J. Fractal Geom. 3 (2016), 119–161
DOI 10.4171/JFG/32

Journal of Fractal Geometry

Asymptotic perturbation of graph iterated function systems

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Abstract. We study an asymptotic perturbation of the limit set generated from a finitely family of conformal contraction maps endowed with a directed graph. We show that if those maps have asymptotic expansions under weak conditions, then the Hausdorff dimension of the limit set behaves asymptotically by the same order. We also prove that the Gibbs measure of a suitable potential and the measure theoretic entropy of this measure have asymptotic expansions under an additional condition. In final section, we demonstrate degeneration of graph iterated function systems.

Mathematics Subject Classification (2010). Primary: 37B10; Secondary: 37C45, 37D35.

Keywords. Thermodynamic formalism, symbolic dynamics, asymptotic perturbation.

Contents

[1] The author is heartily grateful to the referee for his valuable comments.

1. Introduction

Let $D \geq 1$ be an integer. We consider a triplet $(G, (J_v), (T_e))$ satisfying the following conditions.

(1) $G = (V, E, i, t)$ is a finite directed multigraph which consists of a vertices set V, a directed edges set E and two functions $i, t\colon E \to V$. For each $e \in E$, $i(e)$ is called the initial vertex of e and $t(e)$ called the terminal vertex of e. Assume that the graph G is strongly connected and aperiodic.

(2) For each $v \in V$, a subset $J_v \subset \mathbb{R}^D$ is compact and connected so that the interior of J_v is not empty and $J_{v'}$ and J_v are disjoint for $v' \neq v$.

(3) For each $e \in E$, a map $T_e\colon O_{t(e)} \to O_{i(e)}$ is conformal $C^{1+\beta}$-diffeomorphism with $\beta > 0$ and satisfies $0 < \|T_e'(x)\| < 1$ for $x \in O_{t(e)}$, $T_e J_{t(e)} \subset J_{i(e)}$ and $T_e J_{t(e)} \cap T_{e'} J_{t(e')} = \emptyset$ for $e' \in E$ with $e' \neq e$ and $i(e') = i(e)$. Here $O_{t(e)} \subset \mathbb{R}^D$ is an open and connected subset containing $J_{t(e)}$ and $\|T_e'(x)\|$ denotes the operator norm of $T_e'(x)$ on $\mathbb{R}^D$.

It is known that there exist unique non-empty compact subsets $K_v \subset J_v$ for $v \in V$ such that

$$K_v = \bigcup_{e \in E\,:\, i(e)=v} T_e(K_{t(e)})$$

is fulfilled. Put $K = \bigcup_{v \in V} K_v$. In this paper we call the triplet $(G, (J_v), (T_e))$ a graph iterated function system (GIFS) and this set K the limit set of the GIFS. Such a system is studied by many authors ([4, 7, 9, 10, 11]) and they be mainly interested in the calculation of the Hausdorff dimension of K.

Now we state one of our main results. Recall that for an integer $k \geq 0$ and a number $\beta > 0$, a map $f(x)$ from a subset A of a normed space $(X, \|\cdot\|_X)$ to a normed space Y is of class $C^{k+\beta}$ if k-th derivative $f^{(k)}$ of f exists and there is a constant $c > 0$ such that $\|f^{(k)}(x) - f^{(k)}(y)\|_k \leq c\|x - y\|_X^\beta$ for any $x, y \in A$, where $\|\cdot\|_k$ is the usual operator norm on Y^k. Fix a GIFS $(G, (J_v), (T_e))$ and an integer $n \geq 0$. We give a family of GIFSs $(G, (J_v), (T_e(\epsilon, \cdot)))$ with a small parameter $\epsilon > 0$ so that

$(G)_n$ there exist numbers $\beta > 0$ and $\beta(\epsilon) > 0$, and $\mathbb{R}^D$-valued functions T_e of $C^{n+1+\beta}$, $T_{e,1}$ of $C^{n+\beta}$,..., $T_{e,n}$ of $C^{1+\beta}$, and $\widetilde{T}_{e,n}(\epsilon, \cdot)$ of $C^{1+\beta(\epsilon)}$ defined on $O_{t(e)}$ for each $e \in E$ such that $T_e(\epsilon, \cdot)$ has the form

$$T_e + T_{e,1}\epsilon + \cdots + T_{e,n}\epsilon^n + \widetilde{T}_{e,n}(\epsilon, \cdot)\epsilon^n \ \text{ on } J_{t(e)},$$

and, $|\widetilde{T}_{e,n}(\epsilon, \cdot)| \to 0$ and $\|\frac{\partial}{\partial x}\widetilde{T}_{e,n}(\epsilon, \cdot)\| \to 0$ as $\epsilon \to 0$ are satisfied, where $|\cdot|$ is a usual norm on $\mathbb{R}^D$.

We note that $\beta(\epsilon)$ may tend to 0 as $\epsilon \to 0$. Let $K(\epsilon)$ be the limit set of the GIFSs $(G, (J_v), (T_e(\epsilon, \cdot)))$ for $\epsilon > 0$. Now we are in a position to state one of the main result.

Theorem 1.1. *Assume that condition* $(G)_n$ *is satisfied. Then there exist numbers* $s_1, \dots, s_n \in \mathbb{R}$ *such that* $\dim_H K(\epsilon) = \dim_H K + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n)$ *in* $\mathbb{R}$*, where* $\dim_H K(\epsilon)$ *denotes the Hausdorff dimension of* $K(\epsilon)$*.*

Each numbers s_k is explicitly determined for the GIFS $(G, (J_v), (T_e))$ and the maps $T_{e,1}, T_{e,2}, \dots, T_{e,n}$ ([14] for detail).

Remark 1.2. (i) In the system $(G, (J_v), (T_e))$, if the cone property (condition (d) in [11]) is imposed on the set J_v for each $v \in V$, then this system satisfies the condition of conformal graph directed Markov systems (CGDMS) defined in [11] with a finite alphabet. In addition to the property, when V consists of one point v, our system fulfills the condition of conformal iterated function system with a finite alphabet (finite CIFS) defined in [10].

(ii) From each T_e is a conformal C^1-diffeomorphism, this map is either holomorphic or antiholomorphic if $D = 2$, and Liouville's theorem (Theorem A.3.7 in [2]) implies that this map has the form

$$T_e(x) = \zeta_e A_e(i_{\iota_e, \xi_e}(x)) + \rho_e \tag{1.1}$$

if $D \geq 3$, where $\zeta_e \in \mathbb{R}$, $\rho_e \in \mathbb{R}^D$, A_e is a linear isometry on $\mathbb{R}^D$, and i_{ι_e,ξ_e} is either the identity or the inversion with respect to the sphere with the center $\iota_e \in \mathbb{R}^D$ and the radius $\xi_e > 0$. This inversion is defined by

$$i_{\iota_e,\xi_e}(x) = \iota_e + \xi_e^2 \frac{x - \iota_e}{|x - \iota_e|^2}.$$

(iii) When V consists of one point v and the cone property for J_v is fulfilled, our theorem under $n = 0$ contains a similar result of Roy and Urbański (Theorem 5.8 in [10]).

(iv) Assume that $D \geq 3$ and the cone property for J_v, $v \in V$, are satisfied. Let $\mathrm{SO}(D)$ be the totally of linear isometries on $\mathbb{R}^D$ whose determinants are 1, and $\Gamma = (0, \infty) \times (0, \infty) \times \mathbb{R}^D \times \mathbb{R}^D \times \mathrm{SO}(D)$. Note that Γ is an open subset of $\mathbb{R}^{2+D(D+3)/2}$. For the five parameters $z_e = (\zeta_e, \xi_e, \iota_e, \rho_e, A_e) \in \Gamma$ of T_e defined in (1.1), denoted by $K((z_e))$ the limit set of the GIFS $(G, (J_v), (T_e))$. We write $B(z_e, R)$ for the open ball with the center z_e and the radius R. Under this notation, Roy and Urbański (Theorem 7.2 in [11]) showed that there exists a small $R > 0$ such that $B(z_e, R) \subset \Gamma$ for any $e \in E$ and the map

$$\prod_{e\in E} B(z_e, R) \ni (y_e) \longmapsto \dim_H K((y_e))$$

is real-analytic. Our theorem gives an asymptotic version of this result. In fact, it is not hard to check that if the five parameter

$$z_e(\epsilon) = (\zeta_e(\epsilon), \xi_e(\epsilon), \iota_e(\epsilon), \rho_e(\epsilon), A_e(\epsilon))$$

of $T_e(\epsilon, \cdot)$ has n-order asymptotic expansions

$$z_e(\epsilon) = z_e + \sum_{k=1}^{n} (\zeta_{e,k}, \xi_{e,k}, \iota_{e,k}, \rho_{e,k}, A_{e,k})\epsilon^k + \tilde{z}_{e,n}(\epsilon)\epsilon^n$$

with $|\tilde{z}_{e,n}(\epsilon)| \to 0$ as $\epsilon \to 0$, then the GIFSs $(G, (J_v), (T_e(\epsilon, \cdot)))$ satisfies condition $(G)_n$. In particular, each coefficient $T_{e,k}$ $(k = 1, 2, \ldots n)$ is given as C^∞.

For the second result, we also introduce the following condition:

$(G)'_n$ under condition $(G)_n$, the small order parts $\widetilde{T}_{e,n}(\epsilon, x)$ satisfy

$$c_1 = \limsup_{\epsilon\to 0} \max_{e\in E} \sup_{x,y\in O_{t(e)}:x\neq y} \frac{\left\| \frac{\partial}{\partial x}\widetilde{T}_{e,n}(\epsilon, x) - \frac{\partial}{\partial x}\widetilde{T}_{e,n}(\epsilon, y) \right\|}{|x-y|^\beta} < \infty.$$

We give some notation below. We take a number $r \in (0, 1)$ so that $r > \|T'_e\|$ and $r > \sup_{x\in U_{t(e)}} \|T'_e(\epsilon, x)\|$ for any $e \in E$ and for any $\epsilon > 0$, where $U_{t(e)}$ is given by (2.1) in the next section. Denoted by

$$E^{(\infty)} = \left\{ \omega = (\omega_k)_{k=0}^\infty \in \prod_{k=0}^{\infty} E : t(\omega_k) = i(\omega_{k+1}) \text{ for all } k \geq 0 \right\}$$

a code space. The shift transformation

$$\sigma : E^{(\infty)} \longrightarrow E^{(\infty)}$$

is defined by

$$(\sigma\omega)_k = \omega_{k+1} \quad \text{for } \omega = (\omega_k) \in E^{(\infty)}.$$

The pair $(E^{(\infty)}, \sigma)$ is called a *subshift of finite type*. Let

$$\pi : E^{(\infty)} \longrightarrow \mathbb{R}^D$$

be a coding map for the GIFS $(G, (J_v), (T_e))$ defined by

$$\pi\omega = \bigcap_{k=0}^{\infty} T_{\omega_0} \cdots T_{\omega_k} J_{t(\omega_k)} \quad \text{for } \omega \in E^{(\infty)}.$$

We put the function

$$\varphi(\omega) = \log \|T'_{\omega_0}(\pi\sigma\omega)\| \tag{1.2}$$

for $\omega \in E^{(\infty)}$. For each $\epsilon > 0$, $\pi(\epsilon, \omega)$ means the coding map of the GIFS $(G, (J_v), (T_e(\epsilon, \cdot)))$ and $\varphi(\epsilon, \omega)$ the function $\log \left\| \frac{\partial}{\partial x} T_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega)) \right\|$. Denoted by μ the Gibbs measure of $(\dim_H K)\varphi$ on $E^{(\infty)}$ and by $\mu(\epsilon, \cdot)$ the Gibbs measure of $(\dim_H K(\epsilon))\varphi(\epsilon, \cdot)$ on $E^{(\infty)}$. It is known that the $(\dim_H K)$-dimensional Hausdorff measure restricted to K is equivalent to the measure $\mu \circ \pi^{-1}$ by condition (3). For $\theta \in (0, 1)$, a metric d_θ on $E^{(\infty)}$ is defined by

$$d_\theta(\omega, \upsilon) = \theta^{m_0},$$

with $m_0 = \min\{m \geq 0 \colon \omega_m \neq \upsilon_m\}$. For $\mathbb{K} = \mathbb{C}, \mathbb{R}$ or $\mathbb{R}^D$, let $C(E^{(\infty)}, \mathbb{K})$ be the Banach space consisting of all $\mathbb{K}$-valued continuous functions on $E^{(\infty)}$ endowed with the supremum norm

$$\|f\|_\infty = \sup_{\omega \in E^{(\infty)}} |f(\omega)|,$$

and $F_\theta(E^{(\infty)}, \mathbb{K})$ the Banach space consisting of all $\mathbb{K}$-valued d_θ-Lipschitz continuous functions on $E^{(\infty)}$ endowed with the Lipschitz norm

$$\|f\|_\theta = \|f\|_\infty + [f]_\theta,$$

where

$$[f]_\theta = \sup\left\{\frac{|f(\omega) - f(\upsilon)|}{d_\theta(\omega, \upsilon)} \colon \omega_0 = \upsilon_0,\ \omega \neq \upsilon\right\}.$$

If no confusion can arise, we may omit $\mathbb{K}$ from notation of these two spaces. We obtain the second result as follows.

Theorem 1.3. *Assume that condition $(G)'_n$ is satisfied. Choose any $\theta_1 \in (r^\beta, 1)$. Then there exist linear functionals $\mu_1, \mu_2, \ldots, \mu_n \in F^*_{\theta_1}(E^{(\infty)}, \mathbb{R})$, and numbers $H_1, H_2, \ldots, H_n \in \mathbb{R}$ such that for each $f \in F_{\theta_1}(E^{(\infty)}, \mathbb{C})$*

$$\mu(\epsilon, f) = \mu(f) + \mu_1(f)\epsilon + \cdots + \mu_n(f)\epsilon^n + o(\epsilon^n) \quad \text{in } \mathbb{R},$$

$$h(\mu(\epsilon, \cdot)) = h(\mu) + H_1\epsilon + \cdots + H_n\epsilon^n + o(\epsilon^n) \quad \text{in } \mathbb{R},$$

where $h(\mu(\epsilon, \cdot))$ denotes the measure-theoretic entropy of the Gibbs measure $\mu(\epsilon, \cdot)$.

We would prove Theorem 1.1 and Theorem 1.3 in Section 3. We remark that $\mu_1, \ldots, \mu_n$ and $H_1, \ldots, H_n$ are exactly given [14].

One of our motivations for studying those asymptotic expansions is as follows. For a fixed GIFS $(G, (J_v), (T_e))$, we decompose the edge set E of the graph G into $E = E(0) \cup E(1)$ with $E(0) \neq \emptyset$. Consider a family of GIFSs $(G, (J_v), (T_e(\epsilon, \cdot)))$ with a small parameter $\epsilon > 0$ so that

$$\begin{cases} |T_e(\epsilon, \cdot) - T_e| \longrightarrow 0, & e \in E(0), \\ |T_e(\epsilon, \cdot) - a_e| \longrightarrow 0, & e \in E(1), \end{cases}$$

as $\epsilon \to 0$, where a_e is a constant. Put

$$E(0)^{(\infty)} = \left\{ (\omega_k)_{k=0}^{\infty} \in \prod_{k=0}^{\infty} E(0) \colon t(\omega_k) = i(\omega_{k+1}) \text{ for all } k \geq 0 \right\}$$

and

$$\sigma(0) = \sigma|_{E(0)^{(\infty)}}.$$

In this setting, when ϵ tends to 0, the subshift $(E^{(\infty)}, \sigma)$ corresponding to the perturbed system $(G, (J_v), (T_e(\epsilon, \cdot)))$ changes extremely to the subshift $(E(0)^{(\infty)}, \sigma(0))$ corresponding to the unperturbed system $(G(0), (J_v), (T_e))$ at $\epsilon = 0$, where $G(0) = (V, E(0))$. Such a situation is often called a *singular perturbation of symbolic dynamics* ([5, 6, 8]).

One of our interests is to study the behaviour of the dimension $\dim_H K(\epsilon)$, the Gibbs measure $\mu(\epsilon, \cdot)$ and the measure-theoretic entropy of this measure as $\epsilon \to 0$. It is known that $\dim_H K(\epsilon)$ converges to $\dim_H \widetilde{K}(G(0))$ as $\epsilon \to 0$ (as in a special case treated in [8]), where $\widetilde{K}(G(0))$ is the limit set of $(G(0), (J_v), (T_e))$. On the other hand, the continuity of the Gibbs measure $\mu(\epsilon, \cdot)$ depends on the number of strong connected components of $G(0)$ and on the convergence speed (or higher order asymptotic expansion) of each $T_e(\epsilon, \cdot)$. In fact, it is known that when $G(0)$ consists of two strong connected components $\{H_1, H_2\}$ and the two dimensions $\dim_H \widetilde{K}(H_1)$ and $\dim_H \widetilde{K}(H_2)$ are equal, the limit $\lim_{\epsilon \to 0} \mu(\epsilon, \cdot) = \mu$ exists if $T_e(\epsilon, \cdot)$ has a 1-order asymptotic expansion for each $e \in E$ and some conditions are satisfied (Theorem 4.1 and Theorem 4.9 in [15]). In particular, this limit μ has the form $\mu = \gamma_1 \tilde{\mu}_1 + \gamma_2 \tilde{\mu}_2$ with $\gamma_1, \gamma_2 \geq 0$ and $\gamma_1 + \gamma_2 = 1$, and this coefficient is determined with the convergence speed $T_{e,1}$, where each $\tilde{\mu}_k$ is the Gibbs measure corresponding to $(H_k, (J_v), (T_e))$ for $k = 1, 2$. To deal with such a problem, we need to study high order asymptotic behaviors of the dimension $\dim_H K(\epsilon)$ and the Gibbs measure $\mu(\epsilon, \cdot)$ under the case when $E(1) = \emptyset$, and state these in the present paper. This argument is very important in our future work in the case when $E(1) \neq \emptyset$.

