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Hausdorff dimension of the graph of an operator semistable Lévy process

Lina Wedrich[1]

Abstract. Let $X = \{X(t): t \geq 0\}$ be an operator semistable Lévy process in $\mathbb{R}^d$ with exponent E, where E is an invertible linear operator on $\mathbb{R}^d$. For an arbitrary Borel set $B \subseteq \mathbb{R}_+$ we interpret the graph $\mathrm{Gr}_X(B) = \{(t, X(t)): t \in B\}$ as a semi-selfsimilar process on $\mathbb{R}^{d+1}$, whose distribution is not full, and calculate the Hausdorff dimension of $\mathrm{Gr}_X(B)$ in terms of the real parts of the eigenvalues of the exponent E and the Hausdorff dimension of B. We use similar methods as applied in [16] and [8].

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1. Introduction

Let $X = (X(t))_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. Namely, X is a stochastically continuous process with càdlàg paths that has stationary and independent increments and starts in $X(0) = 0$ almost surely. The distribution of X is uniquely determined by the distribution of $X(1)$ which can be an arbitrary infinitely divisible distribution. The process X is called (c^E, c)-operator semistable, if the distribution of $X(1)$ is full, i.e. not supported on any lower dimensional hyperplane, and there exists a linear operator E on $\mathbb{R}^d$ such that

$$\{X(ct)\}_{t\geq 0} \overset{\mathrm{fd}}{=} \{c^E X(t)\}_{t\geq 0} \quad \text{for some } c > 1. \tag{1.1}$$

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Here $\stackrel{\text{fd}}{=}$ denotes equality of all finite dimensional distributions and

$$c^E := \sum_{n=0}^{\infty} \frac{(\log c)^n}{n!} E^n.$$

If for some $\alpha \in (0, 2]$ the exponent E is a multiple of the identity, i.e. $E = \alpha \cdot I$, we call the process $(c^{1/\alpha}, c)$-semistable. The Lévy process is called operator stable if (1.1) holds for all $c > 0$.

The aim of this paper is to calculate the Hausdorff dimension $\dim_H \mathrm{Gr}_X(B)$ of the graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ of an operator semistable Lévy process $X = (X(t))_{t \geq 0}$ for an arbitrary Borel set $B \subseteq \mathbb{R}_+$.

For an arbitrary subset F of $\mathbb{R}^d$ the s-dimensional Hausdorff measure $\mathcal{H}^s(F)$ is defined as

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |F|_i^s : |F_i| \leq \delta \text{ and } F \subseteq \bigcup_{i=1}^{\infty} F_i \right\},$$

where $|F| = \sup\{\|x - y\| : x, y \in F\}$ denotes the diameter of a set $F \subseteq \mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm. It can be shown that the value

$$\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

exists and is unique for all subsets $F \subseteq \mathbb{R}^d$. The critical value $\dim_H F$ is called the Hausdorff dimension of F. Further details on the Hausdorff dimension can be found in [2] and [14].

In the past efforts have been made to generate dimension results for Lévy processes, which fulfill certain scaling properties. An overview can for example be found in [11] or [19]. For an operator semistable Lévy process X and an arbitrary Borel set $B \subseteq \mathbb{R}_+$ Kern and Wedrich [8] calculated the Hausdorff dimension of the range $\dim_H X(B)$ in terms of the real parts of the eigenvalues of the exponent E and the Hausdorff dimension of B. The result is a generalization of the one stated in Meerschaert and Xiao [16], who calculated the Hausdorff dimension $\dim_H X(B)$ for an operator stable Lévy process.

For an arbitrary operator semistable Lévy process X our aim is to adapt the methods used to prove the results above by interpreting the graph $\mathrm{Gr}_X(B) = \{(t, X(t)) : t \in B\}$ as a process on $\mathbb{R}^{d+1}$, which fulfills the scaling property (1.1) for a certain exponent but whose distribution is not full. The method of generating dimension results for a class of Lévy processes by interpreting the graph as a $(d+1)$-dimensional Lévy process has also been employed by Manstavičius in [13].

The most prominent example of a semistable, non-stable distribution is perhaps the limit distribution of the cumulative gains in a series of St. Petersburg games. In this particular case, Kern and Wedrich [9] already calculated the Hausdorff dimension $\dim_H \mathrm{Gr}_X([0,1])$ of the corresponding graph over the interval $[0,1]$ employing the method described above. Dimension results for the graph of a stable Lévy process can be found in [1] and [6]. Furthermore, in the case that X is a dilation stable Lévy process on $\mathbb{R}^d$, i.e. an operator stable Lévy process with a diagonal exponent, Xiao and Lin [12] calculated the Hausdorff dimension $\dim_H \mathrm{Gr}_X(B)$ for an arbitrary Borel set $B \subseteq \mathbb{R}_+$ and Hou [4] determined an exact Hausdorff measure function for $\mathrm{Gr}_X([0,1])$.

This paper is structured as follows: In Section 2.1 we recall spectral decomposition results from [15], which enable us to decompose the exponent E and thereby the operator semistable Lévy process X according to the distinct real parts of the eigenvalues of E. Section 2.2 contains certain uniformity and positivity results from [8] for the density functions of the process X, which will be helpful in the proofs of our main results. The main results on the Hausdorff dimension of the graph of an operator semistable Lévy process are stated and proven in Section 3.

Throughout this paper K denotes an unspecified positive and finite constant that can vary in each occurrence. Fixed constants will be denoted by K_1, K_2, etc.

