J. Fractal Geom. 4 (2017), 43–71
DOI 10.4171/JFG/44

Journal of Fractal Geometry

Structure of the class of iterated function systems that generate the same self-similar set

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Abstract. Let $\mathcal{H}_K$ denote the family of homogenous IFSs that satisfy the open set condition (OSC) and generate the same self-similar set K, we call the IFSs in $\mathcal{H}_K$ *isotopic*, and give the isotopic class $\mathcal{H}_K$ a multiplication operation defined by composition. The finitely generated property of $\mathcal{H}_K$ was first studied by Feng and Wang on $\mathbb{R}$ [FW], and by the authors on $\mathbb{R}^d$ under the strong separation condition [DL]. In this paper, we continue the investigation of the isotopic class on $\mathbb{R}^d$. By using a new technique with the OSC, we prove that $\mathcal{H}_K$ is finitely generated if either (i) K is totally disconnected, or (ii) the convex hull $\mathrm{Co}(K)$ is a polytope, and there exists a line L passing through a vertex of $\mathrm{Co}(K)$ such that $L \cap K$ is a totally disconnected infinite set. The conditions are easy to check and are satisfied by many standard self-similar sets.

Mathematics Subject Classification (2010). 28A80.

Keywords. Equivalence class, IFS, OSC, homogenous, self-similar set, finite generators.

1. Introduction

By an iterated function system (IFS), we mean a finite family of contractive maps $\{\phi_i\}_{i=1}^N$, $N \geq 2$ on $\mathbb{R}^d$. If we apply these maps repeatedly on a seed set, we obtain a unique compact subset K such that $K = \bigcup_{i=1}^N \phi_i(K)$, which is called the *attractor* of the IFS (see [11] and [15]). In particular, if the ϕ_i's are similitudes, i.e.,

$$\phi_i(x) = \rho_i R_i(x + \alpha_i), \quad i = 1, \dots, N \tag{1.1}$$

where $0 < \rho_i < 1$ and the R_i is an orthonormal matrix, then we call K a *self-similar set*.

[1] The corresponding author is supported by NNSF of China (no. 11471075) and NSF of Fujian (no. 2015J01001).

[2] The second author is supported by an HKRGC grant, and NNSF of China (no. 11171100 & 11271148).

This simple setup provides the most fundamental framework in the study of fractals; not only that it has far reaching theoretical aspects, it also offers a very efficient way to generate many fascinating self-similar patterns [3]. As a quote from [4]: *iterated function systems have been at the heart of fractal geometry almost from its origins*. In fact in the main development, there is a vast literature on dimension theory, multifractal structure and dynamical systems of the self-similar sets/measures for given IFSs (see [2], [4], [10], [11], [13], [18], [19], [23], and the references therein). More recently, there is a lot of interest on the study of Lipschitz equivalence of self-similar sets (see [8], [12], [21], [25], and [28]) and topological classifications (see [1], [17], [20], and [27]) of certain fractal sets.

In another direction, it is natural to ask what are the IFSs that produce a given attractor. The problem was originated from the image compression point of view, and had been studied considerably (see, e.g., [5] and [7]). Recently Feng and Wang [14] initiated an interesting investigation on a new aspect on the following problem. Let $\mathcal{H}_K$ be the class of homogenous IFSs on $\mathbb{R}$ that satisfy the open set condition (OSC) and generate the same self-similar set K. Giving $\mathcal{H}_K$ a multiplication operation by composition, they proved the following result

> *Suppose K is not a finite union of intervals, then $\mathcal{H}_K$ has either one or two generators according to K is non-symmetric or symmetric* (*with respect to some point*).

(Note that if $K = [0,1]$, then it is easy to see that $\mathcal{H}_K$ has infinitely many generators.) The proof depends very much on the special properties of $\mathbb{R}$, and they posed the question on extending this to higher dimension. This was considered by the authors [9] under the more restricted *strong separation condition* (SSC) on K (i.e., $\phi_i(K) \cap \phi_j(K) = \emptyset$, for $i \neq j$). In this paper, we will continue the study under the *open set condition* (OSC), which has far richer structure, and include a lot more self-similar sets.

We call an iterated function system (IFS) $\{\phi_i\}_{i=1}^m, m \geq 2$ on $\mathbb{R}^d$ *homogeneous* if the ϕ_i's are as in (1.1) and have identical ρ_i and R_i for all i. For simplicity, we will use Φ (or Ψ) to denote such IFS, and use $\mathcal{I}$ to denote the family of all homogeneous IFSs. For $\Phi_1, \Phi_2 \in \mathcal{I}$, we define

$$\Phi_1\Phi_2 := \Phi_1 \circ \Phi_2 := \{f_1 \circ f_2 \colon f_i \in \Phi_i,\ i = 1,2\},$$

then clearly, $\mathcal{I}$ is a semi-group under this multiplication.

Definition 1.1. Two IFS's Φ, $\Psi \in \mathcal{I}$ are said to be *isotopic* if they generate the same self-similar set K. We use both $[\Phi]$ and $\mathcal{I}_K$ to denote the isotopic classes, and also $\mathcal{I}_K(\rho)$ for the subfamily of $\mathcal{I}_K$ with contraction ratio $\rho > 0$.

It is clear that the isotopic relation is an equivalent relation on $\mathcal{I}$, and $[\Phi]$ is an equivalent class; also, $[\Phi] = \mathcal{I}_K$ is a sub-semigroup of $\mathcal{I}$. We say that $\mathcal{I}_K$ is *finitely generated,* if there exists a finite subset $\mathcal{F} \subset \mathcal{I}_K$ such that every $\Psi \in \mathcal{I}_K$ can be written as $\Psi = \Phi_1 \dots \Phi_k$ for some $\Phi_1, \dots, \Phi_k \in \mathcal{F}$.

We say that an IFS $\{\phi_i\}_{i=1}^N$ satisfies the *open set condition* (OSC) (see [11] and [23]) if there exists a non-empty bounded open set U such that

$$\phi_i(U) \subset U \quad \text{and} \quad \phi_i(U) \cap \phi_j(U) = \emptyset, \quad \text{for } i \neq j.$$

The OSC is one of the most fundamental conditions on an IFS, and it will be assumed throughout the paper unless otherwise stated. We use $\mathcal{H}(\subset \mathcal{I})$ to denote the family of homogeneous self-similar IFSs that satisfy the OSC, and use the same type of notations as the above for the restriction of $\mathcal{I}$ on $\mathcal{H}$ (e.g., $\mathcal{H}_K, \mathcal{H}_K(\rho)$ etc.). Note that $\mathcal{H}$ is not close under multiplication, but $\mathcal{H}_K$ will be (Corollary 6.2), and is hence a sub-semigroup of $\mathcal{I}$. We also assume the self-similar sets under consideration span $\mathbb{R}^d$ without explicitly mentioning. Our main theorems are

Theorem 1.1. *If $K \subset \mathbb{R}^d$ is totally disconnected and spans $\mathbb{R}^d$, then $\mathcal{H}_K$ is finitely generated.*

For self-similar sets that are not confined to totally disconnected sets, we have:

Theorem 1.2. *Suppose the self-similar set $K \subset \mathbb{R}^d$ is such that the convex hull* $\mathrm{Co}(K)$ *is a d-dimensional polytope, and there exists a line L passing through a vertex of* $\mathrm{Co}(K)$ *such that $L \cap K$ is a totally disconnected infinite set, then $\mathcal{H}_K$ is finitely generated.*

The condition that the convex hull of K is a d-dimensional polytope in the above theorem has the following simple characterization.

Proposition 1.3. *Let $K \subset \mathbb{R}^d$ be a self-similar set generated by a homogeneous IFS $\Phi = \{\phi_j(x) = \rho R(x + \alpha_j)\}_{j=1}^m \in \mathcal{I}$ (no OSC is assumed). Assume K spans $\mathbb{R}^d$, then* $\mathrm{Co}(K)$ *is a d-dimensional polytope if and only if $R^k = \mathrm{Id}$, the identity matrix, for some positive integer $k > 0$.*

As a direct consequence of Theorem 1.2 and Proposition 1.3, we have:

Proposition 1.4. *Suppose the orthonormal matrix R in Φ satisfies $R^k = \mathrm{Id}$ for some $k > 0$, and $\dim_H(K) < d$, then $\mathcal{H}_K$ is finitely generated.*

The logarithmic commensurability of the IFSs in $\mathcal{H}_K$ follows from Theorems 1.1 and 1.2.

Theorem 1.5. *Let K be a self-similar set satisfying the assumptions of Theorem 1.1 or Theorem 1.2. If $\Phi = \{\phi_j(x) = \rho R(x + \alpha_j)\}_{j=1}^N$ and $\Psi = \{\psi_j(x) = \varrho S(x + \beta_j)\}_{j=1}^M$ are two IFSs in $\mathcal{H}_K$, then there exist positive integers $k, \ell > 0$ such that $\Phi^k = \Psi^\ell$.*

It follows immediately from Theorem 1.5 that $\mathcal{H}_K$ is closed under multiplication, and is hence a finitely generated semi-group. The above theorems apply to a large number of standard self-similar sets. Theorem 1.2 also include some sets with $\dim_H(K) = d$ (they are self-similar tiles); we can adjust the second condition in the theorem to the *existence of an edge* $[v_1, v_2]$ *of* $\mathrm{Co}(K)$ *such that* $[v_1, v_2] \cap K$ *is not a finite union of intervals*, and obtain the same conclusion (Proposition 5.6). This is an analog of the condition on $\mathbb{R}$ in [14].

The proofs of the two main theorems depend on the fact that the contraction ratios of the IFSs in $\mathcal{H}_K$ are logarithm commensurable as in [14] and [9]. However, the techniques are quite different, we need to bring in two new ingredients in the proofs. In Theorem 1.1, the major technique is from [28]. In their study of Lipschitz equivalence of totally disconnected self-similar sets, Xi and Xiong [28] extended the SSC assumption in [12] to the OSC by devising a subtle "neighborhood decomposition" on the self-similar sets. Their main effort is to show that the Lipschitz equivalence of the self-similar sets (which applies to our case) implies the logarithmic commensurability of the IFS's; but the proof is complicate and lengthy. By adapting their decomposition to our situation, we give a clearer and shorter proof. We show that for a totally disconnected self-similar set K,

$$\ell_K := \sup\{\#(\mathcal{H}_K(\rho^n)) : n \geq 1\} < \infty$$

(Proposition 4.1), which is used to prove the logarithm commensurable property of the contraction ratios (Proposition 4.2), and is a key step in the proof of Theorem 1.1.

For Theorem 1.2, Proposition 1.3 allows us to assume R to be the identity matrix, and reduces our proof of the logarithmic commensurability of the contraction ratios in $\mathcal{H}_K$ to $L \cap K$ (using Proposition 2.3 and Lemma 5.3). For Proposition 1.3, we remark that Kirat [16] has another characterization of $\mathrm{Co}(K)$ to be a polytope, in which the number of vertices can be estimated. Also, Strichartz and Wang [24, Theorem 4.2] gave another characterization that $\mathrm{Co}(K)$ is a polytope by considering the outward unit normal vectors of the $(d-1)$-dimensional faces of the convex hull of set $\{\alpha_j\}_{j=1}^m$.

