J. Fractal Geom. 3 (2016), 187–215
DOI 10.4171/JFG/34

Journal of Fractal Geometry

Quasi-Assouad dimension of fractals

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Abstract. The Assouad dimension of fractals is not invariant under quasi-Lipschitz mappings, even for Ahlfors–David regular sets. In this manuscript, we shall give a new dimension $\dim_{qA}$ of fractals named the quasi-Assouad dimension, which is invariant under any quasi-Lipschitz mapping, satisfying $\overline{\dim}_B E \le \dim_{qA} E \le \dim_A E$ for any compact subset E of a metric space. By virtue of the quasi-Assouad dimension, we show that any Bedford–McMullen carpet F is quasi-Lipschitz Assouad-minimal, i.e., $\dim_A f(F) \ge \dim_A F$ for any quasi-Lipschitz mapping f.

Mathematics Subject Classification (2010). Primary: 28A80; Secondary: 37F35.

Keywords. Assouad dimension, quasi-Lipschitz equivalence, Bedford–McMullen carpet, quasi uniform disconnectedness.

Contents

[1] This work is supported by National Natural Science Foundation of China (Nos. 11071224, 11371329, 11471124) and NSF of Zhejiang Province (Nos. LR13A010001, LY12F02011). Li-Feng Xi is the corresponding author.

1. Introduction

A bijection $f: (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ between two metric spaces is said to be bilipschitz, if there exists a constant $L > 0$ such that for all $x_1, x_2 \in X$,

$$L^{-1}\mathrm{d}_X(x_1, x_2) \le \mathrm{d}_Y(f(x_1), f(x_2)) \le L\mathrm{d}_X(x_1, x_2).$$

Bilipschitz mappings preserve most of the geometric, topological and measure theoretic properties of sets. Determining whether there exists a bilipschitz mapping between two fractals is a topic of interest in geometric measure theory. However, restrictions of this kind of mappings sometimes seem to be too much strict. For example, it was pointed out by Cooper and Pignataro [4], David and Semmes [5] and Falconer and Marsh [7] that there does not exist any bilipschitz mapping between the Cantor ternary set $\mathcal{C}$ and the following self-similar set

$$K = (\beta K) \bigcup \left(\beta K + \tfrac{1}{2} - \tfrac{1}{2}\beta\right) \bigcup (\beta K + (1-\beta)) \text{ with } 3\beta^{\frac{\log 2}{\log 3}} = 1,$$

though at first glance they are so similar. As we know, they have the same fractal dimensions, and both of them are self-similar sets satisfying the strong separate condition (*SSC*). Relaxing the restrictions, it was shown by Xi [27] that they are quasi-Lipschitz equivalent.

Definition 1.1. Two compact metric spaces (X, d_X) and (Y, d_Y) are said to be *quasi-Lipschitz equivalent* if there exists a bijection $g: X \to Y$ called a quasi-Lipschitz mapping, such that for all distinct points $x_1, x_2 \in X$,

$$\frac{\log \mathrm{d}_Y(g(x_1), g(x_2))}{\log \mathrm{d}_X(x_1, x_2)} \longrightarrow 1 \quad \text{uniformly as } \mathrm{d}_X(x_1, x_2) \to 0.$$

It is readily checked that the inverse function g^{-1} is also a quasi-Lipschitz mapping. Therefore, quasi-Lipschitz equivalence is a kind of equivalent relationship between compact metric spaces, under which the following results seem to be more natural.

1 (Xi [27]). Two self-conformal sets (satisfying the SSC) are quasi-Lipschitz equivalent if and only if they have the same Hausdorff dimension.

2 (Wang and Xi [23]). Two uniform disconnected Ahlfors–David s-regular sets ($s > 0$) are quasi-Lipschitz equivalent.

We notice that the most widely used fractal dimensions are invariant under quasi-Lipschitz mappings, e.g., the Hausdorff dimension and the box-counting dimension. That is for any compact metric space X, one has

$$\dim_H g(X) = \dim_H X, \quad \overline{\dim}_B g(X) = \overline{\dim}_B X, \quad \underline{\dim}_B g(X) = \underline{\dim}_B X$$

under any quasi-Lipschitz mapping g. While for the Assouad dimension $\dim_A$, we can find a quasi-Lipschitz mapping $\tilde{g}$ such that

$$1 = \dim_A \tilde{g}(\mathcal{C}) \neq \dim_A \mathcal{C} = \log 2/\log 3,$$

which means that the Assouad dimension is *not invariant* under quasi-Lipschitz mappings. In fact, we have

Proposition 1.2. *Let $E \subset \mathbb{R}$ be a self-similar set satisfying the SSC. Then for any $t \in (\dim_A E, 1]$, there exists a quasi-Lipschitz mapping g_t such that*

$$\dim_A g_t(E) = t.$$

For this reason, we try to find a kind of fractal dimension such that it is as close as possible to the Assouad dimension and invariant under quasi-Lipschitz mappings.

1.1. Quasi-Assouad dimension. We may recall the notion of the *Assouad dimension* as follows. For $0 < r < R$, let $N_r(E)$ denote the least number of balls of radius r required to cover the subset E of a metric space X and

$$N_{r,R}(E) = \sup_{x\in E} N_r(B(x,R)\cap E).$$

Then the Assouad dimension of E is defined as

$$\dim_A E = \inf\Big\{\alpha \geq 0 \;\Big|\; \text{there exists } c > 0 \text{ such that } N_{r,R}(E) \leq c\Big(\frac{R}{r}\Big)^\alpha \text{ for all } 0 < r < R\Big\},$$

which was introduced by Assouad in the late 1970s [1, 2, 3]. When the metric space X is *doubling* (i.e., there exists a positive integer N such that every closed ball in X can be covered by N closed balls of half the radius), the Assouad dimension of E is always finite. The Assouad dimension plays a prominent role in the study of quasiconformal mappings and embeddability problems. Please refer to the textbook [11] and the survey paper [16] for more details.

For any $\delta \in (0,1)$, let

$$h_E(\delta) = \inf \Big\{ \alpha \geq 0 \;\Big|\; \text{there exists } c > 0 \text{ such that } N_{r,R}(E) \leq c\Big(\frac{R}{r}\Big)^{\alpha} \text{ for all } 0 < r < r^{1-\delta} \leq R \Big\}.$$

It is clear that the function $h_E(\delta)$ does not decrease as δ decreasing on $(0,1)$. We define the *quasi-Assouad dimension* of E as the limit

$$\dim_{qA} E := \lim_{\delta\to 0} h_E(\delta).$$

Here we let $\dim_{qA} \varnothing = 0$ for the empty set $\varnothing$.

Proposition 1.3. *Suppose that E is a compact subset of a metric space X. Then for any $\delta \in (0,1)$,*

$$h_E(\delta) = \overline{\lim_{r\to 0}} \sup_{r^{1-\delta} \leq R < 1} \frac{\log N_{r,R}(E)}{\log R - \log r}. \tag{1.1}$$

By the definition above, it is readily checked that the quasi-Assouad dimension defined on subsets of X satisfies the following properties. That is the reason why we can call it a "dimension" (see, e.g., page 40 in [6]).

1 *Monotonicity*. $\dim_{qA} E \leq \dim_{qA} F$ if $E \subset F \subset X$.

2 *Stability*. $\dim_{qA}(E \cup F) = \max(\dim_{qA} E, \dim_{qA} F)$ for any $E, F \subset X$.

3 *Bilipschitz invariance*. $\dim_{qA} E = \dim_{qA} f(E)$ for any subset $E \subset X$ and any bilipschitz transformation f.

4 *Range of values*. $0 \leq \dim_{qA} E \leq \dim_A E$ for any subset $E \subset X$. In particular, for $X = \mathbb{R}^n$, we have $0 \leq \dim_{qA} E \leq n$.

5 *Open sets*. If E is a non-empty open subset of $X = \mathbb{R}^n$, then $\dim_{qA} E = n$.

