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# Conformal graph directed Markov systems: beyond finite irreducibility

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**Abstract.** In this paper, we study CGDMSs which are not necessarily finitely irreducible. We do this from two different perspectives: by investigating irreducible infinite systems and by examining general (i.e. potentially reducible) finite systems. In this latter case, we derive a necessary and sufficient condition under which the Hausdorff measure of the limit set is positive and finite. We further show that if this condition doesn't hold then the Hausdorff measure, though infinite, is $\sigma$-finite. This condition is given in terms of the strongly connected components of the directed graph. We also demonstrate that the pressure and Hausdorff dimension of the limit set are completely determined by the strongly connected components.

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## 1. Introduction

The theory of the limit set generated by the iteration of finitely many similarity maps has been well developed for some time now. A more complicated theory of the limit set generated by the iteration of infinitely many uniformly contracting conformal maps was developed by Mauldin and Urbanski in [7]. Several years after that, they explored the geometric and dynamic properties of a far reaching generalization of conformal iterated function systems, called Graph Directed Markov Systems (GDMS's) (see [9]).

Several concepts are at the core of the analysis of a conformal Graph Directed Markov System (CGDMS); among them: the topological pressure function, the Hausdorff dimension of the limit set and the conformal measure supported on the limit set. The connections between these concepts have been intensively studied by Mauldin and Urbanski, especially in the case when the incidence matrix of the CGDMS is finitely irreducible. In this paper particular attention is paid to the case the system or matrix is finite but not irreducible.

The paper is organized as follows. In the second section we present the most basic concepts for CGDMS's. We show how one can construct the limit set by performing an infinite directed walk through the graph. This leads to a natural map from the coding space to the points of the limit set. We look at several properties that the incidence matrix can have. The most important is when the incidence matrix is finitely irreducible; in this situation most of the results from the theory of the CIFS's can be carried over. In the third section we recall the definition of topological pressure for CGDMS's. In the fourth section we give a precise and alternative definition of the conformal measure supported on the limit set of a CGDMS.

The most well understood of these general systems, besides the iteration of finitely many or a countable infinity of conformal maps, are the CGDMS with a finitely irreducible incidence matrix. We state several of the results concerning them in the fifth section. We further show how these results change in the general case when the incidence matrix is not assumed to be finitely irreducible. Finally, in the sixth section we study the properties of general, i.e. non necessarily irreducible, finite CGDMSs.

## 2. Preliminaries

To introduce graph directed Markov systems we need a directed multigraph $(V, E, i, t)$ and an associated incidence matrix $A$. The multigraph consists of a finite set $V$ of vertices, a (possibly infinitely) countable set $E$ of directed edges and two functions $i, t\colon E \to V$, where $i(e)$ is the initial vertex of edge $e$ and $t(e)$ is the terminal vertex of that edge. There is also a function $A\colon E^2 \to \{0, 1\}$ called an incidence matrix, as this matrix indicates which edge(s) may follow any given edge. This matrix also respects the multigraph since $A_{ef} = 1$ may happen only if $t(e) = i(f)$. The set of one-sided infinite $A$-admissible words is defined by

$$E_A^\infty := \{\omega = \omega_1\omega_2 \cdots \in E^\infty\colon A_{\omega_i\omega_{i+1}} = 1,\ \text{for all } i \ge 1\}.$$

The set of all finite subwords of $E_A^\infty$ will be denoted by $E_A^*$. The set of all subwords of $E_A^\infty$ of length $n$ shall be denoted by $E_A^n$. There is a unique word of length 0 in $E_A^*$ called the *empty word*. The *length* of any word $\omega$ will be denoted by $|\omega|$.

If $\omega \in E_A^\infty$ and $n \geq 1$, then

$$\omega|_n = \omega_1 \omega_2 \dots \omega_n.$$

A *Graph Directed Markov System* (GDMS) consists of a directed multigraph $(V, E, i, t)$, an incidence matrix $A$, a set of non-empty compact metric spaces $\{X_v\}_{v \in V}$ and a set of 1-to-1 contractions $S := \{\varphi_e \colon X_{t(e)} \to X_{i(e)}\}_{e \in E}$ with Lipschitz constant $s$, where $0 < s < 1$. In short, this latter set is called a GDMS.

For each $\omega \in E_A^*$, the map coded by $\omega$ is defined by

$$\varphi_\omega := \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_{|\omega|}} \colon X_{t(\omega)} \longrightarrow X_{i(\omega)},$$

where $t(\omega) := t(\omega_{|\omega|})$ and $i(\omega) := i(\omega_1)$. For $\omega \in E_A^\infty$, the sets $\{\varphi_{\omega|_n}(X_{t(\omega_n)})\}_{n \geq 1}$ form a descending sequence of non-empty compact subsets of $X_{i(\omega_1)}$.

Since $\operatorname{diam}(\varphi|_{\omega_n}(X_{t(\omega_n)})) \leq s^n \operatorname{diam}(X_{t(\omega_n)}) \leq s^n \max\{\operatorname{diam}(X_v) \colon v \in V\}$ for every $n \geq 1$, the intersection

$$\bigcap_{n \geq 1} \varphi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton whose element is denoted by $\pi(\omega)$. The map

$$\pi \colon E_A^\infty \longrightarrow \bigoplus_{v \in V} X_v =: X$$

defined in this way is called the *coding map*. The set $X$ is the disjoint union of the compact sets $X_v$. The set

$$J = J_{E,A} = \pi(E_A^\infty)$$

is called the *limit set* of the GDMS $S$.

From this point on in the paper, we make two simplifying assumptions about the directed graph. First, we assume that for all $e \in E$ there exists $f \in E$ so that $A_{ef} = 1$. Otherwise, if there were $e \in E$ so that $A_{ef} = 0$ for every $f \in E$, then the limit set $J_{E,A}$ would be the same as the limit set $J_{E \setminus \{e\}, A}$ (in the construction of this latter set, $A$ is restricted to $(E \setminus \{e\})^2$). Second, we assume that for every vertex $v \in V$ there exists $e \in E$ so that $i(e) = v$. Otherwise, if there existed $v \in V$ such that no edge has for initial vertex $v$, then the limit set $J$ would be the same if the vertex set were $V \setminus \{v\}$.

If the set of vertices of a GDMS is a singleton and all the entries in the associated incidence matrix are 1, then the GDMS is an *iterated function system* (IFS).

We emphasize that we have two directed graphs that play an important role in our study. The first one is the given multigraph $(V, E, i, t)$. The second one, denoted by $G_{E,A}$, is determined by the matrix $A$. The vertices of $G_{E,A}$ are the edges of the multigraph and $G_{E,A}$ has a directed edge from $e$ to $f$ if and only if $A_{ef} = 1$. Therefore $G_{E,A}$ has infinitely many vertices and edges if and only if $E$ is an infinite set.

The incidence matrix $A$ is said to be *irreducible* if for any two edges $e, f \in E$ there exists a word $\omega \in E_A^*$ so that $e\omega f \in E_A^*$. This is equivalent to saying that the directed graph $G_{E,A}$ is *strongly connected*, i.e. for any two vertices of that graph there exists a path starting from one and ending at the other.

The matrix $A$ is called *primitive* if there exists $p \geq 1$ such that all the entries of $A^p$ are positive (written $A^p > 0$) or, in other words, for any two edges $e, f \in E$ there exists a word $\omega \in E_A^{p-1}$ so that $e\omega f \in E_A^{p+1}$.

The matrix $A$ is said to be *finitely irreducible* if there exists a finite set $\Omega \subset E_A^*$ so that for any two edges $e, f \in E$ there is a word $\omega \in \Omega$ so that $e\omega f \in E_A^*$.

The matrix $A$ is called *finitely primitive* if there exist $p \geq 1$ and a finite set $\Omega \subset E_A^{p-1}$ such that for any two edges $e, f \in E$ there is a word $\omega \in \Omega$ so that $e\omega f \in E_A^{p+1}$.

A GDMS is called *conformal*, and hence a CGDMS, if the following conditions are satisfied.

(1) For every $v \in V$, the set $X_v$ is a compact connected subset of a Euclidean space $\mathbf{R}^d$ (the dimension $d$ common for all vertices) and $X_v = \overline{\mathrm{Int}(X_v)}$.