For the question on the existence of a single generator, or the more specific information of the generators in $\mathcal{H}_K$, we do not have a complete answer like the one dimensional case [14]. We only have some partial results for the case in Theorem 1.2 through Proposition 1.3. We will discuss this through some basic examples such as the Sierpiński gasket, the twin dragon, etc. (Section 6); we show that essentially it is related to the number theoretic properties of the IFSs and the symmetry of the set K. We also give some examples to show that the conditions in the theorems are optimal.

The paper is organized as follows. In Section 2, we define the neighborhood decomposition for the totally disconnected sets, and provide some preliminary results. In Section 3, some essential properties of the neighborhood decomposition are discussed in detail. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 4 and Section 5, respectively. Finally in Section 6, we prove Theorem 1.5, and provide a number of examples to illustrate the theorems. Some remarks and open questions are also discussed.

2. Basic setup

For $E \subset \mathbb{R}^d$, we use $|E|$ to denote the diameter of E, and let $E_\delta = \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) < \delta\}$ be the δ-neighborhood of E. Also we use $\mathrm{Co}(E)$ to denote the closed convex hull of E. For an affine map $f(x) = Ax + b$ on $\mathbb{R}^d$, we consider f as a point in $\mathbb{R}^{d^2+d}$, and define a norm by

$$|||f||| := \sqrt{\mathrm{tr}(A^t A) + \|b\|^2}.$$

Then $\|f(x)\| \le c_K |||f|||$ for $x \in K$, where $c_K = \max\{\|x\| + 1 : x \in K\}$ (use $\|A\|^2 \le \mathrm{tr}(A^t A)$ where $\|A\|$ is the L^2-norm of A). Without loss of generality, we adopt the following convention throughout the paper to simplify notations:

the set K under consideration spans $\mathbb{R}^d$ and $|K| = 1$.

For a totally disconnected compact set K and for fixed $0 < \rho < 1$, we consider K_{ρ^k}, $k \geq 0$, the ρ^k-neighborhood of K, and let $\{C_{k,1}, C_{k,2}, \dots, C_{k,n_k}\}$ be the family of connected components of K_{ρ^k}. Denote

$$K_{k,j} = K \cap C_{k,j}, \quad j = 1, 2, \dots, n_k, \ k \geq 0 \tag{2.1}$$

and

$$\mathcal{K}_k = \{K_{k,1}, \dots, K_{k,n_k}\}, \qquad \mathcal{K} = \bigcup_{k=0}^{\infty} \mathcal{K}_k. \tag{2.2}$$

Since we have assumed $|K| = 1$, it is clear that for $k = 0$, there is only one connected component $C_{0,1}$, hence $\mathcal{K}_0 = \{K\}$. For $k \geq 1$, $\mathcal{K}_k$ is a finite partition of K with distance at least $2\rho^k$ to each other. Moreover, for K a self-similar set of an IFS Φ with N maps and contraction ratio ρ, then $\#\mathcal{K}_k \leq N^k$. We remark that this neighborhood decomposition plays the similar role as the SSC in that the cells in each level of iteration are uniformly separated.

As a simple example, we let $\Phi = \{\phi_i(x)\}_{i=1}^3$ be the IFS on $\mathbb{R}$ with

$$\phi_i(x) = \frac{1}{5}(x + \alpha_i) \quad \text{such that } \alpha_1 = 0,\ \alpha_2 = 3,\ \alpha_3 = 4.$$

The attractor K is the so-called $\{1, 4, 5\}$-self-similar set in view of the selection of the three subintervals in each iteration. The attractor is a totally disconnected set. In Figure 1, the connected components of $\bigcup_{|I|=k} \phi_I([0, 1])$ are $L_{k,j}$, and $C_{k,j}$ are the intervals defined by the dotted ellipses, and $K_{k,j} = L_{k,j} \cap K = C_{k,j} \cap K$.

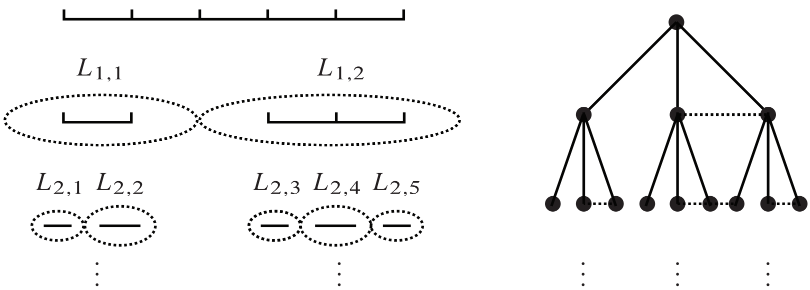

Figure 1. The neighborhood decomposition of the $\{1, 4, 5\}$-self-similar set.

The interest of this example is the surprising result in [25] that the $\{1, 4, 5\}$-self-similar set and the $\{1, 3, 5\}$-Cantor set are Lipschitz equivalent, answering an open question of David and Semmes. It further motivated the deep study of the Lipschitz equivalence of totally disconnected self-similar sets in [28], in which the present neighborhood decomposition was introduced. The right figure is the corresponding graph with the vertices representing the intervals, and the horizontal edges joining the dots, representing the connected components. The graph is called an *augmented tree* for the additional edges, and is a *hyperbolic graph*; it has also been used to study the Lipschitz equivalence problem in the context of hyperbolic graphs (see [21] and [8]).

Lemma 2.1. *Let $K \subset \mathbb{R}^d$ be a totally disconnected compact subset and spans $\mathbb{R}^d$. Then for $0 < \rho < 1$, the following statements hold*:

(i) *for $k \ge 0$,*

$$C_{k,j} = (K_{k,j})_{\rho^k}, \quad j = 1, \dots, n_k;$$

(ii) *for any $K_{k+1,j} \in \mathcal{K}_{k+1}$, there is a unique $K_{k,i} \in \mathcal{K}_k$ such that*

$$K_{k+1,j} \subseteq K_{k,i};$$

(iii) *each $x \in K$ corresponds to a unique sequence $\{j_k\}_k$ with*

$$\bigcap_{k=0}^{\infty} K_{k,j_k} = \{x\};$$

(iv) *there exists an integer $k > 0$ and a neighborhood $V \subset \mathbb{R}^{d^2+d}$ of the identity map* Id *on $\mathbb{R}^d$ such that for $1 \le j \le n_k$,*

  (a) *$f(K_{k,j}) \subset (K_{k,j})_{\rho^k}$ for any $f \in V$, and*

  (b) *$\{y_2 - y_1, y_3 - y_1, \dots, y_{n_k} - y_1\}$ spans $\mathbb{R}^d$ whenever $y_j \in \mathrm{Co}(K_{k,j})$.*

Proof. Statement (i)–(iii) are obvious. For (iv), since K spans $\mathbb{R}^d$, there exist $x_i \in K$, $1 \le i \le d+1$, such that $\{x_2 - x_1,\ x_3 - x_1,\ \dots,\ x_{d+1} - x_1\}$ is a basis of $\mathbb{R}^d$. Hence there exists $\varepsilon > 0$ such that $\{y_2 - y_1,\ y_3 - y_1,\ \dots,\ y_{d+1} - y_1\}$ is also a basis of $\mathbb{R}^d$ whenever $y_i \in B(x_i, \varepsilon)$. As K is a totally disconnected compact set, there exists an integer $k > 0$ such that $|K_{k,j}| < \varepsilon$ for all $K_{k,j} \in \mathcal{K}_k$ (by (iii)); also let

$$V = \{f \text{ affine: } |||f - \mathrm{Id}||| < c_K^{-1}\rho^k, \text{ for all } x \in K\} \tag{2.3}$$

where $c_K = \max\{\|x\| + 1 : x \in K\}$. Hence, for $f \in V$, $\|f(x) - x\| < \rho^k$ for all $x \in K$ (see the first paragraph of this section). As $\{(K_{k,j})_{\rho^k}\}_{j=1}^{n_k}$ is a disjoint cover of K (by (i)), the connected component property implies $f(K_{k,j}) \subseteq (K_{k,j})_{\rho^k}$ for $f \in V$, and (a) follows. Also note that $\mathcal{K}_k$ is a partition of K, each x_j belongs to a unique $K_{k,t_j} \subset B(x_j,\ \varepsilon)$. Hence $\mathrm{Co}(K_{k,t_j}) \subset B(x_j, \varepsilon)$ and (b) follows. □

Proposition 2.2. *Let $K \subset \mathbb{R}^d$ be a totally disconnected compact set and spans $\mathbb{R}^d$. Then for any invertible affine map f on $\mathbb{R}^d$, there is a neighborhood $V \subset \mathbb{R}^{d^2+d}$ of f such that $g(K) \nsubseteq f(K)$ for all $g \in V \setminus \{f\}$.*

Proof. Since f is invertible, we can assume, without loss of generality, that $f = \mathrm{Id}$. We show that the V in (2.3) satisfies the proposition. Suppose otherwise, $g(K) \subseteq K$ for some $g \in V \setminus \{\mathrm{Id}\}$. We let k be as in (2.3) as well, then for any $E \in \mathcal{K}_k$, $g(E) \subseteq E_{\rho^k} \cap K = E$. Hence

$$g(\mathrm{Co}(E)) \subseteq \mathrm{Co}(E), \quad \text{for all } E \in \mathcal{K}_k.$$

By Brouwer's fixed point theorem, there exist $y_j \in \mathrm{Co}(E_{k,j})$ such that

$$g(y_j) = y_j \quad \text{and} \quad y_j \in \mathrm{Co}(E_{k,j}), \quad j = 1, 2, \ldots, n_k.$$

Lemma 2.1(iv)(a) shows that $\{y_2 - y_1, y_3 - y_1, \ldots, y_{n_k} - y_1\}$ contains a basis of $\mathbb{R}^d$. This means that g is the identity map, a contradiction. □

To conclude this section, we give a simple proposition which is used in [14] on $\mathbb{R}$.

Proposition 2.3. *Suppose $\Phi, \Psi \in \mathcal{H}_K$, and assume*

$$\Phi = \{\phi_i(x) = \rho S(x + \alpha_i)\}_{i=1}^N \quad \text{and} \quad \Psi = \{\psi_j(x) = \rho S(x + \beta_j)\}_{j=1}^M,$$

then

$$\Phi = \Psi.$$

Proof. The proof is the same as [14, Proposition 2.1]. The basic idea is that $\log N/|\log \rho| = \dim_H K = \log M/|\log \rho|$ implies $M = N$. The normalized s-dimensional Hausdorff measure μ is the self-similar measure with equal weights for both Φ and Ψ. Hence the Fourier transform of μ satisfies

$$\hat{\mu}(\xi) = P(\rho S^t \xi)\hat{\mu}(\rho S^t \xi) = Q(\rho S^t \xi)\, \hat{\mu}(\rho S^t \xi),$$

where $P(\xi) = \frac{1}{N}\sum_{j=1}^N e^{-2\pi i \alpha_i \cdot S^t \xi}$, $Q(\xi) = \frac{1}{N}\sum_{j=1}^N e^{-2\pi i \beta_i \cdot S^t \xi}$ are the corresponding mask polynomials. This implies $P(\xi) = Q(\xi)$, so that $\{\alpha_i\}_{i=1}^N = \{\beta_j\}_{j=1}^N$. Hence $\Phi = \Psi$. □

3. The neighborhood decomposition

For an IFS $\Phi = \{\phi_j\}_{j=1}^N$, we let $\Sigma = \{1, \ldots, N\}$ and use $\Sigma^* = \bigcup_{k=0}^\infty \Sigma^k$ to denote the set of finite indices, i.e., $J = j_1 \ldots j_k \in \Sigma^k$; we let $|J| = k$ be the length of J, and $\phi_J = \phi_{j_i} \circ \cdots \circ \phi_{j_k}$. We denote

$$\Phi^k = \{\phi_J : |J| = k\}.$$

(As convention $\Phi^0 = \{\mathrm{Id}\}$.) Note that for $\Phi \in \mathcal{H}_K(\rho)$, then $\Phi^k \in \mathcal{H}_K(\rho^k)$. The following lemma shows that the neighborhood decomposition is compatible with the IFS.