Theorem 1.4. *The quasi-Assouad dimension is invariant under any quasi-Lipschitz mapping. More precisely, if two compact metric spaces (X, d_X) and (Y, d_Y) are quasi-Lipschitz equivalent with respect to the map $g\colon X \to Y$, then for any compact subset $E \subset X$,*

$$\dim_{qA} E = \dim_{qA} g(E).$$

It is well-known that for any compact subset E of a metric space, we have

$$\dim_H E \leq \overline{\dim}_B E \leq \dim_A E.$$

Then, what is the relationship between the quasi-Assouad dimension and these fractal dimensions?

Proposition 1.5. *For any compact subset E of a metric space, we have*

$$\dim_H E \leq \overline{\dim}_B E \leq \dim_{qA} E \leq \dim_A E. \tag{1.2}$$

In fact, we can say more about these inequalities in (1.2).

Proposition 1.6. *Given four real numbers a, b, c and d with $0 < a < b < c \leq d \leq 1$, we can find a compact set $F \subset [0, 1]$ such that*

$$\dim_H F = a, \quad \overline{\dim}_B F = b, \quad \dim_{qA} F = c, \quad \dim_A F = d.$$

1.2. Quasi-Lipschitz minimality. A set $E \subset \mathbb{R}^n$ is said to be *quasisymmetrically minimal* if $\dim_H f(E) \geq \dim_H E$ for any n-dimensional quasisymmetric mapping f. For results about quasisymmetric mappings, please refer to [9], [10], [14], [21], [22], et al. In analogy with the definition of quasisymmetric minimality, replacing the quasisymmetric mappings by quasi-Lipschitz mappings, we introduce the notion of *quasi-Lipschitz minimality* for compact sets with respect to the Assouad dimension.

Definition 1.7. A compact subset E of a metric space is said to be quasi-Lipschitz Assouad-minimal if

$$\dim_A g(E) \geq \dim_A E$$

for any quasi-Lipschitz mapping g.

By virtue of the quasi-Assouad dimension, we can show the quasi-Lipschitz minimality of Bedford–McMullen carpets.

Theorem 1.8. *For any compact subset E of a metric space, if $\dim_{qA} E = \dim_A E$, then it is quasi-Lipschitz Assouad-minimal. In particular, $\dim_{qA} E = \dim_A E$ if E is an Ahlfors–David regular set or a Bedford–McMullen carpet. Hence all Ahlfors–David regular sets and Bedford–McMullen carpets are quasi-Lipschitz Assouad-minimal.*

By Proposition 1.6, there exists a compact set $F \subset [0, 1]$ such that

$$\overline{\dim}_B F < \dim_{qA} F = \dim_A F.$$

Therefore, the quasi-Assouad dimension has advantage in determining the quasi-Lipschitz minimality of compact sets than the upper box-counting dimension.

1.3. Quasi uniform disconnectedness. The uniform disconnectedness of subsets in metric spaces is related to their Assouad dimensions (see, e.g., Proposition 5.1.7 of [18]). For any non-empty subset E of a metric space,

$$\dim_A E < 1 \implies E \text{ is uniformly disconnected.}$$

When studying the quasi-Lipschitz equivalence of compact metric spaces, Wang and Xi [24] introduced a weaker notion of *quasi uniform disconnectedness*.

Definition 1.9. We say that a non-empty compact subset E of a metric space (X, d) is *quasi uniformly disconnected,* if there is a constant $r^* > 0$ and a function

$$\psi : (0, r^*) \to (0, r^*)$$

with

$$\psi(r) < r \quad \text{for all } 0 < r < r^*$$

and

$$\lim_{r\to 0} \frac{\log \psi(r)}{\log r} = 1$$

such that for any $x \in E$ and $0 < r < r^*$, we can find a set $E_{x,r} \subset E$ satisfying

$$E \cap B(x, \psi(r)) \subset E_{x,r} \subset B(x, r)$$

and

$$\mathrm{dist}(E_{x,r}, E \backslash E_{x,r}) \geq \psi(r),$$

where

$$\mathrm{dist}(A, B) = \inf_{x\in A, y\in B} \mathrm{d}(x, y)$$

for any subsets $A, B \subset X$.

Finding a smaller fractal dimension to ensure the weaker notion is the other motivation for us to introduce the quasi-Assouad dimension.

Theorem 1.10. *Let E be a non-empty compact subset of a metric space (X, d). If $\dim_{qA} E < 1$, then it is quasi uniformly disconnected.*

The following simple example shows the difference between the uniform disconnectedness and the quasi uniform disconnectedness.

Example 1.11. Fix $0 < \alpha < 1$, let $a_k = \prod_{i=1}^{k}(1 - (i + 1)^{-\alpha})$ for all $k \geq 1$. Then (we will prove in Section 4 that) the countable compact set $E = \{0, 1, a_1, a_2, \ldots\}$ is quasi uniformly disconnected but not uniformly disconnected. Moreover,

$$0 = \dim_{qA} E < \dim_A E = 1.$$

Therefore, one can also get the quasi uniform disconnectedness of the set E by Theorem 1.10.

Remark 1.12. The assumption that the upper box-counting dimension is less than 1 does not imply the quasi uniform disconnectedness of a compact set. In fact, when taking $\alpha = 1$ in Example 1.11, we get $E = \{0, 1, 1/2, 1/3, \ldots\}$. It is well-known that $\overline{\dim}_B E = 1/2$. We can check that E is not quasi uniformly disconnected (with details in Section 4).

Using Theorem 1.10 and Remark 1.12, we have

Claim 1. *The set* $E = \{0, 1, 1/2, 1/3, \ldots\}$ *is quasi-Lipschitz Assouad-minimal, since* $\dim_{qA} E = \dim_A E = 1$.

It is known that for any compact subset $E \subset \mathbb{R}$, we have $\dim_A E < 1$ if and only if E is uniformly disconnected (see, e.g., Theorem 5.2 of [16]). One wonders whether there is a dimensional function $\Phi\colon 2^{\mathbb{R}} \to [0, 1]$ such that $\Phi(E) < 1$ if and only if E is quasi uniformly disconnected. Since $\dim_A E \geq \dim_H E$, we present a reasonable assumption $\Phi(E) \geq \dim_H E$ and obtain the following interesting result:

Proposition 1.13. *For any set function* $\Phi\colon 2^{\mathbb{R}} \to [0, 1]$ *satisfying* $\Phi(E) \geq \dim_H E$ *for all compact sets* $E \subset \mathbb{R}$*, the following equivalence fails for some compact set* $F \subset \mathbb{R}$,

$$\Phi(F) < 1 \text{ if and only if } F \text{ is quasi uniformly disconnected.}$$

1.4. Quasi-Assouad dimensions of Moran sets. Some special cases of Moran sets were first studied by Moran [19]. The later works [12, 13, 25, 26] developed the theory on the geometric structures and the fractal dimensions of Moran sets systematically.

Suppose that $J \subset \mathbb{R}$ is a bounded closed interval with non-empty interior. Let $\{n_k\}_{k\geq 1}$ be a sequence of integers satisfying $n_k \geq 2$ for all k. Let $\{c_k\}_{k\geq 1}$ be a sequence of real numbers such that $c_k \in (0, 1/n_k]$ for all k. Denote

$$\mathcal{D}^k = \{i_1 \dots i_k : i_j \in \mathbb{N} \cap [1, n_j] \text{ for all } 1 \leq j \leq k\}$$

and

$$\mathcal{D}^0 = \{\emptyset\}$$

with the empty word $\emptyset$.

