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# Local dimensions of measures of finite type

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**Abstract.** We study the multifractal analysis of a class of equicontractive, self-similar measures of finite type, whose support is an interval. Finite type is a property weaker than the open set condition, but stronger than the weak separation condition. Examples include Bernoulli convolutions with contraction factor the inverse of a Pisot number and self-similar measures associated with $m$-fold sums of Cantor sets with ratio of dissection $1/R$ for integer $R \leq m$.

We introduce a combinatorial notion called a loop class and prove that the set of attainable local dimensions of the measure at points in a positive loop class is a closed interval. We prove that the local dimensions at the periodic points in the loop class are dense and give a simple formula for those local dimensions. These self-similar measures have a distinguished positive loop class called the essential class. The set of points in the essential class has full Lebesgue measure in the support of the measure and is often all but the two endpoints of the support. Thus many, but not all, measures of finite type have at most one isolated point in their set of local dimensions.

We give examples of Bernoulli convolutions whose sets of attainable local dimensions consist of an interval together with an isolated point. As well, we give an example of a measure of finite type that has exactly two distinct local dimensions.

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**Keywords.** Local dimension, multifractal analysis, finite type, Bernoulli convolution, IFS.

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**Contents**



## 1. Introduction

It is well known that if $\mu$ is a self-similar measure arising from an IFS satisfying the open set condition, then the set of local dimensions of the measure is a closed interval whose endpoints are easily computed. Further, the Hausdorff dimension of the set of points whose local dimension is a given value can be determined using the Legendre transform of the $L^q$-spectrum of the measure. This is known as the multifractal formalism and we refer the reader to [5] for more details.

For measures that do not satisfy the open set condition, the multifractal analysis is more complicated and, in general, much more poorly understood. In [16], Hu and Lau discovered that the 3-fold convolution of the classical middle-third Cantor measure fails the multifractal formalism as there is an isolated point in the set of local dimensions. Subsequently, in [2, 26, 28] further examples of this phenomena were explored and it was shown, for example, that there is always an isolated point in the set of local dimensions of the $m$-fold convolution of the Cantor measure associated with a Cantor set with ratio of dissection $1/R$, when the integer $R \leq m$. More recently, it was proven in [1] that continuous measures satisfying a weak technical condition have the property that a suitably large convolution power admits an isolated point in its set of local dimensions.

In [21], Ngai and Wang introduced the notion of finite type (see Section 2 for the definition). This property is stronger than the weak separation condition introduced in [18], but is satisfied by many self-similar measures which fail to possess the open set condition. Examples include Bernoulli convolutions, $\mu_\varrho$, with contraction factor $\varrho$ equal to the reciprocal of a Pisot number [21] and the Cantor-like measures mentioned above.

Building on earlier work (c.f., [12, 15, 19, 25]), Feng undertook a study of equicontractive, self-similar measures of finite type in [7, 8, 9]. His main results were for Bernoulli convolutions. In particular, he proved that despite the failure of the open set condition, the multifractal formalism still holds for the Bernoulli convolutions whose contraction factor was the reciprocal of a simple Pisot number (meaning, a Pisot number whose minimal polynomial is of the form $x^n - x^{n-1} - \cdots - x - 1$). A particularly interesting example is when the contraction factor is the golden ratio with minimal polynomial $x^2 - x - 1$ (also called the golden mean).

In this paper we study the local dimension theory of equicontractive, self-similar measures $\mu$ of finite type, whose support is a compact interval and for which the underlying probabilities are regular. We first give a simple formula for the value of the local dimension of $\mu$ at any "periodic" point of its support. As a corollary we get that the local dimension exists at "periodic" points. The finite type condition leads naturally to a combinatorial notion we call a "loop class". For a "positive" loop class we prove that the set of attainable local dimensions of the measure is a closed interval and that the set of local dimensions at periodic points in the loop class is a dense subset of this interval. Similar results are also given for upper and lower local dimensions. Given two values $\ell \leq u$ within this interval, we can find an x in this positive loop class with lower local dimension equal to $\ell$ and upper local dimension equal to $u$.

Similar results have been proven in this directions before. In [9], Feng, without the restrictions on the probabilities that we require, constructed a family (finite or countably infinite) of closed intervals $I_j$ with disjoint interiors, where on each of these closed intervals the set of attainable local dimensions of the measure restricted to this interval was a closed interval. These closed intervals, $I_j$, correspond to net intervals within the essential class, a distinguished positive loop class. It is worth observing that, except at the end points, the local dimension of the measure, and the local dimension of the measure restricted to the interval will be the same. At the end points, these may be different. However our results, with the addition of the mild technical assumption on the probabilities, gives this result for the original measure, as well as for positive loop classes other than the essential class. Under the less restrictive assumption of the weak separation condition, Feng and Lau [10] similarly proved that the range of the local dimensions of the measure restricted to a certain open ball is a closed interval.

A consequence of our result is that the set of attainable local dimensions is the union of a closed interval together with the local dimensions at points in finitely many loop classes external to the essential class. We will say that a point is an essential point if it is in the essential class. The set of essential points has full

Lebesgue measure on the support of the measure and in many interesting examples the set of essential points is the interior of the support of the measure. This is the case with many Bernoulli convolutions, $\mu_\varrho$, including when $\varrho^{-1}$ is the golden ratio (c.f. Section 8.1.1), and with the $m$-fold convolution of the Cantor measure on a Cantor set with ratio of dissection $1/R$ when $R \leq m$ (see Section 7).

When the essential set is the interior of the support of the measure $\mu$, then $\mu$ has no isolated point in its set of attainable local dimensions if and only if $\dim_{\text{loc}} \mu(0)$ coincides with the local dimension of $\mu$ at an essential point. In that case, the set of attainable local dimensions of $\mu$ is a closed interval. The Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$ a simple Pisot number, has this property.

However, we construct other examples of Bernoulli convolutions (with contraction factor a Pisot inverse) which do have an isolated point in their set of attainable local dimensions (see Subsection 8.1.2). As far as we are aware, these are the first examples of Bernoulli convolutions known to admit an isolated point. We also construct a Cantor-like measure of finite type, whose set of local dimensions consists of (precisely) two distinct points (see Example 6.1). In all of these examples, the essential set is the interior of the support of the measure.

The convolution square of the Bernoulli convolution, $\mu_\varrho$, with $\varrho^{-1}$ the golden ratio, is another example of a self-similar measure to which our theory applies. It, too, has exactly one isolated point in its set of attainable local dimensions, although in this case the set of non-essential points is countably infinite (see Subsection 8.2).

The computer was used to help obtain some of these results. In principle, the techniques could be applied to other convolutions of Bernoulli convolutions and other measures of finite type, however even with the simple examples given here, the problem can become computationally difficult.

The paper is organized as follows. In Section 2, we detail the structure of self-similar measures of finite type, introduce terminology and describe a number of examples that we will return to throughout the paper. The notion of transition matrices and properties of local dimensions of measures of finite type are discussed in Section 3. In Section 4 we introduce the notion of loop class, essential class and periodic points. A formula is given for the local dimension at a periodic point and we prove that the essential class is always of positive type. In Section 5 we prove that the set of local dimensions at periodic points in a positive loop class is dense in the set of local dimensions at all points in the loop class. We also show that the set of local dimensions at the points of a positive loop class is a closed interval. In particular, this implies that the set of local dimensions at the essential points is a closed interval.

In Section 6 we give a detailed description of our computer algorithm by means of a worked example. We also explain our main techniques for finding bounds on sets of local dimensions and illustrate these by constructing a Cantor-like measure of finite type whose local dimension is the union of two distinct points. In Section 7 we show that with our approach we can partially recover results from [2, 16, 26] about the local dimensions of Cantor-like measures of finite type. We also show that some facts about the endpoints of the interval portion of the local dimension, that are known to be true for Cantor-like measures in the "small" overlap case, do not hold in general. Bernoulli convolutions, $\mu_\varrho$, where $\varrho$ is the reciprocal of a Pisot number of degree at most four, are studied in Section 8 and we see that two of these measures admit an isolated point. We also study the convolution square of the Bernoulli convolution with the golden ratio in this final section.

For the examples in this paper, we present only minimal information. A more detailed analysis of all of these examples can found as supplemental information appended to the arXiv version of the paper [14].

## 2. Terminology and examples

**2.1. Finite type.** Consider the iterated function system (IFS) consisting of the contractions $S_j\colon \mathbb{R} \to \mathbb{R}$, $j = 0, \dots, m$, defined by

$$S_j(x) = \varrho x + d_j \tag{1}$$

where $0 < \varrho < 1$, $d_0 < d_1 < d_2 < \cdots < d_m$ and $m \geq 1$ is an integer. By the associated self-similar set, we mean the unique, non-empty, compact set $K$ satisfying

$$K = \bigcup_{j=0}^{m} S_j(K).$$

Suppose $p_j$, $j = 0, \dots, m$ are probabilities, i.e., $p_j > 0$ for all $j$ and $\sum_{j=0}^{m} p_j = 1$. Our interest is in the self-similar measure $\mu$ associated to the family of contractions $\{S_j\}$ as above, which satisfies the identity

$$\mu = \sum_{j=0}^{m} p_j \mu \circ S_j^{-1}. \tag{2}$$

These measures are sometimes known as *equicontractive*, or *$\varrho$-equicontractive* if we want to emphasize the contraction factor $\varrho$. They are non-atomic, probability measures whose support is the self-similar set.

We put $\mathcal{A} = \{0, \ldots, m\}$. Given an $n$-tuple $\sigma = (j_1, \ldots, j_n) \in \mathcal{A}^n$, we write $S_\sigma$ for the composition $S_{j_1} \circ \cdots \circ S_{j_n}$ and let

$$p_\sigma = p_{j_1} \cdots p_{j_n}.$$

**Definition 2.1.** The iterated function system, $\{S_j(x) = \varrho x + d_j : j = 0, \ldots, m\}$, is said to be of *finite type* if there is a finite set $F \subseteq \mathbb{R}^+$ such that for each positive integer $n$ and any two sets of indices $\sigma = (j_1, \ldots, j_n)$, $\sigma' = (j_1', \ldots, j_n') \in \mathcal{A}^n$, either

$$\varrho^{-n} \, |S_\sigma(0) - S_{\sigma'}(0)| > c \quad \text{or} \quad \varrho^{-n} |S_\sigma(0) - S_{\sigma'}(0)| \in F,$$

where $c = (1 - \varrho)^{-1}(\max d_j - \min d_j)$ is the diameter of $K$.

If $\{S_j\}$ is of finite type and $\mu$ is an associated self-similar measure, we also say that $\mu$ is of finite type.

It is worth noting here that the definition of finite type is independent of the choice of probabilities.

Recall that an algebraic integer greater than 1 is called a *Pisot number* if all its Galois conjugates are less than 1 in absolute value. Examples include integers greater than 1 and the golden ratio, $(1 + \sqrt{5})/2$. In [21, Theorem 2.9], Ngai and Wang showed that if $\varrho^{-1}$ is a Pisot number and all $d_j \in \mathbb{Q}[\varrho^{-1}]$, then the measure $\mu$ satisfying (2) is of finite type. This result allows us to produce many examples of measures of finite type that do not satisfy the open set condition.

The case when the IFS is generated by two contractions is of particular interest.

**Notation 2.2.** We will use the notation $\mu_\varrho$ to denote the self-similar measure

$$\mu_\varrho = \frac{1}{2}\mu_\varrho \circ S_0^{-1} + \frac{1}{2}\mu_\varrho \circ S_1^{-1},$$

where $S_j = \varrho x + j(1 - \varrho)$ for $j = 0, 1$.

**Example 2.3.** When $0 < \varrho \leq 1/2$, the measures, $\mu_\varrho$, are known as *Cantor measures* (or uniform Cantor measures). Their support is the Cantor set with ratio of dissection $\varrho$ and they satisfy the open set condition. When $1/2 < \varrho < 1$, these measures are called *Bernoulli convolutions*. They fail to satisfy the open set condition, but are of finite type whenever $\varrho^{-1}$ is a Pisot number.

Given two probability measures, $\mu, \nu$, the convolution of $\mu$ and $\nu$ is defined as

$$\mu * \nu(E) = \mu \times \nu\{(x, y) : x + y \in E\}.$$

The name "Bernoulli convolution" comes from the fact that

$$\mu_\varrho = *_{n=1}^{\infty}\Big(\frac{\delta_0 + \delta_{(1-\varrho)\varrho^n}}{2}\Big),$$

where the infinite convolution is understood to converge in a weak sense.

Bernoulli convolutions, $\mu_\varrho$, with contraction factor $\varrho$ the inverse of a Pisot number, have been long studied. They have unusual properties and are of interest in fractal geometry, number theory and harmonic analysis. For example, although almost every Bernoulli convolution is absolutely continuous with respect to Lebesgue measure, and even has an $L^2$ density function, those with a Pisot inverse as the contraction factor are not only purely singular, but their Fourier transform, $\widehat{\mu_\rho}(y)$, does not even tend to zero as $y \to \pm\infty$. We refer the reader to [24] and [27] for some of the interesting history of these measures.