In Section 2, we give some notation and auxiliary propositions which need to prove the main theorems. Proofs of the main theorems are shown in Section 3. In the last section 4, we provide concrete examples which satisfy condition $(G)_1$ or $(G)'_1$ in Example 4.1 and Example 4.2. In particular, we demonstrate an example in which the small order part of a function $T_e(\epsilon, \cdot)$ is of $C^{1+\beta(\epsilon)}$ and $\beta(\epsilon)$ tends to 0 as $\epsilon \to 0$. In Example 4.3, we formulate degeneration of graph iterated function systems and calculate the speed of Hausdorff dimension of this limit sets. This example is one of examination in the case when $\sharp E(1) \neq \emptyset$.

2. Auxiliary propositions

In this section, we give some auxiliary propositions which need to prove the main theorems. We begin with the following fact. Let $(G, (J_v), (T_e))$ be a GIFS. Put

$$U_v = \bigcup_{x \in J_v} B(x, \delta) \tag{2.1}$$

for small $\delta > 0$ with $\overline{U_v} \subset O_v$ for all $v \in V$, where $\overline{U_v}$ is the closure of U_v. Then these U_v are open, relative compact and connected subsets of $\mathbb{R}^D$. Furthermore, $T_e \overline{U}_{t(e)} \subset U_{i(e)}$ for any $e \in E$ is satisfied by $T_e J_{t(e)} \subset J_{i(e)}$. We have the next result:

Proposition 2.1 ([9]). *Under the above notation, for any $v \in V$ and for any map T of class C^1 from O_v to a normed space $(Y, \|\cdot\|_Y)$, for each $x, y \in J_v$*

$$\|T(x) - T(y)\|_Y \le c_2 \sup_{z \in U_v} \|T'(z)\||x - y| \tag{2.2}$$

is satisfied, where we put

$$c_2 = \max\left\{1, \max_{v \in V}\left(\frac{\operatorname{diam}(J_v)}{\operatorname{dist}(J_v, \partial U_v)}\right)\right\}.$$

Choose any $\hat{\delta} \in (0, \delta)$ and put

$$\widehat{U}_v = \bigcup_{x \in J_v} B(x, \hat{\delta}) \quad \text{for each } v \in V.$$

We note that even if we replace the set J_v in Proposition 2.1 by $\widehat{U}_v$ and the constant c_2 by $\max\{1, \max_{v \in V}(\operatorname{diam}(\widehat{U}_v)/\operatorname{dist}(\widehat{U}_v, \partial U_v))\}$, the assertion of this proposition is correct.

Proposition 2.2. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces. Assume that a function $f(x)$ from a bounded open set O of X to Y is of $C^{n+\beta}$ with $n \geq 0$ and $\beta > 0$. Then for any $x, y \in O$ with $\{tx + (1-t)y : 0 \leq t \leq 1\} \subset O$ there exists an n-multilinear map $L(n, f, x, y)$ from X^n to Y such that*

$$f(x) = f(y) + \sum_{k=1}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k + L(n, f, x, y)(x-y)^n, \tag{2.3}$$

where $f^{(k)}(y)(x-y)^k$ means $f^{(k)}(y)(\underbrace{x-y, \ldots, x-y}_{k})$ and $L(n, f, x, y)$ is defined by

$$L(0, f, x, y) = f(x) - f(y),$$

$$L(n, f, x, y) = \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!}(f^{(n)}(y + t(x-y)) - f^{(n)}(y))\, dt$$

for $n \geq 1$. In particular, $\|L(n, f, x, y)\|_n / \|x-y\|_X^\beta$ is bounded uniformly in $x, y \in O$ with $\{tx + (1-t)y : 0 \leq t \leq 1\} \subset O$.

Proof. The expansion of $f(x)$ immediately follows from Taylor theorem [1]. It suffices to prove the last assertion. We have

$$\begin{aligned}
\frac{\|L(n, f, x, y)\|_n}{\|x-y\|_X^\beta} &\leq \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!\|x-y\|_X^\beta} \|f^{(n)}(y + t(x-y)) - f^{(n)}(y)\|_n\, dt \\
&\leq \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!\|x-y\|_X^\beta} c\|y + t(x-y) - y\|_X^\beta\, dt \\
&\leq c\int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} t^\beta\, dt = \frac{c}{(\beta+1)\cdots(\beta+n)}.
\end{aligned}$$

Thus we obtain the assertion. □

Proposition 2.3. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces and $O \subset X$ a bounded open set. Assume that a map $f(\epsilon, \cdot) : O \to Y$ with a parameter $\epsilon > 0$ has the form $f(\epsilon, \cdot) = f + f_1\epsilon + \cdots + f_n\epsilon^n + \tilde{f}_n(\epsilon, \cdot)\epsilon^n$ and $\|\tilde{f}_n(\epsilon, \cdot)\|_Y \to 0$ as $\epsilon \to 0$ with maps $f = f_0$ of $C^{n+\beta}$, f_1 of $C^{n-1+\beta}$,..., f_n of C^β and $\tilde{f}_n(\epsilon, \cdot)$ of C^β. Further, $x(\epsilon) \in O$ satisfies $x(\epsilon) = x + x_1\epsilon + \cdots + x_n\epsilon^n + \tilde{x}_n(\epsilon)\epsilon^n$ and $\|\tilde{x}_n(\epsilon)\|_X \to 0$ as $\epsilon \to 0$ for some $x = x_0 \in O$ and $x_1, \ldots, x_n, \tilde{x}_n(\epsilon) \in X$.*

Then $f(\epsilon, x(\epsilon))$ *has the form*

$$f(\epsilon, x(\epsilon)) = y_0 + y_1\epsilon + \cdots + y_n\epsilon^n + \tilde{y}_n(\epsilon)\epsilon^n$$

and $\|\tilde{y}_n(\epsilon)\|_Y \to 0$ *as* $\epsilon \to 0$ *by putting*

$$y_0 = f(x),$$

$$y_j = f_j(x) + \sum_{l=0}^{j-1}\sum_{k=1}^{j-l} \sum_{\substack{i_1,\ldots,i_k \ge 1:\\ i_1+\cdots+i_k=j-l}} \frac{f_l^{(k)}(x)(x_{i_1},\ldots,x_{i_k})}{k!}, \quad 1 \le j \le n,$$

and

$$\begin{aligned}\tilde{y}_n(\epsilon) = &\sum_{l=0}^{n-1}\sum_{k=1}^{n-l}\sum_{i=n-l+1}^{kn} \sum_{\substack{1\le i_1,\ldots,i_k \le n:\\ i_1+\cdots+i_k=i}} \frac{f_l^{(k)}(x)(x_{i_1},\ldots,x_{i_k})}{k!}\epsilon^{i-n+l}\\ &+ \sum_{l=0}^{n}\sum_{k=1}^{n-l} M(l,k,\epsilon,x)\frac{\epsilon^l}{k!}\\ &+ \sum_{l=0}^{n} L(n-l, f_l, x(\epsilon), x)\Big(\frac{x(\epsilon)-x}{\epsilon}\Big)^{n-l} + \tilde{f}_n(\epsilon, x(\epsilon)),\end{aligned}$$

where

$$M(l,k,\epsilon,x) = \sum_{i=1}^{k} f_l^{(k)}(x)(x(\epsilon)-x-\tilde{x}_n(\epsilon)\epsilon^n, \ldots, \underbrace{\tilde{x}_n(\epsilon)}_{i\text{-}th}, \ldots, x(\epsilon)-x).$$

Proof. We have

$$\begin{aligned} f(\epsilon, x(\epsilon)) &= f(x(\epsilon)) + f_1(x(\epsilon))\epsilon + \cdots + f_n(x(\epsilon))\epsilon^n + \tilde{f}_n(\epsilon, x(\epsilon))\epsilon^n\\ &= \sum_{l=0}^{n}\Big(\sum_{k=0}^{n-l}\frac{f_l^{(k)}(x)}{k!}(x(\epsilon)-x)^k \\ &\qquad + L(n-l, f_l, x(\epsilon), x)(x(\epsilon)-x)^{n-l}\Big)\epsilon^l \\ &\quad + \tilde{f}_n(\epsilon, x(\epsilon))\epsilon^n \end{aligned} \tag{2.4}$$

with $(n-l)$-multilinear maps

$$L(n-l, f_l, x(\epsilon), x)\colon X^{n-l} \longrightarrow Y$$

by using (2.3) in Proposition 2.2. Here

$$
\begin{aligned}
f_l^{(k)}(x)(x(\epsilon)-x)^k &= f_l^{(k)}(x)(x_1\epsilon+\cdots+x_n\epsilon^n+\tilde{x}_n(\epsilon)\epsilon^n)^k \\
&= \sum_{i_1,\ldots,i_k=1}^{n} f_l^{(k)}(x)(x_{i_1}\epsilon^{i_1},\ldots,x_{i_k}\epsilon^{i_k}) + M(l,k,\epsilon,x)\epsilon^n \\
&= \sum_{i_1,\ldots,i_k=1}^{n} f_l^{(k)}(x)(x_{i_1},\ldots,x_{i_k})\epsilon^{i_1+\cdots+i_k} + M(l,k,\epsilon,x)\epsilon^n
\end{aligned}
$$

for $k \geq 1$ is satisfied. Thus we obtain the form of $f(\epsilon, x(\epsilon))$. We see the fact $\|\tilde{y}_n(\epsilon)\|_Y \to 0$ as $\epsilon \to 0$ by the definition of $\tilde{y}_n(\epsilon)$. Hence the assertion is given. □

Finally we recall an asymptotic solution of the equation $P(sf(\epsilon,\cdot)) = 0$ for $s \in \mathbb{R}$ [14]. Here $P(f)$ is the topological pressure of a function $f \in C(E^{(\infty)}, \mathbb{R})$ which is defined by

$$
P(f) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\tau} \exp\Big(\sup_{\omega} \sum_{j=0}^{k-1} f(\sigma^j\omega)\Big), \tag{2.5}
$$

where this first summation is over all paths $\tau = \tau_1\cdots\tau_k \in E^k$, i.e.

$$
t(\tau_j) = i(\tau_{j+1}) \quad \text{for } 1 \leq j \leq k-1,
$$

and the supremum is taken over all $\omega \in E^{(\infty)}$ with $\omega_0\cdots\omega_{k-1} = \tau$. It is known that if $f(\epsilon,\cdot) \in C(E^{(\infty)}, \mathbb{R})$ is negative, then the equation $P(\sigma, sf(\epsilon,\cdot)) = 0$ has an unique solution $s = s(\epsilon)$. Suppose that there exist both $\theta, \theta(\epsilon) \in (0,1)$, and $f, f_1, \ldots, f_n \in F_\theta(E^{(\infty)}, \mathbb{R})$ with $f < 0$, as well as $\tilde{f}_n(\epsilon,\cdot) \in F_{\theta(\epsilon)}(E^{(\infty)}, \mathbb{R})$ such that

$$
f(\epsilon,\cdot) = f + f_1\epsilon + \cdots + f_n\epsilon^n + \tilde{f}_n(\epsilon,\cdot)\epsilon^n
$$

and

$$
\|\tilde{f}_n(\epsilon,\cdot)\|_\infty \longrightarrow 0 \quad \text{as } \epsilon \to 0.
$$

Then we obtain the following result.

Theorem 2.4 ([14]). *Under the above condition, there exist* $s_1, \ldots, s_n \in \mathbb{R}$ *such that*

$$
s(\epsilon) = s + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n) \quad \text{in } \mathbb{R}, \tag{2.6}
$$

where s *is an unique solution of the equation* $P(sf) = 0$.

Note that in Theorem 2.6 of [14] this theorem is shown when $\theta(\epsilon)=\theta$. On the other hand, in the case when $\theta(\epsilon)\to 1$, Theorem 2.4 also follows from the proof of the same Theorem 2.6 in [14] with no change at all. In fact, in this Theorem, the asymptotic expansion of the eigenvector $\nu(\epsilon,\cdot)$ corresponding to the Perron eigenvalue of the dual of the Ruelle operator of $s\varphi(\epsilon,\cdot)$ can be proven from the condition $\theta(\epsilon)\in(0,1)$ for $\epsilon>0$.

3. Proofs

In this section, we will show Theorem 1.1 and Theorem 1.3 which are given in Section 1. We use the notation defined in Section 1 and Section 2. For the sake of convenience, we denote the composite map $T_{\omega_0}\cdots T_{\omega_n}$ by $T_{\omega_0\cdots\omega_n}$ and $T_{\omega_0}(\epsilon,\cdot)\cdots T_{\omega_n}(\epsilon,\cdot)$ by $T_{\omega_0\cdots\omega_n}(\epsilon,\cdot)$ for $\omega\in E^{(\infty)}$. Further, we sometimes write $\frac{\partial}{\partial x}T_e(\epsilon,x)$ as $T_e'(\epsilon,x)$ when no confusion is possible. We first prove the following lemma.

Lemma 3.1. *Assume that $(G)_n$ is satisfied. Choose any $\theta_2\in(r,1)$. Then there exist $\pi_1,\dots,\pi_n\in F_{\theta_2}(E^{(\infty)},\mathbb{R}^D)$ such that $\pi(\epsilon,\cdot)$ has the form*

$$\pi(\epsilon,\cdot)=\pi+\pi_1\epsilon+\cdots+\pi_n\epsilon^n+\tilde{\pi}_n(\epsilon,\cdot)\epsilon^n$$

and

$$\|\tilde{\pi}_n(\epsilon,\cdot)\|_\infty\longrightarrow 0\quad \text{as } \epsilon\to 0.$$

Proof. First we show this assertion in the case when $n=0$. For each $\omega\in E^{(\infty)}$ and $k\ge 0$, we have

$$\begin{aligned}
&|\pi(\epsilon,\omega)-\pi\omega|\\
&=|T_{\omega_0\cdots\omega_k}(\epsilon,\pi(\epsilon,\sigma^{k+1}\omega))-T_{\omega_0\cdots\omega_k}(\pi\sigma^{k+1}\omega)|\\
&\le|T_{\omega_0\cdots\omega_k}(\pi(\epsilon,\sigma^{k+1}\omega))-T_{\omega_0\cdots\omega_k}(\pi\sigma^{k+1}\omega)|\\
&\quad+\sum_{i=0}^{k}|T_{\omega_0\cdots\omega_{i-1}}(T_{\omega_i\cdots\omega_k}(\epsilon,\pi(\epsilon,\sigma^{k+1}\omega)))\\
&\qquad\qquad-T_{\omega_0\cdots\omega_i}(T_{\omega_{i+1}\cdots\omega_k}(\epsilon,\pi(\epsilon,\sigma^{k+1}\omega)))|\\
&\le c_2\sup_{z\in U_{t(\omega_k)}}|(T_{\omega_0\cdots\omega_k})'(z)||\pi(\epsilon,\sigma^{k+1}\omega)-\pi\sigma^{k+1}\omega|+|\widetilde{T}_{\omega_0,0}(\epsilon,\cdot)|\\
&\quad+c_2\sum_{i=1}^{k}\sup_{z\in U_{t(\omega_{i-1})}}|(T_{\omega_0\cdots\omega_{i-1}})'(z)||\widetilde{T}_{\omega_i,0}(\epsilon,T_{\omega_{i+1}\cdots\omega_k}(\epsilon,\pi(\epsilon,\sigma^{k+1}\omega)))|\\
&\le c_2\sup_{v\in V}\operatorname{diam}(U_v)r^{k+1}+\sup_{e\in E}|\widetilde{T}_{e,0}(\epsilon,\cdot)|(1+c_2r+\cdots+c_2r^k)
\end{aligned}$$

from inequality (2.2) in Proposition 2.1 by putting $T = T_{\omega_0\cdots\omega_k}$. Letting $k \to \infty$, $\|\pi(\epsilon,\cdot) - \pi\|_\infty \le \sup_{e\in E}|\widetilde{T}_{e,0}(\epsilon,\cdot)|c_2/(1-r)$ is satisfied. Thus we obtain the assertion in the case when $n = 0$.