Let $J_\emptyset = J$. Denote the length of J by $|J|$. Suppose for any $k \geq 1$ and any $i_1 \dots i_{k-1} \in \mathcal{D}^{k-1}$ ($i_1 \dots i_{k-1} = \emptyset$ if $k = 1$), the following $J_{i_1\dots i_{k-1}1}, \dots, J_{i_1\dots i_{k-1}n_k}$ are closed subintervals of $J_{i_1\dots i_{k-1}}$ with their interiors pairwise disjoint, such that

$$\frac{|J_{i_1\dots i_{k-1}j}|}{|J_{i_1\dots i_{k-1}}|} = c_k \quad \text{for all } 1 \leq j \leq n_k.$$

Then we call the following non-empty compact set

$$F = \bigcap_{k=0}^{\infty} \bigcup_{i_1\dots i_k \in \mathcal{D}^k} J_{i_1\dots i_k}$$

a Moran set with the structure $(J, \{n_k\}_k, \{c_k\}_k)$, and denote

$$F \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k).$$

For any $k \geq 1$ and any $i_1 \dots i_k \in \mathcal{D}^k$, we call the closed interval $J_{i_1\dots i_k}$ a basic interval of rank k of the Moran set $F \in \mathcal{M}(J, \{n_k\}, \{c_k\})$.

Further, we call the Moran set $F \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$ a uniform Cantor set, if for any $k \geq 1$ and any $i_1 \dots i_{k-1} \in \mathcal{D}^{k-1}$, the subintervals $\{J_{i_1\dots i_{k-1}j}\}_{j=1}^{n_k}$ are uniformly distributed from left to right in $J_{i_1\dots i_{k-1}}$ such that $J_{i_1\dots i_{k-1}1}$ shares the same left endpoint with $J_{i_1\dots i_{k-1}}$, and $J_{i_1\dots i_{k-1}n_k}$ shares the same right endpoint with $J_{i_1\dots i_{k-1}}$.

Suppose that $F \in \mathcal{M}(J, \{n_k\}, \{c_k\})$ is a Moran set. By Theorem 3.2 of [8], if $\sup_k n_k < \infty$, then

$$\dim_H F = \varliminf_{k\to\infty} \frac{\log(n_1 \dots n_k)}{-\log(c_1 \dots c_k)}, \quad \overline{\dim}_B F = \varlimsup_{k\to\infty} \frac{\log(n_1 \dots n_k)}{-\log(c_1 \dots c_k)}. \tag{1.3}$$

Under the assumption that $\inf_k c_k > 0$, Li, Li, Miao and Xi [15] obtained the Assouad dimension of F

$$\dim_A F = \lim_{m\to\infty} \sup_{k\geq 1} \frac{\log(n_{k+1}\dots n_{k+m})}{-\log(c_{k+1}\dots c_{k+m})}. \tag{1.4}$$

When F is a uniform Cantor set, one recent result by Peng, Wang and Wen [20] is that $\dim_A F = 1$ if $\sup_k n_k = +\infty$.

In this manuscript, we try to compute the quasi-Assouad dimensions of Moran sets, and get the following result.

Theorem 1.14. *If*

$$\lim_{k\to\infty} \frac{\log c_k}{\log c_1 \dots c_k} = 0,$$

then for any Moran set $F \in \mathcal{M}(J, \{n_k\}, \{c_k\})$,

$$\dim_{qA} F = \lim_{\delta\to 0} \overline{\lim_{q\to\infty}} \max_{1\leq p\leq l_{q,\delta}} \frac{\log(n_p\dots n_q)}{-\log(c_p\dots c_q)},$$

where

$$l_{q,\delta} = \max\left\{1 \leq p \leq q \;\middle|\; \frac{\log(c_p\dots c_q)}{\log(c_1\dots c_q)} > \delta\right\}$$

with $\delta \in (0,1)$. *In particular, if* $\inf_k c_k > 0$, *then*

$$\dim_{qA} F = \lim_{\eta\to 0} \overline{\lim_{q\to\infty}} \max_{1\leq p\leq q(1-\eta)} \frac{\log(n_p\dots n_q)}{-\log(c_p\dots c_q)}.$$

From Theorem 1.10 and Theorem 1.14, we have the following corollary.

Corollary 1.15. *If*

$$\lim_{k\to\infty} \frac{\log c_k}{\log(c_1\dots c_k)} = 0 \quad \textit{and} \quad \overline{\lim_{k\to\infty}} \frac{\log n_k}{-\log c_k} < 1,$$

then any Moran set $F \in \mathcal{M}(J, \{n_k\}, \{c_k\})$ *is quasi uniformly disconnected.*

With the above results, we have the following two simple examples. Note that both of the Moran sets given in Example 1.16 and Example 1.17 are quasi uniformly disconnected but not uniformly disconnected. In fact, Example 1.17 is a special case in the proof of Proposition 1.6.

Example 1.16. Let $n_k = 3^k$ and $c_k = 3^{-2k}$ for all $k \geq 1$. Then $\sup_k n_k = +\infty$ and

$$\lim_{k\to\infty} \frac{\log c_k}{\log c_1 \dots c_k} = 0.$$

Let $F \in \mathcal{M}([0,1], \{n_k\}, \{c_k\})$ be a uniform Cantor set. By Theorem 1.14 and the result on Assouad dimension by Peng et al [20], we have

$$\frac{1}{2} = \dim_{qA} F < \dim_A F = 1.$$

Example 1.17. Let $n_k = 2$ for all $k \geq 1$. Let $\{a_i\}_{i\geq 1}$ be a sequence of positive integers with $a_1 = 1$ such that $a_{i+1} > 2a_i + i$ for all i and

$$\lim_{i\to\infty} \frac{i}{a_i} = \lim_{i\to\infty} \frac{a_1 + a_2 + \cdots + a_{i-1}}{a_i} = 0.$$

For any $k \geq 1$, if $k \in [a_i, a_{i+1})$ for some i, then let

$$c_k = \begin{cases} 1/4 & \text{if } k \in [a_i, 2a_i), \\ 1/2 & \text{if } k \in [2a_i, 2a_i + i), \\ 1/8 & \text{otherwise.} \end{cases}$$

Under these assumptions, any Moran set $F \in \mathcal{M}([0,1], \{n_k\}, \{c_k\})$ satisfies

$$\dim_H F = \frac{1}{3} < \overline{\dim}_B F = \frac{2}{5} < \dim_{qA} F = \frac{1}{2} < \dim_A E = 1.$$

The rest of this paper is organized as follows. In the next section, we will give the proof of Propositions 1.3, 1.5 and Theorem 1.4. Section 3 is devoted to the proof of Proposition 1.2 and Theorem 1.8. Results about the quasi uniform disconnectedness will be proved in Section 4. In the last section, we will prove Theorem 1.14 and Proposition 1.6.

2. Properties of the quasi-Assouad dimension

In this section, we try to verify some properties of the quasi-Assouad dimension. More precisely, we will prove Proposition 1.3, 1.5 and Theorem 1.4.

Proof of Proposition 1.3. Fix $\alpha > h_E(\delta)$, we may assume that there exists a constant $c > 0$ such that if $0 < r < r^{1-\delta} \leq R$, then $N_{r,R}(E) \leq c(R/r)^\alpha$. This implies

$$\overline{\lim_{r\to 0}} \sup_{r^{1-\delta}\leq R<1} \frac{\log N_{r,R}(E)}{\log R - \log r} \leq \alpha.$$

Thus

$$\overline{\lim_{r\to 0}} \sup_{r^{1-\delta}\le R<1} \frac{\log N_{r,R}(E)}{\log R - \log r} \le h_E(\delta).$$

On the other hand, for any

$$\alpha' > \overline{\lim_{r\to 0}} \sup_{r^{1-\delta}\le R<1} \frac{\log N_{r,R}(E)}{\log R - \log r},$$

there exists a real number $r_0 \in (0,1)$ such that if $0 < r < r^{1-\delta} \le R < r_0$, then $N_{r,R}(E) \le (R/r)^{\alpha'}$. Since the set E is compact, we can find a constant $c > 0$ such that $N_{r,R}(E) \le c(R/r)^{\alpha'}$ whenever $0 < r < r^{1-\delta} \le R$. Therefore $\alpha' \ge h_E(\delta)$, and thus

$$\overline{\lim_{r\to 0}} \sup_{r^{1-\delta}\le R<1} \frac{\log N_{r,R}(E)}{\log R - \log r} \ge h_E(\delta). \qquad \square$$

Proof of Proposition 1.5. For any $\delta \in (0,1)$, we will verify that $\overline{\dim}_B E \le h_E(\delta)$. Then $\dim_H E \le \overline{\dim}_B E \le \lim_{\delta\to 0} h_E(\delta) = \dim_{qA} E \le \dim_A E$.

