 2.1. Suppose J is a closed interval of length δ. The collection of closed subintervals $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ of J has *homogeneous Moran structure* if

(1) $J_\emptyset = J$;

(2) for all $k \geq 0$, $\sigma \in D_k, J_{\sigma*1}, \dots, J_{\sigma*n_{k+1}}$ are subintervals of J_σ and $\mathring{J}_{\sigma*i} \cap \mathring{J}_{\sigma*j} = \emptyset$ for $i \neq j$;

(3) for all $k \geq 1$, and $\sigma \in D_{k-1}, 1 \leq j \leq n_k$, $c_k = \frac{\lambda(J_{\sigma*j})}{\lambda(J_\sigma)}$.

Suppose that $\mathcal{F}$ is a collection of closed subintervals of J having homogeneous Moran structure. Let

$$E(\mathcal{F}) = \bigcap_{k\geq 1} \bigcup_{\sigma \in D_k} J_\sigma.$$

We say $E(\mathcal{F})$ is a *homogeneous Moran set determined by* $\mathcal{F}$, or it is a *homogeneous Moran set determined by* J, (n_k), (c_k). We will need the following theorem of D. Feng, Z. Wen, and J. Wu from [10].

Theorem 2.2 (D. Feng, Z. Wen, and J. Wu). *If S is a homogeneous Moran set determined by J, (n_k), (c_k), then*

$$\liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}} \leq \dim_H(S) \leq \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}.$$

Given basic sequences $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, sequences of non-negative integers $s = (s_i), t = (t_i), \upsilon = (\upsilon_i)$, and $F = (F_i)$, and a sequence of sets $I = (I_i)$ such that $I_i \subseteq \{0, 1, \dots, \beta_i - 1\}$, define the set $\Theta(\alpha, \beta, s, t, \upsilon, F, I)$ as follows. Let $Q = Q(\alpha, \beta, s, t, \upsilon) = (q_n)$ be the following basic sequence:

$$[[\alpha_1]^{s_1} [\beta_1]^{t_1}]^{\upsilon_1} [[\alpha_2]^{s_2} [\beta_2]^{t_2}]^{\upsilon_2} [[\alpha_3]^{s_3} [\beta_3]^{t_3}]^{\upsilon_3} \dots . \quad (2.1)$$

where $[\alpha]^s$ is the sequence consisting of α repeated s times. Define the function

$$c(n) = \max\left\{t : \sum_{j=1}^{i(n)-1} \upsilon_j(s_j + t_j) + t(s_{i(n)} + t_{i(n)}) > n\right\}$$

Set

$$\Phi_\alpha(i, c, d) = \sum_{j=1}^{i-1} \upsilon_j s_j + c s_i + d$$

where $0 \le c < \upsilon_i$ and $0 \le d < s_i$ and let the functions $i_\alpha(n)$, $c_\alpha(n)$, and $d_\alpha(n)$ be such that

$$\Phi_\alpha^{-1}(n) = (i_\alpha(n), c_\alpha(n), d_\alpha(n)).$$

Note this is possible since Φ_α is a bijection from

$$\mathcal{U} = \left\{(i, c, d) \in \mathbb{N}^3 : 0 \le c < \upsilon_i, 0 \le d < s_i\right\}$$

to $\mathbb{N}$. Define the functions

$$G(n) = \sum_{j=1}^{i_\alpha(n)-1} \upsilon_j(s_j + t_j) + c_\alpha(n)\left(s_{i_\alpha(n)} + t_{i_\alpha(n)}\right) + d_\alpha(n)$$

and

$$g(n) = \min\{t : G(t) \ge n\}.$$

Note that $i_\alpha(g(n)) = i(n)$ and $c_\alpha(g(n)) = c(n)$. Furthermore, define

$$C_\alpha(n) = \left(\sum_{j=1}^{i_\alpha(n)-1} \upsilon_j\right) + c_\alpha(n).$$

We consider the condition on n

$$\left(n - \sum_{j=1}^{i(n)-1} \upsilon_j(s_j + t_j)\right) \mod (s_{i(n)} + t_{i(n)}) \ge s_{i(n)}. \tag{2.2}$$

Define the sets

$$V(n) = \begin{cases} I_{i(n)} & \text{if condition (2.2) holds,} \\ \{F_{G(n)}\} & \text{otherwise.} \end{cases}.$$

That is, we choose digits from $I_{i(n)}$ in positions corresponding to the bases obtained from the sequence β and choose a specific digit from F for the bases obtained from the sequence α. Set

$$\Theta(\alpha,\beta,s,t,\upsilon,F,I)=\{x=0.E_1E_2\ldots \text{ with respect to } Q:\ E_n\in V(n)\}\,.$$

We will need the following basic lemma to prove Lemma 2.4 and elsewhere in this paper.