Next we prove the form $\pi(\epsilon,\cdot) = \pi + \pi_1\epsilon + \tilde{\pi}_1(\epsilon,\cdot)\epsilon$ and $\|\tilde{\pi}_1(\epsilon,\cdot)\|_\infty \to 0$ for some $\tilde{\pi}_1(\epsilon,\cdot) \in C(E^{(\infty)}, \mathbb{R}^D)$ under condition $(G)_1$. To see this, we claim $\limsup_{\epsilon\to 0}\|\pi(\epsilon,\cdot)-\pi\|_\infty/\epsilon < \infty$. We note the inclusion $\pi(\epsilon, E^{(\infty)}) \subset \bigcup_{v\in V} U_v$ for sufficiently small $\epsilon > 0$. In particular, $t\pi\omega + (1-t)\pi(\epsilon,\omega)$ is in $U_{t(\omega_0)}$ for any $\omega \in E^{(\infty)}$, $0 \le t \le 1$ and small $\epsilon > 0$ from uniform convergence of $\pi(\epsilon,\cdot)$. For each $\omega \in E^{(\infty)}$ and small $\epsilon > 0$, we have

$$\begin{aligned}
&|\pi(\epsilon,\omega) - \pi\omega| \\
&\quad = |T_{\omega_0}(\epsilon, \pi(\epsilon,\sigma\omega)) - T_{\omega_0}(\pi\sigma\omega)| \\
&\quad = |T_{\omega_0}(\pi(\epsilon,\sigma\omega)) - T_{\omega_0}(\pi\sigma\omega) \\
&\qquad\quad + T_{\omega_0,1}(\pi(\epsilon,\sigma\omega))\epsilon + \widetilde{T}_{\omega_0,1}(\epsilon,\pi(\epsilon,\sigma\omega))\epsilon| \\
&\quad \le \sup_{t\in[0,1]} ||T'_{\omega_0}(t\pi(\epsilon,\sigma\omega) + (1-t)\pi\sigma\omega)|||\pi(\epsilon,\sigma\omega) - \pi\sigma\omega| \\
&\qquad + \max_{e\in E}(|T_{e,1}| + |\widetilde{T}_{e,1}(\epsilon,\cdot)|)\epsilon \\
&\quad \le r\|\pi(\epsilon,\cdot) - \pi\|_\infty + \max_{e\in E}(|T_{e,1}| + |\widetilde{T}_{e,1}(\epsilon,\cdot)|)\epsilon
\end{aligned}$$

by using condition $(G)_1$ and Mean value theorem. Thus

$$\limsup_{\epsilon\to 0}\frac{\|\pi(\epsilon,\cdot) - \pi\|_\infty}{\epsilon} \le \max_{e\in E}\frac{|T_{e,1}|}{(1-r)} < \infty$$

is satisfied. We consider the expansion

$$\begin{aligned}
\pi(\epsilon,\omega) &= T_{\omega_0}(\pi(\epsilon,\sigma\omega)) + T_{\omega_0,1}(\pi(\epsilon,\sigma\omega))\epsilon + \widetilde{T}_{\omega_0,1}(\epsilon,\pi(\epsilon,\sigma\omega))\epsilon \\
&= \pi\omega + T'_{\omega_0}(\pi\sigma\omega)(\pi(\epsilon,\sigma\omega) - \pi\sigma\omega) + T_{\omega_0,1}(\pi\sigma\omega)\epsilon + \widetilde{R}_1(\epsilon,\omega)\epsilon,
\end{aligned} \tag{3.1}$$

where

$$\widetilde{R}_1(\epsilon,\omega) = L_0(\epsilon,\omega)\Big(\frac{\pi(\epsilon,\sigma\omega) - \pi\sigma\omega}{\epsilon}\Big) + L_1(\epsilon,\omega) + \widetilde{T}_{\omega_0,1}(\epsilon,\pi(\epsilon,\sigma\omega)),$$

$$L_0(\epsilon,\omega) = \int_0^1 T'_{\omega_0}((1-t)\pi\sigma\omega + t\pi(\epsilon,\sigma\omega)) - T'_{\omega_0}(\pi\sigma\omega)\,dt$$

and

$$L_1(\epsilon,\omega) = T_{\omega_0,1}(\pi(\epsilon,\sigma\omega)) - T_{\omega_0,1}(\pi\sigma\omega). \tag{3.2}$$

Note that these last two expressions also satisfy

$$L_0(\epsilon,\omega) = L(1, T_{\omega_0}, \pi(\epsilon,\sigma\omega), \pi\sigma\omega)$$

and

$$L_1(\epsilon,\omega) = L(0, T_{\omega_0,1}, \pi(\epsilon,\sigma\omega), \pi\sigma\omega),$$

where $L(n, f, x, y)$ is defined in Proposition 2.2. We see $\|\widetilde{R}_1(\epsilon,\cdot)\|_\infty \to 0$ from the facts that $\|L(n, f, x, y)\|_n / \|x-y\|_X^\beta$ is uniformly bounded (Proposition 2.2) and $\|\pi(\epsilon,\cdot)-\pi\|_\infty/\epsilon$ is bounded. By using the form (3.1) repeatedly,

$$\begin{aligned}\pi(\epsilon,\omega) - \pi\omega &= \prod_{j=0}^{l} T'_{\omega_j}(\pi\sigma^{j+1}\omega)(\pi(\epsilon,\sigma^{l+1}\omega) - \pi\sigma^{l+1}\omega)\\ &\quad + \sum_{k=0}^{l}\prod_{j=0}^{k-1} T'_{\omega_j}(\pi\sigma^{j+1}\omega)(T_{\omega_k,1}(\pi\sigma^{k+1}\omega)\epsilon + \widetilde{R}_1(\epsilon,\sigma^j\omega)\epsilon)\end{aligned} \tag{3.3}$$

is fulfilled for each l. Letting $l \to \infty$ we obtain the form

$$\pi(\epsilon,\omega) = \pi\omega + \pi_1(\omega)\epsilon + \tilde{\pi}_1(\epsilon,\omega)\epsilon$$

and convergence

$$\|\tilde{\pi}_1(\epsilon,\cdot)\|_\infty \longrightarrow 0 \quad \text{as } \epsilon \to 0,$$

where

$$\pi_1(\omega) = \sum_{k=0}^{\infty} T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(T_{\omega_k,1}(\pi\sigma^{k+1}\omega)) \tag{3.4a}$$

$$\tilde{\pi}_1(\epsilon,\omega) = \sum_{k=0}^{\infty} T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(\widetilde{R}_1(\epsilon,\sigma^k\omega)). \tag{3.4b}$$

We next show that if $\pi(\epsilon,\cdot) = \pi + \pi_1\epsilon + \cdots + \pi_{n-1}\epsilon^{n-1} + \tilde{\pi}_{n-1}(\epsilon,\cdot)\epsilon^{n-1}$ and $\|\tilde{\pi}_{n-1}(\epsilon,\cdot)\|_\infty \to 0$ with some functions $\pi_1,\ldots,\pi_{n-1}$, then so is for n. Consider

$$\pi(\epsilon,\omega) = T_{\omega_0}(\epsilon, x(\epsilon)) = T_{\omega_0}(x(\epsilon)) + \sum_{k=1}^{n} T_{\omega_0,k}(x(\epsilon))\epsilon^k + \widetilde{T}_{\omega_0,n}(\epsilon, x(\epsilon))\epsilon^n$$

$$x(\epsilon) = x + \sum_{k=1}^{n-1} \pi_k(\sigma\omega)\epsilon^k + \tilde{\pi}_{n-1}(\epsilon,\sigma\omega)\epsilon^{n-1},$$

where $x(\epsilon) = \pi(\epsilon, \sigma\omega)$ and $x = \pi\sigma\omega$. A simple result of Proposition 2.3 implies that $\pi(\epsilon, \omega)$ has the form

$$\begin{aligned}\pi(\epsilon, \omega) = \pi\omega + T'_{\omega_0}(\pi\sigma\omega)(\pi(\epsilon, \sigma\omega) - \pi\sigma\omega)\\ + R_1(\omega)\epsilon + \cdots + R_n(\omega)\epsilon^n + \widetilde{R}_n(\epsilon, \omega)\epsilon^n\end{aligned}$$

and

$$\|\widetilde{R}_n(\epsilon, \cdot)\|_\infty \longrightarrow 0 \quad \text{as } \epsilon \to 0,$$

by putting

$$\begin{aligned}R_j(\omega) &= T_{\omega_0, j}(\pi\sigma\omega)\\ &+ \sum_{\substack{0 \le l \le j-1,\\ 1 \le k \le j-l:\\ (l,k) \ne (0,1)}} \sum_{\substack{i_1, \ldots, i_k \ge 1:\\ i_1 + \cdots + i_k = j-l}} \frac{T^{(k)}_{\omega_0, l}(\pi\sigma\omega)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega))}{k!}\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}_n(\epsilon, \omega) = &\sum_{\substack{0 \le l \le n-1,\\ 1 \le k \le n-l:\\ (l,k) \ne (0,1)}} \sum_{i=n-l+1}^{kn} \sum_{\substack{1 \le i_1, \ldots, i_k \le n-1:\\ i_1 + \cdots + i_k = i}} \frac{T^{(k)}_{\omega_0, l}(x)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega))}{k!} \epsilon^{i-n+l}\\ &+ \sum_{\substack{0 \le l \le n-1,\\ 1 \le k \le n-l:\\ (l,k) \ne (0,1)}} \sum_{i=1}^{k} \frac{T^{(k)}_{\omega_0, l}(x)}{k!} (z(\epsilon), \ldots, \underbrace{\tilde{\pi}_{n-1}(\epsilon, \sigma\omega)}_{i\text{-th}}, \ldots, x(\epsilon) - x)\epsilon^{l-1}\\ &+ \sum_{l=0}^{n} L_l(\epsilon, \omega) \left(\frac{x(\epsilon) - x}{\epsilon}\right)^{n-l} + \widetilde{T}_{\omega_0, n}(\epsilon, x(\epsilon)),\end{aligned} \tag{3.5}$$

where

$$z(\epsilon) = \sum_{k=1}^{n-1} \pi_k(\sigma\omega)\epsilon^k$$

and

$$L_l(\epsilon, \omega) = L(n-l, T_{\omega_0, l}, x(\epsilon), x)$$

in Proposition 2.3. Thus by a similar argument in the case when $n = 1$, we obtain the form

$$\pi(\epsilon, \omega) = \pi\omega + \pi_1(\omega)\epsilon + \cdots + \pi_n(\omega)\epsilon^n + \tilde{\pi}_n(\epsilon, \omega)\epsilon^n$$

with

$$\pi_j(\omega) = \sum_{k=0}^{\infty} T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(R_j(\pi\sigma^k\omega)), \tag{3.6a}$$

$$\tilde{\pi}_n(\epsilon, \omega) = \sum_{k=0}^{\infty} T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(\tilde{R}_n(\epsilon, \sigma^k\omega)). \tag{3.6b}$$

Choose any $\theta_2 \in (r, 1)$. We will show $\pi_1, \ldots, \pi_n \in F_{\theta_2}(E^{(\infty)}, \mathbb{R}^D)$. We first prove this assertion in the case when $n = 1$. Let $r_1 \in (r, \theta_2)$ and $\omega, \upsilon \in E^{(\infty)}$ with $\omega_0 \cdots \omega_{m-1} = \upsilon_0 \cdots \upsilon_{m-1}$ and $\omega_m \neq \upsilon_m$. To see $\pi_1 \in F_{r_1}(E^{(\infty)}, \mathbb{R}^D)$, consider

$$\begin{aligned}
&(T_{\omega_0\cdots\omega_{k-1}})'(\pi\sigma^k\omega)(R_1(\sigma^k\omega)) - (T_{\upsilon_0\cdots\upsilon_{k-1}})'(\pi\sigma^k\upsilon)(R_1(\sigma^k\upsilon)) \\
&\quad = ((T_{\omega_0\cdots\omega_{k-1}})'(\pi\sigma^k\omega) - (T_{\upsilon_0\cdots\upsilon_{k-1}})'(\pi\sigma^k\upsilon))(R_1(\sigma^k\omega)) \\
&\qquad + (T_{\upsilon_0\cdots\upsilon_{k-1}})'(\pi\sigma^k\upsilon)(R_1(\sigma^k\omega) - R_1(\sigma^k\upsilon)) = \mathrm{I}_k(\omega, \upsilon) + \mathrm{II}_k(\omega, \upsilon).
\end{aligned}$$

Put $\tau = \omega_0 \cdots \omega_{k-1}$. Recall that for $x, y \in O_{t(\omega_0)}$,

$$|T'_\tau(x) - T'_\tau(y)| \le r^k c_3 |x - y|$$

is satisfied for a constant

$$c_3 = \max_{e\in E} \frac{\|T''_e\| r^{-1}(c_2)^2}{(1 - r)}$$

from the bounded distortion of the GIFS $(G, (J_v), (T_e))$. Moreover, in the case when $k \le m$, we have

$$|\mathrm{I}_k(\omega, \upsilon)| \le c_3 r^k |\pi\sigma^k\omega - \pi\sigma^k\upsilon| \|R_1\|_\infty \le c_3 \|R_1\|_\infty [\pi]_{r_1} (r/r_1)^k d_{r_1}(\omega, \upsilon).$$

Note that since $R_1(\omega) = T_{\omega_0,1}(\pi\sigma\omega)$ holds by the definition, this implies in a straightforward way that $[R_1]_{r_1} < \infty$. Furthermore,

$$|\mathrm{II}_k(\omega, \upsilon)| \le r^k [R_1]_{r_1} d_{r_1}(\sigma^k\omega, \sigma^k\upsilon) \le [R_1]_{r_1} (r/r_1)^k d_{r_1}(\omega, \upsilon).$$

On the other hand, in the case when $k > m$,

$$|\mathrm{I}_k(\omega,\upsilon)|, |\mathrm{II}_k(\omega,\upsilon)| \le 2r^k \|R_1\|_\infty \le 2\|R_1\|_\infty (r/r_1)^k d_{r_1}(\omega,\upsilon)$$

are satisfied. Consequently, we obtain

$$|\pi_1(\omega) - \pi_1(\upsilon)| \le c_4 \sum_{k=0}^{\infty} \left(\frac{r}{r_1}\right)^k d_{r_1}(\omega,\upsilon) = \frac{c_4}{1 - r/r_1} d_{r_1}(\omega,\upsilon)$$

by putting

$$c_4 = \max(c_3\|R_1\|_\infty [\pi]_{r_1} + [R_1]_{r_1}, 2\|R_1\|_\infty).$$

Thus, $\pi_1 \in F_{\theta_1}(E^{(\infty)}, \mathbb{R}^D)$ by $r_1 < \theta_1$ is fulfilled.

We finally show that if we have $\pi_1, \ldots, \pi_{n-1} \in F_{r_{n-1}}(E^{(\infty)}, \mathbb{R}^D)$ for some $r_{n-1} \in (r_{n-2}, \theta_2)$, then π_n is in $F_{r_n}(E^{(\infty)}, \mathbb{R}^D)$ with $r_n \in (r_{n-1}, \theta_2)$. We prove that R_n is in $F_{r_{n-1}}(E^{(\infty)}, \mathbb{R}^D)$ by using the above argument. Note that for $e \in E$ and $l, k \ge 1$ with $l + k \le n$, the function $T^{(k)}_{e,l}$ is at least of C^1 on $O_{t(e)}$. Therefore, for each $\omega, \upsilon \in E^{(\infty)}$ with $\omega_0 = \upsilon_0$,

$$\begin{aligned}
&|T^{(k)}_{\omega_0,l}(\pi\sigma\omega)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega)) - T^{(k)}_{\upsilon_0,l}(\pi\sigma\upsilon)(\pi_{i_1}(\sigma\upsilon), \ldots, \pi_{i_k}(\sigma\upsilon))| \\
&\quad \le |T^{(k)}_{\omega_0,l}(\pi\sigma\omega)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega)) - T^{(k)}_{\omega_0,l}(\pi\sigma\upsilon)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega))| \\
&\qquad + |T^{(k)}_{\omega_0,l}(\pi\sigma\upsilon)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega)) \\
&\qquad\qquad - T^{(k)}_{\omega_0,l}(\pi\sigma\upsilon)(\pi_{i_1}(\sigma\upsilon), \ldots, \pi_{i_k}(\sigma\upsilon))| \\
&\quad \le c_2 |T^{(k+1)}_{\omega_0,l}| |\pi\sigma\omega - \pi\sigma\upsilon| \|\pi_{i_1}\|_\infty \cdots \|\pi_{i_k}\|_\infty \\
&\qquad + \sum_{j=1}^{k} |T^{(k)}_{\omega_0,l}(\pi\sigma\upsilon)(\pi_{i_1}(\sigma\omega), \ldots, \underbrace{\pi_{i_j}(\sigma\omega) - \pi_{i_j}(\sigma\upsilon)}_{j\text{-th}}, \ldots, \pi_{i_k}(\sigma\upsilon))| \\
&\quad \le c_2 |T^{(k+1)}_{\cdot,l}| [\pi]_{r_{n-1}} \|\pi_{i_1}\|_\infty \cdots \|\pi_{i_k}\|_\infty (r_{n-1})^{-1} d_{r_{n-1}}(\omega,\upsilon) \\
&\qquad + |T^{(k)}_{\cdot,l}| \sum_{j=1}^{k} \|\pi_{i_1}\|_\infty \cdots \underbrace{[\pi_{i_j}]_{r_{n-1}}}_{j\text{-th}} \cdots \|\pi_{i_k}\|_\infty (r_{n-1})^{-1} d_{r_{n-1}}(\omega,\upsilon)
\end{aligned} \tag{3.7}$$

is fulfilled by using the inequality (2.2) as

$$T(x) = T^{(k)}_{\omega_0,l}(x)(\pi_{i_1}(\sigma\omega), \ldots, \pi_{i_k}(\sigma\omega)),$$

where

$$|T^{(k)}_{\cdot,l}| = \max_{e \in E} \sup_{x \in U_{t(e)}} |T^{(k)}_{e,l}(x)|.$$

Therefore $R_n \in F_{r_{n-1}}(E^{(\infty)}, \mathbb{R}^D)$ by the definition of R_n. Thus by a similar proof which π_1 is in $F_{r_1}(E^{(\infty)}, \mathbb{R}^D)$, $\pi_n \in F_{r_n}(E^{(\infty)}, \mathbb{R}^D)$ is yielded for $r_n \in (r_{n-1}, \theta_2)$. Hence we obtain the assertion of this lemma. □

Lemma 3.2. *Assume that $(G)_n$ is satisfied. Let $\theta_3 = \theta_2^{\beta}$. Then there exist functions $\varphi_1, \ldots, \varphi_n \in F_{\theta_3}(E^{(\infty)}, \mathbb{R})$ and $\tilde{\varphi}_n(\epsilon, \cdot) \in F_{\theta(\epsilon)}(E^{(\infty)}, \mathbb{R})$ with $\theta(\epsilon) \in (0, 1)$ such that $\varphi(\epsilon, \cdot)$ has the form*

$$\varphi(\epsilon, \cdot) = \varphi + \varphi_1 \epsilon + \cdots + \varphi_n \epsilon^n + \tilde{\varphi}_n(\epsilon, \cdot)\epsilon^n$$

and

$$\|\tilde{\varphi}_n(\epsilon, \cdot)\|_\infty \longrightarrow 0 \quad \text{as } \epsilon \to 0,$$

where φ is defined in (1.2).

