Some properties of energy measures
on Sierpinski gasket type fractals

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*Dedicated to Professor Shinzo Watanabe
on the occasion of his 80th birthday*

Abstract. We confirm, in a more general framework, a part of the conjecture posed by R. Bell, C.-W. Ho, and R. S. Strichartz [Energy measures of harmonic functions on the Sierpiński gasket, *Indiana Univ. Math. J.* **63** (2014), 831–868] on the distribution of energy measures for the canonical Dirichlet form on the two-dimensional standard Sierpinski gasket.

1. Introduction

Energy measures associated with strong local regular Dirichlet forms describe certain local structures of Dirichlet forms. For the standard energy form on a Euclidean space, the energy measure of a function f is given explicitly by $|\nabla f(x)|^2 \, dx$. On the other hand, for canonical Dirichlet forms on fractals, energy measures do not usually have simple expressions and it seems a difficult problem to know how they are distributed in the state space. For example, energy measures are singular with respect to self-similar measures for self-similar Dirichlet forms on most self-similar fractals [7, 12, 14, 2]. Recently, Bell, Ho, and Strichartz [1] studied several properties of energy measures associated with the canonical Dirichlet form on the two-dimensional standard Sierpinski gasket.

In particular, they introduced non-negative coefficients $\{b_j^{(w)}\}_{j=1}^3$ for describing the distribution of the energy measures of harmonic functions on the cells corresponding to each word w (see Section 2 for details), and posed conjectures about properties of the limiting behavior of $\{b_j^{(w)}\}_{j=1}^3$ as w converges to an infinite sequence of words ([1, Conjectures 7.1 and 7.2]). In this paper, we confirm a part of the conjecture in a slightly generalized setting. The proof suggests that the conjectured properties depend strongly on the fractals under consideration having three vertices. Our approach is more straightforward than the original one [1]: we use only primitive linear operators for the analysis and utilize some results on limits of random matrices (cf. [3, 8, 9, 10, 14]).

The remainder of this paper is organized as follows. In Section 2, we provide a framework for Dirichlet forms on self-similar sets and give some preliminary results. The conjectures in [1] are also stated. In Section 3, we prove the main theorem. Section 4 provides some concluding remarks and discussions.

2. Framework and preliminaries

We first introduce a class of self-similar sets and the Dirichlet forms defined on them, following [13]. Let K be a compact, connected, and metrizable space. Let $\{\psi_i\}_{i\in S}$ be a family of continuous injective mappings from K to itself having a finite index set S with $\#S \geq 2$. Denote $S^{\mathbb{N}}$ by Σ and each element of Σ by $\omega_1\omega_2\omega_3\ldots$ with $\omega_n \in S$ for every $n \in \mathbb{N}$. For $i \in S$, a shift operator $\sigma_i\colon \Sigma \to \Sigma$ is defined by $\sigma_i(\omega_1\omega_2\ldots) = i\omega_1\omega_2\ldots$. We assume that there exists a continuous surjective map $\pi\colon \Sigma \to K$ such that $\psi_i \circ \pi = \pi \circ \sigma_i$ for each $i \in S$. The triplet $(K, S, \{\psi_i\}_{i\in S})$ is then called a self-similar structure. Define $W_m = S^m$ for $m \in \mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and denote $\bigcup_{m\in\mathbb{Z}_+} W_m$ by W_*. For $w = w_1w_2\ldots w_m \in W_m \subset W_*$, we define $\psi_w = \psi_{w_1} \circ \psi_{w_2} \circ \cdots \circ \psi_{w_m}$ and $K_w = \psi_w(K)$. For $w = w_1\ldots w_m \in W_m$ and $w' = w'_1\ldots w'_n \in W_n$, ww' denotes $w_1\ldots w_m w'_1\ldots w'_n \in W_{m+n}$. We set

$$\mathcal{P} = \bigcup_{m=1}^{\infty} \sigma^m\Big(\pi^{-1}\Big(\bigcup_{i,j\in S, i\neq j}(K_i \cap K_j)\Big)\Big) \quad \text{and} \quad V_0 = \pi(\mathcal{P}),$$

where $\sigma^m\colon \Sigma \to \Sigma$ is defined by $\sigma^m(\omega_1\omega_2\ldots) = \omega_{m+1}\omega_{m+2}\ldots$. We assume that $\mathcal{P}$ is a finite set. In such a case, $(K, S, \{\psi_i\}_{i\in S})$ is called post-critically finite. Then, from [13, Lemma 1.3.14], each ψ_i has a unique fixed point $\pi(iii\ldots)$.

For a finite set V, let $l(V)$ denote the space of all real-valued functions on V. A canonical inner product $(\cdot, \cdot)_{l(V)}$ on $l(V)$ is defined by

$$(u, v)_{l(V)} = \sum_{p \in V} u(p)v(p).$$

The associated norm is denoted by $| \cdot |_{l(V)}$. Let $D = (D_{pq})_{p,q \in V_0}$ be a symmetric linear operator on $l(V_0)$ with the following properties:

(D1) D is non-positive definite;

(D2) $Du = 0$ if and only if u is constant on V_0;

(D3) $D_{pq} \geq 0$ for all $p, q \in V_0$ with $p \neq q$.

Define
$$\mathcal{E}^{(0)}(u, v) = (-Du, v)_{l(V_0)} \quad \text{for } u, v \in l(V_0).$$

For $m \in \mathbb{N}$, let
$$V_m = \bigcup_{w \in W_m} \psi_w(V_0).$$

For $\mathbf{r} = \{r_i\}_{i \in S}$ with $r_i > 0$ for all $i \in S$, define a bilinear form $\mathcal{E}^{(m)}$ on $l(V_m)$ by

$$\mathcal{E}^{(m)}(u, v) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}^{(0)}(u \circ \psi_w|_{V_0}, v \circ \psi_w|_{V_0}), \qquad u, v \in l(V_m),$$

where $r_w = \prod_{i=1}^{m} r_{w_i}$ for $w = w_1 w_2 \ldots w_m \in W_m$. The pair $(D, \mathbf{r})$ is called a *harmonic structure* if

$$\mathcal{E}^{(0)}(u, u) = \inf\{\mathcal{E}^{(1)}(v, v) \mid v \in l(V_1), \, v|_{V_0} = u\} \quad \text{for every } u \in l(V_0).$$