(2) (Open Set Condition (OSC)) For every $e, f \in E$, $e \neq f$,

$$\varphi_e\big(\mathrm{Int}(X_{t(e)})\big) \bigcap \varphi_f\big(\mathrm{Int}(X_{t(f)})\big) = \emptyset.$$

(3) For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ so that for every $e \in E$ with $t(e) = v$, the map $\varphi_e$ extends to a $C^1$ conformal diffeomorphism of $W_v$ into $W_{i(e)}$.

(4) (Cone property) There exists $\gamma, l > 0$, such that for every $x \in X$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex $x$, central angle of measure $\gamma$, and altitude $l$.

(5) There are two constants $L \geq 1$ and $\alpha > 0$ so that

$$\big||\varphi_e'(y)| - |\varphi_e'(x)|\big| \leq L\|(\varphi_e')^{-1}\|^{-1}\|y - x\|^\alpha$$

for every $e \in E$ and for every pair of points $x, y \in W_{t(e)}$, where $|\varphi_e'(x)|$ represents the norm of the derivative of $\varphi_e$ at $x$. This says that the norms of the derivative maps are all Hölder of order $\alpha$ with Hölder constant depending on the map.

**Remark 2.1.** Condition (5) plays a central role in dimension $d = 1$. If $d \geq 2$ and a GDMS $S = \{\varphi_e \colon X_{t(e)} \to X_{i(e)}\}_{e \in E}$ satisfies conditions (1) and (3), then it also fulfills condition (5) with $\alpha = 1$.

As a straightforward consequence of (5), we get the famous

(6) (Bounded Distortion Property (BDP)) There exists $K \geq 1$ such that for all $\omega \in E_A^*$ and for all $x, y \in W_{t(\omega)}$

$$|\varphi_\omega'(y)| \leq K|\varphi_\omega'(x)|. \tag{1}$$

## 3. Topological pressure for CGDMSs

We now define the topological pressure function, a central object in the theory of CGDMSs.

Given $t \geq 0$ and $n \geq 1$, let

$$Z_{n,E,A}(t) = \sum_{\omega \in E_A^n} \|\varphi_\omega'\|^t.$$

The non-increasing function $Z_n = Z_{n,E,A}$ is called the $n$th-level partition function. The partition functions form a *submultiplicative* sequence of functions since for every $t \geq 0$ and for every $p, q \geq 1$,

$$Z_{p+q,E,A}(t) \leq Z_{p,E,A}(t) Z_{q,E,A}(t).$$

**Remark 3.1.** For a CGDMS with a finitely irreducible incidence matrix, the sequence of partition functions is boundedly supermultiplicative, i.e. for every $t \geq 0$ there exists a constant $C = C(t) > 0$ so that

$$C(t) Z_{p,E,A}(t) Z_{q,E,A}(t) \leq Z_{p+q,E,A}(t)$$

for every $p, q \geq 1$. However, if the incidence matrix is not finitely irreducible, this may not hold anymore. An example is given in Remark 3.3.

We define the *topological pressure function*

$$P = P_{E,A} \colon [0, \infty) \longrightarrow [-\infty, \infty]$$

as follows.

First, let

$$\theta = \theta_{E,A} := \inf\{t \geq 0\colon Z_{n,E,A}(t) < \infty \text{ for all but finitely many } n\text{'s}\}.$$

If $t > \theta_{E,A}$, then the submultiplicativity of the $Z_n$'s allows to define

$$P_{E,A}(t) := \lim_{n\to\infty} \frac{1}{n} \log Z_{n,E,A}(t) = \inf_{n\to\infty} \frac{1}{n} \log Z_{n,E,A}(t) < \infty.$$

If $t < \theta_{E,A}$, we define $P_{E,A}(t) := \infty$.

If $Z_{n,E,A}(\theta_{E,A}) < \infty$ for all but finitely many $n$'s, we put

$$P_{E,A}(\theta_{E,A}) := \lim_{n\to\infty} \frac{1}{n} \log Z_{n,E,A}(\theta_{E,A}) = \inf_{n\to\infty} \frac{1}{n} \log Z_{n,E,A}(\theta_{E,A}).$$

Otherwise, $P_{E,A}(\theta_{E,A}) := \infty$.

The number $\theta_{E,A}$ is thus the *finiteness parameter* for the pressure of the system.

For every $n \geq 1$, let

$$\theta_n = \theta_{n,E,A} := \inf\{t \geq 0\colon Z_{n,E,A}(t) < \infty\}$$

be the finiteness parameter for the $n$th-level partition function of the system.

The next result is a slight generalization of Proposition 4.2.8 in [9].

**Proposition 3.2.** *The following statements hold*:

(*a*) $\theta_{kn} \leq \theta_n$ *for all* $k, n \geq 1$;

(*b*) $\inf_{n\geq 1} \theta_n = \liminf_{n\to\infty} \theta_n \leq \theta \leq \limsup_{n\to\infty} \theta_n$;

(*c*) *if* $A$ *is finitely irreducible, then* $\theta_n = \theta$ *for every* $n \geq 1$;

(*d*) *the pressure function* $P$ *is non-increasing on* $[0, \infty)$, *strictly decreasing to* $-\infty$ *on* $[\theta, \infty)$, *and convex* (*so continuous*) *on* $(\theta, \infty)$;

(*e*) $P(0) = \infty$ *if and only if* $|E| = \infty$;

(*f*) $P(t) = \inf\Big\{u \in \mathbb{R}\colon$ *there exists* $n_u \geq 0$ *so that* $\sum_{\omega\in E_A^*\colon |\omega|\geq n_u} \|\varphi'_\omega\|^t e^{-u|\omega|} < \infty\Big\}$.

**Remark 3.3.** Note that part (c) does not necessarily hold if the matrix is not finitely irreducible. For example, take the standard continued fractions conformal iterated function system $S_{\mathbb{N}}$ (see [8]). We may conceive this system as a graph directed Markov system having a single vertex $v$ with infinitely many self-loops. These self-loops are labeled by the positive integers and to self-loop $n$ is associated the conformal map $\varphi_n(x) = 1/(n+x)$ mapping $X = X_v = [0,1]$ into itself. Let $S$ be the subsystem of $S_{\mathbb{N}}$ generated by $E = \{2,3,4,\ldots\}$ and the edge incidence matrix

$$A_{ef} = 1 \iff |e-f| \le 1.$$

The limit set $J$ of $S$ is the set of all irrational numbers between 0 and 1 whose standard continued fraction expansions have entries at least 2 such that any two adjacent entries differ by at most 1. Clearly, the incidence matrix $A$ is irreducible, though not finitely irreducible. For every $n \ge 1$, one can also show that $\theta_n = 1/2n$. By Proposition 3.2(b), the finiteness parameter for the pressure function is hence $\theta = 0$. Furthermore, if the sequence of partition functions were boundedly supermultiplicative (cf. Remark 3.1), then for every $n \ge 1$ we would have $\theta_n = \theta$. Thus the sequence of partition functions is not boundedly supermultiplicative.

## 4. Conformal measures for CGDMSs

We shall now define conformal measures. For each $\omega \in E_A^*$, let

$$E_{A,\omega}^\infty := \{\tau \in E_A^\infty : \omega\tau \in E_A^\infty\} = \bigcup_{\tilde{e}\in E:\, A_{\omega_{|\omega|}\tilde{e}}=1} [\tilde{e}]$$

be the set of all $A$-admissible infinite words to which the finite word $\omega$ can be prefixed.

**Definition 4.1.** Let $S = \{\varphi_e\}_{e\in E}$ be a CGDMS. A Borel probability measure $m$ on $X$ is said to be *$t$-conformal* provided it is supported on the limit set $J$ and the following two conditions are satisfied.