Lemma 3.1. *Let $\Phi \in \mathcal{H}_K(\rho)$ with K totally disconnected, and let $f \in \Phi^k$. Then $f(K)_{\rho^k}$ is connected; $f(K) \cap K_{k,j} \neq \emptyset$ if and only if $f(K) \subseteq K_{k,j}$; and*

$$K_{k,j} = \bigcup \{f(K) : f \in \Phi^k,\ f(K) \subseteq K_{k,j}\}. \tag{3.1}$$

Moreover, for any $K_{\ell,i}$, $f(K_{\ell,i}) \subseteq K_{k+\ell,j}$ for some j.

Proof. Since we have assumed $|K| = 1$, K_1 is connected, the first statement follows from $f(K)_{\rho^k} = f(K_1)$. For the second statement, assume $f(K) \cap K_{k,j} \neq \emptyset$. Since $C_{k,j}$ is a connected component of K_{ρ^k}, we have $f(K)_{\rho^k} \subset C_{k,j}$. Hence $f(K) \subseteq K_{k,j}$, which verifies the second statement. This also implies that

$$\bigcup\nolimits_j K_{k,j} = K = \bigcup\nolimits_j \{f(K) \subseteq K_{k,j} : f \in \Phi^k\},$$

and the disjointness yields $K_{k,j} = \bigcup \{f(K) \subseteq K_{k,j} : f \in \Phi^k\}$ for each j.

For the last statement, we let $K_{k+\ell,j}$ be such that $f(K_{\ell,i}) \cap K_{k+\ell,j} \neq \emptyset$. Since $C_{\ell,i}$ is connected, so is $f(C_{\ell,i})$. It follows that $\big(f(K_{\ell,i})\big)_{\rho^{k+\ell}} (= f(C_{\ell,i}))$ is also connected. As $C_{k+\ell,j}$ is a connected component, $\big(f(K_{\ell,i})\big)_{\rho^{k+\ell}} \subseteq C_{k+\ell,j}$ for some j, and hence $f(K_{\ell,i}) \subseteq K_{k+\ell,j}$ by the definition of $K_{k+\ell,j}$. □

We remark that the last statement also holds for any similitude h with contraction ratio ρ^k and $h(K) \subset K$. In Lemma 3.3, we will show that for some special $K_{k,j}$, we can actually have equality, $f(K_{\ell,i}) = K_{k+\ell,j}$.

The following proposition contains two crucial estimates of the decomposition.

Proposition 3.2. *Let $\Phi \in \mathcal{H}_K(\rho)$ with K totally disconnected. Then*

(i) $\eta_0 := \sup\{\rho^{-k}|K_{k,j}| : k \geq 0,\ 1 \leq j \leq n_k\} < \infty$, *hence*

$$\rho^k \leq |K_{k,j}| \leq \eta_0 \rho^k;$$

(ii) $\ell_0 := \max_{k,j} \#\{f \in \Phi^k : f(K) \subseteq K_{k,j}\} < \infty$, *hence in* (3.1), *the number of sets in the union* $\leq \ell_0$.

Proof. The main idea is from [28]. For completeness, we provide a simpler proof adapted to our situation. For any $E \subset \mathbb{R}^d$, $\delta > 0$, and $a \in E_\delta$, we use $E_{\delta,a}$ to denote the connected component of E_δ that contains a.

(i) We first give some auxiliary notations. It follows from a basic property of the OSC [11] that

$$M := \sup_{x\in\mathbb{R}^d, k\geq 0} \#\{f \in \Phi^k : f(K) \cap B(x, \rho^k) \neq \emptyset\} < \infty. \tag{3.2}$$

Let $\{g_i\}_{i\in I}$ denote a family of isometries, and let

$$\mathcal{G} = \Big\{\{g_i\}_{i\in I} : \#I \leq M,\ g_i(K) \cap B(0,1) \neq \emptyset, \text{ and } 0 \in \bigcup\nolimits_{i\in I} g_i(K)\Big\}.$$

Let

$$\mathcal{F} = \big\{F = \bigcup\nolimits_{i\in I} g_i(K) : \{g_i\}_{i\in I} \in \mathcal{G}\big\},$$

then $\mathcal{F}$ is compact under the Hausdorff metric. Note that any $F \in \mathcal{F}$ is totally disconnected, as it is the finite union of totally disconnected compact sets; this implies for any fixed $0 < \eta < 1$, there exists a $\delta > 0$ such that $F_{\delta,0}$, the component of F_δ at 0, is contained in $B(0, \eta)$. Denote

$$\delta_F = \sup\{\delta \geq 0 : F_{\delta,0} \subset B(0, \eta)\}.$$

It is direct to check that $\delta_{(\cdot)}$ is a continuous function on $F \in \mathcal{F}$, it implies that $\inf_{F\in\mathcal{F}} \delta_F > 0$. Choose an integer N such that $0 < \rho^{N-1} < \inf_{F\in\mathcal{F}} \delta_F$. Then

$$F_{\rho^N,0} \subset B(0, \eta) \quad \text{for all } F \in \mathcal{F}. \tag{3.3}$$

Now, for any $k > N$ and $E \in \mathcal{K}_k$, we fix an $a \in E$ and define an (expanding) similitude T by $T(y) = \rho^{N-k}(y - a)$. Let

$$\mathcal{T} = \{T \circ f : f \in \Phi^{k-N},\ T \circ f(K) \cap B(0,1) \neq \emptyset\}.$$

As $a \in f(K)$ for some $f \in \Phi^{k-N}$ (by $\Phi \in \mathcal{H}_K$) and $T(a) = 0$, it follows from (3.2) that $\mathcal{T} \in \mathcal{G}$. Let

$$F = \bigcup\{T \circ f(K) : T \circ f \in \mathcal{T}\},$$

then $F \in \mathcal{F}$, and hence $F_{\rho^N,0} \subset B(0, \eta)$ (by (3.3)); also $a \in (T^{-1}(F))_{\rho^N,a}$.

We claim that $E \subseteq T^{-1}(F_{\rho^N,0})$. If $f \in \Phi^{k-N}$ and $T \circ f(K) \cap B(0,1) = \emptyset$, then $T \circ f(K)_{\rho^N}$ is disjoint from $F_{\rho^N,0}$. On the other hand, note that

$$F \cup \bigcup \{T \circ f(K) \colon f \in \Phi^{k-N},\ T \circ f(K) \cap B(0,1) = \emptyset\} = T(K).$$

Therefore $F_{\rho^N,0}$ is the connected component of $(T(K))_{\rho^N}$ containing 0, and $T^{-1}(F_{\rho^N,0})$, which is the connected component of K_{ρ^k} $(= T^{-1}(T(K)_{\rho^N}))$ and contains a. Since $a \in E$ $(\in \mathcal{K}_k)$, we have $E \subseteq T^{-1}(F_{\rho^N,0})$. This proves the claim.

Observe that $x \in F_{\rho^N,0}$ implies $|x| < \eta < 1$ (by (3.3)); also the definition of T implies that $|T^{-1}(F_{\rho^N,0})| \le 2\rho^{k-N}$. These together with the claim verify that $\rho^{-k}|E| \le 2\rho^{-N}$, and statement (i) follows.

(ii) Let

$$\{f_{k,j,\ell}\}_{\ell=1}^{n_{k,j}} = \{f \in \Phi^k \colon f(K) \subseteq K_{k,j}\},$$

and let O be a bounded open set in the OSC such that $O \cap K \neq \emptyset$, see [23]. By considering $O \cap K_1$ where $K_1 := \{x \colon \mathrm{dist}(x,K) < 1\}$, we can assume, without loss of generality, that $O \subset K_1$. As the contraction ratio of $f_{k,j,\ell}$ is ρ^k, we have $\bigcup_{\ell=1}^{n_{k,j}} f_{k,j,\ell}(O) \subseteq (K_{k,j})_{\rho^k}$. Let $\mathcal{L}$ be the d-dimensional Lebesgue measure. Then

$$\begin{aligned}
\mathcal{L}(O) \cdot n_{k,j} \cdot \rho^{dk} &\le \mathcal{L}(O) \sum_{\ell=1}^{n_{k,j}} \rho^{dk} \\
&= \mathcal{L}\Big(\bigcup_{\ell=1}^{n_{k,j}} f_{k,j,\ell}(O) \Big) \\
&\le \mathcal{L}((K_{k,j})_{\rho^k}) \\
&\le \mathcal{L}(B(0,1))(\eta_0 + 2)^d \cdot \rho^{dk}
\end{aligned}$$

(the last inequality is by (i)), and statement (ii) follows from the inequalities. □

Corollary 3.3. *With the assumption and notation as in Proposition 3.2, there exists a constant $c \ge 1$ such that*

$$c^{-1} \le \sum\nolimits_{j=1}^{n_k} |K_{k,j}|^s \le c, \quad k = 0, 1, 2, \ldots$$

where $s = \dim_H(K)$.

Proof. We observe that $f_{k,j,\ell}$ has contraction ratio ρ^k, hence Proposition 3.2(ii) implies

$$1 = \sum_{j=1}^{n_k} \sum_{\ell=1}^{n_{k,j}} \rho^{ks} \le \ell_0 \sum\nolimits_{j=1}^{n_k} |K_{k,j}|^s.$$

(The first identity is the dimension formula under the OSC.) Also by Proposition 3.2(i), we have

$$\sum\nolimits_{j=1}^{n_k} |K_{k,j}|^s \le \sum\nolimits_{j=1}^{n_k} (\eta_0 \rho^k)^s \le \eta_0^s \sum_{j=1}^{n_k} \sum_{\ell=1}^{n_{k,j}} \rho^{ks} = \eta_0^s.$$

The corollary follows. □

The following lemma is another consequence of Proposition 3.2, it allows us to choose a $K_{k,i}$ so that for any multi-index $I \in \Sigma^*$, $\phi_I(K_{k,i})$ is also a full component in the neighborhood decomposition. It will be used in the proof of Propositions 4.1 and 4.2 in the next section.

Lemma 3.4. *Let $\Phi \in \mathcal{H}_K(\rho)$ with K totally disconnected. Then there exist k_0 and $K_{k_0,i_0} \in \mathcal{K}_{k_0}$, such that for any similitude h with contraction ratio ρ^n and $h(K) \subset K$,*

$$h(K_{k_0,i_0}) \in \mathcal{K}_{k_0+n}.$$

In particular, for any multi-index $I \in \Sigma^n$, we have $\phi_I(K_{k_0,i_0}) \in \mathcal{K}_{k_0+n}$.