Then, for $m \geq 0$, the identity

$$\mathcal{E}^{(m)}(u, u) = \inf\{\mathcal{E}^{(m+1)}(v, v) \mid v \in l(V_{m+1}), \, v|_{V_m} = u\}$$

holds for every $u \in l(V_m)$. If, moreover, $0 < r_i < 1$ for all $i \in S$, the harmonic structure is called regular. Henceforth, we assume that a regular harmonic structure $(D, \mathbf{r})$ is given. Let μ be a finite Borel measure on K with full support. Then

$$\mathcal{E}(u, v) = \lim_{m \to \infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}), \quad u, v \in \mathcal{F}$$

with

$$\mathcal{F} = \{u \in C(K) \subset L^2(K, \mu) \mid \sup_{m \in \mathbb{Z}_+} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\}$$

defines a bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ associated with $(D, \boldsymbol{r})$. Under a mild condition on μ, $(\mathcal{E}, \mathcal{F})$ becomes a strong local regular Dirichlet form on $L^2(K, \mu)$ (self-similar measures are adequate, for example; see [13, Theorem 3.4.6]). We will assume such a μ is chosen; the choice is not important for subsequent arguments. We always take continuous functions as μ-versions of elements of $\mathcal{F}$.

Example 2.1. Typical examples are two-dimensional level l Sierpinski gaskets SG_l for $l \geq 2$, which are realized by compact subsets of $\mathbb{R}^2$ (see Figure 1).

They are constructed by $l(l+1)/2$ contraction mappings ψ_i defined as $\psi_i(z) = l^{-1}z + b_{l,i}$ with suitable $b_{l,i} \in \mathbb{R}^2$ and characterized by nonempty compact subsets satisfying $\mathrm{SG}_l = \bigcup_{i=1}^{l(l+1)/2} \psi_i(\mathrm{SG}_l)$. We call SG_2 the two-dimensional standard Sierpinski gasket. The set V_0 consists of the three vertices p_1, p_2, and p_3 of the largest triangle in SG_l. We renumber $\{\psi_i\}_{i=1}^{l(l+1)/2}$ so that ψ_i denotes the contraction mapping whose fixed point is p_i for $i = 1, 2, 3$ and define the matrix D by

$$D = (D_{p_i p_j})_{i,j=1}^3 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$
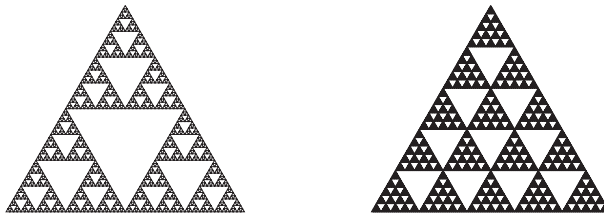

Figure 1. Two-dimensional level l Sierpinski gaskets SG_l ($l = 2, 5$).

Then, there exists a unique number r such that $0 < r < 1$ and $(D, \boldsymbol{r})$ is a regular harmonic structure with $\boldsymbol{r} = \{r, \dots, r\}$. This example satisfies conditions (A1), (A3), and (A4) that are stated later. If we take the normalized Hausdorff measure as μ, the diffusion process associated with the Dirichlet form as stated above is regarded as the Brownian motion on SG_l.

We now resume our discussion of the general situation. For each $x \in l(V_0)$, there exists a unique function $h \in \mathcal{F}$ such that $h|_{V_0} = x$ and h attains the infimum of $\{\mathcal{E}(g, g) \mid g \in \mathcal{F} \text{ and } g|_{V_0} = x\}$. Such a function h is called a harmonic function and the totality of h is denoted by $\mathcal{H}$. The map $\iota\colon l(V_0) \ni x \mapsto h \in \mathcal{H}$ is linear, so we can identify $\mathcal{H}$ with $l(V_0)$ by this map. For $i \in S$, we define a linear operator $A_i\colon l(V_0) \to l(V_0)$ by $(A_i x)(p) = (\iota(x))(\psi_i(p))$ for $x \in l(V_0)$ and $p \in V_0$.

For $w = w_1 w_2 \ldots w_m \in W_m$, we set $A_w = A_{w_m} A_{w_{m-1}} \ldots A_{w_1}$. With an abuse of notation, D and A_w can also be considered as linear maps from $\mathcal{H}$ to $l(V_0)$ by identifying $\mathcal{H}$ with $l(V_0)$. Let $\mathbf{1} \in l(V_0)$ denote a constant function on V_0 with value 1. Set

$$\tilde{l}(V_0) = \{x \in l(V_0) \mid (x, \mathbf{1})_{l(V_0)} = 0\}$$

and let $P : l(V_0) \to l(V_0)$ be the orthogonal projection onto $\tilde{l}(V_0)$.

Lemma 2.2. *The range of D is $\tilde{l}(V_0)$.*

Proof. For $x \in l(V_0)$,

$$(Dx, \mathbf{1})_{l(V_0)} = (x, D\mathbf{1})_{l(V_0)} = 0.$$

Therefore, $D(l(V_0)) \subset \tilde{l}(V_0)$. Since the dimensions of $D(l(V_0))$ and $\tilde{l}(V_0)$ are both $\#V_0 - 1$, we obtain the result. $\qquad\square$

For $f \in \mathcal{F}$, ν_f denotes the energy measure of f (cf. [6]); in our situation, ν_f is the unique finite Borel measure on K such that

$$\int_K g \, d\nu_f = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g) \quad \text{for all } g \in \mathcal{F}.$$

In particular, the energy measures of the constant functions are the zero measure in our framework. From the general theory, it is known that every energy measure for strong local regular Dirichlet forms does not have a point mass (from the energy image density property; see, e.g., [5, Theorem 4.3.8] or [4, Theorem I.7.1.1] for the proof). Therefore, energy measures ν_f do not have a mass on the countable set $V_* := \bigcup_{m \in \mathbb{Z}_+} V_m$ for any $f \in \mathcal{F}$. A concrete expression for ν_f with $f \in \mathcal{H}$ can be provided as follows.