Fix $\delta \in (0,1)$, for any $\alpha > h_E(\delta)$ we may assume that there exists a constant $c > 0$ such that $N_{r,R}(E) \le c(R/r)^{\alpha}$ whenever $0 < r < r^{1-\delta} \le R$. For such r and R, it is readily checked that $N_r(E) \le N_{r,2R}(E) \cdot N_R(E)$. Therefore,

$$\frac{\log N_r(E)}{-\log r} \le \frac{\log N_R(E)}{-\log r} + \frac{\log c}{-\log r} + \frac{\alpha \log(2R)}{-\log r} + \alpha.$$

Since the set E is compact, we know that $N_R(E) < +\infty$. Letting $r \to 0$, we obtain that $\overline{\dim}_B E \le \alpha$, and thus $\overline{\dim}_B E \le h_E(\delta)$. $\square$

Proof of Theorem 1.4. Based on the fact that g is a quasi-Lipschitz mapping, we will show that $\dim_{qA} g(E) \le \dim_{qA} E$. For the same reason, we can obtain that $\dim_{qA} E \le \dim_{qA} g(E)$, since g^{-1} is also a quasi-Lipschitz mapping.

By the definition of quasi-Lipschitz mapping, there are increasing functions

$$\zeta, \phi\colon (0,1) \longrightarrow (0,+\infty)$$

with

$$\lim_{r\to 0} \frac{\log \zeta(r)}{\log r} = \lim_{R\to 0} \frac{\log \phi(R)}{\log R} = 1 \tag{2.1}$$

such that for any $y \in g(E)$ and $r, R > 0$ small enough, one has

$$g(B(g^{-1}y, \zeta(r))) \subset B(y,r) \quad \text{and} \quad B(y,R) \subset g(B(g^{-1}y, \phi(R))). \tag{2.2}$$

Given $\delta \in (0, 1/2)$ and $\varepsilon > 0$, by (2.1) when $R > 0$ is small enough, one has $\zeta(r)^{1-\delta} \leq \phi(R)$ and $\phi(R)/\zeta(r) \leq (R/r)^{1+\varepsilon}$ for all $0 < r < r^{1-2\delta} \leq R$. Then, for any $\alpha > h_E(\delta)$, by (2.2) we have

$$N_{r,R}(g(E)) \leq N_{\zeta(r),\phi(R)}(E) \leq c\Big(\frac{\phi(R)}{\zeta(r)}\Big)^{\alpha} \leq c\Big(\frac{R}{r}\Big)^{\alpha(1+\varepsilon)},$$

where $c = c(\alpha) > 0$ is a constant. Since the set $g(E)$ is compact and the numbers α, ε are arbitrary, we obtain that $h_{g(E)}(2\delta) \leq h_E(\delta)$. Letting $\delta \to 0$, it follows that $\dim_{qA} g(E) \leq \dim_{qA} E$. □

3. Quasi-Lipschitz minimality

In this section, we first show that the Assouad dimensions of sets are not quasi-Lipschitz invariant, even for self-similar sets satisfying the SSC. Then, we will prove that for any Bedford–McMullen carpet F, one has $\dim_{qA} F = \dim_A F$. More precisely, we will prove Proposition 1.2 and Theorem 1.8.

Proof of Proposition 1.2. It is well-known that the self-similar set $E \subset \mathbb{R}$ satisfying the SSC is Ahlfors–David regular with $s := \dim_A E = \dim_H E \in (0, 1)$. Then it is uniformly disconnected. In the light of the result of Wang and Xi [24] that two quasi Ahlfors–David s-regular sets with $s \in (0, 1)$ are quasi-Lipschitz equivalent, we only need to construct a quasi Ahlfors–David s-regular set $F \subset [0, 1]$ satisfying $\dim_A F = t$.

For all $k \geq 1$, let $n_k = 2$ and

$$c_k = \begin{cases} 2^{-(1+1/m)/t} & \text{if } k \in [m^3, m^3 + m] \text{ for some integer } m, \\ 2^{-1/s}, & \text{otherwise.} \end{cases}$$

Let $F_t \in \mathcal{M}([0, 1], \{n_k\}_k, \{c_k\}_k)$ be a uniform Cantor set. By (1.4), we obtain that $\dim_A F_t = t$. We will show that F_t is quasi Ahlfors–David s-regular, i.e., there exists a Borel probability measure μ supported on F_t and a non-decreasing function

$$\Lambda\colon (0, 1) \longrightarrow (0, +\infty)$$

with

$$\lim_{r\to 0} \Lambda(r) = 0$$

such that for all $x \in F_t$ and $0 < r < 1$,

$$s(1 - \Lambda(r)) \leq \frac{\log \mu(B(x, r))}{\log r} \leq s(1 + \Lambda(r)).$$

Let μ be the Moran measure supported on F_t, i.e., μ is the natural measure on F such that for all $k \geq 1$ and any basic interval $J_{i_1 \dots i_k}$ of rank k, one has $\mu(J_{i_1 \dots i_k}) = (n_1 \dots n_k)^{-1}$, see [19]. Let $c_0 = n_0 = 1$. For any $0 < r < 1$, assume that $c_1 \dots c_{k_r+1} \leq r < c_1 \dots c_{k_r}$ for some $k_r \geq 0$. Then

$$\lim_{r \to 0} \frac{\log(c_1 \dots c_{k_r})}{\log r} = \lim_{r \to 0} \frac{\log(c_1 \dots c_{k_r+1})}{\log r} = 1.$$

Thus

$$\lim_{r \to 0} \frac{\log(n_1 \dots n_{k_r}) - \log 3}{\log r} = \lim_{r \to 0} \frac{\log(n_1 \dots n_{k_r+1})}{\log r} = s.$$

Hence, we can find a non-decreasing function

$$\Lambda \colon (0,1) \longrightarrow (0, +\infty)$$

with

$$\lim_{r \to 0} \Lambda(r) = 0$$

such that

$$s - \Lambda(r) \leq \frac{\log(n_1 \dots n_{k_r}) - \log 3}{\log r}, \frac{\log(n_1 \dots n_{k_r+1})}{\log r} \leq s + \Lambda(r). \tag{3.1}$$

For any $x \in F_t$ and $0 < r < 1$, it is readily checked that

$$(n_1 \dots n_{k_r+1})^{-1} \leq \mu(B(x,r)) \leq 3(n_1 \dots n_{k_r})^{-1}.$$

Then by (3.1)

$$s - \Lambda(r) \leq \frac{\log \mu(B(x,r))}{\log r} \leq s + \Lambda(r).$$

Therefore, F_t is quasi Ahlfors–David s-regular. □

Proof of Theorem 1.8. For any quasi-Lipschitz mapping g, by Theorem 1.4 and Proposition 1.5, if $\dim_A E = \dim_{qA} E$, then

$$\dim_A E = \dim_{qA} E = \dim_{qA} g(E) \leq \dim_A g(E).$$

Therefore, E is quasi-Lipschitz Assouad-minimal.

For any Ahlfors–David s-regular set D, we have

$$\dim_A D = \dim_{qA} D = s,$$

since $s = \dim_H D \le \dim_{qA} D \le \dim_A D = s$.