**Example 2.4.** Suppose $\mu$ and $\nu$ are $\varrho$-equicontractive measures, say

$$\mu = \sum_i p_i \mu \circ S_i^{-1} \quad \text{and} \quad \nu = \sum_j q_j \mu \circ T_j^{-1},$$

where

$$S_i(x) = \varrho x + d_i \quad \text{and} \quad T_j(x) = \varrho x + e_j.$$

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$$\{d_i + e_j\}_{i,j} = \{f_t\}_t.$$

Then $\mu * \nu$ is the $\varrho$-equicontractive, self-similar measure satisfying

$$\mu * \nu = \sum_t r_t(\mu * \nu) \circ U_t^{-1}$$

where

$$U_t(x) = \sum_{\substack{i,j \\ d_i+e_j=f_t}} \varrho x + f_t \quad \text{and} \quad r_t = \sum_{\substack{i,j \\ d_i+e_j=f_t}} p_i q_j.$$

It follows directly from Ngai and Wang's result [21] that any $m$-fold convolution power of the Bernoulli convolution or Cantor measure, $\mu_\varrho$, is of finite type when $\varrho^{-1}$ is Pisot.

**Example 2.5.** Another consequence of [21] is that the IFS

$$\Big\{S_j(x) = \frac{1}{R}x + \frac{j}{Rm}(R-1) \colon j = 0, \dots, m\Big\},$$

where $R \geq 2$ is an integer, is of finite type. The convex hull of the self-similar set is $[0, 1]$ and the self-similar set is the full interval $[0, 1]$ when $m \geq R - 1$. When $m \geq R$, the open set condition is not satisfied. The $m$-fold convolutions of Cantor measures with contraction factor $1/R$ are examples of self-similar measures associated with such an IFS.

These Cantor-like measures were studied in [2, 26] using different methods. In Section 7 we will see how our approach relates to some of their results.

**2.2. Standard technical assumptions.** We will refer to the following conditions on a self-similar measure $\mu$ as our standard technical assumptions:

(1) the measure $\mu = \sum_j p_j \mu \circ S_j^{-1}$ is a $\varrho$-equicontractive, self-similar measure, as in equation (2), that is of finite type;

(2) the probabilities, $\{p_j\}_{j=0}^{m}$ satisfy $p_0 = p_m = \min p_j$ (we call these *regular probabilities*);

(3) the support of $\mu$ (equivalently, the underlying self-similar set) is a closed interval. By rescaling the $d_j$ appropriately, we can assume without loss of generality that this interval is $[0, 1]$.

We remark that supp$\mu = [0, 1]$ if and only if (the rescaled) $\{d_j\}$ satisfy $d_0 = 0$, $d_m = 1 - \varrho$ and $d_{i+1} - d_i \leq \varrho$ for all $i = 0, \ldots, m - 1$. In this case, $c = 1$ in the definition of finite type.

Although some of what we say is true more generally for self-similar measures of finite type, we make use of the standard technical assumptions at key points throughout the paper.

The Bernoulli convolutions $\mu_\varrho$ and the $m$-fold convolutions of uniform Cantor measures $\mu_\varrho$ with $\varrho > 1/(m+1)$ are examples of measures satisfying the standard technical assumptions. (See Examples 2.3 and 2.4). The measures of Example 2.5, where $m \geq R - 1$, are also examples of such measures when regular probabilities are chosen.

**2.3. Net intervals and Characteristic vectors.** As we have seen, measures that are of finite type need not satisfy the open set condition. Our primary interest is in this case. The finite type property is, however, stronger than the weak separation condition (see [22] for a proof), and the multifractal analysis of self-similar measures of finite type is somewhat more tractable because of their better structure. This structure is explained in detail in [7, 8, 9], but we will give a quick overview here.

For each integer $n$, let $h_1, \dots, h_{s_n}$ be the collection of elements of the set $\{S_\sigma(0),\ S_\sigma(1)\colon \sigma \in \mathcal{A}^n\}$, listed in increasing order. Put

$$\mathcal{F}_n = \{[h_j, h_{j+1}]\colon 1 \leq j < s_n\}.$$

Elements of $\mathcal{F}_n$ are called *net intervals of level n*. For each $\Delta \in \mathcal{F}_n$, $n \geq 1$, there is a unique element $\widehat{\Delta} \in \mathcal{F}_{n-1}$ which contains $\Delta$. We call $\widehat{\Delta}$ the *parent* of $\Delta$ and $\Delta$ a *child* of $\widehat{\Delta}$. We denote the *normalized length* of $\Delta = [a, b]$ by

$$\ell_n(\Delta) = \varrho^{-n}(b - a).$$

Note that by definition there is no $\sigma \in \mathcal{A}^n$ with $a < S_\sigma(0) < b$, nor can we have $a < S_\sigma(1) < b$. Furthermore, there must be some $\sigma_1, \sigma_2$ with $S_{\sigma_1}(x) = a, S_{\sigma_2}(y) = b$ for suitable choices of $x, y \in \{0, 1\}$.

Next, we consider all $\sigma \in \mathcal{A}^n$ with $\Delta \subseteq S_\sigma[0, 1]$. As $S_\sigma[0, 1]$ is a closed interval of length $\varrho^n$, this is the same as the set of all $\sigma \in \mathcal{A}^n$ with $a - \varrho^n < S_\sigma(0) \leq a$. We suppose

$$\{\varrho^{-n}(a - S_\sigma(0))\colon \sigma \in \mathcal{A}^n,\ \Delta \subseteq S_\sigma[0, 1]\} = \{a_1, \dots, a_k\}$$

and assume $a_1 < a_2 < \cdots < a_k$. We define the *neighbour set* of $\Delta$ as

$$V_n(\Delta) = (a_1, \dots, a_k).$$

Let $\widehat{\Delta} \in \mathcal{F}_{n-1}$ be the parent of $\Delta$, and $\Delta_1, \dots, \Delta_j$ (listed in order from left to right) be all the net intervals of level $n$ which are also children of $\widehat{\Delta}$ and have the same normalized length and neighbour set as $\Delta$. Define $r_n(\Delta)$ to be the integer $r$ with $\Delta_r = \Delta$. The *characteristic vector of* $\Delta$ is the triple

$$\mathcal{C}_n(\Delta) = (\ell_n(\Delta), V_n(\Delta), r_n(\Delta)).$$

We also speak of the pair of characteristic vectors, $\alpha, \beta$, as parent and child if $\alpha = \mathcal{C}_{n-1}(\widehat{\Delta})$ and $\beta = \mathcal{C}_n(\Delta)$ for a parent/child pair $\widehat{\Delta}, \Delta$. The characteristic vector is important because it carries the neighbourhood information about $\Delta$.

Put

$$\Omega = \{\mathcal{C}_n(\Delta)\colon n \in \mathbb{N},\ \Delta \in \mathcal{F}_n\}.$$

If the measure is of finite type, then $\Omega$ will contain only finitely many distinct characteristic vectors.

Suppose the net interval, $\widehat{\Delta}$, has two children, $\Delta_1$ and $\Delta_2$, that differ only in the value of $r_n(\Delta_i)$, that is, they have the same length and the same neighbourhood set. The characteristic vectors for the children of $\Delta_1$ and $\Delta_2$, will be identical as they depend only on $\ell_n(\Delta_i)$ and $V_n(\Delta_i)$, and not on $r_n(\Delta_i)$. For notational reasons, we find it convenient to take advantage of this when drawing the directed graph relating parents to children by suppressing $r_n(\Delta_i)$, equating these two children on the graph, and allowing multiple edges from $\widehat{\Delta}$ to $\Delta_1$. We will call the characteristic vectors where we have suppressed the numbers $r_n(\Delta_i)$ the reduced characteristic vectors, and we will call the resulting graph the reduced transition graph.

**Example 2.6.** In [8, Section 4] and [9, Section 6], Feng studied the Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$the golden ratio, and found that there were seven characteristic vectors. Their normalized lengths and neighbourhood sets are given by

- characteristic vector 1: $(1, (0), 1)$;
- characteristic vector 2: $(\varrho, (0), 1)$;
- characteristic vectors 3a and 3b: $(1-\varrho, (0, \varrho), 1)$ and $(1-\varrho, (0, \varrho), 2)$;
- characteristic vector 4: $(\varrho, (1-\varrho), 1)$;
- characteristic vector 5: $(\varrho, (0, 1-\varrho), 1)$;
- characteristic vector 6: $(2\varrho-1, (1-\varrho), 1)$

Notice there are only six reduced characteristic vectors; we label the two characteristic vectors with identical length and neighbourhood set as 3a and 3b. In [8] these were labelled as 3 and 7. The directed graphs in Figure 1 show the parent/children relationships. The term “essential class”, referred to in the figure, is defined in Section 4.

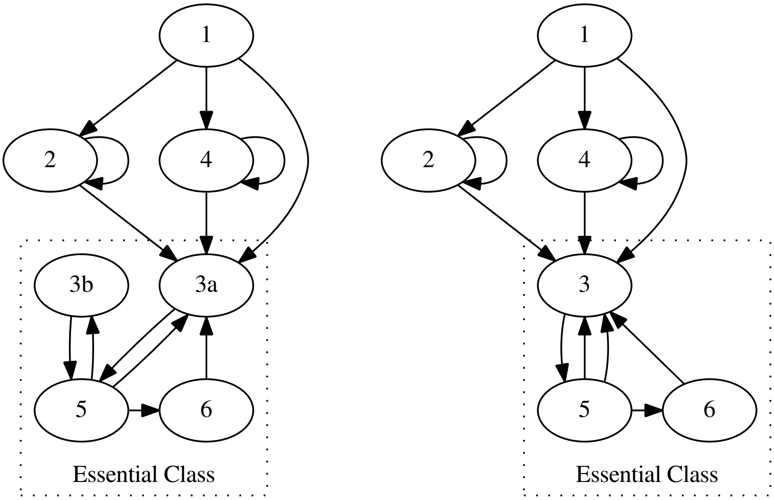
By an *admissible path, $\eta$, of length $L(\eta)=n$*, we will mean an ordered $n$-tuple, $\eta=(\gamma_j)_{j=1}^n$, where $\gamma_j\in\Omega$ for all $j$ and the characteristic vector, $\gamma_j$, is the parent of $\gamma_{j+1}$.

By the *symbolic expression* of $\Delta\in\mathcal{F}_n$ we mean an admissible path of length $n+1$, denoted

$$[\Delta]=(\mathcal{C}_0(\Delta_0),\dots,\mathcal{C}_n(\Delta_n)),$$

where $\Delta=\Delta_n$ and for each $j<n$, $\Delta_j\in\mathcal{F}_j$. Here $\Delta_0$ is $[0,1]$. Feng [7] proved that the symbolic expression uniquely determines $\Delta$.

For $x\in[0,1]$, the symbolic representation for $x$, denoted $[x]$, will mean the sequence $(\mathcal{C}_0(\Delta_0),\mathcal{C}_1(\Delta_1),\dots)$ of characteristic vectors where $x\in\Delta_n$ for all $n$ and $\Delta_j\in\mathcal{F}_j$ is the parent of $\Delta_{j+1}$. We note that unless $x$ is an endpoint of a net interval and not equal to 0 or 1 (in which case there are two representations of $x$), $[x]$ is unique. The notation $[x|N]$ will mean the admissible path consisting of the first $N$ characteristic vectors of $[x]$.




(a) Transition graph. (b) Reduced Transition graph.

Figure 1. Transition graph for $\mu_\varrho$ with $\varrho^{-1}$ the golden ratio.

We frequently write $\Delta_n(x)$ for the net interval of level $n$ containing $x$. Thus $[x]$ is the sequence where the first $n+1$ terms gives the symbolic representation of $\Delta_n(x)$ for each $n$.

## 3. Transition matrices and local dimensions

### 3.1. Local dimensions of measures of finite type

**Definition 3.1.** Given a probability measure $\mu$, by the *upper local dimension* of $\mu$ at $x \in \operatorname{supp}\mu$, we mean the number

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r \to 0^+} \frac{\log \mu([x-r, x+r])}{\log r}.$$

Replacing the lim sup by lim inf gives the *lower local dimension*, denoted $\underline{\dim}_{\mathrm{loc}}\mu(x)$. If the limit exists, we call the number the *local dimension* of $\mu$ at $x$ and denote this by $\dim_{\mathrm{loc}} \mu(x)$.

Multifractal analysis refers to the study of the local dimensions of measures. For a $\varrho$-equicontractive measure $\mu$, it is easy to check that

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{n\to\infty} \frac{\log \mu([x-\varrho^n, x+\varrho^n])}{n \log \varrho} \quad \text{for } x \in \operatorname{supp}\mu, \tag{3}$$

and similarly for the upper and lower local dimensions.

Our first several lemmas will enable us to show that we can replace the interval $[x-\varrho^n, x+\varrho^n]$ by $\Delta_n(x)$.