Lemma 2.3 (cf. [12, Lemma 4]). *For any $f \in \mathcal{H}$ and $w \in W_*$,*

$$\nu_f(K_w) = -\frac{2}{r_w} {}^t(A_w f) D(A_w f). \tag{2.1}$$

We remark that the right-hand side of (2.1) is also described as

$$\frac{2}{r_w} \mathcal{E}(f \circ \psi_w, f \circ \psi_w).$$

This expression justifies an intuitive meaning of $\nu_f(K_w)$ as a "local energy of f on K_w."[1]

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[1] Compare this expression also with the following classical situation: for a Dirichlet form $(Q, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, dx)$, where $Q(f, g) = (1/2) \int_{\mathbb{R}^d} (\nabla f, \nabla g)_{\mathbb{R}^d} \, dx$ for $f, g \in H^1(\mathbb{R}^d)$, the energy measure of $f \in H^1(\mathbb{R}^d)$ is given by $|\nabla f|^2_{\mathbb{R}^d} \, dx$.

The mutual energy measure $\nu_{f,g}$ for f and g in $\mathcal{F}$ is a signed measure on K defined by $\nu_{f,g} = (\nu_{f+g} - \nu_f - \nu_g)/2$. For every Borel subset B of K, the inequality

$$|\nu_{f,g}(B)|^2 \le \nu_f(B)\nu_g(B) \tag{2.2}$$

holds.

We assume the following condition.

(A1) Each point of V_0 is a fixed point of some ψ_j. More precisely, there exists a subset S_0 of S such that $\#S_0 = \#V_0$ and, for each $p \in V_0$, there exists $j \in S_0$ such that $\psi_j(p) = p$. Furthermore, $K \setminus \{p\}$ is a connected set for every $p \in V_0$.

Under this condition, we have the following two lemmas.

Lemma 2.4 (cf. [13, Theorem A.1.2] and [12, Lemma 5]). *Let $j \in S_0$. Take $p \in V_0$ such that $\psi_j(p) = p$ and let u_j denote the column vector $(D_{qp})_{q \in V_0}$. Then*

(1) *r_j is a simple eigenvalue of A_j and ${}^t A_j$. Moreover, the modulus of all the eigenvalues of A_j and ${}^t A_j$ other than 1 and r_j are less than r_j;*

(2) *the vector u_j belongs to $\tilde{l}(V_0)$ and is an eigenvector of ${}^t A_j$ with respect to the eigenvalue r_j;*

(3) *there is a unique eigenvector v_j of A_j with respect to the eigenvalue r_j such that $(u_j, v_j)_{l(V_0)} = 1$. Moreover, every component of v_j is non-negative.*

Lemma 2.5 (cf. [12, Lemma 6]). *For $j \in S_0$ and $x \in l(V_0)$,*

$$\lim_{n \to \infty} r_j^{-n} P A_j^n x = (u_j, x)_{l(V_0)} P v_j.$$

Let $h_1, \dots, h_N$ ($N \in \mathbb{N}$) be a finite number of harmonic functions such that $\mathcal{H}$ is spanned by $h_1, \dots, h_N$ and constant functions. We denote $\sum_{k=1}^{N} \nu_{h_k}$ by ν.

Lemma 2.6. *For every $f \in \mathcal{H}$, ν_f is absolutely continuous with respect to ν.*

Proof. For some $\alpha_i \in \mathbb{R}$ ($i = 1, \dots, N$) and $\beta \in \mathbb{R}$, f can be written as $f = \sum_{i=1}^{N} \alpha_i h_i + \beta$. Then, $\nu_f = \sum_{i,j=1}^{N} \alpha_i \alpha_j \nu_{h_i,h_j}$. By (2.2), this is absolutely continuous with respect to ν. $\qquad \square$

We further assume the following condition.

(A2) For every $i \in S$, A_i is invertible.

For SG_l with the canonical harmonic structure in Example 2.1, (A2) has been confirmed for $l \leq 50$ by numerical computation (cf. [9, p. 297]). It is conjectured that it is true for all $l \geq 2$. See the final note "Added in proof."

Lemma 2.7. *For $j \in S_0$ and $w \in W_*$, ${}^t A_w u_j$ belongs to $\tilde{l}(V_0)$ and is nonzero.*

Proof. This is clear because of the identity $A_w \mathbf{1} = \mathbf{1}$ and condition (A2). $\square$

For $j \in S_0$ and $w \in W_*$, let

$$a_j^{(w)} = \sum_{k=1}^{N} (u_j, A_w h_k)_{l(V_0)}^2.$$

From Lemma 2.7, $a_j^{(w)}$ is strictly positive due to the choice of $\{h_k\}_{k=1}^{N}$.

We have some explicit information on the Radon–Nikodym derivative $d\nu_f / d\nu$ for harmonic functions f.

Lemma 2.8. *For any $f \in \mathcal{H}$, $w \in W_*$, and $j \in S_0$,*

$$\lim_{n \to \infty} \frac{\nu_f(K_{wj^n})}{\nu(K_{wj^n})} = \frac{(u_j, A_w f)_{l(V_0)}^2}{a_j^{(w)}},$$

where K_{wj^n} denotes $K_{w \underbrace{j \ldots j}_{n}}$.

We denote this limit by $\frac{d\nu_f}{d\nu}(wj^\infty)$. Although this precise notation may look like $\frac{d\nu_f}{d\nu}(\pi(wj^\infty))$, it is not clear whether the relation $\pi(wj^\infty) = \pi(\tilde{w}\tilde{j}^\infty)$ implies the identity $\frac{d\nu_f}{d\nu}(wj^\infty) = \frac{d\nu_f}{d\nu}(\tilde{w}\tilde{j}^\infty)$.

Proof of Lemma 2.8. From Lemma 2.3, Lemma 2.5 and the identity $D = {}^t PDP$,

$$\frac{\nu_f(K_{wj^n})}{\nu(K_{wj^n})} = \frac{{}^t(PA_j^n A_w f) D(PA_j^n A_w f)}{\sum_{k=1}^{N} {}^t(PA_j^n A_w h_k) D(PA_j^n A_w h_k)}$$

$$\xrightarrow{n \to \infty} \frac{(u_j, A_w f)_{l(V_0)}^2 \, {}^t(Pv_j) D(Pv_j)}{\sum_{k=1}^{N} (u_j, A_w h_k)_{l(V_0)}^2 \, {}^t(Pv_j) D(Pv_j)} = \frac{(u_j, A_w f)_{l(V_0)}^2}{a_j^{(w)}}.$$

$\square$

We now consider the following rather restrictive condition.

(A3) $D^2 = -\gamma D$ for some $\gamma > 0$. In other words, all the eigenvalues of D are either 0 or $-\gamma$.