For every $e \in E$ and for every Borel set $B \subseteq \pi(E_{A,e}^\infty)$, we have

$$m(\varphi_e(B)) = \int_B |\varphi_e'|^t \, dm, \tag{2}$$

and for all letters $e_1, e_2 \in E$

$$m(\varphi_{e_1}(X_{t(e_1)}) \cap \varphi_{e_2}(X_{t(e_2)})) = 0. \tag{3}$$

By induction the conformality cascades down to all finite words. Indeed, a Borel probability measure $m$ on $X$ is $t$-conformal if and only if it is supported on the limit set $J$, for every $\omega \in E_A^*$ and for every Borel set $B \subseteq \pi(E_{A,\omega}^\infty)$, we have

$$m(\varphi_\omega(B)) = \int_B |\varphi_\omega'|^t \, dm, \tag{4}$$

and for all incomparable words $\omega, \tau \in E_A^*$

$$m(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)})) = 0. \tag{5}$$

Unlike the definition given in (4.28) on p. 77 of [9], we claim that condition (2) can generally be only imposed on the Borel subsets of $\pi(E_{A,e}^\infty)$, $e \in E$. Indeed, assume that the CGDMS $S$ satisfies the strong separation condition, that is, that its first-level sets are mutually disjoint. Condition (3) holds for any measure in such a case. Furthermore, the coding map $\pi$ is injective. By definition, $\pi$ is also surjective and continuous from a compact space to its compact, and thus Hausdorff, image. Therefore it is a homeomorphism. Fix $e \in E$ and let

$$J_{t(e)} = J \cap X_{t(e)} \quad \text{and} \quad E_{A,e}^\infty = \{\tau \in E_A^\infty : e\tau \in E_A^\infty\}.$$

Let us suppose $\pi(\tau) \in \varphi_e(J_{t(e)} \backslash \pi(E_{A,e}^\infty))$ for some $\tau \in E_A^\infty$. If $\tau_1 = e$ then $\sigma\tau \in E_{A,e}^\infty$ and thus $\pi(\sigma\tau) \in \pi(E_{A,e}^\infty)$. Consequently, $\pi(\tau) = \varphi_{\tau_1}(\pi(\sigma\tau)) \in \varphi_e(\pi(E_{A,e}^\infty))$. Hence we deduce that $\pi(\tau) \notin \varphi_e(J_{t(e)} \backslash \pi(E_{A,e}^\infty))$ by the injectivity of $\varphi_e$. So $\tau_1 \neq e$. But then $\varphi_{\tau_1}(X_{t(\tau_1)}) \cap \varphi_e(X_{t(e)}) = \emptyset$. Since $\pi(\tau) \in \varphi_{\tau_1}(X_{t(\tau_1)})$, this implies that $\pi(\tau) \notin \varphi_e(X_{t(e)})$. This contradiction implies that $\varphi_e(J_{t(e)} \backslash \pi(E_{A,e}^\infty)) \cap J = \emptyset$ and consequently $m(\varphi_e(J_{t(e)} \backslash \pi(E_{A,e}^\infty))) = 0$. If $\Phi$ admits a $t$-conformal measure $m$, then

$$\begin{aligned} 0 &= m(\varphi_e(J_{t(e)} \backslash \pi(E_{A,e}^\infty))) \\ &= \int_{J_{t(e)} \backslash \pi(E_{A,e}^\infty)} |\varphi_e'|^t \, dm \geq K^{-t} \|\varphi_e'\|^t m(J_{t(e)} \backslash \pi(E_{A,e}^\infty)). \end{aligned}$$

Therefore $m(J_{t(e)} \backslash \pi(E_{A,e}^\infty)) = 0$ for every $e \in E$. Note that

$$E_{A,e}^\infty = \bigcup_{g \in E:\, A_{eg}=1} [g]$$

and thus

$$\pi(E_{A,e}^\infty) = \bigcup_{g \in E:\, A_{eg}=1} \pi([g]).$$

It follows from the injectivity of $\pi$ that

$$J_{t(e)} \setminus \pi(E^\infty_{A,e}) = \bigcup_{g\in E:\, t(e)=i(g),\, A_{eg}=0} \pi([g]).$$

Suppose additionally that $V$ consists of a single vertex, that $E = \{1, 2, 3\}$, and that the matrix $A$ is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$J_{t(1)} \setminus \pi(E^\infty_{A,1}) = \bigcup_{A_{1,g}=0} \pi([g]) = \pi([2]),$$

$$J_{t(2)} \setminus \pi(E^\infty_{A,2}) = \bigcup_{A_{2,g}=0} \pi([g]) = \pi([3]),$$

$$J_{t(3)} \setminus \pi(E^\infty_{A,3}) = \bigcup_{A_{3,g}=0} \pi([g]) = \pi([1]).$$

Hence

$$m(J) = \sum_{e\in E} m(\pi([e])) = \sum_{e\in E} m(J_{t(e)} \setminus \pi(E^\infty_{A,e})) = 0.$$

This means that such a system cannot admit a $t$-conformal measure as defined in (4.28) on p.77 of [9], i.e. if the Borel sets $B \subset X_{t(e)}$ are allowed to intersect $J_{t(e)} \setminus \pi(E^\infty_{A,e})$. However, any such system is regular as it is finite and primitive ($A^2 > 0$).

Given a word $\omega \in E^*_A$, set

$$X_\omega := \bigcup_{e\in E:\, A_{\omega_{|\omega|}e}=1} \varphi_e(X_{t(e)}).$$

We now enunciate a characterization of conformality in terms of the sets $X_e$, $e \in E$.

**Lemma 4.2.** *A Borel probability measure $m$ on $X$ is a $t$-conformal measure if and only if it is supported on $J$ and is such that* (3) *holds and that* (2) *is satisfied for all $e \in E$ and all Borel sets $B \subseteq X_e$.*

*Proof.* One implication is immediate since $\pi(E^\infty_{A,e})$ is a Borel subset of $X_e$ for all $e \in E$. For the opposite direction, suppose that $m$ is a $t$-conformal measure. Fix $e \in E$. We just need to show that (2) is satisfied for all Borel sets $B \subseteq X_e$. Let $E_e = \{\tilde{e} \in E\colon A_{e\tilde{e}} = 1\}$. We claim that it is sufficient to establish (2) for all $\tilde{e} \in E_e$ and all Borel sets $B \subseteq \varphi_{\tilde{e}}(X_{t(\tilde{e})})$. Indeed, if that is the case and $B \subseteq X_e$, then order $E_e = \{\tilde{e}_n\}_{n\in N}$ as a finite or infinite sequence, let $B_0 = \emptyset$ and define successively

$$B_n = B \cap \varphi_{\tilde{e}_n}(X_{t(\tilde{e}_n)})\setminus \bigcup_{j=0}^{n-1} B_j, \quad n = 1, 2, \ldots.$$

Observe that each set $B_n$ is a Borel subset of $\varphi_{\tilde{e}_n}(X_{t(\tilde{e}_n)})$, that the sets $B_n$, $n \in N$, are mutually disjoint and that their union is $B$. Using the injectivity of $\varphi_e$, we then obtain that

$$m(\varphi_e(B)) = \sum_{n\in N} m(\varphi_e(B_n)) = \sum_{n\in N} \int_{B_n} |\varphi_e'|^t \, dm = \int_B |\varphi_e'|^t \, dm.$$

This proves our claim. So we can fix $\tilde{e} \in E_e$ and assume that $B \subseteq \varphi_{\tilde{e}}(X_{t(\tilde{e})})$. Split $B$ into the two sets

$$B_1 := B \cap \pi(E^\infty_{A,e}) \quad \text{and} \quad B_2 := B\setminus\pi(E^\infty_{A,e}).$$

As $B_1 \subseteq \pi(E^\infty_{A,e})$ and $m$ is $t$-conformal, the Borel set $B_1$ satisfies (2). Regarding $B_2$, note that $B_2 \cap J \subseteq (\varphi_{\tilde{e}}(X_{t(\tilde{e})})\setminus\pi(E^\infty_{A,e})) \cap J$. Let $\pi(\tau) \in \varphi_{\tilde{e}}(X_{t(\tilde{e})})\setminus\pi(E^\infty_{A,e})$ for some $\tau \in E^\infty_A$. If $\tau_1 \in E_e$ then $\tau \in E^\infty_{A,e}$. Hence $\tau_1 \notin E_e$. This implies that

$$(\varphi_{\tilde{e}}(X_{t(\tilde{e})})\setminus\pi(E^\infty_{A,e})) \cap J \subseteq \bigcup_{\bar{e}\notin E_e} \varphi_{\tilde{e}}(X_{t(\tilde{e})}) \cap \varphi_{\bar{e}}(X_{t(\bar{e})})$$

and hence

$$\begin{aligned} m(B_2) = m(B_2 \cap J) &\leq m((\varphi_{\tilde{e}}(X_{t(\tilde{e})})\setminus\pi(E^\infty_{A,e})) \cap J) \\ &\leq \sum_{\bar{e}\notin E_e} m(\varphi_{\tilde{e}}(X_{t(\tilde{e})}) \cap \varphi_{\bar{e}}(X_{t(\bar{e})})) \\ &= 0. \end{aligned}$$