Lemma 2.3. *Let L be a real number and* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be two sequences of positive real numbers such that*

$$\sum_{n=1}^{\infty} b_n=\infty \quad \textit{and} \quad \lim_{n\to\infty}\frac{a_n}{b_n}=L.$$

Then

$$\lim_{n\to\infty}\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}=L.$$

Lemma 2.4. *Given basic sequences* $\alpha=(\alpha_i)$ *and* $\beta=(\beta_i)$*, sequences of non-negative integers* $s=(s_i), t=(t_i), \upsilon=(\upsilon_i)$*, and* $F=(F_i)$*, and a sequence of sets* $I=(I_i)$ *such that* $I_i\subseteq\{0,1,\ldots,\beta_i-1\}$ *such that the following conditions hold*:

$$\lim_{n\to\infty}\frac{s_n\log\alpha_n}{\sum_{i=1}^{n-1}\upsilon_i t_i\log\beta_i}=0 \tag{2.3}$$

and

$$\lim_{n\to\infty}\frac{s_n\log\alpha_n}{t_n\log\beta_n}=0. \tag{2.4}$$

Then

$$\dim_H(\Theta(\alpha,\beta,s,t,\upsilon,F,I))=\gamma:=\lim_{n\to\infty}\frac{\log|I_n|}{\log\beta_n}.$$

Proof. Note that $\Theta(\alpha, \beta, s, t, \upsilon, F, I)$ is a homogeneous Moran set with

$$n_k = \begin{cases} |I_k| & \text{if } q_k = \beta_{i(k)} \\ 1 & \text{if } q_k = \alpha_{i(k)} \end{cases}$$

and

$$c_k = \frac{1}{q_k}.$$

Set

$$\begin{aligned} \mathfrak{N} = &\sum_{j=1}^{i(n)-1} \sum_{k=1}^{\upsilon_j} [t_j \log \beta_j + s_j \log \alpha_j] \\ &+ \sum_{j=1}^{b(n)} [t_{i(n)} \log \beta_{i(n)} + s_{i(n)} \log \alpha_{i(n)}] \\ &+ s_{i(n)} \log \alpha_{i(n)}. \end{aligned}$$

We get

$$\begin{aligned} &\dim_{\mathrm{H}} (\Theta(\alpha, \beta, s, t, \upsilon, F, I)) \\ &\geq \liminf_{k \to \infty} \frac{\log n_1 n_2 \ldots n_k}{-\log c_1 c_2 \ldots c_{k+1} n_{k+1}} \\ &\geq \lim_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} \sum_{k=1}^{\upsilon_j} t_j \log |I_i| + \sum_{j=1}^{b(n)} t_{i(n)} \log |I_{i(n)}|}{\mathfrak{N}} \\ &= \lim_{n \to \infty} \frac{\left(\sum_{j=1}^{i(n)-1} \upsilon_j t_j \gamma \log \beta_j \right) + b(n) t_{i(n)} \gamma \log \beta_{i(n)}}{\mathfrak{N}} \end{aligned}$$

(where we have used Lemma 2.3)

$$= \lim_{n \to \infty} \frac{\left(\sum_{j=1}^{i(n)-1} \upsilon_j t_j \gamma \log \beta_j \right) + b(n) t_{i(n)} \gamma \log \beta_{i(n)}}{\left(\sum_{j=1}^{i(n)-1} \upsilon_j t_j \log \beta_j \right) + b(n) t_{i(n)} \log \beta_{i(n)} + s_{i(n)} \log \alpha_{i(n)}}$$

(which follows from (2.4))

$$= \lim_{n\to\infty} \frac{\left(\sum_{j=1}^{i(n)-1} \upsilon_j t_j \gamma \log \beta_j\right) + b(n) t_{i(n)} \gamma \log \beta_{i(n)}}{\left(\sum_{j=1}^{i(n)-1} \upsilon_j t_j \log \beta_j\right) + b(n) t_{i(n)} \log \beta_{i(n)}}$$

$$= \gamma.$$

which we get from (2.3). The upper bound follows from a similar calculation. □

For a sequence of real numbers $X = (x_n)$ with $x_n \in [0, 1)$ and an interval $I \subseteq [0, 1]$, define

$$A_n(I, X) = \#\{i \le n : x_i \in I\}.$$

We will need the following standard definition and lemma that we quote from [12].