In Example 2.1, (A3) holds with $\gamma = 3$.

The following proposition was proved in [1, Theorem 6.1] for the case of the standard Dirichlet form on SG_2; however, that proof was different from the one presented here.

Proposition 2.9. *Assume conditions* (A1)–(A3). *For each* $w \in W_*$, *there exists a set of positive numbers* $\{b_j^{(w)}\}_{j \in S_0}$ *such that*

$$\frac{\nu_f(K_w)}{\nu(K_w)} = \sum_{j \in S_0} b_j^{(w)} \frac{d\nu_f}{d\nu}(wj^\infty), \qquad f \in \mathcal{H}. \tag{2.3}$$

If $\#V_0 \geq 3$, *the* $\{b_j^{(w)}\}_{j \in S_0}$ *are uniquely determined.*

Proof. Since ${}^t DD = D^2 = -\gamma D$,

$$\begin{aligned}
\frac{\nu_f(K_w)}{\nu(K_w)} &= \frac{{}^t(A_w f)D(A_w f)}{\sum_{k=1}^N {}^t(A_w h_k)D(A_w h_k)} \\
&= \frac{-\gamma^{-1} \cdot {}^t(DA_w f)(DA_w f)}{-\gamma^{-1} \sum_{k=1}^N {}^t(DA_w h_k)(DA_w h_k)} \\
&= \frac{\sum_{j \in S_0}(u_j, A_w f)_{l(V_0)}^2}{\sum_{k=1}^N \sum_{i \in S_0}(u_i, A_w h_k)_{l(V_0)}^2} \\
&= \sum_{j \in S_0} \frac{a_j^{(w)}}{\sum_{i \in S_0} a_i^{(w)}} \frac{d\nu_f}{d\nu}(wj^\infty).
\end{aligned}$$

Therefore, (2.3) holds by letting

$$b_j^{(w)} = \frac{a_j^{(w)}}{\sum_{i \in S_0} a_i^{(w)}}. \tag{2.4}$$

To prove the uniqueness of $\{b_j^{(w)}\}_{j \in S_0}$, it suffices to prove that

$$\sum_{j \in S_0} \beta_j (u_j, A_w f)_{l(V_0)}^2 = 0 \quad \text{for all } f \in \mathcal{H} \tag{2.5}$$

implies $\beta_j = 0$ for all $j \in S_0$. Let j and k be distinct elements of S_0. Denote the fixed points of ψ_j and ψ_k by p_j and p_k, respectively. From Lemma 2.2, there exists an $x \in l(V_0)$ such that $(Dx)(p_j) = 1$, $(Dx)(p_k) = -1$, and $(Dx)(p) = 0$ for $p \in V_0 \setminus \{p_j, p_k\}$; in other words, $(u_j, x)_{l(V_0)} = 1$, $(u_k, x)_{l(V_0)} = -1$, and $(u_i, x)_{l(V_0)} = 0$ for $i \in S_0 \setminus \{j, k\}$. From the surjectiveness of A_w, there exists an $f \in \mathcal{H} \simeq l(V_0)$ such that $A_w f = x$. Then, from (2.5), $\beta_j + \beta_k = 0$. This relation implies that $\beta_j = 0$ for all $j \in S_0$ because $\#S_0 = \#V_0 \geq 3$. $\qquad\square$

From (2.4), we have the identity

$$\sum_{j \in S_0} b_j^{(w)} = 1. \tag{2.6}$$

The coefficients $\{b_j^{(w)}\}_{j \in S_0}$ provide some information on the distribution of the energy measures of harmonic functions. In a typical example, $\{b_j^{(w)}\}_{j \in S_0}$ describes the skewness of ν on the cell K_{wj} relative to K_w as follows, which is due to Bell, Ho, and Strichartz [1].

Theorem 2.10 (cf. [1, Theorem 6.3]). *We consider the standard Dirichlet form on* SG_2 *given in Example* 2.1 *and write* $S = S_0 = \{1, 2, 3\}$. *As a choice of* $\{h_i\}_{i=1}^N \subset \mathcal{H}$, *let* $N = 2$ *and take a pair* $h_1, h_2 \in \mathcal{H}$ *so that* $\mathcal{E}(h_i, h_j) = \delta_{ij}/4$ *for any* $i, j \in \{1, 2\}$, *where* δ_{ij} *represents the Kronecker delta. Accordingly,* $\nu = \nu_{h_1} + \nu_{h_2}$. *Then, the identity*

$$\frac{1}{5}\left(b_j^{(w)} - \frac{1}{3}\right) = \frac{1}{4}\left(\frac{\nu(K_{wj})}{\nu(K_w)} - \frac{1}{3}\right)$$

holds for any $w \in W_*$ *and* $j \in S_0$.

It is easy to see that the measure ν given above is a probability measure on K and is independent of the choice of h_1 and h_2. For SG_l with $l \geq 3$, such a clear interpretation of $\{b_j^{(w)}\}_{j \in S_0}$ as in Theorem 2.10 seems difficult to obtain.

Let λ be a Borel probability measure on $\Sigma = S^{\mathbb{N}}$, which is defined as the infinite product of the uniform probability measure $(\#S)^{-1} \sum_{j \in S} \delta_j$ on S. For $\omega = \omega_1 \omega_2 \cdots \in \Sigma$ and $m \in \mathbb{N}$, $[\omega]_m$ denotes $\omega_1 \omega_2 \ldots \omega_m \in W_m$. For $m \in \mathbb{Z}_+$, let ξ_m denote the image measure of λ by the map $\Sigma \ni \omega \mapsto \{b_j^{([\omega]_m)}\}_{j \in S} \in \mathbb{R}^{\#S}$. Bell, Ho, and Strichartz [1] discuss some properties of ξ_m for the canonical Dirichlet form on SG_2 and posed conjectures, which we call Conjecture 2.12 below.