Moreover,

$$\varphi_e(B_2) \cap J \subseteq \varphi_e(X_{t(e)}\setminus\pi(E^\infty_{A,e})) \cap J.$$

Let $\pi(\tau) \in \varphi_e(X_{t(e)} \backslash \pi(E^\infty_{A,e}))$ for some $\tau \in E^\infty_A$. If $\tau_1 = e$ then $\sigma\tau \in E^\infty_{A,e}$ and thus $\pi(\sigma\tau) \in \pi(E^\infty_{A,e})$. Consequently,

$$\pi(\tau) = \varphi_{\tau_1}(\pi(\sigma\tau)) \in \varphi_e(\pi(E^\infty_{A,e})).$$

Hence $\pi(\tau) \notin \varphi_e(X_{t(e)} \backslash \pi(E^\infty_{A,e}))$ by the injectivity of $\varphi_e$. So $\tau_1 \neq e$ and $\pi(\tau) \in \varphi_{\tau_1}(X_{t(\tau_1)}) \cap \varphi_e(X_{t(e)})$. This implies that

$$\varphi_e(X_{t(e)} \backslash \pi(E^\infty_{A,e})) \cap J \subseteq \bigcup_{\bar{e} \in E \backslash \{e\}} \varphi_{\bar{e}}(X_{t(\bar{e})}) \cap \varphi_e(X_{t(e)})$$

and hence

$$\begin{aligned} m(\varphi_e(B_2)) &= m(\varphi_e(B_2) \cap J) \\ &\le m\Big( \bigcup_{\bar{e} \in E \backslash \{e\}} \varphi_{\bar{e}}(X_{t(\bar{e})}) \cap \varphi_e(X_{t(e)}) \Big) \\ &\le \sum_{\bar{e} \in E \backslash \{e\}} m(\varphi_{\bar{e}}(X_{t(\bar{e})}) \cap \varphi_e(X_{t(e)})) \\ &= 0. \end{aligned}$$

So $m(\varphi_e(B_2)) = 0 = m(B_2)$ and the Borel set $B_2$ satisfies (2). We conclude that the Borel set $B = B_1 \cup B_2$ satisfies (2) since these latter two disjoints sets do. $\square$

The following lemma reveals the behavior of a conformal measure with respect to any Borel set.

**Lemma 4.3.** *Suppose that $m$ is a $t$-conformal measure on $X$. For all $\omega \in E^*_A$ and all Borel sets $B \subseteq X_{t(\omega)}$, we have*

$$m(\varphi_\omega(B)) = m(\varphi_\omega(B \cap X_\omega)) = m(\varphi_\omega(B) \cap \varphi_\omega(X_\omega)).$$

*Moreover,*

$$m(\varphi_\omega(B)) \le \int_B |\varphi'_\omega|^t \, dm.$$

*Proof.* The $n$th-level sets $\varphi_\omega(X_{t(\omega)})$, $\omega \in E_A^n$, cover $J$ and are $m$-measure-theoretically mutually disjoint. Consequently,

$$1 = \sum_{\omega \in E_A^n} m(\varphi_\omega(X_{t(\omega)})).$$

Consider the sets $\varphi_\omega(X_\omega)$, $\omega \in E_A^n$. Since $X_\omega \subseteq X_{t(\omega)}$, we have

$$m(\varphi_\omega(X_\omega)) \le m(\varphi_\omega(X_{t(\omega)})).$$

Observe that

$$\varphi_\omega(X_\omega) = \bigcup_{e \in E\,:\, A_{\omega_{|\omega|}e}=1} \varphi_\omega(\varphi_e(X_{t(e)})) = \bigcup_{\tau \in E_A^{n+1}\,:\, \tau|_n = \omega} \varphi_\tau(X_{t(\tau)}).$$

That is, the family $\varphi_\omega(X_\omega)$, $\omega \in E_A^n$, is a particular grouping of the $(n+1)$th-level sets. Thereafter,

$$1 = \sum_{\tau \in E_A^{n+1}} m(\varphi_\tau(X_{t(\tau)})) = \sum_{\omega \in E_A^n} m(\varphi_\omega(X_\omega)) \le \sum_{\omega \in E_A^n} m(\varphi_\omega(X_{t(\omega)})) = 1.$$

This means the above inequality is in fact an equality. Since

$$m(\varphi_\omega(X_\omega)) \le m(\varphi_\omega(X_{t(\omega)}))$$

for all $\omega \in E_A^n$, it ensues that

$$m(\varphi_\omega(X_\omega)) = m(\varphi_\omega(X_{t(\omega)}))$$

and

$$m(\varphi_\omega(X_{t(\omega)}) \backslash \varphi_\omega(X_\omega)) = 0$$

for all $\omega \in E_A^n$. Now, let $B \subseteq X_{t(\omega)}$. Then

$$\begin{aligned} m(\varphi_\omega(B \cap X_\omega)) &= m(\varphi_\omega(B) \cap \varphi_\omega(X_\omega)) \\ &= m(\varphi_\omega(B) \cap \varphi_\omega(X_\omega)) + m(\varphi_\omega(B) \cap (\varphi_\omega(X_{t(\omega)}) \backslash \varphi(X_\omega))) \\ &= m(\varphi_\omega(B) \cap \varphi_\omega(X_{t(\omega)})) \\ &= m(\varphi_\omega(B)). \end{aligned}$$

Using Lemma 4.2, we conclude that

$$m(\varphi_\omega(B)) = m(\varphi_\omega(B \cap X_\omega)) = \int_{B \cap X_\omega} |\varphi_\omega'|^t \, dm \le \int_B |\varphi_\omega'|^t \, dm. \qquad \square$$

For CIFSs, a stronger statement can be made still.

**Lemma 4.4.** *Suppose that $S$ is a CIFS and that $m$ is a $t$-conformal measure on $X$. For all $\omega \in E_A^*$ and all Borel sets $B \subseteq X$, we have*

$$m(\varphi_\omega(B)) = m(\varphi_\omega(B \cap X_1)) = m(\varphi_\omega(B) \cap \varphi_\omega(X_1)),$$

*where*

$$X_1 := \bigcup_{e \in E} \varphi_e(X)$$

*is the first stage in the construction of the limit set $J$ of $S$. Moreover,*

$$m(\varphi_\omega(B)) = \int_B |\varphi_\omega'|^t \, dm.$$

*Proof.* A CIFS consists of a single vertex and its matrix comprises only ones, i.e. all transitions are allowed. Therefore $X_{t(\omega)} = X$ and $X_\omega = X_1$ for all $\omega \in E_A^*$. The first part of the statement was thus proved in Lemma 4.3. For the last part, since $m$ is supported on $J$, we deduce that $m(J) = m(X_1) = m(X)$. It follows that $m(B) = m(B \cap X_1)$ for all Borel sets $B \subseteq X$. Consequently,

$$m(\varphi_\omega(B)) = m(\varphi_\omega(B \cap X_\omega)) = \int_{B \cap X_1} |\varphi_\omega'|^t \, dm = \int_B |\varphi_\omega'|^t \, dm. \qquad \square$$

## 5. Finitely irreducible vs. irreducible systems

Next, we present some results about finitely irreducible CGDMSs and discuss which part(s) of these results are valid for irreducible systems. We recall the following definition.

**Definition 5.1.** A CGDMS $S$ is *regular* if there is $t \geq 0$ so that $P_{E,A}(t) = 0$. If a CGDMS is not regular, then it is said to be *irregular*.

We shall see shortly that any irreducible CGDMS that admits a conformal measure is regular. First, we prove the following.