Definition 2.5. Let $X = (x_1, \ldots, x_N)$ be a finite sequence of real numbers. The number

$$D_N = D_N(X) = \sup_{0\le\alpha\le\beta\le 1} \left| \frac{A_N([\alpha, \beta), X)}{N} - (\beta - \alpha) \right|$$

is called the *discrepancy* of the sequence ω.

It is well known that a sequence X is uniformly distributed mod 1 if and only if $D_N(X) \to 0$.

Lemma 2.6. *Let $x_1, x_2, \ldots, x_N$ and $y_1, y_2, \ldots, y_N$ be two finite sequences in $[0, 1)$. Suppose $\epsilon_1, \epsilon_2, \ldots, \epsilon_N$ are non-negative numbers such that $|x_n - y_n| \le \epsilon_n$ for $1 \le n \le N$. Then, for any $\epsilon \ge 0$, we have*

$$|D_N(x_1, \ldots, x_N) - D_N(y_1, \ldots, y_N)| \le 2\epsilon + \frac{\overline{N}(\epsilon)}{N},$$

where $\overline{N}(\epsilon)$ denotes the number of n, $1 \le n \le N$, such that $\epsilon_n > \epsilon$.

3. Proofs

We will compute the Hausdorff dimension of difference sets formed by taking unions or intersections of the sets $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$. Note that the non-empty sets which must be considered in Theorem 1.7 are

$$\mathcal{N}(Q) \setminus \mathcal{DN}(Q), \quad \mathcal{RN}(Q) \setminus \mathcal{N}(Q), \quad \mathcal{RN}(Q) \setminus \mathcal{DN}(Q),$$
$$\mathcal{DN}(Q) \setminus \mathcal{N}(Q), \quad \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q), \quad \mathcal{DN}(Q) \setminus \mathcal{RN}(Q),$$
$$\mathcal{RN}(Q) \setminus (\mathcal{N}(Q) \cap \mathcal{DN}(Q)).$$

Note that there are many other ways of writing these sets, but we choose the simplest representation.

The Hausdorff dimension for each of the sets except for $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ will be computed in Theorem 3.1, Theorem 3.3, and Theorem 3.4 or follow from these theorems and Figure 1. For example $\mathcal{RN}(Q)\setminus\mathcal{N}(Q)$ contains $\mathcal{RN}(Q)\cap\mathcal{DN}(Q)\setminus\mathcal{N}(Q)$ which has full Hausdorff dimension by Theorem 3.1. We will compute the Hausdorff dimension of $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ for restricted basic sequences in Theorem 3.5.

Theorem 3.1. *If Q is infinite in limit, then*

$$\dim_H(\mathcal{RN}(Q) \cap \mathcal{DN}(Q)\setminus\mathcal{N}(Q)) = 1.$$

Proof of Theorem 3.1. Let $P = (p_i)$ with $p_i = \lfloor \log i \rfloor + 2$ and $\xi \in \mathcal{N}(P)$ with $\xi = .F_1F_2\cdots$ with respect to P. Fix a sequence $X = (x_n)$ that is uniformly distributed modulo 1. Define the sequences

$$L_0 = 0;$$

$$\nu_n = \inf\left\{ t : \frac{\sum_{m=0}^{n-1} \log q_{L_{n-1}+m}}{\sum_{m=0}^{j-L_{n-1}-1} \log q_{L_{n-1}+m}} < \frac{1}{n}, \text{ for all } j \geq t \right\},$$

$$\upsilon_{n,k} = \inf\left\{ t : \frac{Q_n^{(k)}}{\sum_{m=1}^{j} P_{m-k+1}^{(k)}} < \frac{1}{n}, \text{ for all } j \geq t \right\},$$