Until the end of this section, we consider the Dirichlet form for two-dimensional standard Sierpinski gasket $SG_2 =: K$ given in Example 2.1 and take the measure ν

as in Theorem 2.10. We write $S = S_0 = \{1, 2, 3\}$ and

$$\mathbb{D} := \Big\{ (b_1, b_2, b_3) \in \mathbb{R}^3 \ \Big| \ \sum_{j=1}^{3} b_j = 1 \text{ and } \sum_{j=1}^{3} \Big(b_j - \frac{1}{3} \Big)^2 < \frac{1}{6} \Big\}$$

$$\Big(= \Big\{ (b_1, b_2, b_3) \in \mathbb{R}^3 \ \Big| \ \sum_{j=1}^{3} b_j = 1 \text{ and } \sum_{j=1}^{3} b_j^2 < \frac{1}{2} \Big\} \Big).$$

Theorem 2.11 (cf. [1, Theorem 6.5]). *For all $w \in W_*$, $(b_1^{(w)}, b_2^{(w)}, b_3^{(w)})$ belongs to $\mathbb{D}$; that is,*

$$\sum_{j=1}^{3} \Big(b_j^{(w)} - \frac{1}{3} \Big)^2 < \frac{1}{6}. \tag{2.7}$$

This inequality is sharp. In particular, ξ_m concentrates on $\mathbb{D}$ for all m.

We note that (2.7) can be rewritten as

$$\sum_{j=1}^{3} (b_j^{(w)})^2 < \frac{1}{2}$$

because of (2.6).

Conjecture 2.12 (cf. [1, Conjectures 7.1 and 7.2]). *Let (r, θ) be polar coordinates for the disk $\mathbb{D}$ with center $\boldsymbol{c} = (1/3, 1/3, 1/3)$. More specifically,*

$$r(z) = |z - \boldsymbol{c}|_{\mathbb{R}^3},$$

$$\theta(z) = \mathrm{Arg}((z - \boldsymbol{c}, \boldsymbol{a}_1)_{\mathbb{R}^3} + \sqrt{-1}(z - \boldsymbol{c}, \boldsymbol{a}_2)_{\mathbb{R}^3}) \in (-\pi, \pi]$$

with $\boldsymbol{a}_1 = (1/\sqrt{2}, -1/\sqrt{2}, 0)$, $\boldsymbol{a}_2 = (-1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$, where $(\cdot, \cdot)_{\mathbb{R}^3}$ and $|\cdot|_{\mathbb{R}^3}$ denote the standard inner product and norm on $\mathbb{R}^3$, respectively. For $m \in \mathbb{N}$, let P_m and Q_m denote the image measures of ξ_m by the mappings $\theta(\cdot)$ and $r(\cdot)$, respectively. Then

(1) *P_m converges weakly to an absolutely continuous measure on $(-\pi, \pi]$ as $m \to \infty$;*

(2) *Q_m converges weakly to the delta measure at $1/\sqrt{6}$ as $m \to \infty$.*

Bell, Ho, and Strichartz [1] also conjectured the invariance of the limit of P_m under some rational maps, but we skip the details because we do not discuss such kind of property in this paper.

In the next section we prove Theorem 2.11 and confirm Conjecture 2.12(2) in a slightly more general situation.

3. Main results

We keep the notation used in the previous section and always assume conditions (A1)–(A3).

Fix $w \in W_*$. For $j \in S_0$, let $z_j = {}^t A_w u_j$. Note that $z_j \in \tilde{l}(V_0)$ and $z_j \neq 0$ from Lemma 2.7. Also, since $\sum_{j \in S_0} u_j = 0$ from $D\mathbf{1} = 0$, we have

$$\sum_{j \in S_0} z_j = 0. \tag{3.1}$$

For $x, y \in \tilde{l}(V_0) \subset l(V_0)$, we define

$$\langle x, y \rangle = \sum_{k=1}^{N} (x, h_k)_{l(V_0)} (y, h_k)_{l(V_0)} \quad \text{and} \quad \|x\| = \langle x, x \rangle^{1/2}.$$

Then, $\langle \cdot, \cdot \rangle$ is an inner product on $\tilde{l}(V_0)$ and the identity $\|z_j\|^2 = a_j^{(w)}$ holds. We remark that there exists a positive definite symmetric operator H on $\tilde{l}(V_0)$ such that $\langle x, y \rangle = (Hx, Hy)_{l(V_0)}$ for all $x, y \in \tilde{l}(V_0)$.

Fix an arbitrary $k \in S_0$ and let $S_0' = S_0 \setminus \{k\}$. Then we have the following lemma.

Lemma 3.1. *The following identity holds*:

$$\sum_{j \in S_0} (b_j^{(w)})^2 = \left(2 + \frac{2\sum_{i,j \in S_0', i \neq j} \|z_i\|^2 \|z_j\|^2 - \left(\sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle\right)^2}{\sum_{j \in S_0} \|z_j\|^4}\right)^{-1}. \tag{3.2}$$

Proof. From (2.4),

$$\sum_{j \in S_0} (b_j^{(w)})^2 = \frac{\sum_{j \in S_0} (a_j^{(w)})^2}{\left(\sum_{j \in S_0} a_j^{(w)}\right)^2} = \frac{\sum_{j \in S_0} \|z_j\|^4}{\left(\sum_{j \in S_0} \|z_j\|^2\right)^2}$$

$$= \left(2 + \frac{\left(\sum_{j \in S_0} \|z_j\|^2\right)^2 - 2\sum_{j \in S_0} \|z_j\|^4}{\sum_{j \in S_0} \|z_j\|^4}\right)^{-1}. \tag{3.3}$$

Using the identity $z_k = -\sum_{j \in S_0'} z_j$ from (3.1), we have

$$
\begin{aligned}
\sum_{j \in S_0} \|z_j\|^4 &= \sum_{j \in S_0'} \|z_j\|^4 + \Big\| \sum_{j \in S_0'} z_j \Big\|^4 \\
&= \sum_{j \in S_0'} \|z_j\|^4 + \Big(\sum_{j \in S_0'} \|z_j\|^2 + \sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle \Big)^2 \\
&= 2 \sum_{j \in S_0'} \|z_j\|^4 + \sum_{i,j \in S_0', i \neq j} \|z_i\|^2 \|z_j\|^2 \\
&\quad + 2 \sum_{j \in S_0'} \|z_j\|^2 \sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle + \Big(\sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle \Big)^2
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\Big(\sum_{j \in S_0} \|z_j\|^2 \Big)^2 &- \sum_{j \in S_0} \|z_j\|^4 \\
&= \sum_{i,j \in S_0, i \neq j} \|z_i\|^2 \|z_j\|^2 \\
&= \sum_{i,j \in S_0', i \neq j} \|z_i\|^2 \|z_j\|^2 + 2 \Big\| \sum_{j \in S_0'} z_j \Big\|^2 \sum_{j \in S_0'} \|z_j\|^2 \\
&= \sum_{i,j \in S_0', i \neq j} \|z_i\|^2 \|z_j\|^2 + 2 \Big(\sum_{j \in S_0'} \|z_j\|^2 \Big)^2 + 2 \sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle \sum_{j \in S_0'} \|z_j\|^2 \\
&= 3 \sum_{i,j \in S_0', i \neq j} \|z_i\|^2 \|z_j\|^2 + 2 \sum_{j \in S_0'} \|z_j\|^4 + 2 \sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle \sum_{j \in S_0'} \|z_j\|^2.
\end{aligned}
\tag{3.5}
$$