**Lemma 3.2.** *Suppose $\mu$ satisfies the standard technical assumptions. Let $\Delta = [a,b] \in \mathcal{F}_n$, with $V_n(\Delta) = (a_1, \dots, a_k)$. Then*

$$\mu(\Delta) = \sum_{i=1}^{k} \mu[a_i, a_i + \ell_n(\Delta)] \sum_{\substack{\sigma \in \mathcal{A}^n \\ \varrho^{-n}(a - S_\sigma(0)) = a_i}} p_\sigma .$$

*Proof.* This argument can basically be found in Feng [7], but we give the details here for completeness. Iterating (2) $n$ times gives

$$\mu(\Delta) = \sum_{\sigma \in \mathcal{A}^n} p_\sigma \mu(S_\sigma^{-1}(\Delta)).$$

Since $\mu$ is a non-atomic measure supported on [0, 1], we have

$$\mu(\Delta) = \sum_{\substack{\sigma \in \mathcal{A}^n \\ S_\sigma(0,1) \cap \Delta \neq \emptyset}} p_\sigma \mu(S_\sigma^{-1}(\Delta))$$

Now, $S_\sigma(0,1) \cap \Delta \neq \emptyset$ implies that $\Delta \subset S_\sigma[0,1]$, hence by definition of the neighbourhood set $\varrho^{-n}(a - S_\sigma(0)) = a_i$ for some $i$. Thus $S_\sigma(0) = a - a_i \varrho^n$, so $S_\sigma[0,1] = [a - a_i\varrho^n, a - a_i\varrho^n + \varrho^n]$. This implies that

$$\mu(\Delta) = \sum_{i=1}^{k} \sum_{\substack{\sigma \in \mathcal{A}^n \\ \varrho^{-n}(a - S_\sigma(0)) = a_i}} p_\sigma \mu(S_\sigma^{-1}(\Delta))$$

We observe that $S_\sigma(x) = \varrho^n x + S_\sigma(0)$, and hence

$$\begin{aligned} S_\sigma([a_i, a_i + \ell_n(\Delta)]) &= [a_i\varrho^n + a - a_i\varrho^n, a_i\varrho^n + a - a_i\varrho^n + \ell_n(\Delta)\varrho^n] \\ &= [a, a + \ell_n(\Delta)\varrho^n] \\ &= [a, b] \\ &= \Delta. \end{aligned}$$

Hence

$$\mu(\Delta) = \sum_{i=1}^{k} \sum_{\substack{\sigma \in \mathcal{A}^n \\ \varrho^{-n}(a - S_\sigma(0)) = a_i}} p_\sigma \mu[a_i, a_i + \ell_n(\Delta)]$$

$$= \sum_{i=1}^{k} \mu[a_i, a_i + \ell_n(\Delta)] \sum_{\substack{\sigma \in \mathcal{A}^n \\ \varrho^{-n}(a - S_\sigma(0)) = a_i}} p_\sigma$$

as claimed. □

**Notation 3.3.** For $i = 1, 2, \ldots, \mathrm{card}(V_n(\Delta)) = k$, put

$$P_n^i(\Delta) = p_0^{-n} \sum_{\sigma \in \mathcal{A}^n : \varrho^{-n}(a - S_\sigma(0)) = a_i} p_\sigma$$

and

$$P_n(\Delta) = \sum_{i=1}^{k} P_n^i(\Delta).$$

Here we have chosen to normalize by multiplying by $p_0^{-n}$. This is done so that the minimal non-zero entry in the transition matrices (defined in the next subsection) is at least 1.

**Corollary 3.4.** *There is a constant $c > 0$ such that for any $n$ and any $\Delta \in \mathcal{F}_n$,*

$$c p_0^n P_n(\Delta) \leq \mu(\Delta) \leq p_0^n P_n(\Delta).$$

*Proof.* The upper bound is clear from the lemma. For the lower bound we note that each $\mu[a_i, a_i + \ell_n(\Delta)] > 0$ as the support of $\mu$ is the full interval [0, 1]. The finite type condition ensures there are only finitely many choices for $[a_i, a_i + \ell_n(\Delta)]$. □

**Lemma 3.5.** *Suppose $\mu$ satisfies the standard technical assumptions. There are constants $c_1, c_2 > 0$ such that if $\Delta_1$, $\Delta_2$ are two adjacent net intervals of level $n$, then*

$$c_1 \frac{1}{n} P_n(\Delta_2) \leq P_n(\Delta_1) \leq c_2 n P_n(\Delta_2).$$

*Proof.* The proof is similar to that of [8, Lemma 2.11] and proceeds by induction on $n$. The base case holds as there are only finitely many choices for $P_1(\Delta_j)$ when $\Delta_j \in \mathcal{F}_1$. Now assume the result for level $n - 1$ and we will verify it holds for level $n$.

If $\Delta_1, \Delta_2$ have the same parent $\widehat{\Delta}$, the result follows easily from the observation that

$$P_{n-1}(\widehat{\Delta}) \leq P_n(\Delta_j) \leq m p_0^{-1} \max p_j P_{n-1}(\widehat{\Delta}).$$

Otherwise, $\Delta_1$ and $\Delta_2$ are children of adjacent net intervals of level $n-1$, $\widehat{\Delta_1}, \widehat{\Delta_2}$ respectively, and we can suppose $\Delta_1$ is to the left of $\Delta_2$. As in [8], put

$$D_1 = \{\sigma \in \mathcal{A}^{n-1} \colon \widehat{\Delta_1} \subseteq S_\sigma[0,1] \text{ and they share the same right endpoint}\};$$
$$D_2 = \{\sigma \in \mathcal{A}^{n-1} \colon \widehat{\Delta_2} \subseteq S_\sigma[0,1] \text{ and they share the same left endpoint}\};$$
$$E_j = \{\sigma \in \mathcal{A}^{n-1} \diagdown D_j \colon \widehat{\Delta_j} \subseteq S_\sigma[0,1]\},\ j = 1, 2.$$

The definitions ensure that $E_1 = E_2$,

$$p_0^{n-1} P_{n-1}(\widehat{\Delta_j}) = \sum_{\sigma \in D_j} p_\sigma + \sum_{\sigma \in E_j} p_\sigma$$

and

$$\begin{aligned} p_0^n P_n(\Delta_1) &\leq \sum_{\sigma \in D_1} p_\sigma p_m + \sum_{\sigma \in E_1} p_\sigma \sum_{j=0}^{m} p_j \\ &\leq p_m \sum_{\sigma \in D_1} p_\sigma + p_m \sum_{\sigma \in E_1} p_\sigma + \sum_{\sigma \in E_2} p_\sigma + \sum_{\sigma \in D_2} p_\sigma \\ &\leq p_m p_0^{n-1} P_{n-1}(\widehat{\Delta_1}) + p_0^{n-1} P_{n-1}(\widehat{\Delta_2}) \end{aligned}$$

Applying the induction assumption gives

$$\begin{aligned} p_0 P_n(\Delta_1) &\leq p_m P_{n-1}(\widehat{\Delta_1}) + P_{n-1}(\widehat{\Delta_2}) \\ &\leq p_m c_2 (n-1) P_{n-1}(\widehat{\Delta_2}) + P_{n-1}(\widehat{\Delta_2}) \\ &\leq (p_m c_2 (n-1) + 1) P_{n-1}(\widehat{\Delta_2}). \end{aligned}$$

Taking $c_2 \geq 1/p_0 = 1/p_m \geq 1$ gives

$$\begin{aligned} p_0 P_n(\Delta_1) &\leq (c_2 p_m (n-1) + c_2 p_m) P_{n-1}(\widehat{\Delta_2}) \\ &\leq c_2 p_m n P_{n-1}(\widehat{\Delta_2}) \\ &\leq c_2 p_m n P_n(\Delta_2). \end{aligned}$$

By observing that $p_0 = p_m > 0$ this implies that $P_n(\Delta_1) \leq c_2 n P_n(\Delta_2)$ as required.

The other inequality is similar. □

Note that in the proof the assumption that $\{p_j\}$ were regular probabilities was important.

The following is immediate from the two previous results.

**Corollary 3.6.** *There are constants $C_1, C_2$ such that if $\Delta_1, \Delta_2$ are adjacent net intervals of level $n$, then*

$$C_1 \frac{1}{n} \mu(\Delta_2) \le \mu(\Delta_1) \le C_2 n \mu(\Delta_2).$$

Together these results yield the following useful approach to computing local dimensions.

**Corollary 3.7.** *Suppose $\mu$ satisfies the standard technical assumptions. Let $x \in \operatorname{supp} \mu$ and $\Delta_n(x)$ denote a net interval of level $n$ containing $x$. Then*

$$\begin{aligned} \overline{\dim}_{\mathrm{loc}} \mu(x) &= \limsup_{n \to \infty} \frac{\log \mu(\Delta_n(x))}{n \log \varrho} \\ &= \frac{\log p_0}{\log \varrho} + \limsup_{n \to \infty} \frac{\log P_n(\Delta_n(x))}{n \log \varrho}. \end{aligned} \tag{4}$$

*A similar statement holds for the* (*lower*) *local dimensions.*

*Proof.* Since any net interval of level $n$ has length at most $\varrho^n$, the interval $[x - \varrho^n, x + \varrho^n]$ contains $\Delta_n(x)$. The finite type property ensures it is contained in a union of a uniformly bounded number of $n$'th level net intervals, say $\bigcup_{j=1}^N \Delta_n(x_j)$, where $\Delta_n(x_j)$ is adjacent to $\Delta_n(x_{j+1})$ and for a suitable index $j$, $x_j = x$. Thus for constants $c, C$ (independent of the choice of $n$ and $x$),

$$\begin{aligned} c p_0^n P_n(\Delta_n(x)) \le \mu(\Delta_n(x)) \le \mu([x - \varrho^n, x + \varrho^n]) \le \sum_{j=1}^N \mu(\Delta_n(x_j)) \\ \le \sum_{j=1}^N p_0^n P_n(\Delta_n(x_j)) \le N p_0^n C^N n^N P_n(\Delta_n(x)). \end{aligned}$$

Thus the limiting behaviour of the three expressions

$$\frac{\log \mu([x - \varrho^n, x + \varrho^n])}{n \log \varrho}, \quad \frac{\log \mu(\Delta_n(x))}{n \log \varrho}$$

and

$$\frac{\log p_0}{\log \varrho} + \frac{\log P_n(\Delta_n(x))}{n \log \varrho}$$

coincide. $\square$

**Remark 3.8.** If $\Delta = [0, b]$, then $V_n(\Delta) = \{0\}$ and hence $P_n(\Delta) = 1$. Consequently, $\dim_{\mathrm{loc}} \mu(0) = \log p_0 / \log \varrho$. More generally, since $V_n(\Delta)$ is never empty and $p_0$ is the minimal probability, it follows that $P_n(\Delta_n(x)) \geq 1$ for all $n$ and $x$. Consequently,

$$\dim_{\mathrm{loc}} \mu(x) \leq \dim_{\mathrm{loc}} \mu(0) \text{ for all } x \in \operatorname{supp}\mu.$$

**3.2. Transition matrices.** The results of the previous subsection show that for studying the local dimensions of these measures it will be helpful to make accurate estimates of $P_n(\Delta_n(x))$. Towards this, slightly modifying [7] we define *primitive transition matrices*, $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))$, for a net interval $\Delta = [a, b]$ of level $n$ and parent $\widehat{\Delta} = [c, d]$ as follows:

**Notation 3.9.** Suppose $V_n(\Delta) = (a_1, \dots, a_K)$ and $V_{n-1}(\widehat{\Delta}) = (c_1, \dots, c_J)$. For $j = 1, \dots, J$ and $k = 1, \dots, K$, we set

$$T_{jk} := (T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)))_{jk} = p_0^{-1} p_\ell$$

if $\ell \in \mathcal{A}$ and there exists $\sigma \in \mathcal{A}^{n-1}$ with $S_\sigma(0) = c - \varrho^{n-1} c_j$ and $S_{\sigma\ell}(0) = a - \varrho^n a_k$. This is equivalent to saying

$$T_{jk} = p_0^{-1} p_\ell \text{ if } c - \varrho^{n-1} c_j + \varrho^{n-1} d_\ell = a - \varrho^n a_k.$$

We set $(T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)))_{jk} = 0$ otherwise.

As $\Omega$ is finite for a measure of finite type, there is an upper bound on the size of these matrices. The entries are non-negative and all non-zero entries are at least one. Each column has at least one non-zero entry because $a_k \in V_n(\Delta)$ if and only if there is some $c_j \in V_{n-1}(\widehat{\Delta})$ which “contributes” to it, in the sense defined above.

It is also important to note that the standard technical assumption that $\operatorname{supp}\mu = [0, 1]$ guarantees that given $\sigma \in \mathcal{A}^{n-1}$ such that $S_\sigma(0) = c - \varrho^{n-1} c_j$, there exists $\ell \in \mathcal{A}$ such that $S_{\sigma\ell}(0) = \varrho^{n-1} d_\ell + S_\sigma(0) \in (a - \varrho^n, a]$. This means that each row of the matrix $(T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)))$ also has a non-zero entry.

For $K = \operatorname{card}(V_n(\Delta))$, put

$$Q_n(\Delta) = (P_n^1(\Delta), \dots, P_n^K(\Delta)).$$

The same reasoning as in [7, Theorem 3.3] shows that

$$Q_n(\Delta) = Q_{n-1}(\widehat{\Delta})(T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))).$$

Thus if $[\Delta] = (\gamma_0, \ldots, \gamma_n)$ (that is, $\gamma_j = \mathcal{C}_j(\Delta_j)$ and $\gamma_0 = \mathcal{C}_0([0,1])$), then since $Q_0[0,1] = (1)$ we have

$$P_n(\Delta) = \|Q_n(\Delta)\| = \|T(_0, \gamma_1) \cdots T(\gamma_{n-1}, \gamma_n)\|,$$

where by the norm of matrix $M = (M_{jk})$ we mean

$$\|M\| := \sum_{jk} |M_{jk}|.$$

Given an admissible path $\eta = (\gamma_1, \ldots, \gamma_n)$, we write

$$T(\eta) = T(\gamma_1, \ldots, \gamma_n) = T(_1, \gamma_2) \cdots T(\gamma_{n-1}, \gamma_n)$$

and refer to such a product as a *transition matrix*.