$$L_n = \sup\{\inf\{t : \log(q_j) > n, \text{ for all } j \geq t\}, L_{n-1} + n^2, L_{n-1} + \nu_n, \sup_{k \leq n}\{\upsilon_{n,k}\}\},$$

and set

$$i(n) = \max\{j : L_j \le n\}.$$

Note that ν_n and $\upsilon_{n,k}$ are finite since Q is infinite in limit and P is fully divergent. Define the set

$$S = \bigcup_{n=1}^{\infty} \{L_n, L_n + 1, \dots, L_n + n - 1\}.$$

Note that this set has density 0 since

$$\frac{\#S \cap \{1, \dots, n\}}{n} \le \frac{\sum_{j=1}^{i(n)+1} j}{\sum_{j=1}^{i(n)} L_j - L_{j-1}} \le \frac{\sum_{j=1}^{i(n)+1} j}{\sum_{j=1}^{i(n)} j + j^2} \longrightarrow 0 \quad \text{as } n \text{ goes to infinity.}$$

Define the intervals

$$V(n) = \begin{cases} [F_{n-L_{i(n)}}, F_{n-L_{i(n)}} + 1) & \\ & \text{if } n \in [L_{i(n)}, \dots, L_{i(n)} + i(n)], \\ [x_n q_n - \omega_n, x_n q_n + \omega_n) \cap [\lceil \log i(n) \rceil, q_n - 1] & \\ & \text{otherwise,} \end{cases}$$

where

$$\omega_n = q_n^{1-\epsilon_n} \quad \text{and} \quad \epsilon_n = \frac{\min\{\log q_1 \cdots q_{n-1}, \log q_n\}^{1/2}}{\log q_n}$$

Set

$$\Lambda_Q = \{x = .E_1 E_2 \cdots \text{ with respect to } Q : E_n \in V(n)\}.$$

We claim that $\Lambda_Q \subseteq \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q)$ and $\dim_{\mathrm{H}}(\Lambda_Q) = 1$. Let $x \in \Lambda_Q$ and let B be a block of length k. Note that by the definition of L_n, there are only finitely many values $n \in \mathbb{N} \backslash S$ such that B occurs at position n in the Q-Cantor series expansion of x. This is because all digits E_n with $n \in \mathbb{N} \backslash S$ must be greater than $\lceil \log i(n) \rceil$ by the definition of $V(n)$ and since $i(n)$ tends to infinity as n does. Thus, if m is the maximum digit for the block B, we have that for $n \in \mathbb{N} \backslash S$ with $i(n) > m$, that $E_n > m$. Thus

$$N_n^Q(B, x) = \sum_{i=1}^{i(n)} N_{i-k+1}^P(B, \xi) + O(1).$$

So for any two blocks B_1 and B_2 of length k, we have

$$\lim_{n\to\infty}\frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} = \lim_{n\to\infty}\frac{\sum_{i=1}^{i(n)} N_{i-k+1}^P(B_1,\xi)+O(1)}{\sum_{i=1}^{i(n)} N_{i-k+1}^P(B_2,\xi)+O(1)} = \lim_{n\to\infty}\frac{N_{n-k+1}^P(B_1,\xi)}{N_{n-k+1}^P(B_2,\xi)} = 1.$$

Thus $x \in \mathcal{RN}(Q)$.

Consider the sequence $Y = \left(\frac{E_n}{q_n}\right)$. For $n \in \mathbb{N}\backslash S$, we have

$$\left|\frac{E_n}{q_n} - x_n\right| < \frac{\omega_n}{q_n},$$

which tends to 0 as n goes to infinity. We therefore have for $\epsilon > 0$ that

$$\bar{N}(\epsilon) = O(1) + \#S \cap \{1,\ldots,N\}.$$

Thus by Lemma 2.6

$$|D_N(X) - D_N(Y)| < 2\epsilon + \frac{O(1)}{N} + \frac{\#S \cap \{1,\ldots,N\}}{N} < 3\epsilon$$

if N is sufficiently large. Since the inequality holds for all $\epsilon > 0$, we have that $\left(\frac{E_n}{q_n}\right)$ is uniformly distributed mod 1. Thus $x \in \mathcal{DN}(Q)$.