**Lemma 5.2.** *If an irreducible CGDMS admits a conformal measure $m$, then $m(\varphi_e(X_e)) > 0$ and $m(X_e) > 0$ for all $e \in E$ and*

$$M := \min\{m(X_v) \colon v \in V\} > 0.$$

*Proof.* Let $m$ be a $t$-conformal measure. We first claim that there exists $e \in E$ such that $m(X_e) > 0$. Otherwise, $1 = m(J) \le \sum_{e\in E} m(X_e) = 0$, since $J \subseteq \bigcup_{e\in E} X_e$. We now claim that $m(X_f) > 0$ for all $f \in E$. Indeed, since the incidence matrix is irreducible, there is a word $\alpha \in E_A^*$ so that $f\alpha e \in E_A^*$. Then

$$\begin{aligned} m(X_f) &\ge m(\varphi_{\alpha e}(X_{t(e)})) \\ &\ge m(\varphi_{\alpha e}(X_e)) \\ &= \int_{X_e} |\varphi'_{\alpha e}|^t \, dm \\ &\ge K^{-t} \|\varphi'_{\alpha e}\|^t m(X_e) \\ &> 0. \end{aligned}$$

It also follows that

$$m(\varphi_f(X_f)) = \int_{X_f} |\varphi'_f|^t \, dm \ge K^{-t} \|\varphi'_f\|^t m(X_f) > 0.$$

Now, let $v \in V$. According to our standing assumptions, there is an edge $f_v$ such that $t(f_v) = v$. Then $m(X_v) \ge m(X_{f_v}) > 0$. Since there are finitely many vertices, the result ensues. □

**Theorem 5.3.** *The following statements hold.*

(a) *Let $S$ be a CGDMS for which there is a finite set $F \subseteq E$ such that for every $e \in E$ there is $f \in F$ with $A_{ef} = 1$. If $S$ admits a $t$-conformal measure $m$ such that $m(X_f) > 0$ for all $f \in F$, then there is a constant $C \ge 1$ such that*

$$1 \le Z_n(t) \le C, \quad \textit{for all } n \ge 1. \tag{6}$$

*Therefore $P(t) = 0$ and $S$ is regular.*

(b) *Let $S$ be an irreducible CGDMS for which there is a finite set $F \subseteq E$ such that for every $e \in E$ there is $f \in F$ with $A_{ef} = 1$. If $S$ admits a $t$-conformal measure, then $P(t) = 0$ and $S$ is regular.*

*Proof.* (a) For every $n \geq 1$ observe that

$$\begin{aligned}
1 &= m(J) \\
&= m\Big(\bigcup_{\omega \in E_A^n} \varphi_\omega(X_\omega)\Big) \\
&= \sum_{\omega \in E_A^n} m(\varphi_\omega(X_\omega)) \\
&= \sum_{\omega \in E_A^n} \int_{X_\omega} |\varphi'_\omega|^t \, dm \\
&\leq \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^t \\
&= Z_n(t).
\end{aligned}$$

On the other hand, note that

$$\begin{aligned}
1 &= \sum_{\omega \in E_A^n} \int_{X_\omega} |\varphi'_\omega|^t \, dm \\
&\geq \sum_{\omega \in E_A^n} K^{-t} \|\varphi'_\omega\|^t m(X_\omega) \\
&\geq K^{-t} \inf_{f \in F} m(\varphi_f(X_f)) \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^t \\
&\geq K^{-2t} \inf_{f \in F} \|\varphi'_f\|^t m(X_f) \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^t =: C^{-1} Z_n(t).
\end{aligned}$$

We deduce that

$$1 \leq Z_n(t) \leq C, \quad \text{for all } n \geq 1. \tag{7}$$

We conclude that $P(t) = 0$. Hence the system is regular.

(b) This is a direct consequence of (a) and Lemma 5.2. □

**Remark 5.4.** The contrapositive of part (b) affirms that an irregular, irreducible CGDMS which comprises a finite set $F \subseteq E$ such that for every $e \in E$ there is $f \in F$ with $A_{ef} = 1$ cannot admit a conformal measure.

The next theorem states that for finitely irreducible systems, the Hausdorff dimension of the limit set can be approximated by the Hausdorff dimension of the limit sets generated by the finite subsystems. It further reveals that the Hausdorff dimension (abbreviated HD) of the limit set is equal to the parameter where the pressure function turns from positive to negative. This is a generalization of the well-known Bowen's formula and can be found in Theorem 4.2.13 of [9]. We will return to this theme in the next section.

**Theorem 5.5.** *For any finitely irreducible CGDMS,*

$$\begin{aligned}\mathrm{HD}(J_{E,A}) &= \sup\{\mathrm{HD}(J_{F,A})\colon F \subset E, |F| < \infty\}\\ &= \inf\{t \geq 0\colon P_{E,A}(t) < 0\}\\ &\geq \theta_{E,A}.\end{aligned}$$

*Thus, if $P_{E,A}(t) = 0$ for some parameter $t$, then $t$ is the only zero of the pressure function $P_{E,A}$ and $t = \mathrm{HD}(J_{E,A})$.*

## 6. General finite CGDMSs

One may garner many properties of a general CGDMS from the structure of, and the relationships between, its strongly connected components. We will demonstrate this for the Hausdorff dimension and the Hausdorff measure of the limit set of finite CGDMSs.

### 6.1. Strongly connected components

**Definition 6.1.** Let $S_{E,A}$ be a CGDMS. We say that edge $c_1$ *leads to* edge $c_2$, and denote this by $c_1 \rightsquigarrow c_2$, if there is a word $\omega \in E_A^*$ such that $\omega_1 = c_1$ and $\omega_{|\omega|} = c_2$. Equivalently, $c_1 \rightsquigarrow c_2$ if there is a path starting at $c_1$ and ending at $c_2$ on the associated directed graph $G_{E,A}$. A set $C \subseteq E$ of edges is called a *strongly connected component* of $S_{E,A}$ (or of the directed graph $G_{E,A}$) if $c_1 \rightsquigarrow c_2$ for any $c_1, c_2 \in C$ and $C$ is a largest set, in the sense of inclusion, having this property.

Observe that any strongly connected component gives rise to an irreducible (sub)system.

**Definition 6.2.** We say that a strongly connected component $C_1$ *leads to* a strongly connected component $C_2$, and denote this by $C_1 \rightsquigarrow C_2$, if there is some edge in $C_1$ which leads to some edge in $C_2$.

Note that if a component $C_1$ leads to another component $C_2$, then $C_2$ cannot lead to $C_1$.

**Definition 6.3.** An edge is called *isolated* if it does not belong to any strongly connected component.

**6.2. Pressure and Hausdorff dimension of the limit set.** We first show that the pressure of any finite CGDMS is the maximum of the pressure of its component subsystems.

**Theorem 6.4.** *The pressure function of any finite CGDMS $S_{E,A}$ satisfies*

$$P_{E,A}(t) = \max\{P_{C,A}(t) \colon C \text{ is a strongly connected component of } S_{E,A}\}$$

*for all $t \geq 0$.*

*Proof.* Let $C_1, C_2, \dots, C_k$ be the strongly connected components of $S_{E,A}$ and fix $1 \leq j \leq k$. Let $b$ be the number of isolated edges. If $\tau \in E_A^*$, then each isolated edge can appear at most once in $\tau$. So $\tau$ can be written as a concatenation of subwords from distinct components and no more than $b$ isolated edges. Thus any $A$-admissible word $\tau$ that contains at least one letter (edge) from $C_j$ can be uniquely written as $\beta\alpha_j\gamma$, where $\alpha_j \in (C_j)_A^*$ and $\beta, \gamma \in (E \setminus C_j)_A^*$. (Note that $\beta$ and/or $\gamma$ may be the empty word, while $\alpha_j$ is the longest subword of $\tau$ that has letters from $C_j$ only.) For each $\tau \in E_A^*$ and $1 \leq j \leq k$, let $\alpha_j(\tau)$ be the longest subword of $\tau$ in $(C_j)_A^*$. For any $t \geq 0$, we have $\|\varphi'_\tau\|^t \leq \prod_{j=1}^k \|\varphi'_{\alpha_j(\tau)}\|^t$. Since the map $\tau \mapsto (\alpha_1(\tau), \dots, \alpha_k(\tau))$ is bounded-to-one, say at most $T$-to-1 for some $T \geq 0$, the following inequality holds for all $u \in \mathbb{R}$:

$$\sum_{\omega \in E_A^*} \|\varphi'_\omega\|^t e^{-u|\omega|} \leq T \max\{1, e^{-ub}\} \prod_{j=1}^k \sum_{\omega_j \in (C_j)_A^*} \|\varphi'_{\omega_j}\|^t e^{-u|\omega_j|}.$$

Using Proposition 3.2(f), we deduce that $P_{E,A}(t) \leq \max_{1 \leq j \leq k}\{P_{C_j,A}(t)\}$. The opposite inequality is obvious. $\square$

It follows immediately that the Hausdorff dimension of the limit set of a finite CGDMS is the maximum of the dimension of its component subsystems.