Proof. Note that the equation

$$\varphi(\epsilon, \omega) = \frac{1}{D} \log |\det T'_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))|$$

follows from $T_{\omega_0}(\epsilon, \cdot)$ is conformal.

First we show that the function $\det T'_e(\epsilon, \cdot)$ has an asymptotic expansion. Recall that when we write $T_e(\epsilon, x) = (t_{e,1}(\epsilon, x), t_{e,2}(\epsilon, x), \ldots, t_{e,D}(\epsilon, x))$ in $\mathbb{R}^D$ for $e \in E$ and $x = (x_1, x_2, \ldots, x_D) \in O_{t(e)}$, the function $\det T'_e(\epsilon, x)$ satisfies the form

$$\det T'_e(\epsilon, x) = \sum_{\eta} \varepsilon(\eta) \frac{\partial t_{e,1}(\epsilon, x)}{\partial x_{\eta(1)}} \frac{\partial t_{e,2}(\epsilon, x)}{\partial x_{\eta(2)}} \cdots \frac{\partial t_{e,D}(\epsilon, x)}{\partial x_{\eta(D)}}, \tag{3.8}$$

where η runs through the finite set of all permutations of $\{1, 2, \ldots, D\}$, and $\varepsilon(\eta)$ denotes the signature of the permutation η. Since the expansion

$$T'_e(\epsilon, \cdot) = T'_e + T'_{e,1}\epsilon + \cdots + T'_{e,n}\epsilon^n + \widetilde{T}'_{e,n}(\epsilon, \cdot)\epsilon^n$$

and $\|\widetilde{T}'_{e,n}(\epsilon, \cdot)\| \to 0$ as $\epsilon \to 0$ follow, so has for each element $\frac{\partial}{\partial x_l} t_{e,k}(\epsilon, \cdot)$. We denote $T_{e,k}(x)$ as $(t_{e,k,1}(x), t_{e,k,2}(x), \ldots, t_{e,k,D}(x))$ for $k = 0, 1, \ldots, n$ and $\widetilde{T}_{e,n}(\epsilon, x)$ as $(\tilde{t}_{e,n,1}(\epsilon, x), \tilde{t}_{e,n,2}(\epsilon, x), \ldots, \tilde{t}_{e,n,D}(\epsilon, x))$, where $T_{e,0}(x) = T_e(x)$. We also obtain an n-order asymptotic expansion

$$\det T'_e(\epsilon, \cdot) = \det T'_e + \kappa_{e,1}\epsilon + \cdots + \kappa_{e,n}\epsilon^n + \tilde{\kappa}_{e,n}(\epsilon, \cdot)\epsilon^n \tag{3.9}$$

and $\|\tilde{\kappa}_{e,n}(\epsilon,\cdot)\|_\infty \to 0$ as $\epsilon \to 0$ by the form (3.8), where each $\kappa_{e,k}(x)$ and $\tilde{\kappa}_{e,n}(\epsilon,x)$ have the forms

$$\kappa_{e,k}(x) = \sum_{\eta} \varepsilon(\eta) \sum_{\substack{0\le i_1,i_2,\dots,i_D\le n:\\ i_1+i_2+\cdots+i_D=k}} \frac{\partial t_{e,i_1,1}(x)}{\partial x_{\eta(1)}} \frac{\partial t_{e,i_2,2}(x)}{\partial x_{\eta(2)}} \cdots \frac{\partial t_{e,i_D,D}(x)}{\partial x_{\eta(D)}}, \quad (3.10a)$$

$$\begin{aligned}\tilde{\kappa}_{e,n}(\epsilon,x) &= \sum_{\eta} \varepsilon(\eta) \sum_{i=n+1}^{Dn} \sum_{\substack{0\le i_1,i_2,\dots,i_D\le n:\\ i_1+i_2+\cdots+i_D=i}} \frac{\partial t_{e,i_1,1}(x)}{\partial x_{\eta(1)}} \frac{\partial t_{e,i_2,2}(x)}{\partial x_{\eta(2)}} \cdots \frac{\partial t_{e,i_D,D}(x)}{\partial x_{\eta(D)}} \epsilon^{i-n} \\ &\quad + \sum_{\eta} \varepsilon(\eta) \sum_{j=1}^{D} \Big\{ \Big(\sum_{s=0}^{n} \frac{\partial t_{e,s,1}(x)}{\partial x_{\eta(1)}} \epsilon^s \Big) \cdots \underbrace{\frac{\partial \tilde{t}_{e,n,j}(\epsilon,x)}{\partial x_{\eta(j)}}}_{j\text{-th}} \cdots \frac{\partial t_{e,D}(\epsilon,x)}{\partial x_{\eta(D)}} \Big\}.\end{aligned} \quad (3.10b)$$

In particular, since the function $T'_{e,j}$ is of $C^{n-j+\beta}$ for $j = 0,1,\dots,n$, each $\kappa_{e,k}$ is of $C^{n-k+\beta}$ from this definition.

Next we give the n-order asymptotic expansion of

$$\omega \longmapsto T'_{\omega_0}(\epsilon, \pi(\epsilon,\sigma\omega)).$$

By virtue of the above argument together with Proposition 2.3 and Lemma 3.1, the function $\det T'_{\omega_0}(\epsilon,\pi(\epsilon,\sigma\omega))$ has

$$\begin{aligned}&\det T'_{\omega_0}(\epsilon,\pi(\epsilon,\sigma\omega)) \\ &\quad = \det T'_{\omega_0}(\pi\sigma\omega) + f_{\omega_0,1}(\sigma\omega)\epsilon + \cdots + f_{\omega_0,n}(\sigma\omega)\epsilon^n + \tilde{f}_{\omega_0,n}(\epsilon,\sigma\omega)\epsilon^n\end{aligned}$$

and

$$\|\tilde{f}_{\omega_0,n}(\epsilon,\cdot)\|_\infty \longrightarrow 0$$

by putting $f(\epsilon,\cdot) = \det T'_{\omega_0}(\epsilon,\cdot)$, $x = \pi\sigma\omega$, and $x(\epsilon) = \pi(\epsilon,\sigma\omega)$ in Proposition 2.3. Here each $f_{\omega_0,j}$ and $\tilde{f}_{\omega_0,n}(\epsilon,\cdot)$ are given by

$$f_{\omega_0,j}(\sigma\omega) = \kappa_{\omega_0,j}(\pi\sigma\omega) + \sum_{l=0}^{j-1} \sum_{k=1}^{j-l} \sum_{\substack{i_1,\dots,i_k\ge 1:\\ i_1+\cdots+i_k=j-l}} \frac{\kappa^{(k)}_{\omega_0,l}(x)(\pi_{i_1}(\sigma\omega),\dots,\pi_{i_k}(\sigma\omega))}{k!}$$

and

$$\tilde{f}_{\omega_0,n}(\epsilon,\sigma\omega) = \sum_{l=0}^{n-1}\sum_{k=1}^{n-l}\sum_{i=n-l+1}^{kn}\sum_{\substack{1\le i_1,\ldots,i_k\le n:\\ i_1+\cdots+i_k=i}} \frac{\kappa^{(k)}_{\omega_0,l}(x)(\pi_{i_1}(\sigma\omega),\ldots,\pi_{i_k}(\sigma\omega))}{k!}\epsilon^{i-n+l}$$
$$+\sum_{l=0}^{n}\sum_{k=1}^{n-l}\sum_{i=1}^{k}\kappa^{(k)}_{\omega_0,l}(x)(z(\epsilon),\ldots,\underbrace{\tilde{\pi}_n(\epsilon,\sigma\omega)}_{i\text{-th}},\ldots,x(\epsilon)-x)\frac{\epsilon^l}{k!}$$
$$+\sum_{l=0}^{n} L(n-l,\kappa_{\omega_0,l},x(\epsilon),x)\Big(\frac{x(\epsilon)-x}{\epsilon}\Big)^{n-l} + \tilde{\kappa}_{e,n}(\epsilon,x(\epsilon)), \tag{3.11}$$

where

$$z(\epsilon) = \sum_{k=1}^{n}\pi_k(\sigma\omega)\epsilon^k.$$

Now we show that $\omega \mapsto f_{\omega_0,j}(\sigma\omega)$ is a d_{θ_3}-Lipschitz function. In this form, the function $\kappa_{\omega_0,j}$ is of $C^{n-j+\beta}$ and at least of C^β. Similarity, since $\kappa^{(k)}_{\omega_0,l}$ is of $C^{n-l-k+\beta}$ with $n-l-k\ge 0$, this function is also of C^β. By a similar argument in (3.7), we obtain that for $\omega,\upsilon\in E^{(\infty)}$ with $\omega_0=\upsilon_0$

$$|\kappa_{\omega_0,j}(\pi\sigma\omega)-\kappa_{\omega_0,j}(\pi\sigma\upsilon)| \le c_5(j,0)[\pi]^\beta_{\theta_2}\theta_3^{-1}d_{\theta_3}(\omega,\upsilon) \tag{3.12}$$

and

$$|\kappa^{(k)}_{\omega_0,l}(\pi\sigma\omega)(\pi_{i_1}(\sigma\omega),\ldots,\pi_{i_k}(\sigma\omega)) - \kappa^{(k)}_{\omega_0,l}(\pi\sigma\upsilon)(\pi_{i_1}(\sigma\upsilon),\ldots,\pi_{i_k}(\sigma\upsilon))|$$
$$\le c_5(l,k)[\pi]^\beta_{\theta_2}\theta_3^{-1}\|\pi_{i_1}\|_\infty\cdots\|\pi_{i_k}\|_\infty d_{\theta_3}(\omega,\upsilon)$$
$$+\sum_{i=1}^{k}\max_{e\in E}\sup_{x\in J_{t(e)}}|\kappa^{(k)}_{e,l}(x)|\|\pi_{i_1}\|_\infty\cdots\underbrace{[\pi_{i_i}]_{\theta_2}}_{i\text{-th}}\cdots\|\pi_{i_k}\|_\infty d_{\theta_2}(\omega,\upsilon), \tag{3.13}$$

where

$$c_5(l,k) = \max_{e\in E}\sup_{x,y\in J_{t(e)}:\,x\ne y}\frac{|\kappa^{(k)}_{e,l}(x)-\kappa^{(k)}_{e,l}(y)|}{|x-y|^\beta}.$$

Thus the inequalities (3.12) and (3.13) imply $f_{\omega_0,j}\in F_{\theta_3}(E^{(\infty)})$.

Finally, we prove the assertion of this lemma. For any small $\epsilon > 0$, the signature

$$\mathrm{sign}(\det T'_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))) = \mathrm{sign}(\det T'_{\omega_0}(\pi\sigma\omega)) = s(\omega_0)$$

depends only on ω_0. Therefore

$$\varphi(\epsilon, \omega) = \frac{1}{D} \log(s(\omega_0) \det T'_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega)))$$

is satisfied. By applying Proposition 2.3 with $x(\epsilon) = s(\omega_0) \det T'_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))$ and $f(\epsilon, x) = (1/D) \log x$, we obtain the asymptotic expansion of $\varphi(\epsilon, \omega)$ and also see $\|\tilde{\varphi}_n(\epsilon, \cdot)\|_\infty \to 0$ by the definition of $\tilde{y}_n(\epsilon)$ in Proposition 2.3. Consequently, each φ_k and $\tilde{\varphi}_n(\epsilon, \cdot)$ have the forms

$$\varphi_k(\omega) = \frac{1}{D} \sum_{i=1}^{k} (-1)^{i-1}(i-1)! x^{-i} \sum_{j_1 \cdots j_k} \frac{x_1^{j_1} \cdots x_k^{j_k}}{j_1! \cdots j_k!},$$

$$\begin{aligned}
&\tilde{\varphi}_n(\epsilon, \omega) \\
&= \frac{1}{D} \sum_{k=1}^{n} \sum_{i=n+1}^{kn} \sum_{\substack{1 \le i_1, \ldots, i_k \le n: \\ i_1 + \cdots + i_k = i}} \frac{(-1)^{k-1}}{k x^k} x_{i_1} x_{i_2} \cdots x_{i_k} \epsilon^{i-n} \\
&\quad + \frac{1}{D} \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{(-1)^{k-1}}{k x^k} \Big\{ \Big(\sum_{j=1}^{n} x_j \epsilon^j \Big) \cdots \underbrace{\tilde{f}_{\omega_0, n}(\epsilon, \sigma\omega)}_{i\text{-th}} \cdots (y(\epsilon) - x) \Big\} \\
&\quad + \frac{1}{D} \int_0^1 (1-t)^{n-1} (-1)^{n-1} \Big(\frac{x^n - (x + t(y(\epsilon) - x))^n}{(x + t(y(\epsilon) - x))^n x^n} \Big) \Big(\frac{y(\epsilon) - x}{\epsilon} \Big)^n dt,
\end{aligned} \tag{3.14}$$

where the second summation of $\varphi_k(\omega)$ is taken over all integers $j_1, \ldots, j_k$ so that $0 \le j_1, \ldots, j_k$, $j_1 + \cdots + j_k = i$ and $j_1 + 2j_2 + \cdots + kj_k = k$, and where

$$x = \det T'_{\omega_0}(\pi\sigma\omega),$$

$$x_j = f_{\omega_0, j}(\sigma\omega) \quad \text{for } 1 \le j \le n,$$

and

$$y(\epsilon) = \det T'_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega)).$$

The fact that each φ_k is a d_{θ_3}-Lipschitz function is yielded by this definition. To see that $\tilde{\varphi}_n(\epsilon,\cdot)$ is a $d_{\theta(\epsilon)}$-Lipschitz function, it is sufficient to show that $\varphi(\epsilon,\cdot) \in F_{r(\epsilon)}(E^{(\infty)})$ with $r(\epsilon) = r^{\beta(\epsilon)}$ for each $\epsilon > 0$. For $\omega, \upsilon \in E^{(\infty)}$ with $\omega_0 = \upsilon_0$, we have

$$\begin{aligned}
&|\varphi(\epsilon,\omega) - \varphi(\epsilon,\upsilon)| \\
&\quad \le \frac{1}{\min_{e\in E}\inf_{x\in J_{t(e)}}|T_e'(\epsilon,x)|}|T_{\omega_0}'(\epsilon,\pi(\epsilon,\sigma\omega)) - T_{\omega_0}'(\epsilon,\pi(\epsilon,\sigma\upsilon))| \\
&\quad \le \frac{c_6(\epsilon)}{\min_{e\in E}\inf_{x\in J_{t(e)}}|T_e'(\epsilon,x)|}|\pi(\epsilon,\sigma\omega) - \pi(\epsilon,\sigma\upsilon)|^{\beta(\epsilon)}.
\end{aligned}$$

with

$$c_6(\epsilon) = \max_{e\in E} \sup_{x,y\in J_{t(e)}:\, x\neq y} \frac{|T_e'(\epsilon,x) - T_e'(\epsilon,y)|}{|x-y|^{\beta(\epsilon)}}.$$

Now we will prove

$$\sup_{\epsilon>0}[\pi(\epsilon,\cdot)]_r \le c_7 \tag{3.15}$$

for some constant $c_7 > 0$. Choose any $\omega, \upsilon \in E^{(\infty)}$ so that

$$\tau = \omega_0\cdots\omega_{m-1} = \upsilon_0\cdots\upsilon_{m-1}$$

and $\omega_m \neq \upsilon_m$ for an integer $m \ge 1$. We obtain

$$\begin{aligned}
|\pi(\epsilon,\omega) - \pi(\epsilon,\upsilon)| &= |T_\tau(\epsilon,\pi(\epsilon,\sigma^m\omega)) - T_\tau(\epsilon,\pi(\epsilon,\sigma^m\upsilon))| \\
&\le c_2 \sup_{x\in U_{t(\omega_{m-1})}} |T_\tau'(\epsilon,x)||\pi(\epsilon,\sigma^m\omega) - \pi(\epsilon,\sigma^m\upsilon)| \\
&\le c_7 r^m
\end{aligned}$$

with $c_7 = c_2 \max_{v\in V} \operatorname{diam} J_v$. Therefore (3.15) is satisfied. Consequently, $\varphi(\epsilon,\cdot)$ is a $d_{r(\epsilon)}$-Lipschitz function. Hence $\tilde{\varphi}_n(\epsilon,\cdot)$ is a $d_{\theta(\epsilon)}$-Lipschitz function with $\theta(\epsilon) = \max(\theta_3, r(\epsilon))$. □

Proof of Theorem 1.1. By virtue of Lemma 3.2, the function $\varphi(\epsilon,\cdot)$ fulfills the condition in Theorem 2.4. Thus the assertion follows from Theorem 2.4. □

Lemma 3.3. *Assume that $(G)_n'$ is satisfied. Let $\theta_4 \in (\theta_2, 1)$. Then we have*

$$\limsup_{\epsilon\to 0}[\tilde{\pi}_n(\epsilon,\cdot)]_{\theta_4} < \infty,$$

where $\tilde{\pi}_n(\epsilon,\cdot)$ is defined in Lemma 3.1.