Then,

$$
\Big(\sum_{j \in S_0} \|z_j\|^2 \Big)^2 - 2 \sum_{j \in S_0} \|z_j\|^4 = 2 \sum_{i,j \in S_0', i \neq j} \|z_i\|^2 \|z_j\|^2 - \Big(\sum_{i,j \in S_0', i \neq j} \langle z_i, z_j \rangle \Big)^2
$$

by combining (3.4) and (3.5). This identity and (3.3) imply (3.2). $\qquad\square$

Lastly we consider the following condition.

(A4) $\#V_0 = 3$.

The following extends Theorem 2.11 ([1, Theorem 6.5]) to more general situations, and the proof is more straightforward.

Theorem 3.2. *Under conditions* (A1)–(A4),

$$\sum_{j \in S_0} (b_j^{(w)})^2 < \frac{1}{2} \tag{3.6}$$

for all $w \in W_*$. This inequality is sharp.

Proof. Let $S_0' = \{1, 2\}$. Then, (3.2) can be rewritten as

$$\sum_{j \in S_0} (b_j^{(w)})^2 = \left(2 + 4 \cdot \frac{\|z_1\|^2 \|z_2\|^2 - \langle z_1, z_2 \rangle^2}{\sum_{j \in S_0} \|z_j\|^4}\right)^{-1}. \tag{3.7}$$

Moreover, the inequality $|\langle z_1, z_2 \rangle| \leq \|z_1\| \|z_2\|$ holds with equality if and only if z_1 and z_2 are linearly dependent. Since u_1 and u_2 are linearly independent by the property (D2) of D, the inequality is strict. Therefore, we obtain (3.6). The sharpness of this inequality is confirmed by Theorem 3.6 below, so we omit the proof here. □

Remark 3.3. As can be seen from the proof above, it seems difficult to obtain a good estimate of $\sum_{j \in S_0} (b_j^{(w)})^2$ if $\#V_0 > 3$. Indeed, if $\#V_0 = 4$ and $S_0 = \{1, 2, 3, 4\}$, eq. (3.2) is rewritten as

$$\sum_{j \in S_0} (b_j^{(w)})^2 = \left(2 + \frac{4I}{\sum_{j \in S_0} \|z_j\|^4}\right)^{-1}$$

with

$$I = \sum_{(i,j) \in \{(1,2),\,(2,3),\,(3,1)\}} (\|z_i\|^2 \|z_j\|^2 - \langle z_i, z_j \rangle^2)$$

$$- 2 \sum_{(i,j,k) \in \{(1,2,3),\,(2,3,1),\,(3,1,2)\}} \langle z_i, z_j \rangle \langle z_j, z_k \rangle.$$

We may need other functionals to specify the range of $\{b_j^{(w)}\}_{j \in S_0}$ in such a case.

Let $\widetilde{D}$ denote the restriction of D as a negative definite symmetric operator on $\tilde{l}(V_0)$. For $w \in W_*$, let $\widetilde{A}_w$ denote the restriction of $PA_w(= PA_w P)$ as a linear operator on $\tilde{l}(V_0)$.

For the statement of the main theorem, we recall the concept of strong irreducibility of random matrices.

Definition 3.4. A set $\mathcal{T}$ of invertible linear operators on $\tilde{l}(V_0)$ is called strongly irreducible if there does not exist a finite family $L_1, \ldots, L_k$ of proper linear subspaces of $\tilde{l}(V_0)$ such that $M(L_1 \cup \cdots \cup L_k) = L_1 \cup \cdots \cup L_k$ for all $M \in \mathcal{T}$.

Example 3.5. We again consider a canonical harmonic structure on the two-dimensional level l Sierpinski gasket in Example 2.1. Further, we assume (A2). Then we can prove that $\{\widetilde{A}_i\}_{i \in S}$ is strongly irreducible. Indeed, from (A2) and the fact that the sequence $\{(|\det \widetilde{A}_j|^{-1/2}\widetilde{A}_j)^n\}_{n=1}^{\infty}$ is unbounded for $j \in S_0$, as shown in the proof of Theorem 3.6 below, it suffices to prove the following claim by [3, Part A, Chapter II, Proposition 4.3]:

> for every $x \in \tilde{l}(V_0) \setminus \{0\}$, the set $\{\widetilde{A}_i^n x \mid i \in S, \, n \in \mathbb{Z}_+\}$ has three elements y_1, y_2, y_3 such that y_j and y_k are pairwise linearly independent for $j \neq k$. (3.8)

Let $S_0 = \{1, 2, 3\}$. By the symmetry of the harmonic structure and Lemma 2.4(1), $\widetilde{A}_1$ has two different eigenvalues and the eigenvectors of $\widetilde{A}_1$ (up to multiplicative constants) are $z_1 := {}^t(2, -1, -1)$ and $z_2 := {}^t(0, 1, -1)$. The set of eigenvectors of $\widetilde{A}_1^2$ is the same as that of $\widetilde{A}_1$. The same claims hold for $\widetilde{A}_2$ with z_1 and z_2 replaced by ${}^t(-1, 2, -1)$ and ${}^t(1, 0, -1)$, respectively. Now, let $x \in \tilde{l}(V_0) \setminus \{0\}$. If x and z_i are linearly dependent for $i = 1$ or 2, any two of $\{x, A_2 x, A_2^2 x\}$ are linearly independent. Otherwise, any two of $\{x, A_1 x, A_1^2 x\}$ are linearly independent. Therefore, (3.8) follows.

We now resume our discussion of the general situation. The following is the main theorem of this paper.