We will show that for any Bedford–McMullen carpet F, one has

$$\dim_A F = \dim_{qA} F.$$

Given two positive integers $n > m \ge 2$ and a fixed set

$$A \subset \{0, 1, \dots, (n-1)\} \times \{0, 1, \dots, (m-1)\}$$

with cardinality $\#A \ge 2$, the Bedford–McMullen carpet $F \subset \mathbb{R}^2$ is defined as the unique non-empty compact set satisfying

$$F = \bigcup_{(a,b)\in A} S_{(a,b)}(F),$$

where

$$S_{(a,b)}(x, y) = \Big(\frac{x+a}{n}, \frac{y+b}{m}\Big).$$

Let

$$t_j = \#\{i \in \{0, 1, \dots, (n-1)\} \mid (i, j) \in A\},$$
$$t = \max_j t_j,$$

and

$$s = \#\{j \mid t_j > 0\}.$$

Mackay [17] proved that

$$\dim_A F = \log_m s + \log_n t.$$

For example, if $m = 3$, $n = 4$ and $A = \{(0,0), (2,0), (0,2), (2,2), (2,4)\}$, we have the initial pattern as in Figure 1.

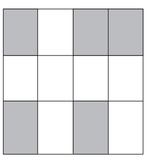

Figure 1. $m = 3, n = 4, s = 2, t = 3$.

We shall verify

$$\dim_{qA} F \geq \log_m s + \log_n t.$$

Note that the boxes of rank $k \geq 1$ in the sense of McMullen are rectangles with width n^{-k} and height $m^{-k'}$ satisfying $m^{-k'} \approx n^{-k}$, where $\approx$ means that k' is the smallest integer such that $m^{-k'} \leq n^{-k}$. For $r = n^{-k_1}$ and $R = n^{-k_2}$ with integers $k_1 \geq k_2 \geq 1$, let $\widetilde{N}_{r,R}$ denote the largest number of sub-boxes of rank k_1 contained in one box of rank k_2. For the usual number $N_{r,R}(F)$, it is readily checked that there exists a constant $c > 0$ such that

$$\widetilde{N}_{r,R} \leq c N_{r,R}(F) \quad \text{for all } r, R \in \{n^{-k} \mid k \in \mathbb{N}\}. \tag{3.2}$$

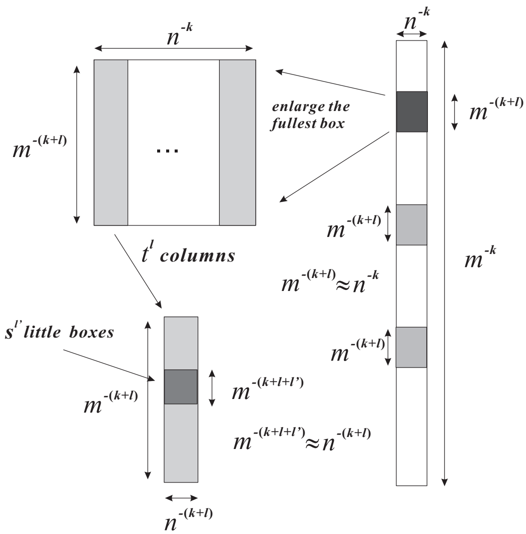

Figure 2. Calculation of $\widetilde{N}_{r_k,R_k}$

For $k \geq 1$ large enough, as in Figure 2, we consider the big boxes with width $R_k = n^{-k}$ and height $m^{-(k+l)}$ ($m^{-(k+l)} \approx n^{-k}$) and the little boxes with width $r_k = n^{-(k+l)}$ and height $m^{-(k+l+l')}$ ($m^{-(k+l+l')} \approx n^{-(k+l)}$). Since each "fullest" box of rank k contains t^l columns of width $n^{-(k+l)}$ and height $m^{-(k+l)}$, and each column contains $s^{l'}$ little boxes, we have

$$\widetilde{N}_{r_k,R_k} = s^{l'} t^l.$$

Then by (3.2),

$$\frac{\log N_{r_k,R_k}(F)}{\log R_k - \log r_k} \geq \frac{l' \log s + l \log t - \log c}{\log R_k - \log r_k}. \tag{3.3}$$

Note that

$$\frac{\log R_k}{\log r_k} = \frac{k}{k+l} \longrightarrow \frac{\log m}{\log n} \quad \text{as } k \to \infty.$$

Thus for k large enough, one has

$$1 - \frac{\log R_k}{\log r_k} \geq \Big(1 - \frac{\log m}{\log n}\Big)/2 =: \delta_0.$$

Since $m^{-(k+l)} \approx n^{-k}$ and $m^{-(k+l+l')} \approx n^{-(k+l)}$, letting $k \to \infty$, by (1.1) and (3.3), we obtain that

$$\begin{aligned} h_F(\delta_0) &= \overline{\lim_{r\to 0}} \sup_{r^{1-\delta_0} \leq R < 1} \frac{\log N_{r,R}(F)}{\log R - \log r} \\ &\geq \overline{\lim_{k\to\infty}} \frac{\log N_{r_k,R_k}(F)}{\log R_k - \log r_k} \\ &\geq \log_m s + \log_n t. \end{aligned}$$

Therefore,

$$\dim_{qA} F = \lim_{\delta\to 0} h_F(\delta) \geq h_F(\delta_0) \geq \log_m s + \log_n t. \qquad \square$$

4. Quasi uniform disconnectedness

In this section, we first give the details of Example 1.11 and Remark 1.12, which will help us to understand the notion of quasi uniform disconnectedness. Then, we turn to its relationship with quasi-Assouad dimension and prove Theorem 1.10 and Proposition 1.13.

4.1. Details of Example 1.11. Fix $\alpha \in (0,1)$, since $\sum_{i=1}^{\infty}(i+1)^{-\alpha} = +\infty$, the product $\prod_{i=1}^{\infty}(1-(i+1)^{-\alpha})$ converges to 0. Therefore,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \prod_{i=1}^{k} (1-(i+1)^{-\alpha}) = 0.$$

We claim that the sequence $\{a_k\}_k$ satisfies

(1) $\lim_{k\to\infty} \dfrac{\log a_{k+1}}{\log a_k} = 1$;

(2) $\lim_{k\to\infty} \dfrac{\log k}{\log a_k} = 0$;

(3) $\lim_{k\to\infty} \dfrac{\log(a_k - a_{k+1})}{\log a_k} = 1$.

For (1), we have

$$\lim_{k\to\infty} \frac{\log a_{k+1}}{\log a_k} = 1 + \lim_{k\to\infty} \frac{\log(1-(k+2)^{-\alpha})}{\log a_k} = 1.$$

For (2), by Stolz theorem, we have

$$\lim_{k\to\infty} \frac{\log k}{-\log a_k} = \lim_{k\to\infty} \frac{\log(1+1/k)}{\log(a_k/a_{k+1})} = \lim_{k\to\infty} \frac{(k+2)^{\alpha}}{k} = 0.$$

Since $a_k - a_{k+1} = (k+2)^{-\alpha} a_k$, by virtue of (2), we obtain (3).

Suppose E is uniformly disconnected, then there are constants $C > 0$ and $r^* > 0$ such that for any $x \in E$ and $0 < r < r^*$, we can find a set $E_{x,r} \subset E$ satisfying

$$E \cap B(x, Cr) \subset E_{x,r} \subset B(x,r) \quad \text{and} \quad \operatorname{dist}(E_{x,r}, E \backslash E_{x,r}) \geq Cr.$$

If we take $k \geq 1$ large enough such that $a_k/2 < r^*$ and

$$a_k - a_{k+1} = (k+2)^{-\alpha} a_k \leq C a_k/3,$$

then one can not find such $E_{x,r}$ for $x = a_k$ and $r = a_k/2$. In fact, since the gap sequence $\{a_k - a_{k+1}\}_k$ is decreasing, for any subset $A \subset E \cap B(x,r)$ containing x, we must have

$$\operatorname{dist}(A, E\backslash A) \leq a_k - a_{k+1} \leq Ca_k/3 < Cr.$$

Therefore, E is not uniformly disconnected.

Let $r^* = 1$. For $0 < r < r^*$, let

$$\psi(r) = (a_k - a_{k+1})/2$$

if $a_k \leq r < a_{k-1}$ for some $k \geq 1$ (with $a_{k-1} = 1$ if $k = 1$). Then $\psi(r) < r$ for all $0 < r < r^*$ and

$$\lim_{r\to 0} \frac{\log \psi(r)}{\log r} = 1$$

by (1)–(3).