**Corollary 6.5.** *Let $S_{E,A}$ be a finite CGDMS. Then*

$$\mathrm{HD}(J_{E,A}) = \max\{\mathrm{HD}(J_{C,A})\colon C \text{ is a strongly connected component of } S_{E,A}\}.$$

*Proof.* This is a consequence of Proposition 3.2(d,e) and Theorems 5.5 and 6.4. □

We also obtain a generalization of Bowen's formula to all finite CGDMSs.

**Corollary 6.6.** *For any finite system, $P_{E,A}(t) = 0$ if and only if $t = \mathrm{HD}(J_{E,A})$.*

*Proof.* This is a consequence of Proposition 3.2(d,e) and Theorems 5.5 and 6.4. □

Theorem 6.4 and Corollary 6.5 suggest making the following definition.

**Definition 6.7.** Let $h = \mathrm{HD}(J_{E,A})$, i.e. let $h$ be the unique parameter $t$ such that $P_{E,A}(t) = 0$. A strongly connected component $C$ is called *Hausdorff-maximal* if $P_{C,A}(h) = 0$. In other words, $C$ is *Hausdorff-maximal* if and only if $\mathrm{HD}(J_{C,A}) = \mathrm{HD}(J_{E,A})$.

Corollary 6.5 can thus be restated as affirming that any finite CGDMS has a Hausdorff-maximal component.

**6.3. Hausdorff measure of the limit set.** Our next goal is to study the restriction of the Hausdorff measure to the limit set of a finite system. It it well known that this measure is positive and finite for any finite irreducible system. We will show that this measure is positive and $\sigma$-finite for any finite system. Moreover, we will find a characterization of the finiteness of this measure.

In general, the $h$-dimensional Hausdorff measure restricted to the limit set of any finite system is positive and $\sigma$-finite.

**Proposition 6.8.** *Let $S_{E,A}$ be a finite CGDMS and let $h = \mathrm{HD}(J_{E,A})$. Then $\mathcal{H}^h|_{J_{E,A}}$ is positive and $\sigma$-finite.*

*Proof.* It is well-known that the limit set of any finite irreducible system has a positive and finite $h$-dimensional Hausdorff measure (for instance, see Theorems 4.5.1 and 4.5.3 in [9]). We deduce from this that $\mathcal{H}^h(J_{E,A}) \geq \mathcal{H}^h(J_{C,A}) > 0$ for all Hausdorff-maximal components $C$ of the finite system $S_{E,A}$.

To prove the $\sigma$-finiteness, we shall need further notation. For every $\omega \in E_A^*$, every $F \subseteq E_A^*$ and every $G \subseteq E_A^\infty$, let

$$\omega F = \{\omega\tau \in E_A^* \colon \tau \in F\} \quad \text{and} \quad \omega G = \{\omega\gamma \in E_A^\infty \colon \gamma \in G\}.$$

Let $I$ denote the set of isolated letters for the system $S_{E,A}$. Let $b = |I| < \infty$ and observe that $I_A^* = \bigcup_{j=0}^b I_A^j$. Let $C_1, C_2, \ldots, C_k$ be the Hausdorff-maximal components of $S_{E,A}$. Note that for every $e \notin I$, there exists a unique $1 \le k(e) \le k$ such that $e \in C_{k(e)}$. For every $\omega \in E_A^*$, let

$$K_\omega = \begin{cases} \omega(C_{k(\omega_{|\omega|})})_A^\infty & \text{if } \omega_{|\omega|} \notin I, \\ \displaystyle\bigcup_{\chi \in \omega I_A^*} \bigcup_{\zeta \in \chi(E \setminus I)} K_\zeta & \text{if } \omega_{|\omega|} \in I. \end{cases}$$

The sequence of compact sets $(K_n)_{n \ge 1}$, where $K_n = \bigcup_{\omega \in E_A^n} K_\omega$, clearly constitutes an exhaustion of $E_A^\infty$. Therefore the sequence of compact sets $(\pi(K_n))_{n \ge 1}$ forms an exhaustion of $J_{E,A}$. Moreover, for every $n \ge 1$,

$$\mathcal{H}^h(\pi(K_n)) \le \sum_{\omega \in E_A^n} \mathcal{H}^h(\pi(K_\omega)),$$

where, if $\omega_{|\omega|} \notin I$ then

$$\begin{aligned} \mathcal{H}^h(\pi(K_\omega)) &= \mathcal{H}^h(\pi(\omega(C_{k(\omega_{|\omega|})})_A^\infty)) \\ &= \mathcal{H}^h(\varphi_\omega(J_{C_{k(\omega_{|\omega|})},A})) \\ &\le \|\varphi'_\omega\|^h \mathcal{H}^h(J_{C_{k(\omega_{|\omega|})},A}) \\ &\le \|\varphi'_\omega\|^h \max_{1 \le j \le k} \mathcal{H}^h(J_{C_j,A}) \\ &< \infty \end{aligned}$$

whereas, if $\omega_{|\omega|} \in I$ then

$$\begin{aligned} \mathcal{H}^h(\pi(K_\omega)) &\le \sum_{\chi \in \omega I_A^*} \sum_{\zeta \in \chi(E \setminus I)} \mathcal{H}^h(\pi(K_\zeta)) \\ &\le \sum_{\chi \in \omega I_A^*} \sum_{\zeta \in \chi(E \setminus I)} \|\varphi'_\zeta\|^h \max_{1 \le j \le k} \mathcal{H}^h(J_{C_j,A}) \\ &< \infty. \end{aligned}$$

Consequently, $\mathcal{H}^h(\pi(K_n)) < \infty$ for every $n \ge 1$ and the $h$-dimensional Hausdorff measure is $\sigma$-finite. □

To identify a characterization of the finiteness of the Hausdorff measure, we need some intermediate results.

The first of these results pertains to the words that do not have any letter from a Hausdorff-maximal component.

**Lemma 6.9.** *Let $S_{E,A}$ be a finite CGDMS, $h = \mathrm{HD}(J_{E,A})$, and $C_1, C_2, \ldots, C_k$ be the Hausdorff-maximal components of $S_{E,A}$. Let $C_0 = E \setminus \bigcup_{1 \le i \le k} C_i$. Then there exists $k_0 > 0$ and $0 < a < 1$ so that*

$$Z_{n,C_0,A}(h) = \sum_{\omega \in (C_0)^n_A} \|\varphi'_\omega\|^h \le k_0 a^n, \quad \text{for all } n \ge 1.$$

*Consequently, there exists $M_0 > 0$ such that*

$$\sum_{\omega \in (C_0)^*_A} \|\varphi'_\omega\|^h \le M_0.$$

*Proof.* If $C_0$ does not contain any strongly connected component of $G_{E,A}$, then it consists solely of the isolated letters and thus $Z_{n,C_0,A}(t) = 0$ for all $n$ sufficiently large. If $C_0$ contains strongly connected components of $G_{E,A}$, then none of those is Hausdorff-maximal. Therefore $P_{C_0,A}(h) < 0$ according to Theorem 6.4 (applied with $E$ replaced by $C_0$). Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{n,C_0,A}(h) = P_{C_0,A}(h) < 0$$

and the lemma follows immediately. □

We now turn our attention to those words which have a letter from a Hausdorff-maximal component, assuming that this latter does not lead to any other Hausdorff-maximal component.