Theorem 3.6. *Assume conditions* (A1)–(A4). *Let κ be a Borel measure on $\Sigma = S^{\mathbb{N}}$. We further suppose **either** of the following cases.*

(I) *κ is an infinite product of a probability measure on S with full support, and $\{\widetilde{A}_i\}_{i \in S}$ is strongly irreducible.*

(II) *The image measure of κ by $\pi \colon \Sigma \to K$ is absolutely continuous with respect to ν.*

Then,

$$\lim_{n\to\infty} \sum_{j\in S_0} (b_j^{([\omega]_n)})^2 = \frac{1}{2} \quad \text{for } \kappa\text{-a.e. } \omega. \tag{3.9}$$

In particular, the image measure of κ by the map

$$\Sigma \ni \omega \longmapsto \{b_j^{([\omega]_n)}\}_{j\in S_0} \in l(S_0) \cong l(V_0)$$

converges weakly as $n \to \infty$ to a measure that concentrates on the set

$$\Big\{ x \in l(V_0) \ \Big| \ \sum_{p\in V_0} x(p) = 1 \text{ and } |x|^2_{l(V_0)} = 1/2 \Big\}.$$

The result for Case (I) gives an affirmative answer to Conjecture 2.12(2). We remark that the strong irreducibility of $\{\widetilde{A}_i\}_{i\in S}$ is not necessary in Case (II).

Proof of Theorem 3.6. First, we note that $\tilde{l}(V_0)$ is two-dimensional because of (A4). Using the same notation as in Theorem 3.2, with $w = [\omega]_n$ for $\omega \in \Sigma$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|z_1\|^2 \|z_2\|^2 - \langle z_1, z_2\rangle^2 &= |Hz_1|^2_{l(V_0)} |Hz_2|^2_{l(V_0)} - (Hz_1, Hz_2)^2_{l(V_0)} \\
&= |Hz_1 \wedge Hz_2|^2_{\bigwedge^2 \tilde{l}(V_0)} \\
&= (\det H)^2 (\det \widetilde{A}_{[\omega]_n})^2 |u_1 \wedge u_2|^2_{\bigwedge^2 \tilde{l}(V_0)}.
\end{aligned}$$

Moreover, since u_1 and u_2 are linearly independent, the map

$$B \longmapsto (\|Bu_1\|^4 + \|Bu_2\|^4)^{1/4}$$

provides a norm on the space $\mathcal{L}(\tilde{l}(V_0))$ of all linear operators on $\tilde{l}(V_0)$. Therefore,

$$\sum_{j\in S_0} \|z_j\|^4 \geq \|\widetilde{A}_{[\omega]_n} u_1\|^4 + \|\widetilde{A}_{[\omega]_n} u_2\|^4 \geq c\|\widetilde{A}_{[\omega]_n}\|^4_{\mathrm{op}},$$

where $\|\cdot\|_{\mathrm{op}}$ represents the operator norm on $\mathcal{L}(\tilde{l}(V_0))$ and c is a positive constant independent of ω and n. Then

$$0 < \frac{\|z_1\|^2 \|z_2\|^2 - \langle z_1, z_2\rangle^2}{\sum_{j\in S_0} \|z_j\|^4} \leq \frac{(\det H)^2 (\det \widetilde{A}_{[\omega]_n})^2 |u_1 \wedge u_2|^2_{\bigwedge^2 \tilde{l}(V_0)}}{c\|\widetilde{A}_{[\omega]_n}\|^4_{\mathrm{op}}}.$$

By virtue of (3.7), eq. (3.9) follows if we can prove that

$$\lim_{n\to\infty} \frac{(\det \widetilde{A}_{[\omega]_n})^2}{\|\widetilde{A}_{[\omega]_n}\|^4_{\mathrm{op}}} = 0 \quad \text{for } \kappa\text{-a.e. } \omega. \tag{3.10}$$

Suppose Case (I) and fix $j \in S_0$. Since $\widetilde{A}_j$ is invertible from (A2) and has two eigenvalues with different moduli from Lemma 2.4(1), the sequence $\{(|\det \widetilde{A}_j|^{-1/2} \widetilde{A}_j)^n\}_{n=1}^{\infty}$ is unbounded. Together with strong irreducibility of $\{A_i\}_{i \in S}$, Furstenberg's theorem (cf. [3, Part A, Chapter II, Theorems 4.1 and 3.6]) implies that for two elements $x_1, x_2 \in \tilde{l}(V_0)$ which are linearly independent,

$$\lim_{n \to \infty} \delta(\widetilde{A}_{[\omega]_n} x_1, \widetilde{A}_{[\omega]_n} x_2) = 0 \quad \text{for } \kappa\text{-a.e. } \omega.$$

Here, $\delta(\cdot, \cdot) \in [0, 1]$ denotes the angular distance, that is,

$$\delta(y_1, y_2) = \sqrt{1 - \left(\frac{y_1}{|y_1|_{l(V_0)}}, \frac{y_2}{|y_2|_{l(V_0)}}\right)^2_{l(V_0)}} \quad \text{for } y_1, y_2 \in \tilde{l}(V_0) \setminus \{0\}.$$

Let x_1, x_2 be an orthonormal basis of the inner product space $(\tilde{l}(V_0), (\cdot, \cdot)_{l(V_0)})$. Then

$$\delta(\widetilde{A}_{[\omega]_n} x_1, \widetilde{A}_{[\omega]_n} x_2) = \frac{|\det \widetilde{A}_{[\omega]_n}|}{|\widetilde{A}_{[\omega]_n} x_1|_{l(V_0)} |\widetilde{A}_{[\omega]_n} x_2|_{l(V_0)}} \geq \frac{|\det \widetilde{A}_{[\omega]_n}|}{\|\widetilde{A}_{[\omega]_n}\|^2_{\text{op}}}.$$

Thus we obtain (3.10).