With this notation, the results of the previous subsection can be stated as

**Corollary 3.10.** *Suppose $\mu$ satisfies the standard technical assumptions. If $x \in$ supp $\mu$, then*

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \frac{\log p_0}{\log \varrho} + \limsup_{n\to\infty} \frac{\log \|T([x|n])\|}{n \log \varrho} \tag{5}$$

*and similarly for the* (*lower*) *local dimension.*

**Example 3.11.** Again, consider the Bernoulli convolution, $\mu_\varrho$, with $\varrho^{-1}$ the golden ratio. Feng [8] showed that 0 has symbolic representation $(1, 2, 2, \ldots)$ and that $T(1, 2, 2, \ldots) = [1]$. Applying Corollary 3.10 gives another proof that $\dim_{\mathrm{loc}} \mu_\varrho(0) = \log 2 / |\log \varrho|$.

Next, we give a useful simple lemma.

**Lemma 3.12.** *Suppose $\mu$ satisfies the standard technical assumptions. Let $A$ and $B$ be transition matrices. Then $\|B\| \leq \|AB\|$ and $\|B\| \leq \|BA\|$.*

*Proof.* We have

$$\|AB\| = \sum_{ij} \Big( \sum_k A_{ik} B_{kj} \Big) = \sum_{jk} \Big( \sum_i A_{ik} \Big) B_{kj}.$$

Since all the entries of each of the matrices is nonnegative and each column of $A$ has an entry $\geq 1$, it follows that $\|AB\| \geq \|B\|$.

The argument for the other inequality is similar, noting that each row of $A$ has an entry $\geq 1$ as a consequence of the standard technical assumptions. □

An important consequence of this result is that the local dimension of $\mu$ at $x$ depends only on the tail of the symbolic representation of $x$.

**Corollary 3.13.** *Suppose* $[x] = (\gamma_0, \gamma_1, \dots)$. *For any* $N$,

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \frac{\log p_0}{\log \varrho} + \limsup_{n\to\infty} \frac{\log \|T(\gamma_N, \gamma_{N+1}, \dots, \gamma_{N+n})\|}{n \log \varrho}$$

*and similarly for the* (*lower*) *local dimension.*

*If* $[y] = (\gamma_0, \beta_1, \dots, \beta_n, \gamma_N, \gamma_{N+1}, \dots)$, *then the* (*upper or lower*) *local dimensions of* $\mu$ *at* $x$ *and* $y$ *agree.*

*Proof.* This holds since

$$\begin{aligned}\|T(\gamma_N, \gamma_{N+1}, \dots, \gamma_{N+n})\| &\le \|T(\gamma_0, \dots, \gamma_N) T(\gamma_N, \gamma_{N+1}, \dots, \gamma_{N+n})\| \\ &= \|T(\gamma_0, \dots, \gamma_N, \gamma_{N+1}, \dots, \gamma_{N+n})\| \\ &\le \|T(\gamma_0, \dots \gamma_N)\| \|T(\gamma_N, \gamma_{N+1}, \dots, \gamma_{N+n})\|. \quad \square\end{aligned}$$

**Notation 3.14.** By $\mathrm{sp}(M)$ we mean the spectral radius of the square matrix $M$, the largest eigenvalue of $M$ in absolute value. Recall that

$$\mathrm{sp}(M) = \lim_n \|M^n\|^{1/n}.$$

We will call a matrix $M$ *positive* if all its entries are strictly positive and write $M > 0$. We record here some elementary facts about positive matrices that will be useful later.

**Lemma 3.15.** *Suppose* $\mu$ *satisfies the standard technical assumptions. Assume* $A, B, C$ *are transition matrices and* $B$ *is positive.*

(1) *Then* $\|ABC\| \ge \|A\| \, \|C\|$.

(2) *There is a constant* $C_1 = C_1(B)$ *such that if* $AB$ *is a square matrix, then*

$$\|AB\| \le C_1 \, \mathrm{sp}(AB).$$

(3) *Suppose* $B$ *is a square matrix. There is a constant* $C_2 = C_2(B)$ *such that*

$$\mathrm{sp}(B^n) \le \|B^n\| \le C_2 \, \mathrm{sp}(B^n) \text{ for all } n.$$

*Proof.* To see (1), let $B = (B_{jk})$. As all $B_{jk} \geq 1$, a simple calculation gives

$$\|ABC\| = \sum_{ijkl} A_{ij} B_{jk} C_{kl} \geq \sum_{ijkl} A_{ij} C_{kl} = \|A\| \, \|C\| \, .$$

For (2), assume $A = (A_{jk})$ is a $q \times p$ matrix. Let $b = \max B_{jk}$. As the entries of $A$ are non-negative and the entries of $B$ are at least 1, it is easy to see that

$$\begin{aligned} \|AB\| &= \sum_{j,k} (AB)_{jk} = \sum_{j,k,l} A_{jl} B_{lk} \\ &\leq b \sum_{j,k=1}^{q} \sum_{l=1}^{p} A_{jl} \leq bq \sum_{j=1}^{q} \sum_{l=1}^{p} A_{jl} B_{lj} \\ &= bq \sum_{j=1}^{q} (AB)_{jj} \leq bq^2 \operatorname{sp}(AB), \end{aligned}$$

with the final inequality holding because the sum of the diagonal entries of $AB$ is the sum of the eigenvalues of $AB$, counted by multiplicity.

For (3), let $B = PJP^{-1}$ be the Jordan decomposition of $B$ and let $\beta = \operatorname{sp}(B)$. By the Perron-Frobenius theory, $\beta$ is a simple root of the characteristic polynomial of $B$ and all other eigenvalues of $B$ are strictly less than $\beta$ in modulus. Since all entries of $B$ are at least 1, it can be easily seen that $\beta > 1$. As $\|B^n\| \leq \|P\| \, \|J^n\| \, \|P^{-1}\|$, it is enough to prove $\|J^n\| \leq C_1 \beta^n$ where $C_1$ depends on $B$, but not $n$.

Assume $B$ is of size $d \times d$. Since the Jordan block for $\beta$ is $1 \times 1$, all entries of $J^n$, other than the $(1, 1)$ entry which is $\beta^n$, are either 0 or of the form $\binom{n}{j} \alpha^{n-j}$ where $j \leq \min(d-1, n)$ and $\alpha$ is an eigenvalue of $B$ with $|\alpha| < \beta$. Thus

$$\|J^n\| \leq \beta^n + d^2 n^d \beta_0^n$$

where $\beta_0 < \beta$ is the maximum of 1 and the modulus of the eigenvalues of $B$ other than $\beta$. As $(\beta/\beta_0)^n \geq d^2 n^d$ for all $n$ sufficiently large depending on $d, \beta, \beta_0$, it follows that for some constant $C_2$, depending on $\beta, \beta_0, d$, (and hence depending only on $B$) we have $\|J^n\| \leq C_2 \beta^n$ for all $n$. This proves the right hand inequality.

The left hand inequality is obvious. □

## 4. Loop classes and periodic points

**4.1. Essential and Loop classes.** Feng in [9] also introduced the notion of an essential class for measures of finite type. Here we introduce the more general definition of a loop class, of which the essential class is a special case.

**Definition 4.1.** (i) A non-empty subset $\Omega' \subseteq \Omega$ is called a *loop class* if whenever $\alpha, \beta \in \Omega'$, then there are characteristic vectors $\gamma_j$, $j = 1, \dots, n$, such that $\alpha = \gamma_1$, $\beta = \gamma_n$ and $(\gamma_1, \dots, \gamma_n)$ is an admissible path with all $\gamma_j \in \Omega'$.

(ii) A loop class $\Omega' \subseteq \Omega$ is called an *essential class* if, in addition, whenever $\alpha \in \Omega'$ and $\beta \in \Omega$ is a child of $\alpha$, then $\beta \in \Omega'$.

(iii) We call a loop class *maximal* if it is not properly contained in any other loop class.

The finite type property ensures that every element in the support of $\mu$ is contained in a loop class. Clearly every loop class is contained in a unique maximal loop class. Feng in [9, Lemma 6.4], proved there is always precisely one essential class. Of course, the essential class is a maximal loop class.

**Notation 4.2.** We will denote the essential class by $\Omega_0$ and here-after speak of "the" essential class.

**Definition 4.3.** If $[x] = (\gamma_0, \gamma_1, \gamma_2, \dots)$ with $\gamma_j \in \Omega_0$ for all large $j$, we will say that $x$ is an *essential point* (or is *in the essential class*) and call $x$ a *non-essential point* otherwise. The phrase, $x$ is *in the loop class* $\Omega'$, will have a similar meaning. An admissible path will be said to be in a given loop class if all its members are in that class.

**Remark 4.4.** Note that if $[x] = (\gamma_j)$ is non-essential, then none of the characteristic vectors $\gamma_j$ belong to $\Omega_0$. A non-essential point necessarily has its tail in some loop class external to the essential class.

We remark that the essential class is dense in $[0, 1]$. This is because the uniqueness of the essential class ensures that every net interval contains a net subinterval of higher level whose characteristic vector is in the essential class. In fact, we show next that the set of essential points has full Lebesgue measure in $[0, 1]$.

**Proposition 4.5.** *Suppose $\mu$ satisfies the standard technical assumptions. Then the set of non-essential points is a subset of a closed set of Lebesgue measure* 0.

*Proof.* As we already observed, every net interval contains a descendent net subinterval whose characteristic vector is in the essential class. (We will abuse notation slightly and call such a net subinterval "essential".) The finite type property ensures we can find such a net subinterval in a bounded number of generations and that there exists some $\lambda > 0$ such that the proportion of the length of the net subinterval to the length of the original interval is $\geq \lambda$.

We now exhibit a Cantor-like construction. We begin with [0, 1]. Consider the first level at which there is a net subinterval that is essential. Remove the interiors of all the net subintervals of this level that are essential. The resulting closed subset of [0, 1] is a finite union of closed intervals, say $\mathcal{U}_1$, whose lengths total at most $1 - \lambda$. We repeat the process of removing the interiors of the essential net subintervals at the next level at which there are essential net subintervals in each of the intervals of $\mathcal{U}_1$. The resulting closed subset now has length at most $(1-\lambda)^2$.

After repeating this procedure $k$ times one can see that the non-essential points are contained in a finite union of closed intervals, $C_k$, whose total length is at most $(1 - \lambda)^k$. It follows that the non-essential points are contained in the closed set $\bigcap_{k=1}^{\infty} C_k$, and this set has measure 0. □

**Remark 4.6.** It is worth remarking that the construction above may leave some essential points within the Cantor-like construction. As the resulting set is measure 0, the smaller set of just the non-essential points will also be measure 0.

**Example 4.7.** From Figure 1 one can see that the Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$ the golden ratio, has seven distinct loop classes: $\{3a, 3b, 5, 6\}$, $\{3a, 3b, 5\}$, $\{3a, 5, 6\}$, $\{3a, 5\}$, $\{3b, 5\}$, $\{2\}$, and $\{4\}$. Of these, $\{2\}$, $[4\}$ and the essential class, $\{3a, 3b, 5, 6\}$ (with 4 elements and 3 reduced elements) are maximal. The two loop classes external to the essential class, $\{2\}$ and $\{4\}$, are associated to the two endpoints, 0 and 1. These are the only two non-essential points, in other words, the set of essential points is $(0, 1)$.

**Definition 4.8.** We will say the loop class $\Omega'$ is of *positive type* if there is an admissible path $\eta$ in $\Omega'$ such that $T(\eta)$ is a positive matrix.

**Remark 4.9.** We note that as there is a non-zero entry in each row and column of each primitive transition matrix, then any loop class $\Omega'$ of positive type has the property that for every $\delta, \delta' \in \Omega'$ there is an admissible path $\eta = (\delta, \delta_1, \ldots, \delta_r, \delta')$ in $\Omega'$ such that $T(\eta)$ is positive.

**Example 4.10.** The loop classes $\{2\}$ and $\{4\}$ of Example 4.7 are of positive type since (as shown in [8, Section 4]) $T(2, 2) = T(4, 4) = [1]$. As $T(5, 6, 3a) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, the essential class, $\{3a, 3b, 5, 6\}$, is also of positive type. However, the loop class, $\{3a, 5\}$, is not of positive type since all transition matrices from this loop class are of the form $\left[\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right]$ for some $n$.

**Remark 4.11.** In Example 8.2 we give a Bernoulli convolution where a maximal loop class is not of positive type.

However, the essential class is always of positive type under our standard technical assumptions.

**Proposition 4.12.** *Suppose $\mu$ satisfies the standard technical assumptions. Then the essential class is of positive type.*

*Proof.* Fix $\delta \in \Omega_0$. We recall that Feng in [9, Lemma 6.4] showed that given any positive integer $k \leq \text{cardinality}(V(\delta))$, there is an admissible path $\gamma$ in the essential class, going from $\delta$ to $\delta$, such that all entries of the $k$'th row of $T(\gamma)$ are non-zero.