**Lemma 6.10.** *Let $S_{E,A}$ be a finite CGDMS, $h = \mathrm{HD}(J_{E,A})$, and $C_1, C_2, \ldots, C_k$ be the Hausdorff-maximal components of $S_{E,A}$. Suppose that $C_i \not\to C_j$ for every $1 \le i \ne j \le k$. For each $1 \le i \le k$, let $C_i^{**}$ be the set of all finite $A$-admissible words with at least one letter from $C_i$. Then there exists $M_i > 0$ such that*

$$\sum_{\omega \in C_i^{**} \cap E^n_A} \|\varphi'_\omega\|^h \le M_i, \quad \text{for all } n \ge 1.$$

*Proof.* Let

$$C_0 = E \setminus \bigcup_{1 \le i \le k} C_i$$

and $n \ge 1$. Fix $1 \le i \le k$. Since the Hausdorff-maximal component $C_i$ generates a finite irreducible CGDMS, it admits an $h$-conformal measure and by Lemma 5.2 and Theorem 5.3(a), there exists, there exists $N_i > 0$ such that

$$\sum_{\omega \in (C_i)_A^n} \|\varphi'_\omega\|^h \le N_i, \quad \text{for all } n \ge 1.$$

For each $j \ge 1$, let $C_{i,j}^{**}$ be the set of all words in $C_i^{**}$ containing exactly $j$ letters from $C_i$. Every word of length $n$ in $C_{i,j}^{**}$ is of the form $\alpha\omega\beta$ with $\alpha, \beta \in C_0^*$ and $\omega \in (C_i)_A^j$, where $|\alpha| + |\beta| = n - j$ (note that $\alpha$ and/or $\beta$ may be the empty word). Based on this observation and on Lemma 6.9, we obtain

$$\sum_{\omega \in C_{i,j}^{**} \cap E_A^n} \|\varphi'_\omega\|^h \le (n - j + 1) k_0^2 a^{n-j} N_i.$$

Therefore

$$\begin{aligned}
\sum_{\omega \in C_i^{**} \cap E_A^n} \|\varphi'_\omega\|^h &= \sum_{j=1}^{n} \sum_{\omega \in C_{i,j}^{**} \cap E_A^n} \|\varphi'_\omega\|^h \\
&\le k_0^2 N_i \sum_{j=1}^{n} (n - j + 1) a^{n-j} \\
&\le k_0^2 N_i \sum_{l=0}^{\infty} (l + 1) a^l \\
&\le \frac{k_0^2 N_i}{(1-a)^2}.
\end{aligned}$$

Set

$$M_i = \frac{N_i k_0^2}{(1-a)^2}.$$ □

We now demonstrate that the partition functions of a finite system, at parameter $h$, are uniformly bounded from above if the Hausdorff-maximal components of the system do not communicate.

**Proposition 6.11.** *Let $S_{E,A}$ be a finite CGDMS and let $h = \mathrm{HD}(J_{E,A})$. Let $C_1, C_2, \dots, C_k$ be the Hausdorff-maximal components of $S_{E,A}$. If $C_i \not\to C_j$ for all $1 \le i \ne j \le k$, then there exists a constant $M > 0$ such that*

$$Z_{n,E,A}(h) \le M, \quad \textit{for all } n \ge 1.$$

*Proof.* Let $n \geq 1$ and

$$C_0 = E \setminus \bigcup_{1 \leq i \leq k} C_i.$$

Since the Hausdorff-maximal components are pairwise non-communicating, we have

$$\begin{aligned} Z_{n,E,A}(h) &= \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^h \\ &= \sum_{\omega \in (C_0)_A^n} \|\varphi'_\omega\|^h + \sum_{i=1}^{k} \sum_{\omega \in C_i^{**} \cap E_A^n} \|\varphi'_\omega\|^h \\ &\leq \sum_{i=0}^{k} M_i =: M, \end{aligned}$$

where the constants $M_i$ originate from Lemmas 6.9 and 6.10. □

We can now establish the finiteness of the $h$-dimensional Hausdorff measure of the limit set of any finite system whose Hausdorff-maximal components do not communicate.

**Proposition 6.12.** *Let $S_{E,A}$ be a finite CGDMS and let $h = \mathrm{HD}(J_{E,A})$. Let $C_1, C_2, \ldots, C_k$ be the Hausdorff-maximal components of the system. If $C_i \not\rightsquigarrow C_j$ for all $1 \leq i \neq j \leq k$, then $0 < \mathcal{H}^h(J_{E,A}) < \infty$.*

*Proof.* By Proposition 6.8, we know that $0 < \mathcal{H}^h(J_{E,A})$. On the other hand, for every $n \geq 1$ the $n$th-level sets $\{\varphi_\omega(X_{t(\omega)})\}_{\omega \in E_A^n}$ form a cover of $J_{E,A}$ whose mesh converges to 0 as $n \to \infty$. Then

$$\sum_{\omega \in E_A^n} \mathrm{diam}(\varphi_\omega(X_{t(\omega)}))^h \leq D^h \sum_{\omega \in E_A^n} \|\varphi'_\omega\|^h = D^h Z_{n,E,A}(h) \leq D^h M,$$

where $M$ is the constant in Proposition 6.11 and $D$ is a constant coming from (4.20) on page 73 of [9]. Thus, $\mathcal{H}^h(J_{E,A}) < \infty$. □

Next, we demonstrate that the partition functions of a finite system, at parameter $h$, are not bounded from above if some Hausdorff-maximal components of the system communicate.

**Proposition 6.13.** *Let $S_{E,A}$ be a finite CGDMS and let $h = \mathrm{HD}(J_{E,A})$. Let $C_1, C_2, \ldots, C_k$ be the Hausdorff-maximal components of the system. If there exists $1 \leq i \neq j \leq k$ so that $C_i \rightsquigarrow C_j$, then*

$$\sup_{n \geq 1} Z_{n,E,A}(h) = \infty.$$

*Proof.* Without losing generality, let us assume that $C_1 = \{e_{1,1}, e_{1,2}, \dots, e_{1,p}\}$ and $C_2 = \{e_{2,1}, e_{2,2}, \dots, e_{2,q}\}$ are Hausdorff-maximal components such that $C_1 \rightsquigarrow C_2$. Reindexing if necessary, we may even assume that $e_{1,p} \rightsquigarrow e_{2,1}$ in such a way that there exists $\omega_0 \in [E \setminus (C_1 \cup C_2)]_A^*$ with $e_{1,p}\omega_0 e_{2,1} \in E_A^*$. For every $n \geq 1$ and $1 \leq l \leq p$, let $C_{1,A}^{l,n}$ be the set of all $A$-admissible words of length $n$ with letters from $C_1$ exclusively that terminate with the letter $e_{1,l}$. Define

$$Z_{n,C_1,A,l}(h) = \sum_{\omega \in C_{1,A}^{l,n}} \|\varphi'_\omega\|^h.$$

We shall prove that for every $1 \leq l \leq p$,

$$\inf_{n \geq 1} Z_{n,C_1,A,l}(h) > 0.$$

First we show that there exists $1 \leq l_0 \leq p$ so that $\inf_{n\geq 1} Z_{n,C_1,A,l_0}(h) > 0$. According to (6), there exists $M \geq 1$ such that

$$1 \leq Z_{n,C_1,A}(h) = \sum_{l=1}^{p} Z_{n,C_1,A,l}(h) \leq M, \quad \text{for all } n \geq 1.$$

Therefore there exists $1 \leq l_0 \leq p$ and an increasing subsequence $(n_j)_{j\geq 1}$ of the sequence of natural numbers $(n)_{n\geq 1}$ such that

$$\inf_{j \geq 1} Z_{n_j,C_1,A,l_0}(h) \geq \frac{1}{p}.$$

For every $n \geq 1$ and $j \geq 1$ such that $n_j > n$ we have

$$\frac{1}{p} \leq Z_{n_j,C_1,A,l_0}(h) \leq Z_{n_j-n,C_1,A}(h) Z_{n,C_1,A,l_0}(h) \leq M Z_{n,C_1,A,l_0}(h).$$

Consequently,

$$\inf_{n \geq 1} Z_{n,C_1,A,l_0}(h) \geq \frac{1}{Mp}.$$

Let $1 \leq l \leq p$. Since the matrix $A$ restricted to $C_1$ is irreducible, there exists $\gamma_l$ such that $e_{1,l_0}\gamma_l e_{1,l} \in (C_1)_A^*$. For every $n \geq 1$ we then obtain that

$$\begin{aligned} Z_{n+|\gamma_l|+1,C_1,A,l}(h) &\geq \sum_{\omega \in C_{1,A}^{l_0,n}} \|\varphi'_{\omega\gamma_l e_{1,l}}\|^h \\ &\geq K^{-h} \|\varphi'_{\gamma_l e_{1,l}}\|^h Z_{n,C_1,A,l_0}(h) \\ &\geq \frac{K^{-h}\|\varphi'_{\gamma_l e_{1,l}}\|^h}{Mp}. \end{aligned}$$