Next, suppose Case (II). Again let x_1, x_2 be an orthonormal basis of $\tilde{l}(V_0)$. Define $\hat{h}_i = \iota(x_i)$ for $i = 1, 2$ and $\hat{v} = v_{\hat{h}_1} + v_{\hat{h}_2}$. Since the linear span of $\hat{h}_1, \hat{h}_2$, and constant functions is $\mathcal{H}$, v and $\hat{v}$ are mutually absolutely continuous from Lemma 2.6. Therefore, we may assume that the image measure of κ by the mapping π coincides with $\hat{v}$ for proving (3.10).[2] In our situation, the index ([9, Definition 2.9]) of the Dirichlet form under consideration is 1 (see [8, Proposition 3.4] or [10, Theorem 4.10] for the proof; see also [14]). In particular, we have[3]

$$\text{rank}\left(\frac{dv_{\hat{h}_i, \hat{h}_j}}{d\hat{v}}\right)^2_{i,j=1} \leq 1 \quad \hat{v}\text{-a.e.}$$

On the other hand, for $i, j \in \{1, 2\}$,

$$\frac{dv_{\hat{h}_i, \hat{h}_j}}{d\hat{v}}(\pi(\omega)) = \lim_{n \to \infty} \frac{v_{\hat{h}_i, \hat{h}_j}(K_{[\omega]_n})}{\hat{v}(K_{[\omega]_n})} = \lim_{n \to \infty} \frac{-{}^t x_i \, {}^t \widetilde{A}_{[\omega]_n} \widetilde{D} \widetilde{A}_{[\omega]_n} x_j}{\text{tr}(-{}^t \widetilde{A}_{[\omega]_n} \widetilde{D} \widetilde{A}_{[\omega]_n})} \quad \text{for } \kappa\text{-a.e. } \omega,$$

---

[2] Since $\hat{v}(V_*) = 0$, such a κ is uniquely identified. More specifically, for $m \in \mathbb{N}$ and $A \subset W_m$, $\kappa(\{\omega \in \Sigma \mid [\omega]_m \in A\})$ is given by $\sum_{k=1}^{2} \sum_{w \in A} 2r_w^{-1} \mathcal{E}(\hat{h}_k \circ \psi_w, \hat{h}_k \circ \psi_w)$ from Lemma 2.3.

[3] In fact, the equality holds from [9, Proposition 2.11].

where the first equality follows from the martingale convergence theorem and the second one is due to Lemma 2.3. Then, for κ-a.e. ω,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \det\Big(\frac{-\,{}^t\widetilde{A}_{[\omega]_n}\widetilde{D}\,\widetilde{A}_{[\omega]_n}}{\operatorname{tr}(-\,{}^t\widetilde{A}_{[\omega]_n}\widetilde{D}\,\widetilde{A}_{[\omega]_n})}\Big) \\
&= \lim_{n\to\infty} \frac{(\det\widetilde{A}_{[\omega]_n})^2 \det(-\widetilde{D})}{\|\sqrt{-\widetilde{D}}\,\widetilde{A}_{[\omega]_n}\|_{\mathrm{HS}}^4} \\
&\geq \overline{\lim_{n\to\infty}} \frac{(\det\widetilde{A}_{[\omega]_n})^2 \det(-\widetilde{D})}{c'\|\widetilde{A}_{[\omega]_n}\|_{\mathrm{op}}^4},
\end{aligned}
$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm on $\mathcal{L}(\tilde{l}(V_0))$ and c' is a positive constant depending only on D. Thus, (3.10) holds.

The last claim of the theorem follows from the general fact that almost sure convergence implies convergence in law. □

4. Concluding remarks

We give some comments as concluding remarks.

(1) As can be seen from the proof, Condition (A4) is crucial for Theorem 3.2 and thus for Theorem 3.6. It may be an interesting problem to provide an appropriate formulation when $\#V_0 > 3$.

(2) In both cases (I) and (II) in Theorem 3.6, κ has no mass on $\pi^{-1}(V_*)$. Therefore, the statements of Theorem 3.6 and Conjecture 2.12 can be rephrased in terms of a measure on K instead of the measure κ on Σ: that is, self-similar measures on K in Case (I) and ν in Case (II), respectively.

(3) In Theorem 3.6, the measure κ of Case (I) and that of Case (II) are mutually singular in many cases (cf. [12, Theorem 2]). Case (II) looks like a more natural formulation in the sense that there is no need for the extra assumption of the strong irreducibility of $\{A_j\}_{j\in S}$ and because the concept of the index of Dirichlet forms, which also has probabilistic interpretations [9, 11], appears naturally in the proof.

(4) At the moment, there are no clues concerning Conjecture 2.12(1). The distribution of P_m with $m = 13$ is given in the left-hand graph of Figure 2.

This figure shows the normalized histogram (2000 slices) of P_{13} on $[-\pi/3, \pi/3]$ for the Dirichlet form on SG_2 assumed in Conjecture 2.12; because of the symmetry it suffices to consider only this interval.[4] This figure supports the validity of Conjecture 2.12(1). On the other hand, the right-hand graph of Figure 2 shows the distribution on $[-\pi/3, \pi/3]$ of the image measure of κ by the map $\Sigma \ni \omega \mapsto \theta(\{b_j^{([\omega]_m)}\}_{j=1}^3) \in (-\pi, \pi]$ with $m = 13$, where κ is taken so that its image measure by the map $\pi \colon \Sigma \to K$ is equal to the measure ν given in Theorem 2.10. Here, the interval $[-\pi/3, \pi/3]$ is again divided into 2000 slices. The distribution looks very different and the possible limit measure as $m \to \infty$ might be singular with respect to the Lebesgue measure.
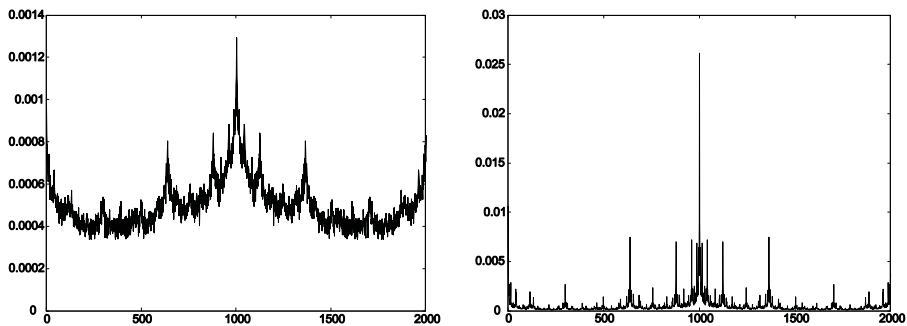

Figure 2. Angular distributions.

Added in proof. After the acceptance of this paper, it was announced that condition (A2) was confirmed for SG_l for all $l \geq 2$ with the canonical harmonic structure in Example 2.1, see [15].

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[4] The histogram of P_{13} with 100 slices is also provided in [1, Figure 7.4]; the figure there corresponds to the right half of ours. However, with only 100 slices the graph does not seem to reflect the irregular behavior of the distribution of P_m very precisely.

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