Given both $x \in E$ and $0 < r < r^*$, suppose $a_k \le r < a_{k-1}$ for some k. If $0 \le x \le a_{k+1}$, then we take $E_{x,r} = [0, a_{k+1}] \cap E$; if $a_{k+1} < x \le 1$, then we take $E_{x,r} = \{x\}$. Then

$$E \cap B(x, \psi(r)) \subset E_{x,r} \subset B(x, r) \quad \text{and} \quad \operatorname{dist}(E_{x,r}, E \backslash E_{x,r}) \ge \psi(r).$$

Therefore, E is quasi uniformly disconnected.

Since E is not uniformly disconnected, we have $\dim_A E = 1$. For any $\varepsilon > 0$ and $\delta \in (0, 1)$, we will verify $h_E(\delta) \le \varepsilon$. Then $\dim_{qA} E = \lim_{\delta \to 0} h_E(\delta) = 0$.

Since the gap sequence $\{a_k - a_{k+1}\}_k$ is decreasing and $\lim_{k\to\infty} a_k = 0$, when k is large enough such that $a_k < (1 - a_1)/2$, we can find a unique positive integer m_k such that $(a_{m_k} - a_{m_k+1})/2 \le a_k < (a_{m_k-1} - a_{m_k})/2$. By (1)–(3), we have

$$\lim_{k\to\infty} \frac{\log(a_{m_k}/a_k)}{\log a_{k-1}} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\log(2m_k)}{\log a_{k-1}} = 0.$$

Therefore, there exists a positive integer k_0 such that for all $k \ge k_0$,

$$\min\left\{\frac{\log(a_{m_k}/a_k)}{\log a_{k-1}}, \frac{\log(2m_k)}{\log a_{k-1}}\right\} \ge -\delta\varepsilon,$$

which means that

$$\max\left\{\frac{a_{m_k}}{a_k}, 2m_k\right\} \le a_{k-1}^{-\delta\varepsilon}. \tag{4.1}$$

For any $0 < r < r^{1-\delta} \le R < a_{k_0}$, assume that $a_n \le r < a_{n-1}$ and $a_l \le R < a_{l-1}$ for some $n \ge l \ge k_0$. Since $E = ([0, a_{m_n}] \cap E) \cup ([a_{m_n-1}, 1] \cap E)$, we have

$$\begin{aligned} N_{r,R}(E) &= \sup_{x\in E} N_r(B(x, R) \cap E) \\ &\le N_{a_n}(E) \\ &\le \frac{a_{m_n}}{2a_n} + m_n \\ &\le \max\left\{\frac{a_{m_n}}{a_n}, 2m_n\right\}. \end{aligned}$$

Then by (4.1), $N_{r,R}(E) \le a_{n-1}^{-\delta\varepsilon} \le r^{-\delta\varepsilon} \le (R/r)^{\varepsilon}$. Since the set E is compact, we can find a constant $c > 0$ such that $N_{r,R}(E) \le c\,(R/r)^{\varepsilon}$ for all $0 < r < r^{1-\delta} \le R$. Therefore, $h_E(\delta) \le \varepsilon$.

4.2. Details of Remark 1.12. Suppose E is quasi uniformly disconnected, then there is a constant $r^* > 0$ and a function $\psi\colon (0, r^*) \to (0, r^*)$ with $\psi(r) < r$ for all $0 < r < r^*$ and

$$\lim_{r\to 0} \frac{\log \psi(r)}{\log r} = 1$$

such that for any $x \in E$ and $0 < r < r^*$, we can find a set $E_{x,r} \subset E$ satisfying

$$E \cap B(x, \psi(r)) \subset E_{x,r} \subset B(x, r) \quad \text{and} \quad \mathrm{dist}(E_{x,r}, E\backslash E_{x,r}) \geq \psi(r).$$

If we take $k \geq 1$ large enough such that $1/(2k) < r^*$ and

$$-\log[k(k+1)] < \log \psi(1/(2k)),$$

then one can not find such $E_{x,r}$ for $x = 1/k$ and $r = 1/(2k)$. In fact, for any subset $A \subset E \cap B(x, r)$ containing x, we must have

$$\mathrm{dist}(A, E\backslash A) \leq 1/k - 1/(k+1) = 1/[k(k+1)] < \psi(1/(2k)).$$

Therefore, E is not quasi uniformly disconnected.

4.3. Quasi uniform disconnectedness of fractal

Proof of Theorem 1.10. Assume that $\dim_{qA} E = s < 1$. For any $\delta \in (0,1)$ and $r \in (0,1)$, let

$$h_E(\delta, r) = \sup_{r^{1-\delta} \leq R < 1} \frac{\log N_{r,R}(E)}{\log R - \log r}.$$

Then by (1.1), we have

$$\overline{\lim_{r\to 0}}\, h_E(\delta, r) = h_E(\delta).$$

For any $\delta \in (0,1)$, there exists a real number $r_\delta \in (0,1)$ such that for any $r \in (0, r_\delta)$, we have

$$h_E(\delta, r) < h_E(\delta) + (1-s)/2 \leq s + (1-s)/2 = (1+s)/2.$$

Then whenever $r \in (0, r_\delta)$ and $r^{1-\delta} \leq R < 1$, we have

$$N_{r,R}(E) \leq (R/r)^{(1+s)/2}. \tag{4.2}$$

For any $0 < R < 1$, let

$$\rho(R) = \inf_{0<\delta<1} \max\left\{\frac{\log r_\delta}{\log R} - 1, \frac{\delta}{1-\delta}, \frac{2\log 6}{-(1-s)\log R}\right\}.$$

Then we have

$$\rho(R) \geq \frac{2\log 6}{-(1-s)\log R} > 0. \tag{4.3}$$

Fix $\delta \in (0, 1)$, we obtain that

$$\overline{\lim_{R\to 0}}\, \rho(R) \leq \overline{\lim_{R\to 0}} \max\left\{\frac{\log r_\delta}{\log R} - 1, \frac{\delta}{1-\delta}, \frac{2\log 6}{-(1-s)\log R}\right\} \leq \frac{\delta}{1-\delta}.$$

Therefore, we have

$$\lim_{R\to 0} \rho(R) = 0.$$

Fix $x \in X$ and $R \in (0, 1)$, let $r = R^{1+2\rho(R)}$. Let $B_0 = \{x\}$ and

$$B_i = B(x, ir)\backslash B(x, (i-1)r) \text{ for all } i \geq 1.$$

Write $n_R = [R/r]$, where $[z]$ denotes the integral part of the real number z. Then

$$n_R = [R^{-2\rho(R)}] \geq [R^{\frac{4\log 6}{(1-s)\log R}}][6^{\frac{4}{1-s}}] > 4.$$

Theorem 1.10 will follow from the claim below.

Claim 2. *If* $R \in (0, 1)$*, then* $B_i \cap E = \varnothing$ *for some* $1 \leq i \leq n_R$.

Otherwise, we may assume that $B_i \cap E$ is non-empty for all $1 \leq i \leq n_R$. Take a point $y_i \in B_i \cap E$ for each $1 \leq i \leq n_R$ and let

$$\Theta = \{y_i \mid 1 \leq i \leq n_R\}.$$

For any $1 \leq i, j \leq n_R$ with $j \geq i + 3$, we have

$$\mathrm{d}(y_i, y_j) \geq \mathrm{d}(x, y_j) - \mathrm{d}(x, y_i) > (j-1)r - ir \geq 2r.$$

Then $B(y_j, r) \cap B(y_i, r) = \varnothing$. Therefore, a ball with radius r covers at most three points in Θ, thus

$$n_R/3 \leq N_{r,R}(E). \tag{4.4}$$

By the definition of $\rho(R)$ and (4.3), there exists a $\delta \in (0,1)$ such that

$$\max\left\{\frac{\log r_\delta}{\log R} - 1, \frac{\delta}{1-\delta}, \frac{2\log 6}{-(1-s)\log R}\right\} < 2\rho(R).$$

Then for such δ, we have $r < r_\delta$, $r^{1-\delta} < R$ and

$$\frac{\log 6}{-(1-s)\log R} < \rho(R). \tag{4.5}$$

Thus, by (4.2) we have $N_{r,R}(E) \le R^{-(1+s)\rho(R)}$. Using (4.4) and the definition of n_R, we obtain that $R/(6r) \le n_R/3 \le R^{-(1+s)\rho(R)}$, which implies

$$\rho(R) \le \frac{\log 6}{-(1-s)\log R}.$$

This is contradictory to (4.5). The claim is proved.