Choose such a path, $\eta_1$, with all the entries of row 1 of $T(\eta_1)$ non-zero. Row 2 of $T(\eta_1)$ has a non-zero entry in some column, say $k$, and, of course, the $(1, k)$ entry is also non-zero. Choose an essential, admissible path $\eta_2$ from $\delta$ to $\delta$, with all entries of row $k$ non-zero. Matrix multiplication shows that all entries of both rows 1 and 2 of $T(\eta_1)T(\eta_2) = T(\eta_1, \eta_2)$ are non-zero. By repeated application of this reasoning we can construct an admissible path $\eta$ in $\Omega_0$, from $\delta$ to $\delta$, such that all entries of $T(\eta)$ are strictly positive. □

### 4.2. Periodic points and their local dimensions

**Definition 4.13.** We call $x$ ∈supp$\mu$ a *periodic point* if $x$ has symbolic representation

$$[x] = (\gamma_0, \ldots, \gamma_s, \theta^-, \theta^-, \ldots)$$

where $\theta$ is an admissible cycle (a non-trivial path with the same first and last letter) and $\theta^-$ is the path with the last letter of $\theta$ deleted. We refer to $\theta$ as a period of $x$.

We will say that periodic $x$ is *positive* if the square transition matrix $T(\theta)$ is positive.

It is worth noting that $\theta$ and $\theta^-$ are not uniquely defined. For example, the path $(1, 3a, 5, 3a, 5, 3a, \ldots)$ from Example 4.7 can be decomposed as $(\gamma_0, \gamma_1) = (1, 3a)$ and $\theta^- = (5, 3a)$, or as $(\gamma_0) = (1)$ and $\theta^- = (3a, 5)$, or as $(\gamma_0) = (1)$ and $\theta^- = (3a, 5, 3a, 5)$, etc. In what follows, it will not make any significant difference as to which choice is made.

Observe that

$$T([x|N]) = T(\gamma_0, \ldots, \gamma_s, \delta_1)(T(\theta))^n$$

when $N = s + 1 + L(\theta^-)n$ and $\delta_1$ is the first letter of $\theta$. (Recall $L(\theta^-)$ is the length of path $\theta^-$.) A periodic $x$ is in the essential class if it has a period that is a path in the essential class.

There is a simple formula for the local dimensions at periodic points.

**Proposition 4.14.** *Suppose $\mu$ satisfies the standard technical assumptions. Let $x$ be the periodic point with period $\theta$. Then the local dimension exists and*

$$\dim_{\text{loc}} \mu(x) = \frac{\log p_0}{\log \varrho} + \frac{\log(\text{sp}(T(\theta)))}{L(\theta^-) \log \varrho}.$$

*Proof.* Suppose $[x] = (\gamma_0, \dots, \gamma_s, \theta^-, \theta^-, \dots)$ and $\theta^- = (\delta_1, \dots, \delta_{L(\theta^-)})$. Let $S = T(\gamma_0, \dots, \gamma_s, \delta_1)$. According to Lemma 3.12, there is a constant $c > 0$ such that

$$\|(T(\theta))^n\| \leq \|S(T(\theta))^n\| \leq \|S(T(\theta))^n T(\delta_1, \dots, \delta_r)\| \leq c\,\|(T(\theta))^n\|$$

for any $r < L(\theta^-)$. Thus Corollary 3.10 implies that

$$\dim_{\text{loc}} \mu(x) = \frac{\log p_0}{\log \varrho} + \lim_{n\to\infty} \frac{\log \|(T(\theta))^n\|}{nL(\theta^-) \log \varrho}.$$

Since the limit exists and $\frac{1}{n} \log \|(T(\theta))^n\| \to \log(\text{sp}(T(\theta)))$, the result follows. $\square$

## 5. Local dimensions of positive loop classes

In [8] and [9], Feng showed that the set of local dimensions for the Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$ the golden ratio, was an interval and determined its endpoints. This is not true, in general, for measures of finite type. For instance, it is known that the set of local dimensions of the $m$-fold convolution of uniform Cantor measures with contraction factor $1/R$, for integer $R \leq m$, is the union of an interval and an isolated point (see [2, 16, 26]).

Feng also proved that the set of attainable local dimensions of $\mu_\varrho$ was the closure of the set of local dimensions at periodic points. In this section we will prove that if a loop class, $\Omega'$, is of positive type, then the set of local dimensions at points in $\Omega'$ is a closed interval. Moreover, this interval is the closure of the set of local dimensions at the periodic points in $\Omega'$. These statements hold, in particular, for the essential class, $\Omega_0$, since the essential class is a positive loop class according to Proposition 4.12.

In Sections 6, 7 and 8 we give examples to illustrate that the set of local dimensions attained at points of a loop class external to the essential class can overlap, or may be disjoint from the local dimensions of the essential class.

### 5.1. Local dimensions at periodic points are dense

**Theorem 5.1.** *Suppose $\mu$ satisfies the standard technical assumptions. Assume that $\Omega'$ is a loop class of positive type. The set of local dimensions of $\mu$ at positive, periodic points in the loop class $\Omega'$ is dense in the set of all local dimensions at points in $\Omega'$. It is also dense in the set of all upper (or lower) local dimensions at points in $\Omega'$.*

**Remark 5.2.** We remark that in this theorem (and other results of this section) the assumption that the loop class is of positive type may not be necessary for particular IFS and particular loops.

*Proof.* We will prove denseness in the set of lower local dimensions at points in the loop class $\Omega'$. The arguments for the (upper) local dimensions are the same.

Fix $x$ in $\Omega'$, say $[x] = (\gamma_0, \gamma_1, \gamma_2, \dots)$ with $\gamma_k \in \Omega'$ for all $k \geq M$. Choose a subsequence $(n_k)$ such that

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \frac{\log p_0}{\log \varrho} + \lim_{k\to\infty} \frac{\log \|T(\gamma_M, \dots, \gamma_{n_k})\|}{n_k \log \varrho}.$$

As $\Omega'$ is finite, by passing to a further subsequence if necessary (not renamed) we can assume all $\gamma_{n_k} = \delta \in \Omega'$. Put

$$D = \lim_{k\to\infty} \frac{\log \|T(\gamma_M, \dots, \gamma_{n_k})\|}{n_k \log \varrho}.$$

Let $\eta$ be an admissible path in $\Omega'$ going from $\delta$ to $\gamma_M$, such that $T(\eta) > 0$. (We remark that it is important that this transition matrix is independent of the choice of $k$.) Of course, then $\theta_k = (\gamma_M, \dots, \gamma_{n_k-1})\eta$ is an admissible cycle in $\Omega'$.

As $T(\eta)$ is a positive matrix, Lemmas 3.12 and 3.15(2) imply that there are constants, $K_k$, bounded above and bounded below away from 0 such that

$$\|T(\gamma_M, \dots, \gamma_{n_k})\| = K_k\, \mathrm{sp}(T(\gamma_M, \dots, \gamma_{n_k})T(\eta)) = K_k\, \mathrm{sp}(T(\theta_k)).$$

Consider the local dimension at the periodic point $y_k$ in $\Omega'$ given by

$$[y_k] = (\gamma_0, \dots, \gamma_{M-1}, \theta_k^-, \theta_k^-, \dots).$$

Since $T(\theta_k) = T(\gamma_M, \dots, \gamma_{n_k})T(\eta)$ is positive, $y_k$ is a positive point. Furthermore, we have

$$\begin{aligned}\dim_{\mathrm{loc}} \mu(y_k) &= \frac{\log p_0}{\log \varrho} + \frac{\log(\mathrm{sp}(T(\theta_k))}{L(\theta_k^-)\log \varrho} \\ &= \frac{\log p_0}{\log \varrho} - \frac{\log K_k}{L(\theta_k^-)\log \varrho} + \frac{\log \|T(\gamma_M, \dots, \gamma_{n_k})\|}{L(\theta_k^-)\log \varrho}.\end{aligned}$$

Fix $\varepsilon > 0$. The choice of $n_k$ and boundedness of $K_k$ ensures we can choose $k$ large enough so that

$$\left|\frac{\log \|T(\gamma_M, \dots, \gamma_{n_k})\|}{n_k \log \varrho} - D\right| < \varepsilon,$$

$$\left|\frac{\log K_k}{L(\theta_k^-)\log \varrho}\right| < \varepsilon \quad \text{and} \quad \left|\frac{n_k}{L(\theta_k^-)} - 1\right| < \varepsilon.$$

It is now easy to see that

$$|\dim_{\mathrm{loc}} \mu(x) - \dim_{\mathrm{loc}} \mu(y_k)| < (D+2)\varepsilon. \qquad \square$$

Since the essential class is of positive type, we immediately deduce the following important fact about these measures of finite type.

**Corollary 5.3.** *The set of local dimensions at essential periodic points is dense in the set of local dimensions at all essential points.*

Here is another needed elementary fact whose proof is an exercise for the reader.

**Lemma 5.4.** *Let $\theta = (\delta_1, \delta_2, \dots, \delta_L, \delta_1)$ be a cycle and let $\theta^* = (\delta_k, \dots, \delta_L, \delta_1, \dots, \delta_k)$ be any cyclic shift of $\theta$. Then*

$$\mathrm{sp}(T(\theta)) = \mathrm{sp}(T(\theta^*))$$

**Theorem 5.5.** *Suppose $\mu$ satisfies the standard technical assumptions. Assume that the loop class, $\Omega'$, is of positive type and suppose $(x_n)$ is a sequence of positive, periodic points in $\Omega'$. Then there is some $x$ in $\Omega'$ such that*

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_n \dim_{\mathrm{loc}} \mu(x_n) \ \text{ and } \ \underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_n \dim_{\mathrm{loc}} \mu(x_n).$$

*Proof.* Assume $x_n$ has period $\theta_n$ in $\Omega'$ and that $T(\theta_n) > 0$. We put

$$S := \limsup_n \frac{\log(\mathrm{sp}(T(\theta_n))}{L(\theta_n^-)} \quad \text{and} \quad I := \liminf_n \frac{\log(\mathrm{sp}(T(\theta_n))}{L(\theta_n^-)}, \tag{6}$$

so

$$\limsup_n \dim_{\mathrm{loc}} \mu(x_n) = \frac{\log p_0}{\log \varrho} + \frac{\log I}{\log \varrho}.$$

and

$$\liminf_n \dim_{\mathrm{loc}} \mu(x_n) = \frac{\log p_0}{\log \varrho} + \frac{\log S}{\log \varrho}.$$

By passing to a subsequence, not renamed, we can assume all

$$\left| \frac{\log(\mathrm{sp}(T(\theta_{2n})))}{L(\theta_{2n}^-)} - S \right| < \varepsilon_{2n} \quad \text{and} \quad \left| \frac{\log(\mathrm{sp}(T(\theta_{2n+1})))}{L(\theta_{2n+1}^-)} - I \right| < \varepsilon_{2n+1},$$

where $\varepsilon_n$ is a decreasing sequence tending to 0. Further, we can assume that all the even labelled paths, $\theta_n^-$, have the same first letter, say $\alpha_S$, and the same last letter, $\beta_S$. Similarly, we can assume all the odd labelled paths have a common first letter $\alpha_I$ and common last letter $\beta_I$.

Since $\Omega'$ is of positive type, we can certainly choose two admissible paths in $\Omega'$, $\lambda_{SI}$ going from $\beta_S$ to $\alpha_I$ and $\lambda_{IS}$ going from $\beta_I$ to $\alpha_S$, so that $T(\lambda_{SI})$ and $T(\lambda_{IS})$ are positive.

We want to inductively define a rapidly increasing subsequence $(k_n)$ such that

$$[x] = (\eta, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \lambda_{IS}, \underbrace{\theta_2^-, \dots, \theta_2^-}_{k_2}, \lambda_{SI}, \underbrace{\theta_3^-, \dots, \theta_3^-}_{k_3}, \dots)$$

has the desired property, where $\eta$ is an admissible path beginning with $\mathcal{C}_0([0, 1])$ and ending with the parent of $\alpha_I$.

Temporarily fix $n$. With abuse of notation we will write the truncated product as

$$[x]_n = \eta \prod_{i=1}^{n} (\theta_i^-)^{k_i} \lambda_i$$

where $\lambda_i = \lambda_{SI}$ if $i$ is even, and $\lambda_{IS}$ if $i$ is odd.

Recall that $C_2(B)$ is a function on a square matrix $B$, as defined in Lemma 3.15(3). Define $K_n$ as the maximal $K_n := C_2(T(\theta_{n+1}^*))$, taken over all cyclic shifts of $\theta_{n+1}^*$ of $\theta_{n+1}$. Let $\lambda_n'$ be any prefix of $\lambda_n$ and $\theta_{n+1}'$ any prefix of $\theta_{n+1}^-$. The elementary lemmas of the Section 3 imply that

$$\lim_{k\to\infty} \left| \frac{\log(\|T([x]_{n-1}(\theta_n^-)^k \lambda_n')\|)}{L([x]_{n-1}(\theta_n^-)^k \lambda_n')} - \frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)} \right| = 0 \tag{7}$$

and

$$\lim_{k\to\infty}\left|\frac{\log(\|T([x]_{n-1}(\theta_n^-)^k\lambda_n\theta'_{n+1})\|)}{L([x]_{n-1}(\theta_n^-)^k\lambda_n\theta'_{n+1})}-\frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)}\right|=0. \tag{8}$$

Obviously,

$$\lim_{k\to\infty}\frac{\log(K_n)}{L([x]_{n-1}(\theta_n^-)^k\lambda_n\theta'_{n+1})}=0 \tag{9}$$

and

$$\lim_{k\to\infty}\frac{L(\lambda_n\theta_{n+1})}{k}\cdot\frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)}=0. \tag{10}$$

We pick $k_n$ such that the left hand side of these four limits are each less than $\varepsilon_n$ for all choices of $\lambda'_n$ and $\theta'_{n+1}$.