Proof. First we show the assertion in the case when $n = 0$. Since $\pi(\epsilon,\cdot)$ satisfies (3.15) and since π is in $F_r(E^{(\infty)}, \mathbb{R}^D)$, we obtain

$$\sup_{\epsilon>0}[\tilde{\pi}_0(\epsilon,\cdot)]_{\theta_4} \le c_7 + [\pi]_{\theta_4} < \infty$$

by $r < \theta_4$.

Next we consider the case when $n = 1$. Since $\tilde{\pi}_1(\epsilon,\cdot)$ has the form

$$\tilde{\pi}_1(\epsilon,\cdot) = \frac{\pi(\epsilon,\cdot) - \pi}{\epsilon} + \pi_1,$$

it is sufficient to show that

$$\limsup_{\epsilon\to 0}[(\pi(\epsilon,\cdot)-\pi)/\epsilon]_{r_1} < \infty \quad \text{with } r_1 \in (\theta_2, \theta_4).$$

We give some notation below. Recall the definition of U_v and δ in (2.1). We take $\hat{\delta} \in (0, \delta/(1 + 1/(1-r)))$ and put

$$\widehat{U}_v = \bigcup_{x\in J_v} B(x, \hat{\delta}) \quad \text{for } v \in V.$$

Note the inclusion

$$J_v \subset \widehat{U}_v \subset \overline{\widehat{U}_v} \subset U_v \subset \overline{U_v} \subset O_v.$$

We see $T_e(\overline{\widehat{U}_{t(e)}}) \subset \widehat{U}_{i(e)}$ and $T_e(\epsilon, \overline{\widehat{U}_{t(e)}}) \subset \widehat{U}_{i(e)}$. We also obtain

$$B(\widehat{U}_v, \hat{\delta}/(1-r)) \subset U_v. \tag{3.16}$$

Indeed, for any $x \in B(\widehat{U}_v, \hat{\delta}/(1-r))$, there exist $y \in \widehat{U}_v$ and $z \in J_v$ such that $|x-y| < \hat{\delta}/(1-r)$, $|y-z| < \hat{\delta}$ and

$$|x-z| < |x-y| + |y-z| < (1/(1-r)+1)\hat{\delta} < \delta.$$

Let $\widehat{T}_e(\epsilon,\cdot) \in C(O_{t(e)}, \mathbb{R}^D)$ be

$$\widehat{T}_e(\epsilon,\cdot) = T_e + T_{e,1}\epsilon + \widetilde{T}_{e,1}(\epsilon,\cdot)\epsilon \quad \text{for each } e \in E.$$

We see $\widehat{T}_e(\epsilon,\cdot) = T_e(\epsilon,\cdot)$ on $J_{t(e)}$ by condition $(G)_1$. We need to show the boundedness of $\widetilde{T}'_{e,1}(\epsilon,\cdot)$ on $U_{t(e)}$. By virtue of condition $(G)'_1$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and for $x \in U_{t(e)}$

$$|\widetilde{T}'_{e,1}(\epsilon,x)| \le |\widetilde{T}'_{e,1}(\epsilon,z)| + (c_1+1)|x-z|^\beta \le 1 + (c_1+1)\delta^\beta = c_8$$

for some element $z \in J_{t(e)}$ with $|x-z| < \delta$ by $\sup_{y\in J_{t(e)}}|\widetilde{T}'_{e,1}(\epsilon,y)| \to 0$ as $\epsilon \to 0$. Therefore, we have that for any $e \in E$, $x \in U_{t(e)}$ and $\epsilon \in (0,\epsilon_0)$

$$|\widehat{T}'_e(\epsilon,x) - T'_e(x)| = |T'_{e,1}(x)\epsilon + \widetilde{T}'_{e,1}(\epsilon,x)\epsilon| \le c_9\epsilon, \tag{3.17}$$

and

$$\begin{aligned}|\widehat{T}_e(\epsilon, x) - T_e(x)| &= |T_{e,1}(x)\epsilon + \widetilde{T}_{e,1}(\epsilon, x)\epsilon| \\ &\le (\sup_{y\in U_{t(e)}} |T_{e,1}(y)| + |\widetilde{T}_{e,1}(\epsilon, z)| + \sup_{y\in U_{t(e)}} |\widetilde{T}'_{e,1}(\epsilon, y)||x - z|)\epsilon \\ &\le (\sup_{y\in U_{t(e)}} |T_{e,1}(y)| + \sup_{y\in J_{t(e)}} |\widetilde{T}_{e,1}(\epsilon, y)| + c_8\delta)\epsilon \le c_{10}\epsilon,\end{aligned} \tag{3.18}$$

with some point $z \in J_{t(e)}$, where we put

$$c_9 = \max_{e\in E} \sup_{y\in U_{t(e)}} |T'_{e,1}(y)| + c_8,$$

and

$$c_{10} = \max_{e\in E}(\sup_{y\in U_{t(e)}} |T_{e,1}(y)| + \sup_{y\in J_{t(e)}} |\widetilde{T}_{e,1}(\epsilon, y)| + c_8\delta).$$

Choose a small number $\epsilon_1 \in (0, \epsilon_0)$ so that

$$\sup_{y\in U_{t(e)}} |T'_e(y)| + c_9\epsilon < r,$$

$$c_{10}\epsilon < \hat{\delta}$$

and

$$(1-t)\pi w + t\pi(\epsilon, w) \in \bigcup_{v\in V} \widehat{U}_v$$

are satisfied for any $0 < \epsilon < \epsilon_1$, $e \in E$, $t \in [0, 1]$ and $w \in E^{(\infty)}$. By inequality (3.17), we see

$$\sup_{x\in U_{t(e)}} |\widehat{T}'_e(\epsilon, x)| < r$$

and therefore $\widehat{T}_e(\epsilon, \overline{\widehat{U}_{t(e)}}) \subset \widehat{U}_{i(e)}$ for $0 < \epsilon < \epsilon_1$.

To see that the map $\omega \mapsto (\pi(\epsilon, \omega) - \pi\omega)/\epsilon$ is in $F_{r_1}(E^{(\infty)})$, we note the following:

$$\begin{aligned}\pi(\epsilon, \omega) &= T_\tau(\epsilon, \cdot)(\pi(\epsilon, \sigma^m\omega)) \\ &= T_\tau(\epsilon, \pi\sigma^m\omega) \\ &\quad + \int_0^1 T'_\tau(\epsilon, (1-t)\pi\sigma^m\omega + t\pi(\epsilon, \sigma^m\omega))(\pi(\epsilon, \sigma^m\omega) - \pi\sigma^m\omega)\, dt.\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\Big|\frac{\pi(\epsilon,\omega)-\pi\omega}{\epsilon}-\frac{\pi(\epsilon,\upsilon)-\pi\upsilon}{\epsilon}\Big| \\
&\quad\le \Big|\frac{(T_\tau(\epsilon,\cdot)-T_\tau)(\pi\sigma^m\omega)}{\epsilon}-\frac{(T_\tau(\epsilon,\cdot)-T_\tau)(\pi\sigma^m\upsilon)}{\epsilon}\Big| + 2r^m\Big\|\frac{\pi(\epsilon,\cdot)-\pi}{\epsilon}\Big\|_\infty \\
&\quad= I_1(\epsilon,\omega,\upsilon) + 2\Big\|\frac{\pi(\epsilon,\cdot)-\pi}{\epsilon}\Big\|_\infty d_r(\omega,\upsilon)
\end{aligned}$$

and

$$\begin{aligned}
I_1(\epsilon,\omega,\upsilon) &= \Big|\frac{(\widehat{T}_\tau(\epsilon,\cdot)-T_\tau)(\pi\sigma^m\omega)}{\epsilon}-\frac{(\widehat{T}_\tau(\epsilon,\cdot)-T_\tau)(\pi\sigma^m\upsilon)}{\epsilon}\Big| \\
&\le c_{11}\sup_{x\in\widehat{U}_{t(\omega_{m-1})}}\Big|\Big(\frac{\widehat{T}_\tau(\epsilon,\cdot)-T_\tau}{\epsilon}\Big)'(x)\Big|\,|\pi(\sigma^m\omega)-\pi(\sigma^m\upsilon)| \\
&\le c_{12}\sup_{x\in\widehat{U}_{t(\omega_{m-1})}}\Bigg|\frac{\prod_{i=0}^{m-1}\widehat{T}'_{\omega_i}(\epsilon,x_i(\epsilon,x))-\prod_{i=0}^{m-1}T'_{\omega_i}(y_i(x))}{\epsilon}\Bigg| \qquad (3.19)\\
&\le c_{12}\sup_{x\in\widehat{U}_{t(\omega_{m-1})}}\sum_{i=0}^{m-1}r^{m-1}\Big|\frac{\widehat{T}'_{\omega_i}(\epsilon,x_i(\epsilon,x))-T'_{\omega_i}(\epsilon,y_i(x))}{\epsilon}\Big|,
\end{aligned}$$

with

$$c_{11}=\max\Big\{1,\max_{v\in V}\frac{\mathrm{diam}(J_v)}{\mathrm{dist}(J_v,\partial\widehat{U}_v)}\Big\},$$

$$c_{12}=c_{11}\max_{v\in V}\mathrm{diam}(J_v),$$

and

$$x_i(\epsilon,x)=\widehat{T}_{\omega_{i+1}\cdots\omega_{m-1}}(\epsilon,x)$$

$$y_i(x)=T_{\omega_{i+1}\cdots\omega_{m-1}}(x)$$

for $i=0,1,\ldots,m-2$, and

$$x_{m-1}(\epsilon,x)=y_{m-1}(x)=x.$$

We note $y_i(x) \in \widehat{U}_{t(\omega_{i+1})}$ for $x \in \widehat{U}_{t(w_{m-1})}$. Now we show

$$|x_i(\epsilon, x) - y_i(x)| < c_{10}\frac{\epsilon}{1-r}$$

for each i. By (3.18), we have

$$|x_{m-2}(\epsilon, x) - y_{m-2}(x)| = |\widehat{T}_{\omega_{m-1}}(\epsilon, x) - T_{\omega_{m-1}}(x)| \le c_{10}\epsilon < \hat{\delta} < \frac{\hat{\delta}}{(1-r)}$$

for $0 < \epsilon < \epsilon_1$ and therefore $tx_{m-2}(\epsilon, x) + (1-t)y_{m-2}(x) \in U_{t(\omega_{m-1})}$ for all $t \in [0, 1]$ from (3.16). When we assume

$$|x_i(\epsilon, x) - y_i(x)| < c_{10}\epsilon(1 + r + \cdots + r^{m-2-i}) \quad \text{for each } 1 \le i \le m-2,$$

$x_i(\epsilon, x)$ also satisfies $tx_i(\epsilon, x) + (1-t)y_i(x) \in U_{t(\omega_{i+1})}$ for all $t \in [0, 1]$ and

$$\begin{aligned}
&|x_{i-1}(\epsilon, x) - y_{i-1}(x)| \\
&\quad \le |\widehat{T}_{\omega_i}(\epsilon, x_i(\epsilon, x)) - T_{\omega_i}(x_i(\epsilon, x))| + |T_{\omega_i}(x_i(\epsilon, x)) - T_{\omega_i}(y_i(x))| \\
&\quad \le c_{10}\epsilon + \sup_{t\in[0,1]} |T'_{\omega_i}(tx_i(\epsilon, x) + (1-t)y_i(x))||x_i(\epsilon, x) - y_i(x)| \\
&\quad \le c_{10}\epsilon + r(c_{10}\epsilon(1 + r + \cdots + r^{m-2-i})) \\
&\quad < c_{10}\frac{\epsilon}{1-r}.
\end{aligned}$$

Thus we see

$$|x_i(\epsilon, x) - y_i(x)| < c_{10}\frac{\epsilon}{1-r} \quad \text{for } i = 0, 1, \ldots, m-2.$$

We obtain

$$\begin{aligned}
&\left|\frac{\widehat{T}'_{\omega_i}(\epsilon, x_i(\epsilon, x)) - T'_{\omega_i}(\epsilon, y_i(x))}{\epsilon}\right| \\
&\quad \le \left|\frac{\widehat{T}'_{\omega_i}(\epsilon, x_i(\epsilon, x)) - T'_{\omega_i}(\epsilon, x_i(\epsilon, x))}{\epsilon}\right| + \left|\frac{T'_{\omega_i}(\epsilon, x_i(\epsilon, x)) - T'_{\omega_i}(\epsilon, y_i(x))}{\epsilon}\right| \\
&\quad \le c_9 + \sup_{t\in[0,1]} |T''_{\omega_i}(tx_i(\epsilon, x) + (1-t)y_i(x))| \left|\frac{x_i(\epsilon, x) - y_i(x)}{\epsilon}\right| \\
&\quad \le c_9 + \frac{c_{13}c_{10}}{1-r},
\end{aligned}$$

with

$$c_{13} = \max_{e\in E} \sup_{x\in U_{t(e)}} |T''_e(x)|.$$

Therefore we see

$$I_1(\epsilon,\omega,\upsilon) \le c_{12}\Big(c_9 + \frac{c_{13}c_{10}}{1-r}\Big) m r^{m-1}.$$

Choose any $r_1 \in (r, \theta_4)$. Since the equation

$$m r^{m-1} = m(r/r_1)^{m-1}(r_1)^{m-1}$$

and the inequality

$$m(r/r_1)^{m-1} \le -\frac{1}{\left(\exp(1)\frac{r}{r_1}\log\left(\frac{r}{r_1}\right)\right)} = c_{14}$$

follow for any $m \ge 1$, we have

$$I_1(\epsilon,\omega,\upsilon) \le c_{12}\left(c_9 + \frac{c_{13}c_{10}}{1-r}\right) c_{14} r_1^{-1}(r_1)^m.$$

Thus, for any $0 < \epsilon < \epsilon_1$

$$\left[\frac{\pi(\epsilon,\cdot)-\pi}{\epsilon}\right]_{r_1} \le c_{12}\Big(c_9 + \frac{c_{13}c_{10}}{1-r}\Big) c_{14} r_1^{-1} + 2c_{15} = c_{16}$$

holds with

$$c_{15} = \sup_{\epsilon>0}\left\|\frac{\pi(\epsilon,\cdot)-\pi_\infty}{\epsilon}\right\|_\infty.$$

Consequently we obtain

$$[\tilde{\pi}_1(\epsilon,\cdot)]_{r_1} \le c_{16} + [\pi_1]_{r_1} \quad \text{for any } 0<\epsilon<\epsilon_1.$$

We have the assertion in the case $n = 1$.