Proof. For any $k \ge 1$, Lemma 3.1 and Proposition 3.2(ii) imply that each $E := K_{k,j}$ can be written as the union of $\phi_J(K)$, $|J| = k$, and the number in the union is at most ℓ_0; moreover, the union is $\mathcal{H}^s$-measure non-overlap by OSC (s is the Hausdorff dimension of K). Observe that $\mathcal{H}^s(\phi_J(K)) = \rho^{sk}\mathcal{H}^s(K)$, hence $\{\rho^{-sk}\mathcal{H}^s(E): E \in \mathcal{K}_k\}$ takes at most ℓ_0 possible values. We conclude that there is an integer $k_0 > 0$ and $K_{k_0,i_0} \in \mathcal{K}_{k_0}$ such that

$$\rho^{-sk_0}\mathcal{H}^s(E_{k_0,i_0}) = \sup_{k>0}\{\rho^{-sk}\mathcal{H}^s(E): E \in \mathcal{K}_k\}. \tag{3.4}$$

Now for the given h, $h(K_{k_0,i_0})$ is contained in some $K_{k_0+t,\ell} \in \mathcal{K}_{k_0+t}$ (by Lemma 3.1 and the remark). By (3.4), we have

$$\rho^{-s(k_0+t)}\mathcal{H}^s(h(K_{k_0,i_0})) = \sup_{k>0}\{\rho^{-sk}\mathcal{H}^s(E): E \in \mathcal{K}_k\}.$$

Therefore, $h(K_{k_0,i_0}) = K_{k_0+t,\ell} \in \mathcal{K}_{k_0+t}$, and the lemma follows. □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 depends on the following two propositions.

Proposition 4.1. *For $0 < \rho < 1$ and K is totally connected, we have*

$$\ell_K := \sup\{\#\mathcal{H}_K(\rho^n) \colon n \geq 1\} < \infty. \tag{4.1}$$

Proof. To prove (4.1), we assume on the contrary that $\ell_K = \infty$. Then there exists a sequence of positive integers $\{m_k\}_k$ such that $\#\mathcal{H}_K(\rho^{m_k}) > k$, and we can choose distinct IFSs $\Phi_{k,j} \in \mathcal{H}_K(\rho^{m_k})$, $1 \leq j \leq k$. Fix an $x_0 \in K$. For every k, there exist similitudes $h_{k,j} \in \Phi_{k,j}$, $1 \leq j \leq k$, such that $x_0 \in h_{k,j}(K)$. Let $x_{k,j} \in K$ such that $x_0 = h_{k,j}(x_{k,j})$, then $h_{k,j}$ can be written as

$$h_{k,j}(x) = \rho^{m_k} T_{k,j}(x - x_{k,j}) + x_0, \quad 1 \leq j \leq k,\ k \geq 1$$

where $\rho^{m_k} T_{k,j}$ is the linear part of maps in $\Phi_{k,j}$. Furthermore, Proposition 2.3 shows that $\{T_{k,j}\}_j$ are distinct orthogonal matrices, so $h_{k,j}$ are all distinct. Let $S_{k,j} = h_{k,1}^{-1} \circ h_{k,j}$, then $S_{k,2}, S_{k,3}, \ldots, S_{k,k}$ are all distinct and

$$S_{k,j}(x) = T_{k,1}^{-1} T_{k,j}(x - x_{k,j}) + x_{k,1}, \quad 2 \leq j \leq k,\ k \geq 1.$$

Since $T_{k,1}^{-1} T_{k,j}$ is an orthogonal matrix, and $x_{k,j}$ belongs to K, the sets $\{S_{k,j}\}_j$ and $\{S_{k,j}^{-1}\}_j$ are contained in a compact set of $\mathbb{R}^{d^2+d}$. Note that the number of maps in $\{S_{k,2}, S_{k,3}, \ldots, S_{k,k}\}$ tend to infinity, this implies

$$\lim_{k\to\infty} \min\{|||S_{k,i} - S_{k,j}||| \colon 2 \leq i < j \leq k\} = 0.$$

Hence there exist $2 \leq i_k < j_k \leq k$ so that $\lim_{k\to\infty} |||S_{k,i_k} - S_{k,j_k}||| = 0$. As $\{S_{k,j}^{-1}\}_j$ and $\{S_{k,j}\}_j$ are uniformly bounded, we have $\lim_{k\to\infty} |||S_{k,i_k}^{-1} \circ S_{k,j_k} - \mathrm{Id}||| = 0$. This implies

$$\lim_{k\to\infty} \rho^{-m_k} \|h_{k,j_k}(x) - h_{k,i_k}(x)\| = \lim_{k\to\infty} \|S_{k,i_k}^{-1} \circ S_{k,j_k}(x) - x\| = 0 \tag{4.2}$$

uniformly for all $x \in K$.

Next, by Lemma 2.1(iv), there exists an integer τ such that $\{y_2 - y_1, y_3 - y_1, \ldots, y_{n_\tau} - y_1\}$ contains a basis of $\mathbb{R}^d$ for any $y_j \in \mathrm{Co}(K_{\tau,j})$. By Lemma 3.4, there exist τ_0, K_{τ_0,i_0}, such that for the multi-indices I_j with $|I_j| = \tau$ and $\phi_{I_j}(K) \subset K_{\tau,j}$, we have

$$\phi_{I_j}(K_{\tau_0,i_0}) \in \mathcal{K}_{\tau_0+\tau}, \text{ for all } 1 \leq j \leq n_\tau.$$

Since $h_{k,i}(K) \subset K$, by Lemma 3.4 again, we have $h_{k,i}(\phi_{I_j}(K_{\tau_0,i_0})) \in \mathcal{K}_{\tau_0+\tau+m_k}$ for $1 \le j \le n_\tau$. By (4.2), we see that there exists a large k such that

$$h_{k,i_k}(\phi_{I_j}(K_{\tau_0,i_0})) \subset (h_{k,j_k}(\phi_{I_j}(K_{\tau_0,i_0})))_{\rho^{\tau_0+\tau+m_k}}, \text{ for all } 1 \le j \le n_\tau.$$

Hence by the connected component property,

$$h_{k,i_k}(\phi_{I_j}(K_{\tau_0,i_0})) = h_{k,j_k}(\phi_{I_j}(K_{\tau_0,i_0}))$$

necessarily. This implies

$$h_{k,i_k}(\mathrm{Co}(\phi_{I_j}(K_{\tau_0,i_0}))) = h_{k,j_k}(\mathrm{Co}(\phi_{I_j}(K_{\tau_0,i_0}))), \quad 1 \le j \le n_\tau,$$

and Brouwer's fixed point theorem yields a $y_j \in \mathrm{Co}(\phi_{I_j}(K_{\tau_0,i_0}))$ such that

$$h_{k,i_k}(y_j) = h_{k,j_k}(y_j), \quad 1 \le j \le n_\tau.$$

Since $\phi_{I_j}(K_{\tau_0,i_0}) \subset \phi_{I_j}(K) \subseteq K_{\tau,j}$ for each j, so are their convex hulls. Hence from our choice of τ, $\{y_2 - y_1, \ y_3 - y_1, \ \dots, \ y_{n_{\hat{k}}} - y_1\}$ contains a basis of $\mathbb{R}^d$. It follows that $h_{k,i_k,} = h_{k,j_k}$. This contradicts the distinctness of $h_{k,j}$, and completes the proof. □

Next we verify the logarithmic commensurability of the contraction ratios in $\mathcal{H}_K$. It is instructive to compare this with Proposition 2.3 where the contraction ratios and the orthogonal matrices are the same.

Proposition 4.2. *Let K be a totally disconnected self-similar set, and let $\Phi, \Psi \in \mathcal{H}_K$ with*

$$\Phi = \{\phi_i(x) = \rho R(x + \alpha_i)\}_{i=1}^N$$

and

$$\Psi = \{\psi_j(x) = \varrho S(x + \beta_j)\}_{j=1}^M.$$

Then $\{\log \rho, \ \log \varrho\}$ are commensurable.

Proof. Let $s = \dim_H(K)$, fix the K_{k_0,i_0} as defined in Lemma 3.4 satisfying (3.4). Since $\frac{\log \rho}{\log \varrho}$ is the limit of a sequence of rational numbers, there exist two sequences of integers $\{n_k\}$ and $\{m_k\}$ such that $\lim\limits_{k\to\infty} \rho^{n_k}\varrho^{-m_k} = 1$. We can assume, without loss of generality, that

$$\varrho^{m_k} \le \rho^{n_k} < 2\varrho^{m_k}, \qquad k \ge 1. \tag{4.3}$$

By Lemma 3.4, we have

$$\phi_1^{n_k}(K_{k_0,i_0}) \in \mathcal{K}_{k_0+n_k}, \quad k \ge 1. \tag{4.4}$$

Now consider $\Phi^{k_0}\Psi^{m_k}$. For any $f \in \Phi^{k_0}\Psi^{m_k}$, $f(K)$ must intersects some $K_{n_k+k_0,j}$. By Lemma 3.1 and (4.3), we see that $(f(K))_{\rho^{n_k+k_0}}$ is connected, and the assumption that $f(K)$ intersects $K_{n_k+k_0,j}$ implies that $f(K)$ must be contained in $K_{n_k+k_0,j}$. Therefore, by (4.4)

$$\phi_1^{n_k}(K_{k_0,i_0}) = \bigcup\{f(K): f \in \Phi^{k_0}\Psi^{m_k},\ f(K) \cap \phi_1^{n_k}(K_{k_0,i_0}) \neq \emptyset\}.$$

Let $\ell_k = \#\{f(K): f \in \Phi^{k_0}\Psi^{m_k},\ f(K) \cap \phi_1^{n_k}(K_{k_0,i_0}) \neq \emptyset\}$, then the OSC implies ($s = \dim_H(K)$)

$$\rho^{-(n_k+k_0)s}\mathcal{H}^s(\phi_1^{n_k}(K_{k_0,i_0})) = \rho^{-(n_k+k_0)s}\ell_k\rho^{k_0 s}\varrho^{m_k s}\mathcal{H}^s(K).$$

On the other hand, by Proposition 3.2(ii) and (3.4), the left hand side of the equality equals $\ell_0\mathcal{H}^s(K)$. Hence

$$\ell_0 = \ell_k\rho^{-n_k s}\varrho^{m_k s}.$$

Note that ℓ_k are integers, the assumption $\lim_{k\to\infty}\rho^{n_k}\varrho^{-m_k} = 1$ implies $\ell_0 = \ell_k$ for all large k. Therefore, $\rho^{n_k} = \varrho^{m_k}$ for all large k. $\{\log\rho, \log\varrho\}$ are commensurable. □

Now we prove the first main theorem (Theorem 1.1)

Theorem 4.3. *If K is totally disconnected, then $\mathcal{H}_K$ is finitely generated.*

Proof. We fix a $\Phi = \{\phi_i(x) = \rho S(x + \alpha_j)\}_{j=1}^N \in \mathcal{H}_K$ such that N is the smallest number of maps for all the IFSs in $\mathcal{H}_K$. Write N as $N = N_1^{t_1}\dots N_\iota^{t_\iota}$, where $N_1 \dots N_\iota$ are distinct primes. We first consider the case $\gcd\{t_1, \dots, t_\iota\} = 1$. Let $s = \dim_H(K)$, then the OSC implies

$$\rho^s = N^{-1},$$

and ρ is the largest contraction ratio of IFS's in $\mathcal{H}_K$.