Certainly this process defines an $x$ belonging to $\Omega'$. We need to check that the upper and lower dimensions of $\mu$ at $x$ are correct. By construction, the limiting behaviour (as $n\to\infty$) of

$$\frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda'_n))\|)}{L([x]_{n-1}(\theta_n^-)^{k_n}\lambda'_n))}\quad\text{and}\quad\frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1})\|)}{L([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1})}$$

approach the values $I$ and $S$ along the odd and even $n$ respectively. Hence it remains to consider the case

$$\frac{\log(\left\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n(\theta_{n+1}^-)^{p_{n+1}}\theta'_{n+1})\right\|)}{L([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n(\theta_{n+1}^-)^{p_{n+1}}\theta'_{n+1})}$$

for $0<p_{n+1}<k_{n+1}$, and $\theta'_{n+1}$ some prefix of $\theta_{n+1}$. This is equivalent to

$$\frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1}(\theta_{n+1}^{*-})^{p_{n+1}})\|)}{L([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1}(\theta_{n+1}^{*-})^{p_{n+1}})}$$

for some cyclic shift $\theta^*_{n+1}$ of $\theta_{n+1}$.

For ease of notation, let

$$L_0(n) = L_0 := L([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1}(\theta_{n+1}^{*-})^{p_{n+1}}).$$

First, observe that

$$\begin{aligned}
E_n &:= \frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1}(\theta_{n+1}^{*-})^{p_{n+1}})\|)}{L_0} \\
&\leq \frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1}\|)}{L_0} + \frac{\log(\|T(\theta_{n+1}^{*})^{p_{n+1}})\|)}{L_0} \\
&\leq \frac{L([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)} \\
&\quad + \frac{\log(K_n\,\mathrm{sp}((T(\theta_{n+1}))^{p_n+1}))}{L_0} + \varepsilon_n \\
&\leq \frac{L_0 - L((\theta_{n+1}^-)^{p_n+1})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)} \\
&\quad + \frac{L((\theta_{n+1}^-)^{p_n+1})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_{n+1}))^{p_n+1})}{L((\theta_{n+1}^-)^{p_n+1})} + \frac{\log K_n}{L_0} + \varepsilon_n \\
&\leq \frac{L_0 - L((\theta_{n+1}^-)^{p_n+1})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)} \\
&\quad + \frac{L((\theta_{n+1}^-)^{p_n+1})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_{n+1})))}{L(\theta_{n+1}^-)} + 2\varepsilon_n \\
&= (1-t_n)\Big(\frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)}\Big) + t_n\Big(\frac{\log(\mathrm{sp}(T(\theta_{n+1})))}{L(\theta_{n+1}^-)}\Big) + 2\varepsilon_n
\end{aligned}$$

for

$$t_n = \frac{L((\theta_{n+1}^-)^{p_n+1})}{L_0}.$$

Here the second inequality comes from (8), Lemma 3.15(3) and Lemma 5.4, and the final inequality from (9).

For the opposite inequality, we use the fact that $T(\lambda_n\theta'_{n+1}) > 0$ and Lemma 3.15 part 1 for the first inequality below, and (7) and (10) for the second and third inequalities to obtain

$$
\begin{aligned}
E_n &= \frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\lambda_n\theta'_{n+1}(\theta_{n+1}^{*-})^{p_{n+1}})\|)}{L_0} \\
&\geq \frac{\log(\|T([x]_{n-1}(\theta_n^-)^{k_n}\|)}{L_0} + \frac{\log(\|T((\theta_{n+1}^*)^{p_{n+1}})\|)}{L_0} \\
&\geq \frac{L([x]_{n-1}(\theta_n^-)^{k_n})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)} \\
&\quad + \frac{L((\theta_{n+1}^-)^{p_n+1})}{L_0} \cdot \frac{\log(\mathrm{sp}(T(\theta_{n+1}))^{p_n+1})}{L((\theta_{n+1}^-)^{p_n+1})} - \varepsilon_n \\
&\geq (1-t_n)\Big(\frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)}\Big) + t_n\Big(\frac{\log(\mathrm{sp}(T(\theta_{n+1})))}{L(\theta_{n+1}^-)}\Big) - 2\varepsilon_n
\end{aligned}
$$

Together these estimates prove $E_n$ lies within $2\varepsilon_n$ of the same convex combination of $\frac{\log(\mathrm{sp}(T(\theta_n)))}{L(\theta_n^-)}$ and $\frac{\log(\mathrm{sp}(T(\theta_{n+1})))}{L(\theta_{n+1}^-)}$, and hence $\limsup E_n$ and $\liminf E_n$ both belong to the interval $[I, S]$. That completes the proof. □

Combining Theorems 5.1 and 5.5, it follows that these measures satisfy the following.

**Corollary 5.6.** *The set of local dimensions at essential points coincides with the set of lower (or upper) local dimensions at essential points. Moreover,*

$$
\begin{aligned}
&\{\dim_{\mathrm{loc}} \mu(x) \colon x \text{ essential}\} \\
&\quad = \mathrm{closure}\{\dim_{\mathrm{loc}} \mu(x) \colon x \text{ essential, positive periodic}\}.
\end{aligned}
$$

**5.2. Set of local dimensions at points in a positive loop class is an interval.** The final result of this section will be to show that the set of local dimensions at points in a loop class of positive type is a closed interval. Of course, in particular, this applies to the essential class.

**Theorem 5.7.** *Suppose $\mu$ satisfies the standard technical assumptions. Further, suppose that the loop class $\Omega'$ is of positive type. Assume $y$ and $z$ are periodic, positive points in $\Omega'$. Then the set of local dimensions of $\mu$ contains the closed interval with endpoints* $\dim_{\mathrm{loc}} \mu(y)$ *and* $\dim_{\mathrm{loc}} \mu(z)$.

*Proof.* Suppose $y$ has period $\varphi$ and $z$ has period $\theta$ where $A = T(\varphi)$ and $B = T(\theta)$ are positive matrices with spectral radii $\alpha$ and $\beta$, respectively. With this notation,

$$\dim_{\text{loc}} \mu(y) = \frac{\log p_0}{\log \varrho} + \frac{\log \alpha}{L(\varphi^-) \log \varrho}$$

and

$$\dim_{\text{loc}} \mu(z) = \frac{\log p_0}{\log \varrho} + \frac{\log \beta}{L(\theta^-) \log \varrho}.$$

Let $0 < t < 1$. We want to prove that there exists a $x$ such that

$$\dim_{\text{loc}} \mu(x) = t \dim_{\text{loc}} \mu(y) + (1-t) \dim_{\text{loc}} \mu(z).$$

Appealing to Theorem 5.5, we see it will be enough to show that there is some sequence of periodic, positive points $x_k$, in the loop class $\Omega'$, such that

$$\lim_k \dim_{\text{loc}} \mu(x_k) = \frac{\log p_0}{\log \varrho} + \frac{t \log \alpha}{L(\varphi^-) \log \varrho} + \frac{(1-t) \log \beta}{L(\theta^-) \log \varrho}. \tag{11}$$

To do this, we start by choosing admissible paths in $\Omega'$, $\eta_1$ joining the last letter of $\theta$ to the first letter of $\varphi$ and $\eta_2$ doing the opposite, such that $T(\eta_j) > 0$. Then for any positive integers, $n, m$, $B^m T(\eta_1) A^n T(\eta_2)$ is a square transition matrix.

Select two sequences of integers, $(n_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, tending to infinity, with

$$\frac{L(\varphi^-) n_k}{L(\theta^-) m_k + L(\varphi^-) n_k} \to t.$$

We will prove that the periodic points with period $\theta_k$ satisfying

$$T(\theta_k) = B^{m_k} T(\eta_1) A^{n_k} T(\eta_2)$$

work.

As $T(\eta_j) > 0$, Lemmas 3.12 and 3.15 combine to imply that for all $k$,

$$\begin{aligned} \operatorname{sp}(B^{m_k} T(\eta_1) A^{n_k} T(\eta_2)) &\leq \|T(\eta_1)\| \, \|T(\eta_2)\| \, \|B^{m_k}\| \, \|A^{n_k}\| \\ &\leq C(A, B, \eta_1, \eta_2) \alpha^{n_k} \beta^{m_k}, \end{aligned}$$

and

$$\begin{aligned} sp(B^{m_k} T(\eta_1) A^{n_k} T(\eta_2)) &\geq C(\eta_2) \|B^{m_k} T(\eta_1) A^{n_k} T(\eta_2)\| \\ &\geq C(\eta_2) \|B^{m_k}\| \, \|A^{n_k}\| \\ &\geq C(\eta_2) \alpha^{n_k} \beta^{m_k}. \end{aligned}$$

Thus

$$\lim_k \frac{\log(\mathrm{sp}(B^{m_k}T(\eta_1)A^{n_k}T(\eta_2)))}{L(\theta^-)m_k + L(\varphi^-)n_k} = \lim_k \frac{n_k \log\alpha + m_k \log\beta}{L(\theta^-)m_k + L(\varphi^-)n_k}$$
$$= \frac{t\log\alpha}{L(\varphi^-)} + \frac{(1-t)\log\beta}{L(\theta^-)}.$$

It follows that (11) is satisfied. □

Combining the three theorems we deduce that the set of local dimensions at the essential points of such measures is a closed interval whose endpoints are given by the infimum and the supremum of the local dimensions at essential, periodic points.

**Corollary 5.8.** *Let*

$$I = \inf\{\dim_{\mathrm{loc}} \mu(x) \colon x \textit{ essential, positive periodic}\}$$

*and*

$$S = \sup\{\dim_{\mathrm{loc}} \mu(x) \colon x \textit{ essential and positive periodic}\}.$$

*Then*

$$\{\dim_{\mathrm{loc}} \mu(x) \colon x \textit{ essential}\} = [I, S].$$

Another immediate corollary is that if the set of essential points is $(0,1)$, then the measure admits at most one isolated point.

**Corollary 5.9.** *If the set of essential points of* $\mu$ *is* $(0,1)$*, then the set of local dimensions of* $\mu$ *consists of a closed interval together with* $\dim_{\mathrm{loc}} \mu(0) = \dim_{\mathrm{loc}} \mu(1)$.

**Example 5.10.** As observed in Example 4.7, this is the situation for the Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$ the golden ratio. Further, since

$$\mathrm{sp}(T(5,3a,5)) = 1 = \mathrm{sp}(T(2,2)) = \mathrm{sp}(T(4,4)),$$

and $\{3a, 5\}$ is in the essential class, it follows that $\dim_{\mathrm{loc}} \mu(0)$ coincides with the local dimension at an essential point. Consequently, the set of local dimensions of $\mu_\varrho$ is equal to the closed interval $[I, S]$ consisting of the local dimensions at the essential points of $\mu_\varrho$. In [8,15] it is shown that $I = \log 2/|\log\rho|$ and $S = I + 1/2$.

More generally, we have the following result.

**Corollary 5.11.** *If every maximal loop class is of positive type, then the set of local dimensions of $\mu$ is a finite union of closed intervals.*

These intervals can be disjoint. Indeed, in Example 6.1 we construct a measure whose set of local dimensions consists of two points (degenerate intervals).

## 6. Algorithm

**6.1. Algorithm for finding characteristic vectors, the essential set and transition matrices.** Given a family of contractions, $S_j(x) = \varrho x + d_j$, and probabilities $p_j$, we have implemented an algorithm to find $\Omega$ and all of the associated transition matrices. We use a modified version of [21] to perform these calculations. We give an overview here, by means of a worked example.

For this we consider the six contractions, the maps $S_j(x) = \frac{1}{3}x + d_j$ for $d_j = 2j/15$, for $j = 0, 1, \ldots, 5$. The corresponding self-similar set is the 5-fold sum of the middle-third Cantor set rescaled to $[0, 1]$. Suppose the maps have normalized probabilities $1, p_1, p_2, p_3, p_4, 1$.

One can see that $\Delta = [0, 1] \in \mathcal{F}_0$ has characteristic vector $(1, (0), 1)$. We will call this characteristic vector 1. Here the first 1 is the normalized length of the interval. The sequence $(0)$ is $V_0(\Delta) = (a_1, a_2, \ldots, a_k)$ is the neighbourhood set. The last 1 is $r_0(\Delta)$. For the questions we are interested in, $r_n(\Delta)$ is not needed, hence we will suppress it in the future, instead allowing multiple edges between nodes in the graph. This gives a reduced characteristic vector of $(1, (0))$.

Instead of considering all intervals in $\mathcal{F}_1$, we only consider those that arise from new reduced characteristic vectors found to be in $\Omega$. So, initially, we look at those children of reduced characteristic vector 1. In this case, these two things are the same, but we will see later on that this is not always the case. We first subdivide $[0, 1]$ by considering the maps $S_j(0) - a_i$ and and $S_j(1) - a_i$ for all $a_i \in V(1)$ and $j = 0, 1, 2, 3, 4, 5$. This partitions the interval at the points $\{k/15\colon k = 0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15\}$.