Note that

$$\lim_{n\to\infty}\frac{N_n^Q(B,x)}{\sum_{i=1}^{i(n)} P_{i-k+1}^{(k)}} = 1.$$

However,

$$\lim_{n\to\infty}\frac{Q_n^{(k)}}{\sum_{i=1}^{i(n)} P_{i-k+1}^{(k)}} = 0$$

by the definition of L_n, so $x \notin \bigcup_{k=1}^{\infty} \mathcal{N}_k(Q)$ and in particular $x \notin \mathcal{N}(Q)$. Thus $\Lambda_Q \subseteq \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q)$.

Evidently Λ_Q is a homogeneous Moran set with $n_k = |V(k)|$ and $c_k = \frac{1}{q_k}$. Thus

$$\begin{aligned}
\dim_{\mathrm{H}}\left(\Lambda_Q\right) &\geq \liminf_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_{k+1} n_{k+1}} \\
&= \liminf_{n\to\infty} \frac{\sum_{i=1}^{k} \chi_{\mathbb{N}\setminus S}(i)\,(1-\epsilon_i)\log q_i}{\sum_{i=1}^{k} \log q_i + \log q_{k+1}} \\
&= \liminf_{n\to\infty} \left(1 - \frac{\sum_{j=1}^{i(n)} \sum_{k=0}^{j-1} \log q_{L_j+k}}{\sum_{j=1}^{i(n)} \sum_{k=0}^{i(j)-i(j-1)} \log q_{L_j+k}} \right) \\
&= \liminf_{n\to\infty} \left(1 - \frac{\sum_{i=0}^{n-1} \log q_{L_n+i}}{\sum_{i=0}^{L_n-L_{n-1}} \log q_{L_n+i}} \right) \\
&= 1
\end{aligned}$$

by the definition of ν_n and L_n. Thus

$$\dim_{\mathrm{H}}\left(\Lambda_Q\right) = 1 \text{ and } \dim_{\mathrm{H}}\left(\mathcal{RN}(Q) \cap \mathcal{DN}(Q)\setminus\mathcal{N}(Q)\right) = 1. \qquad \square$$

Corollary 3.2. *If Q is infinite in limit, then* $\dim_H(\mathcal{RN}(Q)) = 1$.

Theorem 3.3. *If Q is infinite in limit, then*

$$\dim_H\Big(\mathcal{RN}(Q)\setminus\Big(\bigcup_{j=1}^{\infty} \mathcal{N}_j(Q) \cup \mathcal{DN}(Q)\Big)\Big) = 1.$$

Proof. The proof is the same as Theorem 3.1, but with $X = (x_n)$ a sequence that is not uniformly distributed mod 1. $\square$

Theorem 3.4. *If Q is infinite in limit, then*

$$\dim_H\Big(\mathcal{DN}(Q)\backslash\bigcup_{j=1}^{\infty}\mathcal{RN}_j(Q)\Big)=1.$$

Proof. The proof is the same as Theorem 3.1, but we choose $\xi=.E_1E_2\ldots$ with respect to P such that the digit 0 never occurs. □

We will need to refer to the following four conditions.

$$\lim_{n\to\infty}\frac{t_n\alpha_n^k}{s_n\beta_n^k}=0, \tag{3.1}$$

$$\lim_{n\to\infty}\frac{t_n\alpha_n^k}{s_n\beta_n^k}>0, \tag{3.2}$$

$$\lim_{n\to\infty}\frac{\alpha_n^k}{s_n}=0, \tag{3.3}$$

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\upsilon_i s_i}{\sum_{i=1}^{n}\upsilon_i(s_i+t_i)}=0. \tag{3.4}$$

Theorem 3.5. *Suppose that $Q=Q(\alpha,\beta,s,t,\upsilon)$ is infinite in limit, t-divergent (resp. fully divergent), and satisfies conditions (2.3), (2.4), (3.1) for all $k\le t$, (3.3), and (3.4). If $\alpha_i=o(\beta_i)$, then*

$$\dim_H\Big(\bigcap_{j=1}^{t}\mathcal{N}_j(Q)\backslash\mathcal{DN}(Q)\Big)=1 \quad (\textit{resp.}\ \dim_H(\mathcal{N}(Q)\backslash\mathcal{DN}(Q))=1).$$

Proof. We will prove the statement for when $Q=(q_n)$ is fully divergent. The proof for when Q is k-divergent follows similarly. Define the basic sequence P by

$$P=[\alpha_1]^{s_1\upsilon_1}[\alpha_2]^{s_2\upsilon_2}[\alpha_3]^{s_3\upsilon_3}[\alpha_4]^{s_4\upsilon_4}\ldots.$$