Let $n \ge 2$. Finally we show that if $\limsup_{\epsilon\to 0}[\tilde{\pi}_{n-1}(\epsilon,\cdot)]_{\theta(n-1)} < \infty$ is satisfied for some $\theta(n-1) \in (\theta(n-2), \theta_4)$, then so is $\tilde{\pi}_n(\epsilon,\cdot)$. Recall the form of $\tilde{\pi}_n(\epsilon,\cdot)$ defined in (3.6b). We will prove

$$\limsup_{\epsilon\to 0}[\widetilde{R}_n(\epsilon,\cdot)]_{r_n} < \infty \quad \text{for } r_n \in (\theta(n-1), \theta_4),$$

where $\widetilde{R}_n(\epsilon,\cdot)$ is given in (3.5). The boundedness of Lipschitz constant of

$$\omega \longmapsto T^{(k)}_{\omega_0,l}(\pi\sigma\omega)(\pi_{i_1}(\sigma\omega),\ldots,\pi_{i_k}(\sigma\omega))$$

immediately follows from the inequality (3.7). Put

$$t(\epsilon,\omega) = T^{(k)}_{\omega_0,l}(\pi\sigma\omega)(\underbrace{z(\epsilon,\sigma\omega),\ldots,z(\epsilon,\sigma\omega)}_{i-1}, \underbrace{\tilde{\pi}_{n-1}(\epsilon,\sigma\omega)}_{i\text{-th position}}, \underbrace{\pi(\epsilon,\sigma\omega)-\pi\sigma\omega,\ldots,\pi(\epsilon,\sigma\omega)-\pi\sigma\omega}_{k-i})$$

and

$$z(\epsilon,\omega)=\sum_{k=1}^{n-1}\pi_k(\omega)\epsilon^k.$$

By a similar argument of the inequality (3.7), we obtain that for $\omega,\upsilon\in E^{(\infty)}$ with $\omega_0=\upsilon_0$

$$\begin{aligned}
&|t(\epsilon,\omega)-t(\epsilon,\upsilon)|\\
&\quad=\Big\{c_2|T_{\cdot,l}^{(k+1)}|[\pi]_{r_n}\|z(\epsilon,\cdot)\|_\infty^{i-1}\|\pi(\epsilon,\cdot)\\
&\qquad-\pi\|_\infty^{k-i}\|\tilde{\pi}_n(\epsilon,\cdot)\|_\infty\\
&\qquad+(i-1)|T_{\cdot,l}^{(k)}|\|z(\epsilon,\cdot)\|_\infty^{i-2}\|\pi(\epsilon,\cdot)\\
&\qquad-\pi\|_\infty^{k-i}\|\tilde{\pi}_{n-1}(\epsilon,\cdot)\|_\infty[z(\epsilon,\cdot)]_{r_n}\\
&\qquad+(k-i)|T_{\cdot,l}^{(k)}|\|z(\epsilon,\cdot)\|_\infty^{i-1}\|\pi(\epsilon,\cdot)\\
&\qquad-\pi\|_\infty^{k-i-1}\|\tilde{\pi}_{n-1}(\epsilon,\cdot)\|_\infty([\pi(\epsilon,\cdot)]_{r_n}+[\pi]_{r_n})\\
&\qquad+|T_{\cdot,l}^{(k)}|\|z(\epsilon,\cdot)\|_\infty^{i-1}\|\pi(\epsilon,\cdot)-\pi\|_\infty^{k-i}[\tilde{\pi}_{n-1}(\epsilon,\cdot)]_{r_n}\Big\}(r_n)^{-1}d_{r_n}(\omega,\upsilon),
\end{aligned}$$

where $|T_{\cdot,l}^{(k+1)}|$ appears by (3.7). By virtue of $\limsup_{\epsilon\to 0}[\tilde{\pi}_{n-1}(\epsilon,\cdot)]_{\theta(n-1)}<\infty$, we get $\limsup_{\epsilon\to 0}[t(\epsilon,\cdot)]_{r_n}<\infty$. We also have

$$\begin{aligned}
&\Big|L_l(\epsilon,\omega)\Big(\frac{\pi(\epsilon,\sigma\omega)-\pi\sigma\omega}{\epsilon}\Big)^{n-l}-L_l(\epsilon,\upsilon)\Big(\frac{\pi(\epsilon,\sigma\upsilon)-\pi\sigma\upsilon}{\epsilon}\Big)^{n-l}\Big|\\
&\quad\le|L_l(\epsilon,\omega)-L_l(\epsilon,\upsilon)|\Big\|\frac{\pi(\epsilon,\cdot)-\pi}{\epsilon}\Big\|_\infty^{n-l}\\
&\quad+(n-l)|L_l(\epsilon,\upsilon)|\Big\|\frac{\pi(\epsilon,\cdot)-\pi}{\epsilon}\Big\|_\infty^{n-l-1}\Big[\frac{\pi(\epsilon,\cdot)-\pi}{\epsilon}\Big]_{r_n}(r_n)^{-1}d_{r_n}(\omega,\upsilon)
\end{aligned}$$

and

$$\begin{aligned}
&|L_l(\epsilon,\omega)-L_l(\epsilon,\upsilon)|\\
&\quad\le\int_0^1\frac{(1-t)^{n-l-1}}{(n-l-1)!}\Big(|T_{\omega_0,l}^{(n-l)}(\pi\sigma\omega)-T_{\omega_0,l}^{(n-l)}(\pi\sigma\upsilon)|\\
&\qquad+|T_{\omega_0,l}^{(n-l)}((1-t)\pi\sigma\omega+t\pi(\epsilon,\sigma\omega))\\
&\qquad\quad-T_{\omega_0,l}^{(n-l)}((1-t)\pi\sigma\upsilon+t\pi(\epsilon,\sigma\upsilon))|\Big)dt\\
&\quad\le\Big(\frac{c_2}{(n-l)!}[\pi]_{r_n}+\frac{c_{17}}{(n-l)!}([\pi]_{r_n}+[\pi(\epsilon,\cdot)]_{r_n})\Big)|T_{\cdot,l}^{(n-l+1)}|(r_n)^{-1}d_{r_n}(\omega,\upsilon),
\end{aligned}$$

for any $0 < \epsilon < \epsilon_1$, where

$$c_{17} = \max\left\{1, \max_{v \in V} \frac{\operatorname{diam}(\widehat{U}_v)}{\operatorname{dist}(\widehat{U}_v, \partial U_v)}\right\}.$$

Moreover,

$$\begin{aligned}
&|\widetilde{T}_{\omega_0,n}(\epsilon, \pi(\epsilon, \sigma\omega)) - \widetilde{T}_{\omega_0,n}(\epsilon, \pi(\epsilon, \sigma\upsilon))| \\
&\quad \le c_2 \sup_{x \in U_{t(\omega_0)}} |\widetilde{T}'_{\omega_0,n}(\epsilon, x)||\pi(\epsilon, \sigma\omega) - \pi(\epsilon, \sigma\upsilon)| \\
&\quad \le c_2 (\max_{e \in E} \sup_{z \in J_{t(e)}} |\widetilde{T}'_{e,n}(\epsilon, z)| + (c_1 + 1)\delta^\beta)[\pi(\epsilon, \cdot)]_{r_n} (r_n)^{-1} d_{r_n}(\omega, \upsilon)
\end{aligned}$$

holds. Consequently, $\limsup_{\epsilon\to 0}[\widetilde{R}_n(\epsilon, \cdot)]_{r_n} < \infty$ is fulfilled. By the proof of $\pi_1 \in F_{\theta_2}(E^{(\infty)})$ in Lemma 3.1, we have the assertion

$$\limsup_{\epsilon\to 0}[\tilde{\pi}_n(\epsilon, \cdot)]_{\theta(n)} < \infty \quad \text{for } \theta(n) \in (r_n, \theta_4).$$

Hence the proof of this Lemma is complete. □

Lemma 3.4. *Assume that $(G)'_n$ are satisfied. Then*

$$\limsup_{\epsilon\to 0}[\tilde{\varphi}_n(\epsilon, \cdot)]_{\theta_5} < \infty$$

with

$$\theta_5 = \theta_4^\beta,$$

where $\tilde{\varphi}_n(\epsilon, \cdot)$ is defined in Lemma 3.2 and θ_4 is given in Lemma 3.3.

Proof. Recall the small order part $\tilde{\kappa}_{e,n}(\epsilon, \cdot)$ defined in (3.9). By virtue of the assumption $(G)'_n$, the function

$$\widetilde{T}_{e,n}(\epsilon, \cdot) = (\tilde{t}_{e,n,1}(\epsilon, \cdot), \tilde{t}_{e,n,2}(\epsilon, \cdot), \dots, \tilde{t}_{e,n,D}(\epsilon, \cdot))$$

fulfills the condition

$$\limsup_{\epsilon\to 0} \sup_{x,y \in O_{t(e)}:\, x \neq y} \frac{\left|\dfrac{\partial}{\partial x_j}\tilde{t}_{e,n,i}(\epsilon, x) - \dfrac{\partial}{\partial x_j}\tilde{t}_{e,n,i}(\epsilon, y)\right|}{|x-y|^\beta} < \infty$$

for each $i, j \in \{1, 2, \ldots, D\}$. This condition implies that $\tilde{\kappa}_{e,n}(\epsilon, \cdot)$ is of C^β and

$$\limsup_{\epsilon \to 0} \sup_{x,y \in J_{t(e)}:\, x \neq y} \frac{|\tilde{\kappa}_{e,n}(\epsilon, x) - \tilde{\kappa}_{e,n}(\epsilon, y)|}{|x - y|^\beta} < \infty$$

by the form (3.10). Thus the small order part of the map $\omega \mapsto \det T'_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))$ defined by (3.11) is bounded uniformly in any small $\epsilon > 0$ with respect to $[\cdot]_{\theta_5}$. The definition of $\tilde{\varphi}_n(\epsilon, \cdot)$ in (3.14) yields the assertion. □

Proof of Theorem 1.3. The map $\epsilon \mapsto (\dim_H K(\epsilon))\varphi(\epsilon, \cdot)$ has an n-order asymptotic expansion by Theorem 1.1 and Lemma 3.2. Moreover, it is not hard to check that this small order part is bounded uniformly in any small $\epsilon > 0$ with respect to $[\cdot]_{\theta_5}$ from Lemma 3.4. Hence the assertion is yielded from Theorem 2.4 in [14] by replacing $\varphi(\epsilon, \cdot)$ as $(\dim_H K(\epsilon))\varphi(\epsilon, \cdot)$. □

4. Examples

In the final section, we will give two concrete examples of asymptotic perturbed GIFSs in Example 4.1 and Example 4.2. We also formulate degeneration of graph iterated function systems and calculate the Hausdorff dimension of this limit sets in Example 4.3.

Example 4.1. Let

- $\beta \in (0, 1)$,
- $G = (V = \{v\}, E = \{1, 2\}, i, t)$ with $i(1) = t(1) = i(2) = t(2) = v$, and
- $J_v = [0, 1] \subset \mathbb{R}^1$.

We define two maps $T_1(\epsilon, \cdot)$ and $T_2(\epsilon, \cdot)$ in $C(J_v, J_v)$ by

$$T_1(\epsilon, x) = \frac{x^{2+\beta}}{6} + \frac{x}{6} + x^{1+\beta}\epsilon + x^{1+\epsilon}\epsilon^2,$$

and

$$T_2(\epsilon, x) = T_2(x) = \frac{x^{2+\beta}}{6} + \frac{x}{6} + \frac{1}{2}.$$

It is easy to see that the triplet $(G, (J_v), (T_e(\epsilon, \cdot)))$ satisfies the condition of GIFS. In this case, the map $T_1(\epsilon, x)$ has the form

$$T_1(\epsilon, \cdot) = T_1 + T_{1,1}\epsilon + \widetilde{T}_{1,1}(\epsilon, \cdot)\epsilon$$

if we put

$$T_1(x) = \frac{x^{2+\beta}}{6} + \frac{x}{6},$$

$$T_{1,1}(x) = x^{1+\beta}$$

and

$$\widetilde{T}_{1,1}(\epsilon, x) = x^{1+\epsilon}\epsilon.$$

Therefore this GIFS fulfills condition $(G)_1$ by choosing $\beta(\epsilon) = \epsilon$. In particular, $T_1(\epsilon, \cdot)$ is of $C^{1+\epsilon}$ for each $\epsilon > 0$. Theorem 1.1 implies that the limit set $K(\epsilon)$ has the form

$$\dim_H K(\epsilon) = \dim_H K + s_1\epsilon + o(\epsilon)$$

in $\mathbb{R}^1$, where s_1 is given by

$$s_1 = -(\dim_H K)\frac{\mu(\varphi_1)}{\mu(\varphi)}$$

(for example, Section 5.1 in [14]) and μ is the Gibbs measure of $(\dim_H K)\varphi$.

Example 4.2. We use the notation $\beta, G, J_v, T_2(\epsilon, \cdot)$ defined in Example 4.1. Put

$$T_1(\epsilon, x) = \frac{|x-\epsilon|^{2+\beta}}{6} + \frac{x}{6} \quad \text{for } x \in J_v.$$

This map yields the expansion

$$T_1(\epsilon, x) = T_1 + T_{1,1}\epsilon + \widetilde{T}_{1,1}(\epsilon, \cdot)\epsilon$$

and convergence

$$|\widetilde{T}_{1,1}(\epsilon, \cdot)| \longrightarrow 0 \quad \text{and} \quad \left|\frac{\partial}{\partial x}\widetilde{T}_{1,1}(\epsilon, \cdot)\right| \longrightarrow 0 \quad \text{as } \epsilon \to 0,$$

where

$$T_1(x) = \frac{x^{2+\beta}}{6} + \frac{x}{6},$$

$$T_{1,1}(x) = -\frac{(\beta+2)x^{1+\beta}}{6},$$

and $\widetilde{T}_{1,1}(\epsilon, \cdot)$ is the remainder. Furthermore, we obtain

$$c_1 \le \frac{(\beta+1)(\beta+2)}{6},$$

where c_1 is defined in condition $(G)'_1$. Therefore the GIFS $(G,(J_v),(T_e(\epsilon,\cdot)))$ satisfies $(G)'_1$. Let $f \in F_\theta(E^{(\infty)},\mathbb{C})$ with

$$\theta = (\sup_x |T_1'(x)|)^\beta = \Big(\frac{1}{3}+\frac{\beta}{3}\Big)^\beta.$$

It follows from Theorem 1.3 and the results in [14] that the Gibbs measure $\mu(\epsilon,\cdot)$ of $(\dim_H K(\epsilon))\varphi(\epsilon,\cdot)$ has the form

$$\mu(\epsilon,f) = \mu(f) + \mu_1(f)\epsilon + o(\epsilon) \quad \text{in } \mathbb{R}^1$$

and

$$h(\mu(\epsilon,\cdot)) = h(\mu) + H_1\epsilon + o(\epsilon) \qquad \text{in } \mathbb{R}^1$$

by putting

$$\mu_1(f) = -\nu(f\mathcal{S}(\mathcal{L}_{E,s\varphi}(\varphi_1 h))) - \nu(\varphi_1\mathcal{S}(hf))$$

and

$$H_1 = -\mu_1(\varphi)$$

(see Section 5.1 in [14]). Here the operator $\mathcal{L}_{E,s\varphi}$ and the triplet $(1,h,\nu)$ are defined in the next example, and

$$\mathcal{S} = (\mathcal{L}_{E,s\varphi} - \mathcal{P} - \lambda\mathcal{I})^{-1}(\mathcal{I}-\mathcal{P}),$$

with $\mathcal{P}f = \nu(hf)$.

Example 4.3. Let $\epsilon_1 = \epsilon_1(\epsilon)$ and $\epsilon_2 = \epsilon_2(\epsilon)$ be positive functions with the conditions

$$\lim_{\epsilon\to 0}\epsilon_1(\epsilon) = \lim_{\epsilon\to 0}\epsilon_2(\epsilon) = 0.$$

We consider a family of GIFSs

$$\{(G,(J_v),(T_e(\epsilon,\cdot))) : \epsilon > 0\}$$

as follows.

(a) There exists a decomposition of the edge set E into $E(0)$ and $E(1)$ with $E(0) \neq \emptyset$ such that $(T_e(\epsilon,\cdot))_{e\in E(0)}$ satisfy condition $(G)_1$ by putting $\epsilon = \epsilon_1$ and $(T_e(\epsilon,\cdot))_{e\in E(1)}$ condition $(G)_1$ by putting $\epsilon = \epsilon_2$ and $T_e \equiv a_e$. Namely

$$T_e(\epsilon,\cdot) = \begin{cases} T_e + T_{e,1}\epsilon_1 + \widetilde{T}_{e,1}(\epsilon,\cdot)\epsilon_1, & e \in E(0) \\ a_e + T_{e,1}\epsilon_2 + \widetilde{T}_{e,1}(\epsilon,\cdot)\epsilon_2, & e \in E(1). \end{cases}$$

Here the triplet $(G,(J_v),(T_e)_{e\in E(0)})$ is a GIFS and each a_e, $e \in E(1)$ is a constant with $a_e \in J_{i(e)}$.

(b) The subgraph $G(0) = (V, E(0))$ of G has exactly one strongly connected component $G_0 = (V_0, E_0)$. The Hausdorff dimension $s = \dim_H \widetilde{K}(G_0)$ of the limit set $\widetilde{K}(G_0)$ of the GIFS $(G_0, (J_v)_{v\in V_0}, (T_e)_{e\in E_0})$ is positive. Furthermore, either $s < 1$ or condition $(G)'_0$ is satisfied.