For any $\Psi = \{\psi_j(x) = \varrho T(x + \beta_j)\}_{j=1}^M \in \mathcal{H}_K$, we have $\varrho^s = M^{-1}$. Proposition 4.2 shows that $\varrho^m = \rho^n$ for some positive integers n, m. Hence $M^m = N^n$, and $M = N_1^{\frac{n}{m}t_1}\dots N_\iota^{\frac{n}{m}t_\iota}$. Therefore $u = \frac{n}{m}$ is a positive integer and $\varrho = \rho^u$ by using the assumption $\gcd\{t_1, \dots, t_\iota\} = 1$. Since $\Psi = \{\psi_j(x)\} \in \mathcal{H}_K$ is arbitrary, we have

$$\mathcal{H}_K \subset \bigcup_{u=1}^{\infty}\{\{\rho^u T(x + \beta_j)\}_{j=1}^M : M = N^u,\ T \text{ is orthogonal}\}. \tag{4.5}$$

By Proposition 4.1, we can write

$$\mathcal{H}_K(\rho^u) = \{\Psi_{u,1}, \Psi_{u,2}, \ldots, \Psi_{u,\ell_u}\}$$

with $1 \le \ell_u \le \ell_K$. (Note that $\mathcal{H}_K(\rho^u) \neq \emptyset$ as it always contain Φ^u.) Hence by (4.5),

$$\mathcal{H}_K = \bigcup_{u=1}^{\infty} \mathcal{H}_K(\rho^u) = \bigcup_{u=1}^{\infty} \{\Psi_{u,j} : 1 \le j \le \ell_u\}. \tag{4.6}$$

Let $\kappa > 0$ so the $\ell_\kappa = \sup_u \ell_u$. Then we see that for $u = \kappa + v$,

$$\mathcal{H}_K(\rho^u) \supseteq \{\Psi_{1,1}^v \circ \Psi_{\kappa,j} : 1 \le j \le \ell_\kappa\}. \tag{4.7}$$

The maximality of ℓ_κ implies that the above "$\supseteq$" is actually "$=$." Therefore, (4.6) shows that every IFS in $\mathcal{H}_K$ is a composition of the IFS's in the set $\bigcup_{\ell=1}^{\kappa} \mathcal{H}_K(\rho^\ell)$. Hence $\mathcal{H}_K$ is finitely generated as a semi-group.

If $\gcd\{t_1, \ldots, t_\iota\} = t_0 > 1$, then $N = N_1^{t_0 s_1} \ldots N_\tau^{t_0 s_\iota}$. We let $N_0 = N_1^{s_1} \ldots N_\tau^{s_\iota}$, $u = \frac{t_0 n}{m}$ (an integer). Then $\varrho = \rho^{n/m} = \rho^{u/t_0} := r^u$, and we have, analogous to (4.5),

$$\mathcal{H}_K \subset \bigcup\nolimits_{u \in A} \{\{r^u T(x + \beta_j)\}_{j=1}^M : M = N_0^u,\ T \text{ is orthogonal}\}.$$

where A is the set of integers such that the subfamilies on the right side is non-empty. (Note that there is no guarantee, say, $u = 1$ is in A, and we have to adjust the $\Psi_{1,1}$ in the above proof.) Consider the congruence class of A modulus κ. It is easy to see that there exists a $\tau \ge \kappa$ so that $A_\tau := \{n \in A : n \le \tau\} \equiv A \pmod{\kappa}$. Hence any $u \in A$ satisfying $u > \tau$ can be written as $u = k\kappa + \ell$ with $\ell \in A_\tau$ and $k > 0$. Then we can replace the $\Phi_{1,1}^v \circ \Psi_{\kappa,j}$ in (4.7) by $\Psi_{\ell,1} \circ \Psi_{\kappa,1}^{k-1} \circ \Psi_{\kappa,j}$. □

5. The polytope case

In this section we consider the self-similar set K in connection with its convex hull $\mathrm{Co}(K)$. Recall that F is a *face* of a convex set E if for $x, y \in E$, $\lambda x + (1-\lambda) y \in F$, $0 < \lambda < 1$ implies $x, y \in F$.

Proposition 5.1. *Let K be a self-similar set generated by an IFS $\Phi = \{\phi_j(x) = \rho(x + \alpha_j)\}_{j=1}^N \in \mathcal{K}_K$ (no OSC is assumed). Suppose F is a face of* $\mathrm{Co}(K)$, *then $F \cap K$ is the self-similar set generated by the subfamily $\{\phi_j : \phi_j(K) \cap F \neq \emptyset\}$.*

In particular each vertex of $\mathrm{Co}(K)$ *is the fixed point of a unique $\phi_i \in \Phi$.*

Proof. For $\phi \in \Phi$, we can write $\phi(x) = \rho(x-b)+b$, and $b \in K$ is the fixed point of ϕ. Hence $\phi(x) = \rho x + (1-\rho)b$ is a convex combination of x and b. Let $\Phi_F = \{\phi \in \Phi : \phi(K) \cap F \neq \emptyset\}$. Then for $\phi \in \Phi_F$, there exist $x \in K$ such that $\phi(x) \in F$. By the convex combination and the face property of F, $\phi(x) \in F$ if and only if both $x, b \in F$. In this case, $f(F \cap K) \subseteq F \cap K$. Hence $F \cap K$ is the attractor of Φ_F, and the conclusion follows. □

Proposition 5.2. *Let K be a self-similar set that spans $\mathbb{R}^d$ (no OSC is assumed). Then* $\mathrm{Co}(K)$ *is a d-dimensional polytope if and only if there is an integer k_0 such that $R^{k_0} = \mathrm{Id}$ for any orthogonal matrix R in $\Phi = \{\rho R(x+\alpha_j)\}_{j=1}^N \in \mathcal{K}_K$. Furthermore, there are only finitely many such R.*

Proof. Necessity. Assume $\mathrm{Co}(K)$ is a d-dimensional polytope. Let $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$ be the set of vertices of $\mathrm{Co}(K)$. For each vertex υ_i of $\mathrm{Co}(K)$, let

$$C_i = \left\{ \sum_{\ell=1}^{m} p_\ell(\upsilon_\ell - \upsilon_i) : p_\ell \geq 0 \right\}, \quad 1 \leq i \leq m \tag{5.1}$$

be the smallest convex cone (with vertex at 0) containing $K - \upsilon_i := \{x - \upsilon_i : x \in K\}$. Without loss of generality, we assume that C_1 is such that the Lebesgue measure of $B(0,1) \cap C_1$ is minimal i.e.,

$$\mathcal{L}(B(0,1) \cap C_1) = \min_{1 \leq j \leq m} \mathcal{L}(B(0,1) \cap C_j) \tag{5.2}$$

For each vertex υ_i of $\mathrm{Co}(K)$, there exists $\alpha_i \in \mathbb{R}^d$ such that

$$\alpha_i \cdot (x - \upsilon_i) > 0, \text{ for all } x \in K \setminus \{\upsilon_i\}, \quad 1 \leq i \leq m. \tag{5.3}$$

For any $\Phi = \{\phi_j(x) = \lambda R(x + \alpha_j)\}_{j=1}^N$ that generate K, there exist $\omega \in K$ and j such that $\upsilon_1 = \phi_j(\omega)$. Then (5.3) implies

$$\lambda(R^t \alpha_1) \cdot (x - \omega) = \alpha_1 \cdot (\phi_j(x) - \upsilon_1) > 0, \text{ for all } x \in K \setminus \{\omega\}.$$

Hence ω is also a vertex of $\mathrm{Co}(K)$. Let $\upsilon_{i_1} = \omega$, then by $\phi_j(K) \subset K$ and the cone property, we have $RC_{i_1} \subseteq C_1$. The minimality in (5.2) implies $RC_{i_1} = C_1$.

Denote

$$\mathcal{O}_{i,j} = \{S \text{ orthogonal}: SC_i = C_j\}, \quad \mathcal{O} = \bigcup_{i,j=1}^{m} \mathcal{O}_{i,j}. \tag{5.4}$$

Since C_i is a convex cone with non-empty interior and finitely many edges, $\mathcal{O}_{i,j}$ and $\mathcal{O}$ are finite sets. Note that the above $R \in \mathcal{O}$. Since Φ^n also generates K, the above argument implies $R^n \in \mathcal{O}$ for all $n > 0$. Therefore, $R^k = \mathrm{Id}$ for some k by the finiteness of $\mathcal{O}$. Hence we can choose the least common multiple k_0 of such $k > 0$, and the necessity follows.

Sufficiency. Assume that there is an integer $k > 0$ such that, for any $\Phi = \{\phi_j(x) = \lambda R(x + \alpha_j)\}_{j=1}^N$ generating K, $R^k = \mathrm{Id}$ holds. Then Φ^k has the form $\{g_j(x) = \lambda^k(x + \beta_j)\}_{j=1}^M$. Let ω_j be the fixed point of g_j, and $\Omega = \mathrm{Co}(\{\omega_1, \omega_2, \dots, \omega_M\})$. Then it is easy to see that $g_j(\Omega) \subset \Omega$ for all j. Note that all vertices of Ω are points of K that spans $\mathbb{R}^d$, hence $\Omega = \mathrm{Co}(K)$ and is a d-dimensional polytope. □

By taking $R = \mathrm{Id}$, it follows from the above proposition that the $\mathrm{Co}(K)$ in Proposition 5.1 is a convex polytope. Next we will prove a lemma for the commensurability of the contraction ratios, which will be a key step to prove Theorem 1.2. It is also an extension of [14, Theorem 1.1(i)], as the K in the following lemma need not be a self-similar set.

Lemma 5.3. *Let $K \subset [a, b]$ be a totally disconnected compact set and is not a singleton. Suppose $a \in K$, and there are $\phi_i(x) = \rho_i(x-a)+a$, $i = 1, 2$ satisfying*

$$[a, a + \delta] \cap K \subset \phi_i(K) \subset K, \quad i = 1, 2 \tag{5.5}$$

for some $\delta > 0$. Then ρ_1 and ρ_2 are logarithmic commensurable.

Proof. Suppose $\rho_i, i = 1, 2$ are not logarithm commensurable, then there exist two sequences of positive integers $\{s_n^1\}_n$, $\{s_n^2\}_n$ such that $\tau_n := \rho_1^{s_n^1}/\rho_2^{s_n^2}$ converges to 1, and we can assume, without loss of generality, $\rho_1^{s_n^1} < \rho_2^{s_n^2}$. Note that (5.5) implies $[a, a + \delta] \cap K = [a, a + \delta] \cap \phi_i(K), i = 1, 2$. By applying ϕ_i inductively, we have

$$\phi_i^{s_n^i}([a, a + \delta] \cap K) = [a, a + \rho_i^{s_n^i}\delta] \cap K, \quad i = 1, 2.$$

As $\rho_1^{s_n^1} < \rho_2^{ts_n^2}$, we have $\rho_2^{s_n^2}([0, \delta] \cap (K - a)) \supseteq \rho_1^{s_n^1}([0, \delta] \cap (K - a))$. Therefore

$$[0, \delta] \cap (K - a) \supseteq \tau_n([0, \delta] \cap (K - a)), \quad n \geq 1.$$

This is a contradiction in view of Proposition 2.2 applying to $[0, \delta] \cap (K - a)$, $f = \mathrm{Id}$, and $g(x) = \tau_n x$ for n sufficiently large. Hence ρ_1 and ρ_2 are logarithm commensurable. □

We now prove Theorem 1.2 in the Introduction.