Consider first $\Delta = [0, 2/15]$ coming from this subdivision. We see that its normalized length is $3 \cdot 2/15 = 2/5$ and it has neighbourhood set $(0)$. We label $(2/5, (0))$ as the reduced characteristic vector 2. The $d_j$ that contributes to 0 is $d_0$ and is associated to the normalized probability of 1. Hence the transition matrix is $T(1, 2) = [\,1\,]$.

Now, consider $\Delta = [2/15, 4/15]$. This again has normalized length $2/5$ and neighbourhood set $(0, 2/5)$. We label $(2/5, (0, 2/5))$ as the reduced characteristic vector 3. The 0 comes from $d_1$ and the $2/5$ from $d_0$, hence the transition matrix is $T(1,3) = [\, p_1 \;\; 1\,]$.

We continue in this fashion, noting that for $\Delta = [2j/15, (2j+1)/15]$, where $j = 2, \ldots, 5$, the characteristic vector is $(1/5, (0, 2/5, 5/4))$, which we label as 4. In Feng's notation, we would distinguish these as four different characteristic vectors by use of their third component, $r_1(\Delta)$. We will not distinguish these, but will instead allow multiple maps from the reduced characteristic vector 1 to the reduced characteristic vector 4. These are

$$\begin{aligned} T(1,4) = [\, p_2 \;\; p_1 \;\; 1\,] \quad &\text{or} \\ [\, p_3 \;\; p_2 \;\; p_1\,] \quad &\text{or} \\ [\, p_4 \;\; p_3 \;\; p_2\,] \quad &\text{or} \\ [\, p_5 \;\; p_4 \;\; p_3\,]. & \end{aligned}$$

Continuing in this fashion, we can compute the finite set of reduced characteristic vectors obtaining

- reduced characteristic vector 1: $(1, (0))$;
- reduced characteristic vector 2: $(2/5, (0))$;
- reduced characteristic vector 3: $(2/5, (0, 2/5))$;
- reduced characteristic vector 4: $(1/5, (0, 2/5, 4/5))$;
- reduced characteristic vector 5: $(1/5, (1/5, 3/5))$;
- reduced characteristic vector 6: $(2/5, (1/5, 3/5))$;
- reduced characteristic vector 7: $(2/5, (3/5))$.

The characteristic vectors 4 and 5 comprise the essential set.

For a complete list of transition matrices, see [14].

**6.2. Algorithm for finding bounds on local dimensions.** Given a loop class of positive type, there are two main ways we obtain good estimates on the set of local dimensions associated to this loop class. The first is to find explicit examples of periodic points within the loop class and calculate their local dimensions by determining the spectral radius of the transition matrix of the cycle. Applying Theorem 5.1, this will produce an interval contained in the set of local dimensions of the loop class.

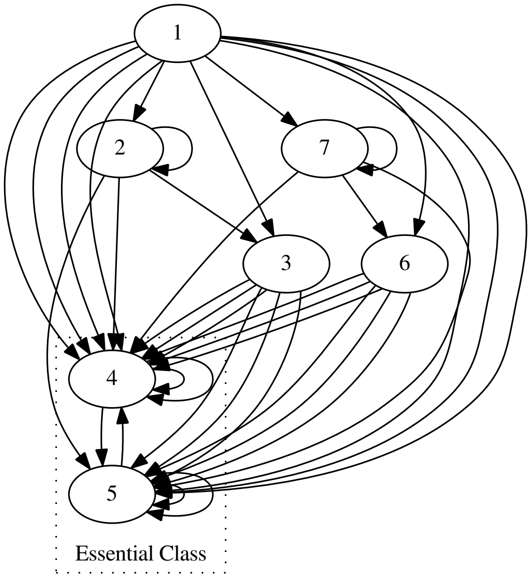



Figure 2. Transition graph for $3x-1$, with $d_j \in \{2j/15\colon j=0,1,\ldots,5\}$.

To find upper and lower bounds on the set of local dimensions we use the family of pseudo-norms:

$$\|T\|_{C,\min} = \min\Big\{\sum_{j\in C} |T_{jk}|\colon k\in C\Big\},$$

$$\|T\|_{\min} = \min\Big\{\sum_{k} |T_{jk}|\colon j\Big\}$$

$$\|T\|_{\max} = \max\Big\{\sum_{k} |T_{jk}|\colon j\Big\}.$$

These are the sub-norm on indices $C$, the total sub-norm and the total sup-norm respectively. The total sup-norm is actually a norm. Here $C$ is a subset of the indices of the column vectors. Care must be take here that the subset $C$ is valid for all matrices within the loop class one is considering, as different transition matrices may have different dimensions. For all matrices, $T_1, T_2 \geq 0$ we have

$$\|T_1T_2\|_{C,\min} \geq \|T_1\|_{C,\min}\,\|T_2\|_{C,\min}\,,$$

$$\|T_1T_2\|_{\min} \geq \|T_1\|_{\min}\,\|T_2\|_{\min}\,,$$

$$\|T_1T_2\|_{\max} \leq \|T_1\|_{\max}\,\|T_2\|_{\max}$$

and

$$\|T\|_{\min}, \|T\|_{C,\min} \le \|T\| \le \eta \|T\|_{\max}$$

where $\eta$ is the number of columns in $T$. We can thus obtain upper and lower bounds for the local dimensions at points in the loop class by calculating these pseudo-norms for all admissible products of primitive transition matrices from the loop class up to some fixed length and using formula (5).

We remark that pseudo-norms based on a subset of the indices of the row vectors would serve, as well.

**Example 6.1.** A measure whose local dimension is two isolated points. We continue with the example described above, but now take the normalized uniform weights $p_i = 1$. Let $\mu$ denote the associated self-similar measure.

The essential class, as mentioned above, is $\{4, 5\}$ and its primitive transition matrices are

$$T(4,4) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad T(4,5) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$T(4,4) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T(5,5) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$T(5,4) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad T(5,5) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

As the total column sub and sup-norms for all these matrices is 2, it follows that all $k$-fold products of primitive transition matrices in the essential class will have their (usual) norm in the interval $[2^k, 3 \cdot 2^k]$. It follows that the local dimension at any point in the essential class is $(\log 6 - \log 2)/\log 3 = 1$.

As the local dimension at 0 and 1 is equal to $\log 6/\log 3$, the set of local dimensions of $\mu$ consists of the two distinct points,

$$\{\dim_{\text{loc}} \mu(x) \colon x \in [0,1]\} = \{1\} \cup \{1 + \log 2/\log 3\}.$$

## 7. Cantor-like measures

In Example 2.5 we considered the IFS

$$\left\{S_j(x) = \frac{1}{R}x + \frac{j}{mR}(R-1)\colon j = 0, \ldots, m\right\} \tag{12}$$

for integers $R \geq 2$, and the self-similar Cantor-like measures $\mu$ associated with this IFS and probabilities $\{p_j\colon j = 0, \ldots, m\}$. This class of measures includes, for example, the $m$-fold convolution of the uniform Cantor measure with contraction factor $1/R$, rescaled to $[0,1]$. The self-similar set is the full interval $[0,1]$ when $m \geq R-1$.

The results in [2] and [26], which extend [16], together imply that if $m \geq R \geq 3$ and $p_0, p_m < \min p_j$, $j \neq 0, m$, then the set of local dimensions of $\mu$ consists of a closed interval and one (or two) isolated points, the local dimensions at 0 and 1. These facts can be recovered by our methods as well, when, in addition, $p_0 = p_m$.

**Proposition 7.1.** *Let $\mu$ be the self-similar measure associated with the IFS (12) with $m \geq R \geq 2$ and probabilities satisfying $p_0 = p_m < \min_{j\neq 0,m} p_j$. Then the set of essential points is the open interval $(0,1)$ and $\dim_{\mathrm{loc}} \mu(0)$ is an isolated point in the set of all local dimensions. Indeed, for $x \neq 0, 1$,*

$$\dim_{\mathrm{loc}} \mu(x) \leq \frac{\left|\log(\min\{2p_0, p_j\colon j \neq 0, m\})\right|}{\log R} < \frac{|\log p_0|}{\log R} = \dim_{\mathrm{loc}} \mu(0).$$

*Proof.* We note, first, that the iterates of 0 at level $n$ (meaning the real values $S_\sigma(0)$ for $\sigma \in \mathcal{A}^n$) occur every $(R-1)/(R^n m)$, beginning at 0 and ending at $1 - R^{-n}$. Similarly, the iterates of 1 are spaced the same distance apart, but start at $R^{-n}$ and end at 1. In the subinterval $[R^{-n}, 1-R^{-n}]$ they alternate, except in the special case that $m \equiv 0 \bmod (R-1)$, when they coincide. We will assume $m \not\equiv 0 \bmod (R-1)$ and leave the easier case for the reader.

Adjacent $n$'th level net intervals contained in $[R^{-n}, 1-R^{-n}]$ are of the form $[a,b]$, $[b,c]$ where if $a$ is an iterate of 0, then

$$a = \frac{j(R-1)}{R^n m}, \quad b = \frac{1}{R^n} + \frac{k(R-1)}{R^n m} \quad \text{and} \quad c = \frac{(j+1)(R-1)}{R^n m}$$

for suitable integers $j, k$. There is a similar formula when $a$ is an iterate of 1. If $m = L(R-1) + r$ for integer $L$ and $r \in \{1, \ldots, R-2\}$, then it is easily seen that $L + k = j$. Thus $n$'th level net intervals have either (non-normalized) length

$$\frac{r}{R^n m} \quad \text{or} \quad \frac{R-1-r}{R^n m},$$

and, in either case, have length between $1/(R^n m)$ and $1/R^n$.

Let $x \in (0,1)$ and assume $n$ is chosen so large that $3R^{-n} < x < 1 - 3R^{-n}$. Let $[A, B]$ denote the $n$'th level net interval containing $x$. The choice of $n$ ensures that all the numbers of the form $i(R-1)/(R^n m)$ for integer $i$ with

$$0 < A - i(R-1)/(R^n m) < R^{-n}$$

are iterates of 0 at level $n$ and hence comprise the neighbour set of $[A, B]$. Thus the neighbour set depends only upon whether $A$ is an iterate of 0 or an iterate of 1. It follows that there are only two reduced characteristic vectors associated with these net intervals.

Furthermore, the length of these net intervals is sufficiently large to ensure that that the interior of each such interval contains at least one $n+1$-iterate of 0 and one $n+1$-iterate of 1. Consequently, each such $n$'th level net interval contains both styles of net intervals of level $n+1$ (and no other children). It follows that their characteristic vectors belong to the essential class and that proves the set of essential points is $(0, 1)$.

To prove the upper bound given in the statement of the proposition for $\dim_{\text{loc}} \mu(x)$ with $0 < x < 1$, we will show that if $[x] = (\gamma_0, \gamma_1, \ldots)$ and $n$ is sufficiently large, then $\|T(\gamma_{n-1}, \gamma_n)\|_{\min} \geq \min\{2, p_j p_0^{-1} : j \neq 0, m\}$ where $\|\cdot\|_{\min}$ is the total column sub-norm introduced in the preceding section.

We can assume $3R^{-N} < x < 1-3R^{-N}$ and take $n > N$. Suppose $\gamma_n = \mathcal{C}(\Delta_n)$, $\Delta_n = [a, b]$, $\widehat{\Delta_n} = [c, d]$, and the neighbour sets are $V_n(\Delta_n) = (a_1, \ldots, a_K)$ and $V_{n-1}(\Delta_{n-1}) = (c_1, \ldots, c_J)$ respectively, where $c_1 < \cdots < c_J$.

Temporarily fix $k$. By definition, $T(\gamma_{n-1}, \gamma_n)_{jk} = p_\ell p_0^{-1}$ when there is some $\sigma \in \mathcal{A}^n$ and $\ell \in \mathcal{A}$ such that $S_\sigma(0) = c - R^{-(n-1)} c_j$ and $S_{\sigma\ell}(0) = a - R^{-n} a_k$, and $T(\gamma_{n-1}, \gamma_n)_{jk} = 0$ otherwise.

Of course, there is some choice of $j$ with a valid choice of $\ell$. If $\ell \neq 0, m$ we are done. So assume this $\ell = 0$. That means $a - R^{-n} a_k = c - R^{-(n-1)} c_j$. The bounds on $x$ ensure that for some $\tau \in \mathcal{A}^{n-1}$,

$$S_\tau(0) = S_\sigma(0) - (R-1)/(R^{n-1} m).$$

Furthermore,

$$0 < c - S_\tau(0) \leq a - S_\sigma(0) + \frac{R-1}{R^{n-1} m} < \frac{1}{R^n} + \frac{R-1}{R^{n-1} m} \leq \frac{1}{R^{n-1}}.$$

This implies there is some $i$ such that $c_i = R^{-(n-1)}(c - S_\tau(0))$; indeed, $i = j+1$.

Another routine calculation shows $S_{\tau R}(0) = a - R^{-n} a_k$, thus

$$T(\gamma_{n-1}, \gamma_n)_{j+1,k} = p_R p_0^{-1}.$$

If, instead, $\ell = m$, similar arguments prove $T(\gamma_{n-1}, \gamma_n)_{j-1,k} = p_{m-R} p_0^{-1}$. This completes the proof. □

**Remark 7.2.** We are not claiming that these bounds on $\dim_{\mathrm{loc}} \mu(x)$ for $x \neq 0, 1$ are sharp, merely illustrating that we can recover the property that $\dim_{\mathrm{loc}} \mu(0)$ is an isolated point with our approach.