It follows from Theorem 1.1 that the limit set $\widetilde{K}(\epsilon)$ of $(G_0, (J_v)_{v\in V_0}, (T_e(\epsilon, \cdot))_{e\in E_0})$ gives the form

$$\dim_H \widetilde{K}(\epsilon) = s + s_1\epsilon_1 + o(\epsilon_1). \tag{4.1}$$

Denoted by $K(\epsilon)$ the limit set of the GIFS $(G, (J_v), (T_e(\epsilon, \cdot)))$. We put

$$s(\epsilon) = \dim_H K(\epsilon).$$

Then we obtain the next theorem.

Theorem 4.4. *Assume conditions* (a) *and* (b). *Then we have the form*

$$s(\epsilon) = s + s_1\epsilon_1 + s_1'(\epsilon_2)^s + o(\max(\epsilon_1, (\epsilon_2)^s)) \quad \textit{in } \mathbb{R},$$

where s_1 and s_1' are defined in this proof.

We will show this theorem by using a transfer operator method. Let $\mathcal{L}(\mathcal{X})$ be the totality of bounded linear operators acting on a Banach space $\mathcal{X}$. Denoted by $M(E^{(\infty)})$ the totally of Borel probability measures on $E^{(\infty)}$. For a subset $F \subset E$ and $\varphi \in F_\theta(E^{(\infty)}, \mathbb{R})$, we define a bounded linear operator $\mathcal{L}_{F,\varphi}$ in $\mathcal{L}(F_\theta(E^{(\infty)}, \mathbb{C}))$ by

$$\mathcal{L}_{F,\varphi} f(\omega) = \sum_{e\in F\,:\, t(e)=i(\omega)} e^{\varphi(e\cdot\omega)} f(e\cdot\omega),$$

where $e\cdot\omega$ is the concatenation of e and ω, i.e.

$$e\cdot\omega = e\omega_0\omega_1\cdots.$$

Assume that a graph (V, F) has only one strongly connected component $H = (V_H, E_H)$. Note that $F^{(\infty)} \neq \emptyset$ is satisfied by $E_H^{(\infty)} \neq \emptyset$. Let $\varphi \in F_\theta(E^{(\infty)}, \mathbb{R})$. It is known (Theorem 3.1 in [8] and Theorem 4.1 in [13]) that there exists an unique triplet $(\lambda, h, \nu) \in \mathbb{R} \times F_\theta(E^{(\infty)}) \times M(E^{(\infty)})$ such that λ is the positive eigenvalue of the operator $\mathcal{L}_{F,\varphi}$ with maximal modulus, h is the corresponding nonnegative eigenfunction and ν is the corresponding eigenvector of the dual $\mathcal{L}^*_{F,\varphi}$ with $\nu(h) = 1$. Moreover, $\operatorname{supp} h = \{\omega \in E^{(\infty)} : \omega_0 \in E_H\}$ and $\operatorname{supp} \nu = F^{(\infty)}$ are satisfied. It also see that $h\nu$ becomes the Gibbs measure of $\varphi|_{E_H^{(\infty)}}$ on $E_H^{(\infty)}$ and the equality $\log\lambda = P(\varphi|_{E_H^{(\infty)}})$ holds. For the sake of convenience, we call the triplet (λ, h, ν) a thermodynamic spectral characteristics (TSC for a short) of $\mathcal{L}_{F,\varphi}$.

Assume condition (a). We take the coding maps π and $\pi(\epsilon,\cdot)$ defined in Section 1. We set

$$\pi_1(\omega)=\sum_{k=0}^{\infty}(T_{\omega_0\cdots\omega_{k-1}})'(\pi\sigma^k\omega)T_{\omega_k,1}(\pi\sigma^{k+1}\omega)\quad \text{if } \omega\in E(0)^{(\infty)}$$

and $\pi_1(\omega)=0$ otherwise. We define $\pi_{1,1}$ and $\pi_{1,2}$ on $E^{(\infty)}$ by

$$\begin{aligned}\pi_{1,1}(\omega)&=\sum_{i=0}^{k-1}T'_{\omega_0\cdots\omega_{i-1}}(\pi\sigma^i\omega)T_{\omega_i,1}(\pi\sigma^{i+1}\omega),\\ \pi_{1,2}(\omega)&=T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)T_{\omega_k,1}(\pi\sigma^{k+1}\omega)\end{aligned}$$

if $\omega_0\cdots\omega_{k-1}\in E(0)^k$ and $\omega_k\in E(1)$ for some $k\geq 0$, and $\pi_{1,1}(\omega)=\pi_{1,2}(\omega)=0$ otherwise. In this setting, we have the following lemma.

Lemma 4.5. *Assume condition* (a) *and* $s>0$. *Then*

$$\pi(\epsilon,\cdot)=\pi+(\pi_1+\pi_{1,1})\epsilon_1+\pi_{1,2}\epsilon_2+o(\max(\epsilon_1,\epsilon_2))\quad \textit{in } C(E^{(\infty)}).$$

Proof. By Lemma 3.1, the expansion $\pi(\epsilon,\cdot)=\pi+\pi_1\epsilon_1+o(\epsilon_1)$ in $C(E(0)^{(\infty)})$ is satisfied. For $\omega\in E^{(\infty)}\setminus E(0)^{(\infty)}$, there exists $k\geq 0$ such that we have $\omega_0\dots\omega_{k-1}\in E(0)^k$ and $\omega_k\in E(1)$. By using (3.3) in the proof of Lemma 3.1, we have the equation

$$\begin{aligned}\pi(\epsilon,\omega)-\pi\omega&=\epsilon_1\sum_{i=0}^{k-1}\prod_{j=0}^{i-1}T'_{\omega_j}(\pi\sigma^{j+1}\omega)(T_{\omega_i,1}(\pi\sigma^{i+1}\omega)+\widetilde{R}_1(\epsilon,\sigma^i\omega))\\ &\quad+\prod_{j=0}^{k-1}T'_{\omega_j}(\pi\sigma^{j+1}\omega)(\pi(\epsilon,\sigma^k\omega)-\pi\sigma^k\omega)\\ &=\epsilon_1\pi_{1,1}(\omega)\\ &\quad+\epsilon_1\sum_{i=0}^{k-1}T'_{\omega_0\cdots\omega_{i-1}}(\pi\sigma^i\omega)\widetilde{R}_1(\epsilon,\sigma^i\omega)\\ &\quad+\epsilon_2T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(T_{\omega_k,1}(\pi(\epsilon,\sigma^{k+1}\omega))\\ &\quad+\widetilde{T}_{\omega_k,1}(\epsilon,\pi(\epsilon,\sigma^{k+1}\omega))).\end{aligned}$$

We note that the maps $\pi_1+\pi_{1,1}$ and $\pi_{1,2}$ become continuous functions in $E^{(\infty)}$. It is not hard to verify that this equation implies the assertion. ☐

We give a decomposition

$$E^{(\infty)} = \Sigma(0) \cup \Sigma(1)$$

into

$$\Sigma(i) = \{\omega \in E^{(\infty)} \colon \omega_0 \in E(i)\} \quad \text{for } i = 0, 1.$$

We define a function

$$\varphi \colon E^{(\infty)} \longrightarrow \mathbb{R}$$

by

$$\varphi(\omega) = \begin{cases} \log \|T'_{\omega_0}(\pi\sigma\omega)\|, & \omega \in \Sigma(0) \\ 0, & \omega \in \Sigma(1). \end{cases}$$

Set

$$\Phi(\epsilon, \omega) = \log \left\| \frac{\partial}{\partial x} T_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega)) \right\|,$$

$$\varphi(\epsilon, \cdot) = \Phi(\epsilon, \cdot)\chi_{\Sigma(0)},$$

and

$$\psi(\epsilon, \cdot) = \Phi(\epsilon, \cdot)\chi_{\Sigma(1)},$$

$$\psi_1(\omega) = |T'_{\omega_0,1}(\pi\sigma\omega)|^s,$$

where χ_Σ denotes an indicator function of a set Σ.

Lemma 4.6. *Assume conditions* (a) *and* (b)*. Then there exists a function* φ_1 *such that*

$$\varphi(\epsilon, \cdot) = \varphi + \varphi_1\epsilon_1 + o(\max(\epsilon_1, \epsilon_2^s))$$

in $C(E^{(\infty)})$ *if* $s < 1$*, and in* $C(E(0)^{(\infty)})$ *if* $s \geq 1$.

Proof. First we give the form of $\varphi(\epsilon, \cdot)$. For $e \in E(0)$, $T'_e(\epsilon, \cdot)$ has the form

$$T'_e(\epsilon, \cdot) = T'_e + T'_{e,1}\epsilon_1 + \widetilde{T}'_{e,1}(\epsilon, \cdot)\epsilon_1$$

with

$$\sup_{x \in J_{t(e)}} |\widetilde{T}'_{e,1}(\epsilon, x)| \longrightarrow 0 \quad \text{as } \epsilon \to 0.$$

For $F \subset E$ and $e \in F$, let

$$F_e^{(\infty)} = \{\omega \in F^{(\infty)} \colon \omega_0 = e\}.$$

Equation (2.4) in Proposition 2.3 implies that

$$\begin{aligned}
T'_e(\epsilon, \pi(\epsilon,\omega)) &= T'_e(\pi(\epsilon,\omega)) + T'_{e,1}(\pi(\epsilon,\omega))\epsilon_1 + \widetilde{T}'_{e,1}(\pi(\epsilon,\omega))\epsilon_1 \\
&= T'_e(\pi\omega) \\
&\quad + T''_e(\pi\omega)(\pi(\epsilon,\omega) - \pi\omega) \\
&\quad + N_0(\epsilon,\omega)(\pi(\epsilon,\omega) - \pi\omega)|\pi(\epsilon,\omega) - \pi\omega|^\beta + T'_{e,1}(\pi\omega)\epsilon_1 \\
&\quad + N_1(\epsilon,\omega)|\pi(\epsilon,\omega) - \pi\omega|^\beta \epsilon_1 \\
&\quad + \widetilde{T}'_{e,1}(\pi(\epsilon,\omega))\epsilon_1,
\end{aligned}$$

where we define

- if $\pi(\epsilon,\omega) \neq \pi\omega$,

$$N_0(\epsilon,\omega) = \frac{L(1, T'_e, \pi(\epsilon,\omega), \pi\omega)}{|\pi(\epsilon,\omega) - \pi\omega|^\beta}$$

  and

$$N_1(\epsilon,\omega) = \frac{L(0, T'_{e,1}, \pi(\epsilon,\omega), \pi\omega)}{|\pi(\epsilon,\omega) - \pi\omega|^\beta},$$

  and

- if $\pi(\epsilon,\omega) = \pi\omega$,

$$N_0(\epsilon,\omega) = N_1(\epsilon,\omega) = 0.$$

Note that these functions are bounded uniformly in $\epsilon > 0$ and $\omega \in E_e^{(\infty)}$. Lemma 4.5 implies for $e \in E(0)$

$$T'_e(\epsilon, \pi(\epsilon,\cdot)) = T'_e(\pi\cdot) + T''_e(\pi\cdot)(\pi_1 + \pi_{1,1})\epsilon_1 + T'_{e,1}(\pi\cdot)\epsilon_1 + o(\max(\epsilon_1, \epsilon_2^s))$$

in $C(E_e^{(\infty)}, \mathcal{L}(R^D, R^D))$ if $s < 1$ and in $C(E(0)_e^{(\infty)}, \mathcal{L}(R^D, R^D))$ if $s \geq 1$. Indeed, the term $T''_e(\pi\cdot)\pi_{1,2}\epsilon_2$ is a part of $o(\max(\epsilon_1, \epsilon_2^s))$ if $s < 1$ and is equal to 0 on $E(0)^{(\infty)}$. Therefore the proof in Lemma 3.2 yields the form

$$\det T'_e(\epsilon, \pi(\epsilon,\cdot)) = \det T'_e(\pi\cdot) + a_{e,1}(\cdot)\epsilon_1 + o(\max(\epsilon_1, \epsilon_2^s))$$

in $C(E_e^{(\infty)})$ if $s < 1$ and in $C(E(0)_e^{(\infty)})$ if $s \geq 1$. We see that the sign of $\det T'_e(\epsilon, \pi\cdot)$ is equal to the sign of $\det T'_e(\pi\cdot)$ for any small $\epsilon > 0$, and depends only on e from $0 < \|T'_e\|$. We obtain the assertion by putting

$$\varphi_1(\omega) = \frac{a_{\omega_0,1}(\sigma\omega)}{(D \det T'_{\omega_0}(\pi\sigma\omega))}. \qquad \square$$

Note that the role of s in the statement of Lemma 4.6 and in this proof is independent of $\dim_H \widetilde{K}(G_0)$. The essential role of $s = \dim_H \widetilde{K}(G_0)$ comes later in Lemma 4.8 and Lemma 4.9. For $\epsilon > 0$, let $(1, h(\epsilon,\cdot), \nu(\epsilon,\cdot))$ be the TSC of $\mathcal{L}_{E,s(\epsilon)\Phi(\epsilon,\cdot)}$, $(\tilde{\lambda}(\epsilon), \tilde{h}(\epsilon,\cdot), \tilde{\nu}(\epsilon,\cdot))$ the TSC of $\mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)}$, and $(1, h, \nu)$ the TSC of $\mathcal{L}_{E(0),s\varphi}$. We remark that $\mu = h\nu$ is the Gibbs measure of $s\varphi|_{E_0^{(\infty)}}$.

Lemma 4.7. *Assume conditions* (a) *and* (b). *Then* $s(\epsilon)$ *converges to* s.

Proof. First we prove

$$\liminf_{\epsilon\to 0} s(\epsilon) \geq s.$$

To see this, we need to show

$$0 = P(s(\epsilon)\Phi(\epsilon,\cdot)) \geq P(s(\epsilon)\varphi(\epsilon,\cdot)|_{E_0^{(\infty)}}).$$

Recall the definition of the topological pressure (2.5). For any $k \geq 1$ and for any path $\tau \in E_0^k$ on the graph G_0, we have

$$\begin{aligned}\sup_{\substack{\omega\in E_0^{(\infty)}:\\ \omega_0\cdots\omega_{k-1}=\tau}} \sum_{j=0}^{k-1} s(\epsilon)\varphi(\epsilon,\sigma^j\omega) &= \sup_{\substack{\omega\in E_0^{(\infty)}:\\ \omega_0\cdots\omega_{k-1}=\tau}} \sum_{j=0}^{k-1} s(\epsilon)\Phi(\epsilon,\sigma^j\omega)\\ &\leq \sup_{\substack{\omega\in E^{(\infty)}:\\ \omega_0\cdots\omega_{k-1}=\tau}} \sum_{j=0}^{k-1} s(\epsilon)\Phi(\epsilon,\sigma^j\omega).\end{aligned}$$

This implies $P(s(\epsilon)\varphi(\epsilon,\cdot)|_{E_0^{(\infty)}}) \leq P(s(\epsilon)\Phi(\epsilon,\cdot)) = 0$. Since the map

$$\mathbb{R} \ni t \longmapsto P(t\varphi(\epsilon,\cdot)|_{E_0^{(\infty)}})$$

is monotone decreasing and

$$P((\dim_H \widetilde{K}(\epsilon))\varphi(\epsilon,\cdot)|_{E_0^{(\infty)}}) = 0$$

is satisfied from Bowen's formula, we obtain $s(\epsilon) \geq \dim_H \widetilde{K}(\epsilon)$. The form (4.1) yields $\liminf_{\epsilon\to 0} s(\epsilon) \geq s$.

Next we show the assertion. We consider the equation

$$\begin{aligned} \mathcal{L}_{E,s(\epsilon)\Phi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)} &= \mathcal{L}_{E(1),s(\epsilon)\Phi(\epsilon,\cdot)} \\ &= \mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}(e^{s(\epsilon)\psi(\epsilon,\cdot)}\cdot). \end{aligned} \tag{4.2}$$

Choose any positive sequence $(\epsilon(n))$ so that $\lim_{n\to\infty}\epsilon(n) = 0$ and $s(\epsilon(n))$ converges to a number s^*. Note that s^* satisfies $0 < s \le s^* \le D$. Let (λ^*, h^*, ν^*) be the TSC of the operator $\mathcal{L}_{E(0),s^*\varphi}$. Since $\mathcal{L}_{E(0),s(\epsilon(n))\Phi(\epsilon(n),\cdot)} \to \mathcal{L}_{E(0),s^*\varphi}$ in $\mathcal{L}(C(E^{(\infty)}))$ is fulfilled, we have $\nu(\epsilon(n),\cdot) \to \nu^*$ in sense of weakly convergence by Proposition 4.8(2) in [13]. Equation (4.2) implies

$$\begin{aligned} \nu(\epsilon, (\mathcal{L}_{E,s(\epsilon)\Phi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)})h^*) &= \nu(\epsilon, (\mathcal{I} - \mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)})h^*) \\ &= \nu(\epsilon, \mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}(e^{s(\epsilon)\psi(\epsilon,\cdot)}h^*)). \end{aligned}$$

Letting as $\epsilon \to 0$ running through $(\epsilon(n))$, we obtain

$$\nu^*((\mathcal{I} - \mathcal{L}_{E(0),s^*\varphi})h^*) = (1-\lambda^*)\nu^*(h^*) = 0$$

by $\mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)} \to \mathcal{L}_{E(0),s^*\varphi}$ in $\mathcal{L}(C(E^{(\infty)}))$ and by $e^{s(\epsilon)\psi(\epsilon,\omega)} \to 0$ uniformly in $\omega \in E^{(\infty)}$. This yields $\lambda^* = 1$ from $\nu^*(h^*) > 0$ and therefore

$$\log \lambda^* = P(s^*\varphi|_{E_0^{(\infty)}}) = 0.$$

By Bowen's formula, we get $s^* = \dim_H \tilde{K}(G_0) = s$. Hence $s(\epsilon) \to s$. □

Since $\mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)} \to \mathcal{L}_{E(0),s\varphi}$ in $\mathcal{L}(C(E^{(\infty)}))$ and $e^{s(\epsilon)\psi(\epsilon,\omega)} \to 0$ uniformly in $\omega \in E^{(\infty)}$, we obtain $\mathcal{L}_{E(0),s(\epsilon)\Phi(\epsilon,\cdot)} \to \mathcal{L}_{E(0),s\varphi}$ in $\mathcal{L}(C(E^{(\infty)}))$ by using equation (4.2). Therefore we see $\nu(\epsilon,\cdot) \to \nu$ and $\tilde{\nu}(\epsilon,\cdot) \to \nu$ from Proposition 4.8(2) in [13].