Since this is true for all $1 \le l \le p$ and $n \ge 1$, we deduce that

$$\mu_1 := \inf_{n\ge 1, 1\le l\le p} Z_{n,C_1,A,l}(h) > 0.$$

Similarly, for every $n \ge 1$ and $1 \le m \le q$, let $C_{2,A}^{m,n}$ be the set of all $A$-admissible words of length $n$ with letters from $C_2$ exclusively which begin with the letter $e_{2,m}$. Define

$$Z_{n,C_2,A,m}(h) = \sum_{\omega\in C_{2,A}^{m,n}} \|\varphi'_\omega\|^h.$$

By a similar argument as above,

$$\mu_2 := \inf_{n\ge 1, 1\le m\le q} Z_{n,C_2,A,m}(h) > 0.$$

Thus,

$$\begin{aligned}
Z_{n+|\omega_0|,E,A}(h) &= \sum_{\omega\in E_A^{n+|\omega_0|}} \|\varphi'_\omega\|^h \\
&\ge \sum_{1\le k<n} \sum_{\alpha\in C_{1,A}^{p,k}} \sum_{\beta\in C_{2,A}^{1,n-k}} \|\varphi'_{\alpha\omega_0\beta}\|^h \\
&\ge K^{-2h}\|\varphi'_{\omega_0}\|^h \sum_{1\le k<n} \sum_{\alpha\in C_{1,A}^{p,k}} \|\varphi'_\alpha\|^h \sum_{\beta\in C_{2,A}^{1,n-k}} \|\varphi'_\beta\|^h \\
&= K^{-2h}\|\varphi'_{\omega_0}\|^h \sum_{1\le k<n} Z_{k,C_1,A,p}(h)\, Z_{n-k,C_2,A,1}(h) \\
&\ge K^{-2h}\|\varphi'_{\omega_0}\|^h \mu_1\mu_2(n-1),
\end{aligned}$$

where $K$ is a constant of bounded distortion. The result ensues. □

We can similarly prove the infiniteness of the $h$-dimensional Hausdorff measure of the limit set of any finite system whose Hausdorff-maximal components communicate.

**Proposition 6.14.** *Let $S_{E,A}$ be a finite CGDMS and let $h = \mathrm{HD}(J_{E,A})$. Let $C_1, C_2, \dots, C_k$ be the Hausdorff-maximal components of $S_{E,A}$. If there exists $1 \le i \ne j \le k$ so that $C_i \rightsquigarrow C_j$, then $\mathcal{H}^h(J_{E,A}) = \infty$.*

*Proof.* We shall use the notation, ideas and facts from the proof of Proposition 6.13. Without loss of generality, assume that

$$C_1 = \{e_{1,1}, e_{1,2}, \ldots, e_{1,p}\}$$

and

$$C_2 = \{e_{2,1}, e_{2,2}, \ldots, e_{2,q}\}$$

are Hausdorff-maximal components such that $C_1 \rightsquigarrow C_2$. Reindexing if necessary, we may even assume that $e_{1,p}$ leads to $e_{2,1}$ in such a way that there exists $\omega_0 \in [E \setminus (C_1 \cup C_2)]_A^*$ with $e_{1,p}\omega_0 e_{2,1} \in E_A^*$. For every $1 \le j \le q$, let

$$J_{C_2,A}^{(j)} = \{\pi(\omega) \colon \omega \in (C_2)_A^\infty,\ \omega_1 = e_{2,j}\}.$$

Since

$$J_{C_2,A} = \bigcup_{1 \le j \le q} J_{C_2,A}^{(j)}$$

and

$$\mathcal{H}^h(J_{C_2,A}) > 0,$$

there exists $1 \le j_0 \le q$ such that $\mathcal{H}^h(J_{C_2,A}^{(j_0)}) > 0$. Since $C_2$ is irreducible, for every $e \in C_2$ choose $\gamma_e$ such that $e\gamma_e e_{2,j_0} \in (C_2)_A^*$. For every word $\omega$ whose last letter is in $C_2$, define

$$\gamma_\omega := \gamma_{\omega_{|\omega|}}.$$

For every $n \ge 1$ and $1 \le k < n$, let

$$C_{n,k} = \{\alpha\omega_0\beta\gamma_\beta \in E_A^* \colon \alpha \in C_{1,A}^{p,k},\ \beta \in C_{2,A}^{1,n-k}\}$$

and

$$C_n = \bigcup_{1 \le k < n} C_{n,k}.$$

Observe that any two words in $C_n$ are incomparable if and only if they are distinct. Of course, for every $n \ge 1$,

$$J_{E,A} \supseteq \bigcup_{\tau \in C_n} \varphi_\tau(J_{C_2,A}^{(j_0)}) = \bigcup_{1 \le k < n} \ \bigcup_{\alpha \in C_{1,A}^{p,k}} \ \bigcup_{\beta \in C_{2,A}^{1,n-k}} \varphi_{\alpha\omega_0\beta\gamma_\beta}(J_{C_2,A}^{(j_0)}).$$

Since $C_1$ is a Hausdorff-maximal strongly connected component, $\mathcal{H}^h$ and the unique $h$-conformal measure $m_{C_1}$ for the (finitely) irreducible subsystem $S_{C_1,A}$ are equivalent when viewed as measures supported on $J_{C_1,A}$ according to Theorems 4.5.1 and 4.5.2 in [9]. Thus, for every $1 \le i \ne j \le p$ we know that

$$\mathcal{H}^h(\varphi_{e_{1,i}}(X_{t(e_{1,i})}) \cap \varphi_{e_{1,j}}(X_{t(e_{1,j})})) = 0.$$

Similarly, for every $1 \leq i \neq j \leq q$ we know that

$$\mathcal{H}^h(\varphi_{e_{2,i}}(X_{t(e_{2,i})}) \cap \varphi_{e_{2,j}}(X_{t(e_{2,j})})) = 0.$$

Thus,

$$\mathcal{H}^h(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)})) = 0$$

for every incomparable (i.e. distinct) words $\omega, \tau \in C_n$. Therefore

$$\begin{aligned}
&\mathcal{H}^h(J_{E,A}) \\
&\quad \geq \sum_{1\leq k<n} \sum_{\alpha \in C_{1,A}^{p,k}} \sum_{\beta \in C_{2,A}^{1,n-k}} \mathcal{H}^h(\varphi_{\alpha\omega_0\beta\gamma_\beta}(J_{C_2,A}^{(j_0)})) \\
&\quad \geq \sum_{1\leq k<n} \sum_{\alpha \in C_{1,A}^{p,k}} \sum_{\beta \in C_{2,A}^{1,n-k}} K^{-h}\|\varphi'_{\alpha\omega_0\beta\gamma_\beta}\|^h \mathcal{H}^h(J_{C_2,A}^{(j_0)}) \\
&\quad \geq K^{-4h}\|\varphi'_{\omega_0}\|^h \sum_{1\leq k<n} \sum_{\alpha \in C_{1,A}^{p,k}} \|\varphi'_\alpha\|^h \sum_{\beta \in C_{2,A}^{1,n-k}} \|\varphi'_\beta\|^h \|\varphi'_{\gamma_\beta}\|^h \mathcal{H}^h(J_{C_2,A}^{(j_0)}) \\
&\quad \geq K^{-4h}\|\varphi'_{\omega_0}\|^h \sum_{1\leq k<n} Z_{k,C_1,A,p}(h)\, Z_{n-k,C_2,A,1}(h) \min_{e\in C_2} \|\varphi'_{\gamma_e}\|^h \mathcal{H}^h(J_{C_2,A}^{(j_0)}) \\
&\quad \geq K^{-4h}\|\varphi'_{\omega_0}\|^h (n-1)\mu_1\mu_2 \min_{e\in C_2} \|\varphi'_{\gamma_e}\|^h \mathcal{H}^h(J_{C_2,A}^{(j_0)}).
\end{aligned}$$

In conclusion, $\mathcal{H}^h(J_{E,A}) = \infty$. □

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