Theorem 5.4. *Suppose the self-similar set $K \subset \mathbb{R}^d$ is such that the convex hull* $\mathrm{Co}(K)$ *is a polytope, and there exists a line L passing through a vertex of* $\mathrm{Co}(K)$ *and $L \cap K$ is totally disconnected and infinite, then $\mathcal{H}_K$ is finitely generated.*

Proof. We first prove the logarithmic commensurability of contraction ratios. Let $\{\upsilon_1, \dots, \upsilon_m\}$ be the vertices of $\mathrm{Co}(K)$. For any $\Phi = \{\phi_i(x) = \rho R(x + \alpha_i)\}_{j=1}^N \in \mathcal{H}_K$ and $\Psi = \{\psi_i(x) = \varrho S(x + \beta_i)\}_{j=1}^M \in \mathcal{H}_K$. Proposition 5.2 implies that there is k such that $R^k = S^k = \mathrm{Id}$. Hence without loss of generality, we assume that $R = S = \mathrm{Id}$. Also, note that each vertex of $\mathrm{Co}(K)$ is the fixed point of a unique $\phi_i \in \Phi$ (Proposition 5.1), we can assume $\phi_i(\upsilon_i) = \upsilon_j$ for $i = 1, 2, \dots, m$. It follows easily from the contraction of ϕ_i and the vertices of $\mathrm{Co}(K)$ as fixed points that $\upsilon_j \notin \phi_i(K)$ when $i \neq j$. Similarly, we can assume $\psi_i(\upsilon_i) = \upsilon_i$ for $i = 1, 2, \dots, m$ and $\upsilon_j \notin \psi_i(K)$ when $i \neq j$.

Let $L_K = K \cap L$, without loss of generality, we assume that L passes υ_1. Since $\upsilon_1 \notin \phi_i(K)$ and $\upsilon_1 \notin \phi_i(K)$ when $i > 1$, there is $\delta > 0$ such that

$$B(\upsilon_1, \delta) \cap \phi_i(K) = B(\upsilon_1, \delta) \cap \psi_j(K) = \emptyset, \text{ for all } i, j > 1.$$

Hence $B(\upsilon_1, \delta) \cap L_K = B(\upsilon_1, \delta) \cap \phi_1(L_K) = B(\upsilon_1, \delta) \cap \psi_1(L_K)$. This implies

$$\phi_1(B(\upsilon_1, \delta) \cap L_K) \subset B(\upsilon_1, \delta) \cap L_K, \quad \psi_1(B(\upsilon_1, \delta) \cap L_K) \subset B(\upsilon_1, \delta) \cap L_K.$$

By considering $B(\upsilon_1, \delta) \cap L_K$ in $\mathbb{R}$, Lemma 5.3 shows that ϱ and ρ are logarithm commensurable.

By using Proposition 5.2, it is direct to show that the ℓ_K in Proposition 4.1 is finite; a similar proof as in Theorem 4.3 verifies that $\mathcal{H}_K$ is finitely generated. □

Proposition 5.5. *Suppose the self-similar set $K \subset \mathbb{R}^d$ is such that* $\dim_H(K) < d$, *and* $\mathrm{Co}(K)$ *is a d-dimensional polytope, then $\mathcal{H}_K$ is finitely generated.*

Proof. We need only show that there is a line L satisfies the condition in Theorem 5.4. In view of Proposition 5.2, we can assume $R = \mathrm{Id}$, i.e., $\phi_j(x) = \rho(x + \alpha_j)$. Also, from the proof of the sufficiency of the proposition, we can assume $\{\upsilon_1, \upsilon_2, \dots, \upsilon_m\}$ is the set of vertices of $\mathrm{Co}(K)$ and $\upsilon_j = \phi_j(\upsilon_j)$ for all $1 \le j \le m$. Furthermore we can assume $\mathrm{Co}(\{\upsilon_1, \upsilon_2, \dots, \upsilon_j\})$ is a $(j-1)$-dimensional face of $\mathrm{Co}(K)$ for $1 \le j \le d$, and $\mathrm{Co}(\{\upsilon_1, \upsilon_2, \dots, \upsilon_{d+1}\})$ is a d-dimensional polytope (otherwise we can make a rearrangement of the vertices, and group the consecutive vertices together to form an increasing sequence of faces).

For a vertex υ_j of $\mathrm{Co}(K)$, we can write $\phi_j(x) = \rho(x - \upsilon_j) + \upsilon_j$. Hence for the L passes υ_j and intersects K at $x(\neq \upsilon_j)$, then $\upsilon_j + \rho_j^n(x - \upsilon_j) = \phi_j^n(x) \in L \cap K$ for all integer $n > 0$. It follows that L intersects K either at a single point or infinitely many points.

Now we assume on the contrary that $L \cap K$ is not a totally disconnected infinite set for any such lines L. Then the above assertion implies that the edge (i.e., one dimensional face) $[\upsilon_1, \upsilon_2]$ of $\mathrm{Co}(K)$ intersects K in a segment with positive length, and Proposition 5.1 implies that $[\upsilon_1, \upsilon_2] \cap K$ is a 1-dimensional self-similar set with non-empty interior.

Consider the segments linking υ_3 and points of $[\upsilon_1, \ \upsilon_2] \cap K$, then as the above, the intersection of K and such segment has positive length. Therefore $\mathrm{Co}(\{\upsilon_1, \upsilon_2, \upsilon_3\}) \cap K$ has positive 2-dimensional Lebesgue measure. Therefore $\dim_H(K) \geq 2$ and $d > 2$, again by Proposition 5.1, $\mathrm{Co}(\{\upsilon_1, \upsilon_2, \upsilon_3\}) \cap K$ is a 2-dimensional self-similar set with non-empty interior.

We carry out the same argument for υ_4 and points of $\mathrm{Co}(\{\upsilon_1, \upsilon_2, \upsilon_3\}) \cap K$, and continue. Eventually we conclude that K is a d-dimensional self-similar set with non-empty interior, a contradiction to $\dim_H(K) < d$, and completes the proof. □

The following result is a variant of Theorem 5.4, it does not assume the totally disconnectedness on $L \cap K$ (and allows $\dim_H(K) = d$), but needs the line $L \cap K$ to be an edge (one dimensional face) of $\mathrm{Co}(K)$.

Proposition 5.6. *Suppose the self-similar set $K \subset \mathbb{R}^d$ is such that $\mathrm{Co}(K)$ is a d-dimensional polytope, and there exists an edge $[\upsilon_i, \upsilon_j]$ of $\mathrm{Co}(K)$ such that $[\upsilon_i, \upsilon_j] \cap K$ has infinitely many connected components, then $\mathcal{H}_K$ is finitely generated.*

Proof. We first prove the logarithmic commensurability of contraction ratios. Let $\Phi = \{\phi_j(x) = \varrho R(x + \alpha_j)\}_{j=1}^N$ and $\Psi = \{\psi_j(x) = \rho S(x + \beta_j)\}_{j=1}^M$ be in $\mathcal{H}_K$. By Proposition 5.2, we can assume $R = S = \mathrm{Id}$. Let $F = [\upsilon_i, \upsilon_j]$ be the given edge, then $F \cap K$ is the attractor of Φ_F, Ψ_F (Proposition 5.1). Since $F \cap K$ is a one dimensional self-similar set and has infinitely many connected components, it follows from [14, Theorem 1.1(ii)] that ϱ and ρ are logarithm commensurable.

By using Proposition 5.2, we can show that the ℓ_K in Proposition 4.1 is finite; a similar proof as that of Theorem 4.3 verifies that $\mathcal{H}_K$ is finitely generated. □

6. Consequences and examples

We have the following conclusion on the logarithmic commensurability of the isotopic IFSs in $\mathcal{H}_K$.

Theorem 6.1. *Let K be a self-similar set as in Theorem 1.1, Theorem 1.2, or Proposition 5.6. If*

$$\Phi = \{\phi_j(x) = \rho R(x + \alpha_j)\}_{j=1}^N \quad \text{and} \quad \Psi = \{\psi_j(x) = \varrho S(x + \beta_j)\}_{j=1}^M$$

are two IFSs in $\mathcal{H}_K$, then

$$\Phi^k = \Psi^\ell$$

for some $k,\ \ell > 0$.

Proof. If K is totally disconnected, Proposition 4.2 shows that $\rho^n = \varrho^m$ for some positive integers n, m. Consider the IFSs $\Phi^{in}\Psi^{(k-i)m}$, $1 \le i < k$. Then they all belong to $\mathcal{H}_K(\rho^{kn})$. For k large, Proposition 4.1 implies that there exist i, j with $1 \le i < j < k$ such that $\Phi^{in}\Psi^{(k-i)m} = \Phi^{jn}\Psi^{(k-j)m}$. Comparing the orthonormal matrices yields

$$R^{(j-i)n} = S^{(j-i)m}.$$

This implies $\Phi^{(j-i)n}$ and $\Psi^{(j-i)m}$ have the same contraction ratio and orthogonal matrix. Proposition 2.3 then implies that $\Phi^{(j-i)n} = \Psi^{(j-i)m}$.

For the other two cases, that $\mathrm{Co}(K)$ is a convex polytope implies that $R^k = S^k = \mathrm{Id}$ for some k. That $\rho^n = \varrho^m$ for some positive integers n, m is contained in the proof of Theorem 1.2 (i.e., Theorem 5.4) and Proposition 5.6. Therefore by applying Proposition 2.3 again, we have $\Phi^{kn} = \Psi^{km}$. □

The following two corollaries are straight forward.

Corollary 6.2. *Under the same assumption as in Theorem 6.1, $\mathcal{H}_K$ is a finitely generated semi-group.*

Corollary 6.3. *Under the same assumption as in Theorem 6.1, if one of the IFSs in $\mathcal{H}_K$ satisfies the SSC, then all the IFS in $\mathcal{H}_K$ satisfy the SSC.*

For the more specific description of the generators of $\mathcal{H}_K$, there is a rather precise statement for the one-dimensional case, which is an improvement of [14, Theorem 3.1]. A set $E \subset \mathbb{R}^d$ is said to be symmetric if there exists c such that $(E - c) = -E$.

Proposition 6.4. *Let K be the self-similar set in $\mathbb{R}$ generated by*

$$\Phi = \{\rho(x + \alpha_j)\}_{j=1}^N$$

with $0 < |\rho| < 1$. Assume that Φ satisfies the OSC with an open interval U, and K is not a finite union of intervals. Then

(i) *$\mathcal{H}_K$ is generated by a unique generator if and only if the digit set $\{\alpha_j\}_{j=1}^N$ is not symmetric.*

(ii) *If $\{\alpha_j\}_{j=1}^N$ is symmetric, then there are two generators $\Phi_1 = \{\varrho(x + a_j)\}_{j=1}^M$ and $\Phi_2 = \{-\varrho(x + b_j)\}_{j=1}^M$ such that $\mathcal{H}_K = \{\Phi_1^{n-1}\Phi_2,\ \Phi_1^n : n \in \mathbb{N}\}$.*

We will omit the proof as the main part is as in [14, Theorem 3.1]; the additional part is that $\{\alpha_j\}_{j=1}^N$ is symmetric if and only if K is symmetric, which is an easy consequence that $x \in K$ can be expressed as $\sum_{j=1}^\infty \rho^i \alpha_{i_j}$, $1 \le i_j \le N$. Instead, we use the simple example of $\{1, 4, 5\}$-self-similar set in Section 2 to illustrate the basic idea.