Lemma 4.8. *Assume conditions* (a) *and* (b)*. Then the form*

$$e^{s(\epsilon)\psi(\epsilon,\cdot)} = \psi_1\epsilon_2^{s(\epsilon)} + o(\epsilon_2^{s(\epsilon)}) \quad \text{in } C(E^{(\infty)})$$

is satisfied.

Proof. We consider the function $\psi(\epsilon,\cdot)$. Since for $e \in E(1)$, $T_e'(\epsilon,\cdot)$ has the form

$$T_e'(\epsilon,\cdot) = T_{e,1}'\epsilon_2 + \tilde{T}_{e,1}'(\epsilon,\cdot)\epsilon_2,$$

we obtain

$$\det T_e'(\epsilon,\cdot) = (\epsilon_2)^D \det T_{e,1}'(\cdot) + o((\epsilon_2)^D),$$

and therefore

$$|\det T_e'(\epsilon,\cdot)| = (\epsilon_2)^D |\det T_{e,1}'(\cdot)| + o((\epsilon_2)^D).$$

Now we consider convergence of $e^{s(\epsilon)\psi(\epsilon,\omega)}/\epsilon_2^{s(\epsilon)}$. We have that when ϵ is sufficiently small,

$$\begin{aligned}
&\Big\| \frac{e^{s(\epsilon)\psi(\epsilon,\cdot)}}{\epsilon_2^{s(\epsilon)}} - \psi_1 \Big\|_\infty \\
&\le \sup_{\omega\in\Sigma(1)} \Big| \Big(\frac{|\det T_{\omega_0}'(\epsilon,\pi(\epsilon,\sigma\omega))|}{\epsilon_2^D} \Big)^{\frac{s(\epsilon)}{D}} - |\det T_{\omega_0,1}'(\pi(\epsilon,\sigma\omega))|^{\frac{s(\epsilon)}{D}} \Big| \\
&\quad + \sup_{\omega\in\Sigma(1)} \big| |\det T_{\omega_0,1}'(\pi(\epsilon,\sigma\omega))|^{\frac{s(\epsilon)}{D}} - |\det T_{\omega_0,1}'(\pi\sigma\omega)|^{\frac{s(\epsilon)}{D}} \big| \\
&\quad + \sup_{\omega\in\Sigma(1)} \big| |\det T_{\omega_0,1}'(\pi\sigma\omega)|^{\frac{s(\epsilon)}{D}} - |\det T_{\omega_0,1}'(\pi\sigma\omega)|^{\frac{s}{D}} \big| \\
&= \mathrm{I}(\epsilon) + \mathrm{II}(\epsilon) + \mathrm{III}(\epsilon).
\end{aligned}$$

We note that for numbers $a \in (0,1]$ and $b > 0$, there exists $c > b$ such that $|x^a - y^a| \le |x-y|^a$ for any $x, y \in (b,c)$. Thus we see

$$\mathrm{I}(\epsilon) \le \Big| \frac{o(\epsilon_2^D)}{\epsilon_2^D} \Big|^{s(\epsilon)/D} \longrightarrow 0 \quad \text{with } 0 < \frac{s(\epsilon)}{D} \le 1.$$

We also have $\mathrm{II}(\epsilon) \to 0$ by the same argument. Finally, since $\Sigma(1)$ is compact, we obtain that $\mathrm{III}(\epsilon)$ vanishes. □

Lemma 4.9. *Assume conditions* (a) *and* (b)*, and $s \ge 1$. Then the form*

$$\tilde{\lambda}(\epsilon) = 1 + \nu(\mathcal{L}_{E(1),s\varphi}(\psi_1 h))\epsilon_2^{s(\epsilon)} + o(\epsilon_2^{s(\epsilon)}) \quad \textit{in } \mathbb{R}$$

is satisfied.

Proof. By virtue of condition $(G)_0'$, $\limsup_{\epsilon\to 0}[\varphi(\epsilon,\cdot)]_\theta < \infty$ is yielded for some $\theta \in (0,1)$. Thus we have that $\tilde{h}(\epsilon,\cdot)$ converges to h in $C(E^{(\infty)})$ (Proposition 4.3 in [8]). Equation (4.2) implies

$$(1-\tilde{\lambda}(\epsilon))\nu(\epsilon,\tilde{h}(\epsilon,\cdot)) = \nu(\epsilon,\mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}(e^{s(\epsilon)\psi(\epsilon,\cdot)}\tilde{h}(\epsilon,\cdot))).$$

Hence we obtain the assertion by Lemma 4.6 and by $\nu(h) = 1$. □

Lemma 4.10. *Assume conditions* (a) *and* (b). *Then we have*

$$|s(\epsilon)-s| = O(\max(\epsilon_2^s, \epsilon_2^{s(\epsilon)}, \epsilon_1)).$$

Moreover, if either $\epsilon_1 = O(\epsilon_2^{s(\epsilon)})$ *or* $\epsilon_1 = O(\epsilon_2^s)$ *is satisfied then* $\epsilon_2^{s(\epsilon)}/\epsilon_2^s \to 1$ *holds.*

Proof. First we assume $s < 1$. We consider the equation

$$\begin{aligned}
&\mathcal{L}_{E,s(\epsilon)\Phi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi}\\
&\quad= \mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)} + \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi} + \mathcal{L}_{E(1),s(\epsilon)\Phi(\epsilon,\cdot)}\\
&\quad= \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\varphi(\epsilon,\cdot)\cdot)(s(\epsilon)-s) + \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(G(\epsilon,\cdot)\cdot)(s(\epsilon)-s)^2\\
&\qquad+ \mathcal{L}_{E(0),s\varphi}(s(\varphi(\epsilon,\cdot)-\varphi)\cdot) + \mathcal{L}_{E(0),s\varphi}(H(\epsilon,\cdot)(\varphi(\epsilon,\cdot)-\varphi)^2\cdot)\\
&\qquad+ \mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}(e^{s(\epsilon)\psi(\epsilon,\cdot)}\cdot),
\end{aligned}$$

where we define

$$G(\epsilon,\cdot) = \sum_{k=0}^{\infty}(s(\epsilon)-s)^k(\varphi(\epsilon,\cdot)^{k+2})/(k+2)!$$

and

$$H(\epsilon,\cdot) = \sum_{k=0}^{\infty}(s^{k+2}(\varphi(\epsilon,\cdot)-\varphi)^k)/(k+2)!.$$

We have

$$\begin{aligned}
0 &= \nu(\epsilon,\mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\varphi(\epsilon,\cdot)h))(s(\epsilon)-s)\\
&\quad+ \nu(\epsilon,\mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(G(\epsilon,\cdot)h))(s(\epsilon)-s)^2\\
&\quad+ \nu(\epsilon,\mathcal{L}_{E(0),s\varphi}(s(\varphi(\epsilon,\cdot)-\varphi)h))\\
&\quad+ \nu(\epsilon,\mathcal{L}_{E(0),s\varphi}(H(\epsilon,\cdot)(\varphi(\epsilon,\cdot)-\varphi)^2h))\\
&\quad+ \nu(\epsilon,\mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}(e^{s(\epsilon)\psi(\epsilon,\cdot)}h))
\end{aligned} \tag{4.3}$$

by using $\nu(\epsilon, (\mathcal{L}_{E,s(\epsilon)\Phi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi})h) = 0$. In this equation, we note that the number $\nu(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\varphi(\epsilon,\cdot)h))$ converges to $\nu(\varphi h) = \mu(\varphi) < 0$ as $\epsilon \to 0$. It follows from Lemma 4.6 and Lemma 4.8 that this equation implies the former assertion by dividing equation (4.3) by $\max(\epsilon_2^s, \epsilon_2^{s(\epsilon)}, \epsilon_1)$ and by letting as $\epsilon \to 0$.

Next we assume $s \geq 1$. Let $\chi_0 = \chi_{E(0)^{(\infty)}}$. By the decomposition

$$\begin{aligned}&\mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi}\\&\quad = \mathcal{L}_{E(0),s(\epsilon)\varphi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)} + \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)} - \mathcal{L}_{E(0),s\varphi},\end{aligned}$$

we have

$$\begin{aligned}(\tilde{\lambda}(\epsilon) - 1)\tilde{\nu}(\epsilon, h) = &\ \tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\chi_0\varphi(\epsilon,\cdot)h))(s(\epsilon) - s)\\&+ \tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\chi_0 G(\epsilon,\cdot)h))(s(\epsilon) - s)^2\\&+ \tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi}(\chi_0 s(\varphi(\epsilon,\cdot) - \varphi)h))\\&+ \tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi}(\chi_0 H(\epsilon,\cdot)(\varphi(\epsilon,\cdot) - \varphi)^2 h))\end{aligned} \tag{4.4}$$

from a similar argument above and $\operatorname{supp}\tilde{\nu}(\epsilon,\cdot) = E(0)^{(\infty)}$. By Lemma 4.6, Lemma 4.8, and Lemma 4.9, we obtain the former assertion again.

Finally we assume either $\epsilon_1 = O(\epsilon_2^{s(\epsilon)})$ or $\epsilon_1 = O(\epsilon_2^s)$. Then we have the inequality

$$|s(\epsilon) - s| \leq c \max(\epsilon_2^s, \epsilon_2^{s(\epsilon)}) = c\epsilon_2^{t(\epsilon)}$$

with a constant c and

$$t(\epsilon) = \min(s(\epsilon), s).$$

Therefore

$$1 \leq e^{(t(\epsilon) - \max(s(\epsilon),s))\log\epsilon_2} = e^{-|s(\epsilon)-s|\log\epsilon_2} \leq e^{-c\exp(t(\epsilon)\log\epsilon_2)\log\epsilon_2} \longrightarrow 1$$

as $\epsilon \to 0$ is satisfied. This gives $\epsilon_2^{t(\epsilon)}/\epsilon_2^{\max(s(\epsilon),s)} \to 1$. Hence in particular, $\epsilon_2^{s(\epsilon)}/\epsilon_2^s \to 1$ follows. □

Proof of Theorem 4.4. Let

$$\epsilon_3 = \max(\epsilon_1, \epsilon_2^s).$$

Put

$$s_1 = -s\mu(\varphi_1)/\mu(\varphi)$$

and

$$s_1' = -\nu(\mathcal{L}_{E(1),s\varphi}(\psi_1 h))/\mu(\varphi).$$

First we assume $s < 1$. Equation (4.3) implies the form

$$\begin{aligned}
&\frac{s(\epsilon) - s - s_1\epsilon_1 - s_1'\epsilon_2^s}{\epsilon_3} \\
&= \frac{1}{a(\epsilon)}\nu\Big(\epsilon, \mathcal{L}_{E(0),s\varphi}\Big(s\frac{\varphi(\epsilon,\cdot) - \varphi - \varphi_1\epsilon_1}{\epsilon_3}h\Big)\Big) \\
&\quad + \Big(\frac{1}{a(\epsilon)}\nu(\epsilon, \mathcal{L}_{E(0),s\varphi}(s\varphi_1 h)) - s_1\Big)\frac{\epsilon_1}{\epsilon_3} \\
&\quad + \frac{1}{a(\epsilon)}\nu(\epsilon, \mathcal{L}_{E(0),s\varphi}(H(\epsilon,\cdot)(\varphi(\epsilon,\cdot) - \varphi)^2 h))\frac{1}{\epsilon_3} \\
&\quad + \Big(\frac{1}{a(\epsilon)}\nu\Big(\epsilon, \mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}\frac{e^{s(\epsilon)\psi(\epsilon,\cdot)}}{\epsilon_2^s}h\Big) - s_1'\Big)\frac{\epsilon_2^s}{\epsilon_3} \\
&= J_1(\epsilon) + J_2(\epsilon) + J_3(\epsilon) + J_4(\epsilon),
\end{aligned}$$

with

$$a(\epsilon) = -\nu(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\varphi(\epsilon,\cdot)h)) - \nu(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(G(\epsilon,\cdot)h))(s(\epsilon) - s).$$

We see $a(\epsilon) \to -\nu(\mathcal{L}_{E(0),s\varphi}(\varphi h)) = -\mu(\varphi)$ as $\epsilon \to 0$. We will consider the two cases:

(I) $\epsilon_1 = O(\epsilon_2^{s(\epsilon)})$ and

(II) $\epsilon_2^{s(\epsilon)} = o(\epsilon_1)$.

In case (I), we have $\epsilon_2^{s(\epsilon)}/\epsilon_2^s \to 1$ by virtue of Lemma 4.10. We obtain $J_1(\epsilon) \to 0$ by Lemma 4.6 and $J_2(\epsilon) \to 0$ by $\epsilon_1 = O(\epsilon_3)$. From Lemma 4.6 again, $(\varphi(\epsilon,\cdot) - \varphi)/\epsilon_3$ is bounded uniformly in $\epsilon > 0$ and thus $J_3(\epsilon) \to 0$. Finally, $J_4(\epsilon) \to 0$ follows from $e^{s(\epsilon)\psi(\epsilon,\cdot)}/\epsilon_2^s$ converges to ψ_1 in $C(E^{(\infty)})$ with Lemma 4.8.

In case (II), we have $\epsilon_2^s = o(\epsilon_1)$. Indeed, we suppose $\epsilon_1 = O(\epsilon_2^s)$. Then Lemma 4.10 implies $\epsilon_2^{s(\epsilon)}/\epsilon_2^s \to 1$ and therefore $\epsilon_1 = O(\epsilon_2^{s(\epsilon)})$. This contradicts with the fact (II). Thus we see $\epsilon_1 = \epsilon_3$ for any small $\epsilon > 0$. By a similar argument in the case (I), we obtain $J_1(\epsilon), J_2(\epsilon), J_3(\epsilon) \to 0$. It remains to show $J_4(\epsilon) \to 0$. We notice

$$J_4(\epsilon) = \frac{1}{a(\epsilon)}\nu\Big(\epsilon, \mathcal{L}_{E(1),s(\epsilon)\varphi(\epsilon,\cdot)}\frac{e^{s(\epsilon)\psi(\epsilon,\cdot)}}{\epsilon_2^{s(\epsilon)}}h\Big)\frac{\epsilon_2^{s(\epsilon)}}{\epsilon_1} - s_1'\frac{\epsilon_2^s}{\epsilon_1} \longrightarrow 0$$

from Lemma 4.8.

Next we assume $s \geq 1$. By equation (4.4), we have

$$\begin{aligned}
&\frac{s(\epsilon) - s - s_1\epsilon_1 - s_1'\epsilon_2^s}{\epsilon_3} \\
&= \frac{1}{b(\epsilon)}\tilde{\nu}\Big(\epsilon, \mathcal{L}_{E(0),s\varphi}\Big(\chi_0 s\frac{\varphi(\epsilon,\cdot) - \varphi - \varphi_1\epsilon_1}{\epsilon_3}h\Big)\Big) \\
&\quad + \Big(\frac{\tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi}(\chi_0 s\varphi_1 h))}{b(\epsilon)} - s_1\Big)\frac{\epsilon_1}{\epsilon_3} \\
&\quad + \frac{1}{b(\epsilon)}\tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi}(\chi_0 H(\epsilon,\cdot)((\varphi(\epsilon,\cdot) - \varphi)^2 h)))\frac{1}{\epsilon_3} \\
&\quad + \Big(\frac{1}{b(\epsilon)}\frac{1 - \tilde{\lambda}(\epsilon)}{\epsilon_2^s}\tilde{\nu}(\epsilon, h) - s_1'\Big)\frac{\epsilon_2^s}{\epsilon_3},
\end{aligned}$$

where

$$b(\epsilon) = -\tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\chi_0\varphi(\epsilon,\cdot)h)) - \tilde{\nu}(\epsilon, \mathcal{L}_{E(0),s\varphi(\epsilon,\cdot)}(\chi_0 G(\epsilon,\cdot)h))(s(\epsilon) - s)$$

is given. By a similar argument in the case $s < 1$ and by using Lemma 4.9, the assertion is fulfilled. □

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Received Septemer 5, 2014; revised November 9, 2015

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