In [2, Theorem 6.1], the minimum and maximum local dimensions, other than at $0, 1$, were investigated for the case of the $m$-fold convolution of the uniform Cantor measure for small $m$. We show that there exists $m$ outside of these ranges where these formula do not hold.

**Example 7.3.** Consider the $m$-fold convolution of the uniform Cantor measure with contraction factor $1/R$, for integer $R \geq 3$. It is shown in [2] that if $R \leq m \leq 2R - 2$, then

$$\min_x \dim_{\mathrm{loc}} \mu(x) = \dim_{\mathrm{loc}} \mu(x_{\min}) = \frac{m \log 2 - \log \binom{m}{\lfloor \frac{m}{2} \rfloor}}{\log R} \tag{13}$$

where

$$x_{\min} = \frac{1}{m} \left\lfloor \frac{m}{2} \right\rfloor.$$

It was also shown there that for all even $m$, $\min_x \dim_{\mathrm{loc}} \mu(x) = \dim_{\mathrm{loc}} \mu(x_{\min})$.

Using the computer, we have checked these formulas for $R = 3$ and $3 \leq m \leq 10$. We have found that the right hand side of (13) is not $\dim_{\mathrm{loc}} \mu(x_{\min})$ and is not the minimal local dimension for $m = 5, \ldots, 10$. Moreover, for $m = 5, 7, 9$ the minimal local dimension does not occur at the point $x_{\min}$. In fact, the predicted value of the minimum local dimension is greater than the maximum local dimension other than for $x = 0, 1$. See Tables 1 and 2. We note that in Tables 1 and 2, when the formula is known to hold for theoretical reasons, we put the precise value, otherwise we put a range, coming from the techniques of Section 6.

Table 1. Minimal local dimensions

| $m$ | Left hand side of Formula 13 | Actual min | $\dim_{\text{loc}} \mu(x_{\min})$ |
|---|---|---|---|
| 3 | .892790 | .892790 | .892790 |
| 4 | .892790 | .892790 | .892790 |
| 5 | 1.05875 | [.972382, .972639] | .984145 |
| 6 | 1.05875 | .976628 | .976628 |
| 7 | 1.18029 | [.993576, .993848] | .997991 |
| 8 | 1.18029 | .995246 | .995246 |
| 9 | 1.27620 | [.998541, .998658] | .999739 |
| 10 | 1.27620 | .999022 | .999022 |

Table 2. Maximal local dimensions

| $m$ | Left hand side of Formula 14 | Actual max | $\dim_{\text{loc}} \mu(x_{max})$ |
|---|---|---|---|
| 3 | 1.13355 | 1.13354 | 1.13354 |
| 4 | 1.05875 | 1.05874 | 1.05874 |
| 5 | 1.02757 | 1.02757 | 1.02757 |
| 6 | 1.01434 | 1.01434 | 1.01434 |
| 7 | 1.01434 | [1.00605, 1.00736] | 1.00605 |
| 8 | 1.01434 | [1.00342, 1.00346] | 1.00343 |
| 9 | 1.02721 | [1.00133, 1.00171] | 1.00133 |
| 10 | 1.03074 | [1.00079, 1.00082] | 1.00079 |

In [2], there was also a formula given for the maximum local dimension other than at $x = 0, 1$. Let $r = \lfloor (m - R)/2 \rfloor$, $\ell_{2j} = r + 1$ and $\ell_{2j+1} = m - r - R$. Put

$$x_{\max} = \frac{1}{m} \sum_{j=1}^{\infty} (R - 1) R^{-j} \ell_j .$$

It was shown that for $R \leq m \leq 2R - 1$,

$$\max_{x \neq 0,1} \dim_{\text{loc}} \mu(x) = \dim_{\text{loc}} \mu(x_{\max})$$

and

$$\dim_{\text{loc}} \mu(x_{\max}) = -\frac{\log((p_{r+R+1} + p_r + \sqrt{(p_{r+R+1} - p_r)^2 + 4p_{r+1} p_{r+R}})/2)}{\log R}. \tag{14}$$

Here $p_j$ should be understood as 0 if $j \notin \{0, 1, \ldots, m\}$.

Using our methods, we can show that for $R = 3$, $m = 6$, these statements continue to be true, that is, $\dim_{loc} \mu(x_{max}) = \max_{x \neq 0,1} \dim_{loc} \mu(x)$ and $\dim_{loc} \mu(x_{max})$ is the value specified by the right hand side of formula (14). But for $R = 3$ and $m = 7, \dots, 10$, the predicted maximum local dimension from the right hand side of (14) is too big. We have not been able to determine if $x_{max}$ is the point where the maximum dimension occurs.

We refer the reader to [14] where this example is worked out in full detail.

## 8. Bernoulli convolutions with contraction factors Pisot inverses

**8.1. Bernoulli convolutions with contraction factors Pisot inverses.** In Table 3, we list all Pisot numbers in the open interval $(1, 2)$ of degree less than or equal to 4. For each of these, we give the number of vertices of the reduced transition graph and the size of the reduced essential class for the Bernoulli convolution with contraction factor the inverse of this Pisot number. In the case where the size is listed as 'Unknown', there are more than 10000 reduced characteristic vectors.

**8.1.1. Bernoulli convolutions with no isolated point.** It can be shown for Bernoulli convolutions that whenever there are precisely three more elements in the reduced transition graph than in the essential set, then the non-essential set consists of the characteristic vector of $[0, 1]$ and the characteristic vectors of the right-most and left-most net intervals of level 1. The latter two are maximal loop classes corresponding to the two endpoints of $[0, 1]$, the only two non-essential points. Thus, with the exception of $x^3-x^2-1$, $x^3-x-1$ and possibly $x^4-x^3-1$, for the examples listed in Table 3 the open interval $(0, 1)$ is the set of essential points. We have checked that in all of these examples (where the essential set is known to be $(0, 1)$), the value of the local dimension of the measure at 0 is also the local dimension at an essential point, hence there is no isolated point.

**8.1.2. Bernoulli convolutions with an isolated point**

**Example 8.1** (minimal polynomial $P_1(x) = x^3 - x^2 - 1$). The uniform Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$ the Pisot number with minimal polynomial $P_1$, has five maximal loop classes. In addition to the essential class, there are two singletons (corresponding to the points $0, 1$), one doubleton and one of size 23. All are of positive type. The set of local dimensions of $\mu_\varrho$ consists of an isolated point ($\dim_{loc} \mu_\varrho(0)$) and a closed interval which is the union of the closed intervals

Table 3. Facts about Bernoulli convolutions with $\varrho^{-1}$ a small degree Pisot number.

| Minimal Polynomial of Pisot Number | Approx value of $\varrho$ | Size of reduce Transition graph | Size of Essential set (reduced) |
|---|---|---|---|
| $x^2 - x - 1$ | .618034 | 6 | 3 |
| $x^3 - x^2 - 1$ | .682328 | 152 | 46 |
| $x^3 - x - 1$ | .754878 | 1809 | 1207 |
| $x^3 - 2x^2 + x - 1$ | .569840 | 30 | 27 |
| $x^3 - x^2 - x - 1$ | .543689 | 11 | 8 |
| $x^4 - x^3 - 1$ | .724492 | Unknown | |
| $x^4 - x^3 - 2x^2 + 1$ | .524889 | 538 | 535 |
| $x^4 - 2x^3 + x - 1$ | .535687 | 190 | 187 |
| $x^4 - x^3 - x^2 - x - 1$ | .518790 | 14 | 11 |

generated by the three non-trivial maximal loop classes. In particular, we have

$$[.970222, 1.07770] \subset \{\dim_{\mathrm{loc}} \mu(x) : x \text{ essential}\} \subset [.848302, 1.53266]$$

and

$$[.970221, 1.07771] \cup \{1.81336\} \subset \{\dim_{\mathrm{loc}} \mu(x) : x \in [0,1]\} \subset [.848302, 1.53265] \cup \{1.81336\}$$

See [14] for this example worked out in full.

**Example 8.2** (minimal polynomial $P_2(x) = x^3 - x - 1$). The uniform Bernoulli convolution $\mu_\varrho$, with $\varrho^{-1}$ the Pisot number with minimal polynomial $P_2$, has six maximal loop classes. In addition to the essential class, there are four singletons (two corresponding to the points 0, 1) and one of size 6. The two corresponding to the points 0 and 1, as well as the maximal loop of size 6, are of positive type. The other two singletons, although not positive type, are easy to handle as each has only one transition matrix. The set of local dimensions of $\mu_\varrho$ consists of an isolated point ($\dim_{\mathrm{loc}} \mu_\varrho(0)$) and a closed interval which is the union of the closed intervals generated by the four non-trivial maximal loop classes. In particular, we have

$$[.997949, 1.00853] \subset \{\dim_{\mathrm{loc}} \mu(x) : x \text{ essential}\} \subset [.747924, 1.97198]$$

and

$$[.997949, 1.00853] \cup \{2.46497\} \subset \{\dim_{\mathrm{loc}} \mu(x) : x \in [0,1]\} \subset [.747923, 1.97198] \cup \{2.46497\}$$

See [14] for this example worked out in full.

**8.2. The 2-fold convolution of $\mu_\varrho$.** In this subsection we study the rescaled measure $\nu_\varrho = \mu_\varrho * \mu_\varrho$, where $\mu_\varrho$ is the Bernoulli convolution with $\varrho^{-1}$ the golden ratio. This is the self-similar measure associated with the IFS of contractions $S_1(x) = \varrho x$, $S_2(x) = \varrho x + 1/2 - \varrho/2$ and $S_3(x) = \varrho x + 1 - \varrho$, with corresponding (regular) probabilities $(1/4, 1/2, 1/4)$. It is of finite type and has support [0, 1].

The reduced transition diagram has 40 reduced characteristic vectors. The essential class can be naturally identified with those labelled by $\{28, 29, 30, 33, 34, \ldots, 40\}$. Two cycles in the essential class are $\eta_1 = (29, 35, 39, 29)$ and $\eta_2 = (28, 33, 28)$. The spectral radius of $T(\eta_1)$ is approximately 2.46916, while the spectral radius of $T(\eta_2)$ is approximately 2.48119. This shows that

$$[.992400, 1.00250] \subseteq \{\dim_{\text{loc}} \nu_\rho(x) \colon x \text{ essential}\}.$$

We have also been able to find upper and lower bounds on the local dimensions from the essential class using the method described in Subsection 6.2. We obtain an upper bound by taking the column sub-norm with the subset $C = \{3, 4\}$ and taking admissible products of up to 20 primitive transition matrices. We obtain a lower bound by using the total column sup-norm with products of up to 10 primitive transition matrices. These calculations give

$$\{\dim_{\text{loc}} \nu_\rho(x) \colon x \text{ essential}\} \subseteq [.815721, 1.40091].$$

There are four non-essential maximal loops, each of which is a singleton. The maximal loop classes $\{2\}$ and $\{6\}$ correspond to the two endpoints of the support, 0, 1. The transition matrix in both cases is the $1 \times 1$ identity matrix and the points have local dimension

$$\begin{aligned} \dim_{\text{loc}} \nu_\rho(0) &= \dim_{\text{loc}} \nu_\rho(1) \\ &= \frac{\log 4}{|\log \rho|} \sim 2.88084. \end{aligned}$$

The other two maximal loop classes are $\{25\}$ and $\{19\}$. The characteristic vector of 25 is $(\varrho - 1/2, (1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 3/2 - 3/2\varrho, 1 - 1/2\varrho, 3/2 - \varrho))$. Its transition matrix has the same spectral radius as $T(\eta_2)$, hence the local dimension at any point in the loop class $\{25\}$ coincides with the local dimension at some essential point. Similar statements hold for the loop class $\{19\}$.

Thus the set of local dimensions of $\nu_\rho$ consists of an interval and an isolated point, $\dim_{\text{loc}} \nu_\varrho(0)$.

We recall that a point $x$ can have a most two symbolic representations, and that this will occur only if $x \in \mathcal{F}_n$. Let $x^{(n)}$ be the point with symbolic representation

$$x^{(n)} = (1, \underbrace{2, \dots, 2}_{n}, 7, 8, 10, 19, 19, 19, \dots).$$

This point also has symbolic representation

$$x^{(n)} = (1, \underbrace{2, \dots, 2}_{n+1}, 7, 9, 12, 25, 25, 25, \dots).$$

We see that both representations of these points are external to the loop class, and hence we have a countable number of non-essential points.

Not all points with a symbolic representation in a loop class external to the essential set need be non-essential. To see this we observe that

$$(1, 4, 14, 22, 30, 37, 30, 37, 30, 37, \dots) = (1, 6, 18, 16, 13, 19, 19, 19, 19, \dots)$$

are two symbolic representations for the same point, one of which is in the essential class, and one of which is not. Hence this point is an essential point.

**Remark 8.3.** The transition matrices of the cycles $\eta_1$ and $\eta_2$ give the extreme values of spectral radii over all transition matrices of essential cycles of length up to 10. It would be interesting to know if these give the endpoints of the interval portion of the set of local dimensions.

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