We note that P is fully divergent since Q is fully divergent. By Theorem 1.6, there exists a real number $\xi=E_0.E_1E_2\cdots$ with respect to P that is an element of $\mathcal{N}(P)$. Set

$$I_i=\{\alpha_i,\alpha_i+1,\ldots,\lfloor\beta_i^{1-(1/\log\beta_i)^{1/2}}\rfloor+1\}$$

and $F_i = E_i$. Note that

$$\lim_{n\to\infty} \frac{\log |I_n|}{\log \beta_n} = 1,$$

so $\dim_H(\Theta(\alpha, \beta, s, t, \upsilon, F, I)) = 1$ by Lemma 2.4. We now wish to show that

$$\Theta(\alpha, \beta, s, t, \upsilon, F, I) \subseteq \mathcal{N}(Q) \backslash \mathcal{DN}(Q).$$

Let both k and n be natural numbers, B be a block of length k, and take $x \in \Theta(\alpha, \beta, s, t, \upsilon, F, I)$. We wish to show that

$$N^P_{g(n)}(B, \xi) - kC_\alpha(g(n)) \le N^Q_n(B, x) \le N^P_{g(n)}(B, \xi) + O(1).$$

Let m be the maximum digit in the block B. Since $\min I_i \to \infty$, we know that there are only finitely many indices i such that $m > \min I_i$. Thus, there are at most finitely many occurrences of B starting at position n when $q_n = \beta_{i(n)}$. If every occurrence of B in ξ occurs at the corresponding place in x, then we have

$$N^P_{g(n)}(B, \xi) + O(1) = N^Q_n(B, x).$$

If some of the occurrences of B in ξ do not occur in the corresponding places in x, then we have $N^Q_n(B, x) \le N^P_{g(n)}(B, \xi)$.

On the other hand, the total number of places up to position n where B can occur in the P-Cantor series expansion of ξ but B does not occur in the corresponding positions in the Q-Cantor series expansion of x is at most $kC_\alpha(n)$, the total length of the last k terms of the substrings $[\alpha_i]^{s_i}$ of P. Thus

$$N^P_{g(n)}(B, \xi) - kC_\alpha(g(n)) \le N^Q_n(B, x) \le N^P_{g(n)}(B, \xi) + O(1).$$

Many of the following calculations use Lemma 2.3. Note that

$$P^{(k)}_n = \sum_{j=1}^{i_\alpha(n)-1} \frac{s_j \upsilon_j}{\alpha^k_j} + \frac{s_{i(n)} b_\alpha(n)}{\alpha^k_{i_\alpha(n)}}$$

and

$$Q^{(k)}_n = \Big(\sum_{j=1}^{i(n)-1} \frac{(s_j - k)\upsilon_j}{\alpha^k_j} + \frac{(t_j - k)\upsilon_j}{\beta^k_j} + \Big(\sum_{l=1}^{k-1} \frac{\upsilon_j}{\beta^l_j \alpha^{k-l}_j} + \frac{\upsilon_j}{\alpha^l_j \beta^{k-l}_j} \Big) \Big)$$
$$+ \frac{c(n)(s_{i(n)} - k)}{\alpha^k_{i(n)}} + \frac{c(n)(t_{i(n)} - k)}{\beta^k_{i(n)}} + \Big(\sum_{l=1}^{k-1} \frac{\upsilon_{i(n)}}{\beta^l_{i(n)} \alpha^{k-l}_{i(n)}} + \frac{\upsilon_j}{\alpha^l_{i(n)} \beta^{k-l}_{i(n)}} \Big).$$

Note that by (2.3) and (2.4), we have that

$$\lim_{n\to\infty} \frac{Q_n^{(k)}}{\mathfrak{S}} = 1,$$

where

$$\mathfrak{S} = \left(\sum_{j=1}^{i(n)-1} \frac{(s_j - k)\upsilon_j}{\alpha_j^k} + \frac{(t_j - k)\upsilon_j}{\beta_j^k} \right) + \frac{c(n)(s_{i(n)} - k)}{\alpha_{i(n)}^k} + \frac{c(n)(t_{i(n)} - k)}{\beta_{i(n)}^k}$$