For all $R \in (0,1)$, let $\psi(R) = R^{1+2\rho(R)}$ and $1 \le i_R \le n_R$ so that $B_{i_R} \cap E = \varnothing$. For any $x \in E$ and $0 < R < 1$, by virtue of Claim 1, if we let

$$E_{x,R} = E \cap B(x, (i_R - 1)\psi(R)),$$

then

$$E \cap B(x, \psi(R)) \subset E_{x,R} \subset B(x,R) \quad \text{and} \quad \mathrm{dist}(E_{x,R}, E \backslash E_{x,R}) \ge \psi(R).$$

Therefore, the quasi uniform disconnectedness of the set E is obtained. □

Before the proof of Proposition 1.13. We give the following lemma first.

Lemma 4.1. *Let $n_k = 2$ and $c_k = \frac{k+1}{2(k+2)}$ for all $k \ge 1$. Then the uniform Cantor set $F \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$ is quasi uniformly disconnected.*

Proof. Let $r^* = \frac{1}{3}$. For any $0 < r < r^*$, let

$$\psi(r) = c_1 \dots c_k \cdot (1 - 2c_{k+1}),$$

where $k \ge 1$ is the unique integer such that $c_1 \dots c_{k+1} \le r < c_1 \dots c_k$. Then

$$\psi(r) < c_1 \dots c_{k+1} \le r \quad \text{for all } 0 < r < r^*$$

and

$$\lim_{r\to 0} \frac{\log \psi(r)}{\log r} = 1.$$

Fix $x \in F$ and $0 < r < r^*$. Suppose $k \geq 1$ is the unique integer such that $c_1 \dots c_{k+1} \leq r < c_1 \dots c_k$. Note that the length of every basic interval I of rank $k+1$ of F is $c_1 \dots c_{k+1}$ and $\mathrm{dist}(I, F \backslash I) \geq \psi(r)$. Then the closed ball $B(x,r)$ contains a basic interval $I_{x,r}$ of rank $k+1$ of F such that $x \in I_{x,r}$. Take $E_{x,r} = I_{x,r} \cap F$, one can check that

$$F \cap B(x, \psi(r)) \subset E_{x,r} \subset B(x,r) \quad \text{and} \quad \mathrm{dist}(E_{x,r}, F \backslash E_{x,r}) \geq \psi(r). \qquad \square$$

Proof of Proposition 1.13. Let F be the uniform Cantor set given in Lemma 4.1. Then by (1.3), we have $\dim_H F = 1$.

Suppose $\Phi\colon 2^{\mathbb{R}} \to [0,1]$ is a set function such that $\Phi(E) \geq \dim_H E$ for any compact set $E \subset \mathbb{R}$. If it satisfies for all compact sets $E \subset \mathbb{R}$, one has $\Phi(E) < 1$ if and only if E is quasi uniformly disconnected. Then since F is quasi uniformly disconnected, we should have $\dim_H F < 1$, which is impossible. $\square$

5. Quasi-Assouad dimensions of Moran sets

In this section, we first compute the quasi-Assouad dimensions of Moran sets under the assumption that

$$\lim_{k\to\infty} \frac{\log c_k}{\log(c_1 \dots c_k)} = 0.$$

Then, we will give the proof of Proposition 1.6.

Proof of Theorem 1.14. For $1 \leq p \leq q$, let

$$s_{p,q} = \frac{\log(n_p \dots n_q)}{-\log(c_p \dots c_q)}.$$

Notice that $\dim_{qA} F = \lim_{\delta\to 0} h_F(\delta)$. It suffices to show

$$h_F(\delta) = \overline{\lim_{q\to\infty}} \max_{1\leq p\leq l_{q,\delta}} s_{p,q} \qquad \text{for any } \delta \in (0,1).$$

Step 1. Fix $\delta \in (0,1)$, we will verify

$$h_F(\delta) \leq \overline{\lim}_{q\to\infty} \max_{1\leq p\leq l_{q,\delta}} s_{p,q}.$$

Fix $s > \overline{\lim}_{q\to\infty} \max_{1\leq p\leq l_{q,\delta}} s_{p,q}$, we are going to find two constants $b, c > 0$ such that $N_{r,R}(F) \leq c(R/r)^s$ whenever $0 < r < r^{1-\delta} \leq R < b$.

Since

$$s > \overline{\lim_{q\to\infty}} \max_{1\le p\le l_{q,\delta}} s_{p,q},$$

one can find a constant $\sigma > 0$ small enough such that

$$s(1-\sigma) > \overline{\lim_{q\to\infty}} \max_{1\le p\le l_{q,\delta}} s_{p,q}.$$

Then there exists a positive integer N_1 such that for all $q \ge N_1$,

$$s(1-\sigma) > \max_{1\le p\le l_{q,\delta}} s_{p,q}. \tag{5.1}$$

Since

$$\lim_{k\to\infty} \frac{\log c_k}{\log(c_1\ldots c_k)} = 0,$$

there exists a positive integer N_2 such that for all $k \ge N_2$,

$$\frac{\log c_k}{\log(c_1\ldots c_k)} < \frac{\delta\sigma}{2}. \tag{5.2}$$

Let $N = \max\{N_1, N_2\}$. Take $b = c_1\ldots c_N$ and $c = 3$. For $0 < r < r^{1-\delta} \le R < b$, we may assume that

$$c_1\ldots c_p \le R < c_1\ldots c_{p-1} \quad \text{and} \quad c_1\ldots c_q \le r < c_1\ldots c_{q-1}$$

for some $q \ge p \ge N+1$. Then a ball of radius R intersects at most three basic intervals of rank $(p-1)$ of F, thus

$$N_{r,R}(F) \le 3n_p\ldots n_q = 3(R/r)^{\frac{\log(n_p\ldots n_q)}{\log(R/r)}} \le 3(R/r)^{\frac{\log(n_p\ldots n_q)}{-\log(c_p\ldots c_q)+\log(c_p c_q)}}. \tag{5.3}$$

Since $(c_1\ldots c_q)^{1-\delta} \le r^{1-\delta} \le R < c_1\ldots c_{p-1}$, we have

$$\frac{\log(c_p\ldots c_q)}{\log(c_1\ldots c_q)} > \delta. \tag{5.4}$$

That means $1 \le p \le l_{q,\delta}$. By (5.1), we have

$$s_{p,q} = \frac{\log(n_p\ldots n_q)}{-\log(c_p\ldots c_q)} < s(1-\sigma). \tag{5.5}$$

On the other hand, by (5.2) and (5.4), we obtain that

$$\begin{aligned}\frac{\log(c_p c_q)}{\log(c_p\dots c_q)} &= \frac{\log c_p + \log c_q}{\log(c_1\dots c_q)}\cdot\frac{\log(c_1\dots c_q)}{\log(c_p\dots c_q)}\\ &< \Big(\frac{\log c_p}{\log(c_1\dots c_p)} + \frac{\log c_q}{\log(c_1\dots c_q)}\Big)\cdot\frac{1}{\delta}\\ &< 2\cdot\frac{\delta\sigma}{2}\cdot\frac{1}{\delta}\\ &= \sigma.\end{aligned}$$

Then by (5.5),

$$\frac{\log(n_p\dots n_q)}{-\log(c_p\dots c_q + \log c_p c_q)} \le \frac{\log(n_p\dots n_q)}{-(1-\sigma)\log(c_p\dots c_q)} = \frac{s_{p,q}}{1-\sigma} < s.$$