Example 6.5. Let K be the self-similar set generated by the IFS

$$\Phi = \left\{\frac{1}{5}(x + a) : a = 0, 3, 4\right\}$$

as in Section 2. Then Φ is the unique generator of $\mathcal{H}_K$.

Proof. Obviously, Φ satisfies the OSC with open interval $(0, 1)$. If $\Psi = \{\varrho(x + b_j)\}_{j=1}^M$ $(0 < |\varrho| < 1)$ belongs to $\mathcal{H}_K$, then the logarithmic commensurability of IFSs shows that $\Phi^n = \Psi^m$ for some integers $n, m > 0$. Hence $(\frac{1}{5})^n = \varrho^m$, and the dimension formulas for Φ and Ψ imply $3^n = M^m$. Since 3 is a prime, $M = 3^k$, and so $|\varrho| = (\frac{1}{5})^k$ for some integer $k > 0$. Proposition 2.3 implies $\Psi^2 = \Phi^{2k}$. As Φ satisfies the OSC with the open interval $(0, 1)$, so is Ψ.

If $\varrho < 0$, then there is a map $\psi_j(x) := \varrho(x + b_j)$ such that $\psi_j(1) = 0$, so $\psi_j(x) = \varrho(x - 1)$. Also, there is a $g = (\frac{1}{5})^k(\cdot + c) \in \Phi^k$ such that $g(0) = 0$. As $|\varrho| = (\frac{1}{5})^k$, hence $g(x) = |\varrho|x$. This implies $K = -(K - 1)$, and K is symmetry, a contradiction. Therefore, $\varrho > 0$, and Proposition 2.3 implies $\Psi = \Phi^k$, the assertion follows. □

A non-trivial example for Proposition 5.6 is the IFS $\{\phi_i\}_{i=1}^3$ where $\phi_i(x) = \frac{1}{3}(x + a_i)$, and $\{a_1, a_2, a_3\} \equiv \{0, 1, 2\} (\mathrm{mod}\ 3)$ (but $\neq \{0, 1, 2\}$). Then the self-similar set K is a tile in $\mathbb{R}$ that have infinitely many connected components. It follows from Proposition 6.4 that $\mathcal{H}_K$ has a unique generator. For higher dimensional tile, a simple minded example of this sort can be the product $K \times [0, 1]$;

the more interesting examples are Examples 6.7-6.9. Note that the condition of infinitely many connected components cannot be omitted; it is seen that for [0, 1], $\mathcal{H}_{[0,1]}$ has infinitely many generators, as it contains all the IFS $\Phi_n = \{\frac{1}{n}(\cdot + i)\}_{i=0}^{n-1}$; another example of similar nature is in [14].

Example 6.6. Let $\Phi_n = \{\frac{1}{4n}(x + \alpha) \colon \alpha \in \{0, 1\} \oplus \{0, 8n\} \oplus 4\{0, \dots, n-1\}\}$, $n = 1, 2, \dots$. Then all Φ_n satisfy the OSC with the same attractor $K := [0,\ 1] \cup [2,\ 3]$, and $\mathcal{H}_K$ has infinitely many generators.

Next we consider some two dimensional examples in regard to the generators of $\mathcal{H}_K$. We first make some remarks on Proposition 5.2 with $\mathrm{Co}(K)$ a convex polytope. Let $\Phi = \{\rho R(x + \alpha_i)\}_{i=1}^N$, we showed that this condition is equivalent to $R^{k_0} = \mathrm{Id}$ for some $k_0 \geq 0$, which is determined by $R(C_i) = C_j$ for some i, j (see (5.4)). Hence if $R = \mathrm{Id}$ is the only one satisfies the identity, than $\mathcal{H}_K$ has a unique generator $\Phi = \{\rho(x + \alpha_i)\}_{i=1}^N$ where $N \neq M^k$ for any $k > 1$.

To determine the generators, we make some more observation on the geometry of $\mathrm{Co}(K)$. Let $\{v_1, \dots, v_m\}$ be the vertices of $\mathrm{Co}(K)$, they are fixed points of some $\phi_i \in \Phi$, namely, $\phi_i(v_i) = v_i$ (Proposition 5.1). Let $\{i_1, \dots i_m\}$ be a permutation of $\{1, \dots, m\}$ such that $\phi_{k_j}(v_{i_j}) = v_j$ for each $1 \leq j \leq m$. Then it is not difficult to show that

$$R(C_{i_j}) = C_i \quad \text{and} \quad R((K - v_{i_j}) \cap B(0, \varepsilon)) = (K - v_i) \cap B(0, \varepsilon) \tag{6.1}$$

for some $\varepsilon > 0$. (We can take $\varepsilon > 0$ such that $\phi_\ell(K) \cap B(v_j, \varepsilon) = \emptyset$ for all $\ell \neq k_j$.) This symmetric property can be used to find the admissible ρR of $\Phi \in \mathcal{H}_K$. We illustrate this idea by the following example of Sierpiński gasket.

Example 6.7. Let K be the self-similar set generated by the IFS $\Phi = \{\phi_j\}_{j=1}^3$ on $\mathbb{R}^2$, where $\phi_i(x) = \rho(x - v_i) + v_i$, $0 < \rho \leq \frac{1}{2}$. Let Δ denote the triangle determined by v_1, v_2, v_3. Then K satisfies the conditions in Theorem 1.2, and

(i) if Δ is not isosceles, then Φ is the unique generator of $\mathcal{H}_K$;

(ii) if Δ is isosceles but not equilateral, then $\mathcal{H}_K$ has two generators;

(iii) if Δ is an equilateral triangle, then $\mathcal{H}_K$ has six generators.

Proof. Assume $\Psi = \{\psi_j = \varrho R(\cdot + \beta_j)\}_{j=1}^M$ belongs to $\mathcal{H}_K$. It is clear the assumptions in Theorem 1.2 are satisfied. Similar to Example 6.5, the commensurability of Φ, Ψ and the dimension formula imply $M = 3^n$ and $\varrho = \rho^n$ for some positive integer $n > 0$.

(i) Since Δ is not isosceles, so (6.1) hold only if $R = \mathrm{Id}$. Hence $\Psi = \{\psi_j(x) = \rho^n(x + \beta_j)\}_{j=1}^{3^n}$ for some β_j, and Proposition 2.3 shows $\Psi = \Phi^n$, the assertion follows.

(ii) Since Δ is isosceles but not equilateral. Then there is a reflection R satisfies (6.1), and there are two generators : Φ as given, and $\widetilde{\Phi} = \{\rho R(\cdot + \beta_i)\}_{i=1}^3$ for some β_i.

(iii) It is easy to check that there are six orthonormal matrices that satisfies (6.1), and similar to (ii), there are six generators of $\mathcal{H}_K$. □

Example 6.8. Consider the twin dragon tile $K = T(A, \mathcal{D}_1)$ generated by $\Phi = \{A^{-1}(\cdot + d_i)\}_{i=1}^2, d_i \in \mathcal{D}_1$, where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{D}_1 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Then Φ is a homogenous IFS with contraction ratio $\rho = \frac{1}{\sqrt{2}}$, $S = \sqrt{2}A^{-1}$. Moreover K satisfies the assumptions of Theorem 1.2 (or Proposition 5.6), and $\mathcal{H}_K$ has two generators: the given one Φ, and

$$\widetilde{\Phi} = \{-A^{-1}(\cdot + d_i)\}_{i=1}^2, \quad d_i \in \mathcal{D}_2 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

Proof. It is direct to check that $T(-A, \mathcal{D}_2) = T(A, \mathcal{D}_1) = K$, hence $\widetilde{\Phi} \in \mathcal{H}_K$. Consider the IFS $\widetilde{\Phi}\Phi^3$, since $-A^4 = 4\mathrm{Id}$, Proposition 5.1 shows that $\mathrm{Co}(T(A, \mathcal{D}_1))$ is the polygon, and the vertices are (see Figure 2):

$$\frac{1}{3}\left\{ \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}.$$

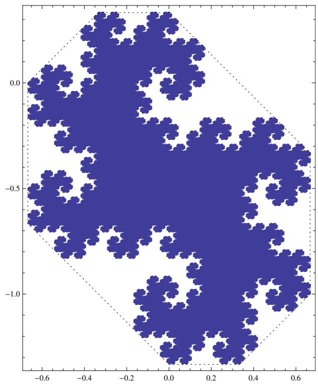

Figure 2. The convex hull of twin dragon.

Denote these vertices by υ_j, $j = 1, 2, \dots, 8$ in the same order as the above, and let $\upsilon_0 = \upsilon_8$ and $\upsilon_9 = \upsilon_1$. It is easy to check that the set $[\upsilon_i, \upsilon_{i+1}] \cap K$ is the Cantor set generated by the IFS $\{\frac{1}{4}(\cdot + 3\upsilon_i), \frac{1}{4}(\cdot + 3\upsilon_{i+1})\}$ (apply Proposition 5.1 to $\{f \in \widetilde{\Phi}\Phi^3 : f(K) \cap [\upsilon_i, \upsilon_{i+1}] \neq \emptyset\}$, which has exactly two maps with fixed points υ_i and υ_{i+1} respectively). Hence the line L determined by υ_i and υ_{i+1} satisfies the assumption in Theorem 1.2 (also Proposition 5.6).

On $\mathrm{Co}(K)$, it is easy to see that each angle is $\frac{3\pi}{4}$. Let C_i be the cone generated by $\mathrm{Co}(K)$ with vertex at υ_i as in (5.1), and let $L_{i,i\pm1} = \{t(\upsilon_{i\pm1} - \upsilon_i) : t \geq 0\}$, be the two sides of the cone. It is clear that all the cones are isometric.

For any IFS $\Psi = \{\varrho R(x + \beta_j)\}_{j=1}^N \in \mathcal{H}_K$, let $RC_1 = C_\ell$ (by (6.1)), then $RL_{1,2} = L_{\ell,\ell+1}$ and $RL_{1,0} = L_{\ell,\ell-1}$ (or the other way round). It follows that $R = S_1 S_2 \dots S_k$ for some $k > 0$, with $S_i = \pm\sqrt{2}A^{-1}$. We show that we can choose the k such that $\varrho R = \pm A^{-k}$, this will imply Ψ is a composition of IFSs from $\{\Phi, \widetilde{\Phi}\}$ by applying Proposition 2.3.