Thus

$$\begin{aligned} \lim_{n\to\infty} \frac{Q_n^{(k)}}{P_{g(n)}^{(k)}} &= \lim_{n\to\infty} \frac{\mathfrak{S}}{\left(\sum_{j=1}^{i(n)-1} \frac{s_j \upsilon_j}{\alpha_j^k} \right) + \frac{c(n) s_{i(n)}}{\alpha_{i(n)}^k}} \\ &= \lim_{n\to\infty} \frac{s_n - k}{s_n} + \frac{(t_n - k)\alpha_n^k}{s_n \beta_n^k} \\ &= 1 + \lim_{n\to\infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} \\ &= 1. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} \lim_{n\to\infty} \frac{C_\alpha(g(n))}{P_{g(n)}^{(k)}} &= \lim_{n\to\infty} \frac{\left(\sum_{j=1}^{i(n)-1} \upsilon_j \right) + c(n)}{\left(\sum_{j=1}^{i(n)-1} \frac{s_j \upsilon_j - k}{\alpha_j^k} \right) + \frac{c(n) s_{i(n)} - k}{\alpha_{i(n)}^k}} \\ &= \lim_{n\to\infty} \frac{\alpha_n^k}{s_n - k/\upsilon_n} \\ &= \lim_{n\to\infty} \frac{\alpha_n^k}{s_n} \\ &= 0. \end{aligned}$$

Since $\xi \in \mathcal{N}(P)$, we have that

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = \lim_{n\to\infty} \frac{N_n^Q(B,x)}{P_{g(n)}^{(k)}} = 1.$$

Therefore, $x \in \mathcal{N}(Q)$.

For n where $q_n = \beta_{i(n)}$, we have

$$\frac{E_n}{q_n} \leq \frac{\beta_{i(n)}^{1-\log^{-1/2}\beta_{i(n)}}}{\beta_{i(n)}} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Up to position n there are at least $\sum_{j=1}^{i(n)} \upsilon_i t_i + c(n)t_{i(n)}$ such places where (3.5) holds. By (3.4), we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{i(n)} \upsilon_i t_i + c(n)t_{i(n)}}{n} = 1,$$

so the sequence $\left(\frac{E_n}{q_n}\right)$ is not uniformly distributed mod 1. Thus $x \notin \mathcal{DN}(Q)$ and

$$\Theta(\alpha, \beta, s, t, \upsilon, F, I) \subseteq \mathcal{N}(Q)\backslash\mathcal{DN}(Q),$$

which implies that $\dim_{\mathrm{H}}(\mathcal{N}(Q)\backslash\mathcal{DN}(Q)) = 1$. □

Theorem 3.6. *Suppose that $Q = Q(\alpha, \beta, s, t, \upsilon)$ is infinite in limit, fully divergent, and satisfies conditions* (2.3), (2.4), (3.1) *for $k \geq \ell$,* (3.2) *for $\ell < k$, and* (3.3). *Then*

$$\dim_H\Big(\bigcap_{j=\ell}^{\infty} \mathcal{N}_j(Q)\Big\backslash \bigcup_{j=1}^{\ell-1} \mathcal{N}_j(Q)\Big) = 1.$$

Proof. Define the same basic sequence P and sequences I and F as in the proof of Theorem 3.5. The same arguments regarding the asymptotics of $N_n^Q(B,x)$ for $x \in \Theta(\alpha, \beta, s, t, F, I)$ hold, so

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{P_{g(n)}^{(k)}} = 1.$$

But since (3.1) holds for $k \geq \ell$, we have that

$$\lim_{n\to\infty} \frac{Q_n^{(k)}}{P_{g(n)}^{(k)}} = 1 + \lim_{n\to\infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} = 1.$$

Thus x is Q-normal of orders greater than or equal to ℓ. □

Example 3.7. Set

$$\begin{aligned}\alpha_n &= \lfloor \log\log(n+2) \rfloor + 2,\\ \beta_n &= \lfloor \log n \rfloor + 2,\\ s_n &= \lfloor \log n \rfloor,\\ t_n &= n,\end{aligned}$$

and

$$\upsilon_n = 2^n.$$

Then the conditions of Theorem 3.5 are satisfied.