Therefore, by (5.3), we have

$$N_{r,R}(F) < 3(R/r)^s = c(R/r)^s.$$

STEP 2. Fix $\delta \in (0,1)$, we will verify

$$h_F(\delta) \ge \overline{\lim}_{q\to\infty} \max_{1\le p\le l_{q,\delta}} s_{p,q}.$$

Fix $\alpha > h_F(\delta)$, we may assume that there are two constants $b, c > 0$ such that $N_{r,R} \le c(R/r)^s$ whenever $0 < r < r^{1-\delta} \le R < b$. Without loss of generality, we may assume that there exists a positive integer M such that $c_1\dots c_M < b \le c_1\dots c_{M-1}$. Fix $\varepsilon \in (0, 2(1-\delta)/\delta)$, we can take a positive integer $N \ge M$ large enough such that for all $q \ge N$,

$$\frac{\log(c_1\dots c_M)}{\log(c_1\dots c_q)} < \frac{\varepsilon}{2+\varepsilon} \quad\text{and}\quad \frac{\log 4c}{-\log(c_1\dots c_q)} < \frac{\delta\varepsilon}{2}. \tag{5.6}$$

For any $q \ge N$, let $r = c_1\dots c_q$. Since

$$\frac{\log(c_{M+1}\dots c_q)}{\log(c_1\dots c_q)} = 1 - \frac{\log(c_1\dots c_M)}{\log(c_1\dots c_q)} > 1 - \frac{\varepsilon}{2+\varepsilon} > \delta,$$

then $M+1 \le l_{q,\delta}$. For any integer $p \in [M+1, l_{q,\delta}]$, let $R = c_1\dots c_{p-1}$. Then $0 < r < r^{1-\delta} \le R < b$, and thus

$$N_{r,R}(F) \le c(R/r)^\alpha = c(c_p\dots c_q)^{-\alpha}. \tag{5.7}$$

Since a ball of radius r intersects at most four basic intervals of rank q, we have

$$N_{r,R}(F) \geq (n_p \dots n_q)/4.$$

Hence, by (5.6) and (5.7), for any integer $p \in [M+1, l_{q,\delta}]$,

$$\begin{aligned}\frac{\log(n_p \dots n_q)}{-\log(c_p \dots c_q)} &\leq \frac{\log 4c}{-\log(c_p \dots c_q)} + \alpha \\ &= \frac{\log 4c}{-\log(c_1 \dots c_q)} \cdot \frac{\log(c_1 \dots c_q)}{\log(c_p \dots c_q)} + \alpha \\ &< \frac{\delta\varepsilon}{2} \cdot \frac{1}{\delta} + \alpha \\ &= \alpha + \frac{\varepsilon}{2}.\end{aligned}$$

Therefore, since $c_k \leq 1/n_k$ for all k, for any integer $1 \leq p \leq M$, by (5.6), we have

$$\begin{aligned}\frac{\log(n_p \dots n_q)}{-\log(c_p \dots c_q)} &= \frac{\log(n_p \dots n_M)}{-\log(c_p \dots c_q)} + \frac{\log(n_{M+1} \dots n_q)}{-\log(c_p \dots c_q)} \\ &\leq \frac{\log(c_p \dots c_M)}{\log(c_p \dots c_q)} + \frac{\log(n_{M+1} \dots n_q)}{-\log(c_{M+1} \dots c_q)} \\ &\leq \frac{\log(c_1 \dots c_M)}{\log(c_{M+1} \dots c_q)} + \alpha + \frac{\varepsilon}{2} \\ &= \frac{\log(c_1 \dots c_M)}{\log(c_1 \dots c_q)} \cdot \frac{\log(c_1 \dots c_q)}{\log(c_{M+1} \dots c_q)} + \alpha + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2+\varepsilon} \cdot \frac{2+\varepsilon}{2} + \alpha + \frac{\varepsilon}{2} \\ &= \alpha + \varepsilon.\end{aligned}$$

In a word, for any $q \geq N$ and any $1 \leq p \leq l_{q,\delta}$, we have $s_{p,q} \leq \alpha + \varepsilon$. Then

$$\varlimsup_{q \to \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q} \leq \alpha + \varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$\varlimsup_{q \to \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q} \leq \alpha.$$

Therefore, we obtain that

$$\overline{\lim_{q\to\infty}} \max_{1\le p\le l_{q,\delta}} s_{p,q} \le h_F(\delta).$$

When $\inf_k c_k > 0$, note that

$$\frac{(q-p+1)\log\sup_k c_k}{q\log\inf_k c_k} \le \frac{\log(c_p\ldots c_q)}{\log(c_1\ldots c_q)} \le \frac{(q-p+1)\log\inf_k c_k}{q\log\sup_k c_k},$$

we have

$$q - \frac{q\delta\log\inf_k c_k}{\log\sup_k c_k} \le l_{q,\delta} \le q+1-\frac{q\delta\log\sup_k c_k}{\log\inf_k c_k}.$$

Therefore, when δ is small and q is large, we obtain that

$$q\Big(1-\frac{\delta\log\inf_k c_k}{\log\sup_k c_k}\Big) \le l_{q,\delta} \le q\Big(1-\frac{\delta\log\sup_k c_k}{2\log\inf_k c_k}\Big). \qquad \square$$

Now, we turn to the proof of Proposition 1.6.

Proof of Proposition 1.6. Let $n_k = 2$ for all $k \ge 1$. Let $\{a_i\}_{i\ge 1}$ be a sequence of positive integers with $a_1 = 1$ such that

$$a_{i+1} > \frac{b(c-a)}{a(c-b)}a_i + i \quad \text{for all } i$$

and

$$\lim_{i\to\infty}\frac{i}{a_i} = \lim_{i\to\infty}\frac{a_1+a_2+\cdots+a_{i-1}}{a_i} = 0.$$

For any $k \ge 1$, if $k \in [a_i, a_{i+1})$ for some i, then let

$$c_k = \begin{cases} 2^{-1/c} & \text{if } k \in \Big[a_i, \frac{b(c-a)}{a(c-b)}a_i\Big), \\ 2^{-1/d} & \text{if } k \in \Big[\frac{b(c-a)}{a(c-b)}a_i, \frac{b(c-a)}{a(c-b)}a_i + i\Big), \\ 2^{-1/a} & \text{otherwise.} \end{cases}$$

Under these assumptions, it is clear that $\inf_k c_k > 0$ and $\frac{\log n_k}{-\log c_k} \in \{a, c, d\}$ for all $k \ge 1$. We will verify that any Moran set $F \in \mathcal{M}([0,1], \{n_k\}_k, \{c_k\}_k)$ satisfies

$$\dim_H F = a, \quad \overline{\dim}_B F = b, \quad \dim_{qA} F = c, \quad \dim_A F = d.$$

First, by (1.4), we have

$$\dim_A F = \lim_{m\to\infty} \sup_{k\geq 1} \frac{\log(n_{k+1}\dots n_{k+m})}{-\log(c_{k+1}\dots c_{k+m})} = d.$$

Then, since $\frac{b(c-a)}{a(c-b)} > 1$, by Theorem 1.14, we obtain that

$$\dim_{qA} F = \lim_{\eta\to 0} \overline{\lim_{q\to\infty}} \max_{1\leq p\leq q(1-\eta)} \frac{\log(n_p\dots n_q)}{-\log(c_p\dots c_q)} = c.$$

Finally, since

$$\underline{\lim_{k\to\infty}} \frac{\log(n_1\dots n_k)}{-\log(c_1\dots c_k)} = a$$

and

$$\overline{\lim_{k\to\infty}} \frac{\log(n_1\dots n_k)}{-\log(c_1\dots c_k)} = b,$$

by (1.3) we have $\dim_H F = a$ and

$$\overline{\dim}_B F = b.$$

□

Acknowledgment. The authors are very grateful to the reviewers for their kindness and patience to improve the quality of the manuscript.

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Received November 24, 2014

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