To this end, we only consider the case $R = -2A^{-2}$, all other cases are similar. Consider $\Psi\Phi^2$, the relation $-A^4 = 4\mathrm{Id}$ implies that each map in $\Psi\Phi^2$ has the form $\frac{1}{2}\varrho(x + \beta)$. It follows that $[\upsilon_1, \upsilon_2] \cap K$ is a self-similar set generated by the sub family $\{f \in \Psi\Phi^2 : f(K) \cap [\upsilon 1, \upsilon_2] \neq \emptyset\}$ (Proposition 5.1). Denote this subfamily by $\{\frac{1}{2}\varrho(x + d_j)\}_{j=1}^M$. Similar to Example 6.5, the commensurability of Φ, Ψ and the dimension formula imply $M = 2^k$, and $\varrho = (\frac{1}{2})^{2k-1}$ for some $k > 0$. If $k = 2\ell$ then $\varrho R = -A^{-8\ell-2}$, so the linear parts of maps in Ψ and $\Phi^{8\ell+1}\widetilde{\Phi}$ are the same. Hence $\Psi = \Phi^{8\ell+1}\widetilde{\Phi}$ by Proposition 2.3. If $k = 2\ell + 1$, then $\varrho R = -A^{-8\ell-6}$, so the linear parts of the maps in Ψ and $\Phi^{8\ell+5}\widetilde{\Phi}$ are the same, and $\Psi = \Phi^{8\ell+5}\widetilde{\Phi}$. Therefore, Φ and $\widetilde{\Phi}$ generate the isotopic class. □

We consider one more example from [22] that not all the fixed points of Φ are vertices of $\mathrm{Co}(K)$.

Example 6.9. Let K be the self-similar set generated by the IFS

$$\Phi = \left\{\frac{1}{3}(x + \alpha) : \alpha \in \mathcal{D}\right\}$$

with

$$\mathcal{D} = \left\{\begin{pmatrix} i + a|j| \\ j \end{pmatrix} : i, j = -1, 0, 1\right\}.$$

There are five fixed points of the $\phi_i \in \Phi$ that are vertices of $\mathrm{Co}(K)$. Φ satisfies the condition in Theorem 1.2 or Proposition 5.6, and $\mathcal{H}_K$ has two generators: Φ and $\widetilde{\Phi} = \{\frac{1}{3}R(\cdot + \alpha) : \alpha \in \mathcal{D}\}$ when $a \neq 0$, where $R = \mathrm{diag}(1, -1)$, the reflection along the x-axis.

Proof. The K is a $\mathbb{Z}^2$-tile as showed in the following Figure 3.

$$\mathrm{Co}(K) = \mathrm{Co}\left(\left\{\begin{pmatrix} \frac{a+1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{a-1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{a+1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{a-1}{2} \\ -\frac{1}{2} \end{pmatrix}\right\}\right) \quad \text{if } a > 0.$$

(for $a < 0$, we replace the middle one by $(\frac{1}{2}, 0)$). The vertical line $x = \frac{a+1}{2}$ passes two vertices of $\mathrm{Co}(K)$ and intersects K a Cantor set when $a > 0$ ($x = \frac{a-1}{2}$ for $a < 0$). Actually, in the five edges of $\mathrm{Co}(K)$, except the two horizontal edges, the other three edges all intersect K a Cantor set. Hence the assumptions of Theorem 1.2 or Proposition 5.6 are satisfied.

It is easy to see that $R = \mathrm{diag}(1, -1)$ is the only orthogonal matrix ($\neq \mathrm{Id}$) satisfies (6.1). For $\Psi = \{\varrho R(\cdot + \beta_j)\}_{j=1}^N$, we can use the same argument as Example 6.8 to conclude that $\Psi = \Phi^k$ or $\Psi = \Phi^{k-1}\widetilde{\Phi}$. We omit the detail. □

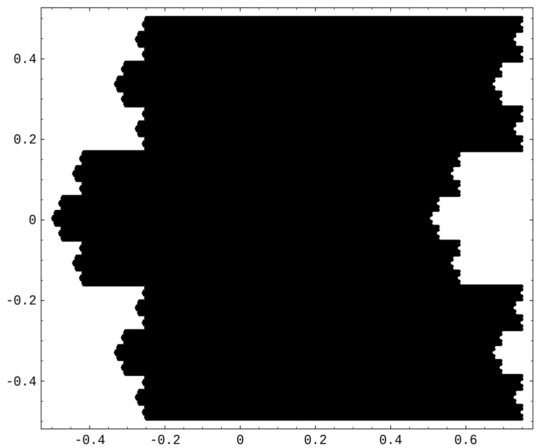

Figure 3. The tile K with $a = 0.5$.

We see from the examples that the two dimensional connected tiles can be very different from the one-dimensional tiles as far as the isotopic property is concern ($\mathcal{H}_{[0,1]}$ has infinitely many generators). In fact a well-known open problem related to this is whether there is a 2-reptile that is also a 3-reptile, see [6], it is a difficult question in the plane, but is trivial on $\mathbb{R}$.

If K is totally disconnected, but $\mathrm{Co}(K)$ is not a polytope, then we do not have a suitable condition to ensure the uniqueness of generator, nor an efficient way to find the generators. We conjecture that if $\Phi = \{\rho R(\cdot + \alpha_j)\}_{j=1}^N \in \mathcal{H}_K$, N is not a power of any integer and 1 is not an eigenvalue of any power R^n ($n > 0$), then Φ is the unique generator of $\mathcal{H}_K$.

In Corollary 6.2, we conclude that $\mathcal{H}_K$ is a semigroup for those cases in this paper. We do not know if this is true without the additional assumptions.

In our consideration, we only deal with the homogenous IFS. We have no knowledge on the non-homogenous case. Note that if the contraction ratios are logarithmic commensurable, then the neighborhood decomposition is still valid with more work, see [28]. This may offer a way to consider this more general case.

References

[1] R. Atkins, M. Barnsley, A. Vince, and D. Wilson, A characterization of hyperbolic affine iterated function systems. *Topology Proc.* **36** (2010), 189–211. MR 2609349 Zbl 1196.28011

[2] C. Bandt and S. Graf, Self-similar sets 7. A characterization of self-similar fractals with positive Hausdorff measure. *Proc. Amer. Math. Soc.* **114** (1992), no. 4, 995–1001. MR 1100644 Zbl 0823.28003

[3] M. Barnsley, *Fractals everywhere.* Second edition. Revised with the assistance of and with a foreword by H. Rising, III. Academic Press Professional, Boston, MA, 1993. MR 1231795 Zbl 0784.58002

[4] M. Barnsley and A. Vince, Developments in fractal geometry. *Bull. Math. Sci.* **3** (2013), no. 2, 299–348. MR 3084010 Zbl 1275.28006

[5] M. F. Barnsley and L. P. Hurd, *Fractal image compression.* Illustrations by L. F. Anson. A. K. Peters, Wellesley, MA, 1993. MR 1203998 Zbl 0796.68186

[6] H. Croft, K. Falconer, and R. Guy, *Unsolved problems in geometry.* Corrected reprint of the 1991 original. Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, New York, 1994. MR 1316393 Zbl 0748.52001

[7] A. Deliu, J. Geronimo, and R. Shonkwiler, On the inverse fractal problem for two-dimensional attractors. *Philos. Trans. Roy. Soc. London Ser. A* **355** (1997), no. 1726, 1017–1062. MR 1448589 Zbl 0886.58068

[8] G. T. Deng, K. S. Lau, and J. Luo, Lipschitz equivalence of self-similar sets and hyperbolic boundaries II. *J. Fractal Geom.* **2** (2015), no. 1, 53–79. MR 3338641 Zbl 1318.28020

[9] Q. R. Deng and K. S. Lau, On the equivalence of homogeneous iterated function systems. *Nonlinearity* **26** (2013), no. 10, 2767–2775. 7–2775. MR 3105513 Zbl 1276.28015

[10] Q. R. Deng and S. M. Ngai, Conformal iterated function systems with overlaps. *Dyn. Syst.* **26** (2011), no. 1, 103–123. MR 2772167 Zbl 1217.28006

[11] K. Falconer, *The geometry of fractal sets.* Cambridge Tracts in Mathematics, 85. Cambridge University Press, Cambridge, 1986. MR 0867284 Zbl 0587.28004

[12] K. Falconer and D. Marsh, On the Lipschitz equivalence of Cantor sets. *Mathematika* **39** (1992), no. 2, 223–233. MR 1203278 Zbl 0791.28006

[13] D. J. Feng and H. Y. Hu, Dimension theory of iterated function systems. *Comm. Pure Appl. Math.* **62** (2009), no. 11, 1435–1500. MR 2560042 Zbl 1230.37031

[14] D. J. Feng and Y. Wang, On the structures of generating iterated function systems of Cantor sets. *Adv. Math.* **222** (2009), no. 6, 1964–1981. MR 2562770 Zbl 1184.28013

[15] J. Hutchinson, Fractals and self similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747. MR 0625600 Zbl 0598.28011

[16] I. Kirat, Remarks on self-affine fractals with polytope convex hulls. *Fractals* **18** (2010), no. 4, 483–498. MR 2741287 Zbl 1209.28014

[17] M. Kwieciński, A locally connected continuum which is not an IFS attractor. *Bull. Polish Acad. Sci. Math.* **47** (1999), no. 2, 127–132. MR 1686674 Zbl 0931.54028

[18] K. S. Lau and S. M. Ngai, Multifractal measures and a weak separation condition. *Adv. Math.* **141** (1999), no. 1, 45–96. MR 1667146 Zbl 0929.28007

[19] K. S. Lau and S. M. Ngai, A generalized finite type condition for iterated function systems. *Adv. Math.* **208** (2007), no. 2, 647–671. MR 2304331 Zbl 1113.28006

[20] K. S. Lau, J. Luo, and H. Rao, Topological fractal squares. *Math. Proc. Cambridge Philos. Soc.* **155** (2013), no. 1, 73–86. MR 3065260 Zbl 1272.28007

[21] J. Luo and K. S. Lau, Lipschitz equivalence of self-similar sets and hyperbolic boundaries. *Adv. Math.* **235** (2013), 555–579. MR 3010069 Zbl 1267.28011

[22] J. C. Lagarias and Y. Wang, Self-affine tiles in $\mathbb{R}^n$. *Adv. Math.* **121** (1996), no. 1, 21–49. MR 1399601 Zbl 0893.52013

[23] A. Schief, Separation properties for self-similar sets. *Proc. Amer. Math. Soc.* **122** (1994), no. 1, 111–115. MR 1191872 Zbl 0807.28005

[24] R. Strichartz and Y. Wang, Geometry of self-affine tiles I. *Indiana Univ. Math. J.* **48** (1999), no. 1, 1–23. MR 1722192 Zbl 0938.52017

[25] H. Rao, H. J. Ruan, and L. F. Xi, Lipschitz equivalence of self-similar sets. *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 3, 191–196. MR 2198192 Zbl 1086.28007

[26] H. Rao, H. J. Ruan, and Y. Wang, Lipschitz equivalence of Cantor sets and algebraic properties of contraction ratios. *Trans. Amer. Math. Soc.* **364** (2012), no. 3, 1109–1126. MR 2869169 Zbl 1244.28015

[27] L. F. Xi and Y. Xiong, Self-similar sets with initial cubic patterns. *C. R. Math. Acad. Sci. Paris* **348** (2010), no. 1-2, 15–20. MR 2586736 Zbl 1225.28008

[28] L. F. Xi and Y. Xiong, Lipschitz equivalence class, ideal class and the Gauss class number problem. Preprint 2013. arXiv:1304.0103 [math.MG]

Received July 8, 2015

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