Example 3.8. Fix some integer ℓ. Set

$$\begin{aligned}\alpha_n &= \lfloor \log\log(n+2) \rfloor + 2,\\ \beta_n &= \lfloor \log n \rfloor + 2,\\ s_n &= \lfloor \log n \rfloor,\\ t_n &= \left\lfloor \left(\frac{\beta_n}{\alpha_n}\right)^{\ell+1} s_n \right\rfloor,\end{aligned}$$

and

$$\upsilon_n = 2^n.$$

Then the conditions of Theorem 3.6 are satisfied.

Proof of Theorem 1.11. Let $\gamma = \dim_{\mathrm{M}}(S)$, $\alpha_i = 2$, $\beta_i = q_i$, $s_i = 0$, $t_i = 1$, $\upsilon_i = 1$, $F_i = 0$, and

$$I_i = S \cap \{0, \ldots, q_i - 2\}.$$

Then (2.3) and (2.4) clearly hold. Note that

$$\mathcal{W}_Q(S) \subseteq \Theta(\alpha, \beta, s, t, \upsilon, F, I),$$

so $\dim_{\mathrm{H}}(\mathcal{W}_Q(S)) \leq \gamma$.

To get a lower bound, we construct a subset of $\mathcal{W}_Q(S)$ with Hausdorff dimension γ. To do this, let $T \subset \mathbb{N}$ be an infinite set that is sparse enough such that

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} \chi_T(i) \log \#(S \cap \{0, \dots, q_i - 2\})}{\sum_{i=1}^{k} \log \#(S \cap \{0, \dots, q_i - 2\})} = 0.$$

Note that such a T exists since

$$\lim_{k\to\infty} \sum_{i=1}^{k} \log \#(S \cap \{0, \dots, q_i - 2\}) = \infty.$$

Let $f : T \to S$ be a surjective function such that for all $t \in T$, we have $q_t > f(t)$. Such an f exists since $\min S < \min Q$, T is infinite, and Q is infinite in limit. Consider the homogeneous Moran set C with

$$n_k = \begin{cases} 1 & \text{if } k \in T \\ \#S \cap \{0, \dots, q_k - 2\} & \text{otherwise} \end{cases}$$

and $c_k = \frac{1}{q_k}$ described as follows: If $k \in T$, then for any $x \in C$, $E_k(x) = f(k)$. Otherwise, $E_k(x) \in S \cap \{0, \dots, q_k - 2\}$. Since f is surjective, we have that for any $x \in C$ that $\mathcal{S}_Q(x) = S$, so $C \subseteq \mathcal{W}_Q(S)$. But

$$\begin{aligned}
\dim_{\mathrm{H}}(C) &\geq \liminf_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_{k+1} n_{k+1}} \\
&= \lim_{k\to\infty} \frac{\sum_{i=1}^{k} \chi_{\mathbb{N}\setminus T}(i) \log \#(S \cap \{0, \dots, q_i - 2\})}{\sum_{i=1}^{k} \log q_i + \log q_{k+1}} \\
&= \lim_{k\to\infty} \frac{\sum_{i=1}^{k} \log \#(S \cap \{0, \dots, q_i - 2\})}{\sum_{i=1}^{k} \log q_i} \\
&= \lim_{k\to\infty} \frac{\log \#(S \cap \{0, \dots, q_k - 2\})}{\log q_k} \\
&= \gamma.
\end{aligned}$$

Thus $\dim_{\mathrm{H}}(\mathcal{W}_Q(S)) \geq \gamma$, so we have $\dim_{\mathrm{H}}(\mathcal{W}_Q(S)) = \gamma$. ☐

4. Further problems

Problem 4.1. For which irrational x does there exist a basic sequence Q where $x \in \mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q)$. The same question may be asked about several of the other sets discussed in this paper. We remark that it is already known that for every irrational x there exist uncountably many basic sequences Q where $x \in \mathcal{DN}(Q)$. See [13].

Problem 4.2. Prove that the conclusions of Theorem 3.5 and Theorem 3.6 hold for all Q that are infinite in limit and fully divergent.

Problem 4.3. In [18] sufficient conditions are given under which countable intersections of sets of the form $\mathcal{DN}(Q) \backslash \bigcup_{j=1}^{\infty} \mathcal{RN}_j(Q)$ have full Hausdorff dimension. Surely a similar result holds for many of the sets described in this paper. Necessary and sufficient conditions similar to conditions found in the paper of W. M. Schmidt [22] may be possible.

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