2. Preliminaries

2.1. Spectral decomposition. Let X be a (c^E, c)-operator semistable Lévy process. Factor the minimal polynomial of E into $q_1(x) \cdot \cdots \cdot q_p(x)$ where all roots of q_i have real parts equal to a_i and $a_i < a_j$ for $i < j$. Let $\alpha_j = a_j^{-1}$ so that $\alpha_1 > \cdots > \alpha_p$, and note that $0 < \alpha_j \leq 2$ by Theorem 7.1.10 in [15]. Define $V_j = \mathrm{Ker}(q_j(E))$. According to Theorem 2.1.14 in [15] $V_1 \oplus \cdots \oplus V_p$ is then a direct sum decomposition of $\mathbb{R}^d$ into E invariant subspaces. In an appropriate basis, E is then block-diagonal and we may write $E = E_1 \oplus \cdots \oplus E_p$ where $E_j : V_j \to V_j$ and every eigenvalue of E_j has real part equal to a_j. Especially, every V_j is an E_j-invariant subspace of dimension $d_j = \dim V_j$ and $d = d_1 + \cdots + d_p$. Write $X(t) = X^{(1)}(t) + \cdots + X^{(p)}(t)$ with respect to this direct sum decomposition, where by Lemma 7.1.17 in [15], $\{X^{(j)}(t), t \geq 0\}$ is a (c^{E_j}, c)-operator semistable Lévy process on V_j. We can now choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ such that the V_j, $j \in \{1, \ldots, p\}$, are mutually orthogonal and throughout this paper we will let $\|x\| = \sqrt{\langle x, x\rangle}$ be the associated Euclidean norm. In particular we have for $t = c^r m > 0$ that

$$\|X(t)\|^2 \stackrel{\mathrm{d}}{=} \|c^{rE} X(m)\|^2 = \|c^{rE_1} X^{(1)}(m)\|^2 + \cdots + \|c^{rE_p} X^{(p)}(m)\|^2, \qquad (2.1)$$

with $r \in \mathbb{Z}$ and $m \in [1, c)$.

The following lemma states a result on the growth behavior of the exponential operators t^{E_j} near the origin $t = 0$. It is a variation of Lemma 2.1 in [16] and a direct consequence of Corollary 2.2.5 in [15].

Lemma 2.1. *For every $j \in \{1, \dots, p\}$ and every $\epsilon > 0$ there exists a finite constant $K \geq 1$ such that for all $0 < t \leq 1$ we have*

$$K^{-1} t^{a_j+\epsilon} \leq \|t^{E_j}\| \leq K t^{a_j-\epsilon} \qquad (2.2)$$

and

$$K^{-1} t^{-(a_j-\epsilon)} \leq \|t^{-E_j}\| \leq K t^{-(a_j+\epsilon)}. \qquad (2.3)$$

2.2. Properties of the density function. The following three lemmas state uniformity results of operator semistable Lévy processes. They will be very helpful in the proofs of our main theorems. The lemmas are taken from Kern and Wedrich [8]. Let $X = \{X(t)\}_{t\geq 0}$ be a full dimensional operator semistable Lévy process on $\mathbb{R}^d$ and $g_t, t > 0$, the corresponding continuous density functions. Lemma 2.2 in [8] states the following:

Lemma 2.2. *The mapping $(t, x) \mapsto g_t(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$ and we have*

$$\sup_{t\in[1,c)} \sup_{x\in\mathbb{R}^d} |g_t(x)| < \infty. \qquad (2.4)$$

As a consequence we get a result on the existence of negative moments of an operator semistable Lévy process $X = \{X(t)\}_{t\geq 0}$ on $\mathbb{R}^d$ given in Lemma 2.3 of [8].

Lemma 2.3. *For any $\delta \in (0, d)$ we have*

$$\sup_{t\in[1,c)} \mathbb{E}[\|X(t)\|^{-\delta}] < \infty. \qquad (2.5)$$

Furthermore, we will need a uniform positivity result for the density functions taken from Lemma 2.4 of [8].

Lemma 2.4. *Let $\{X(t)\}_{t\geq 0}$ be an operator semistable Lévy process with maximal index $\alpha_1 > 1$, $d_1 = 1$ and with density g_t as above. Then there exist constants $K > 0$, $r > 0$ and uniformly bounded Borel sets $J_t \subseteq \mathbb{R}^{d-1} \cong V_2 \oplus \cdots \oplus V_p$ for $t \in [1, c)$ such that*

$$g_t(x_1, \ldots, x_p) \geq K > 0 \quad \textit{for all } (x_1, \ldots, x_p) \in [-r, r] \times J_t. \tag{2.6}$$

Further, we can choose $\{J_t\}_{t\in[1,c)}$ such that $\lambda^{d-1}(J_t) \geq R$ for every $t \in [1, c)$. Note that the constants K, r and R do not depend on $t \in [1, c)$.

Remark 2.5. Note that $\alpha_1 > 1$ is a necessary condition in Lemma 2.4. To see that, take for example the α_1-stable subordinator with $0 < \alpha_1 < 1$. Here the support of the density function is a subset of $\mathbb{R}_+$, so that (2.6) does not hold for any $r > 0$.

3. Main results

The following two Theorems are the main results of this paper. The constants α_1, α_2 and d_1 are defined as in Section 2.1 by means of the spectral decomposition.

Theorem 3.1. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator semistable Lévy process on $\mathbb{R}^d$ with $d \geq 2$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) = \begin{cases} \dim_H B \cdot \max(\alpha_1, 1) & \textit{if } \alpha_1 \dim_H B \leq d_1, \\ 1 + \max(\alpha_2, 1) \cdot \left(\dim_H B - \dfrac{1}{\alpha_1}\right) & \textit{if } \alpha_1 \dim_H B > d_1. \end{cases}$$

The dimension result for the one-dimensional case reads as follows:

Theorem 3.2. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a $(c^{1/\alpha}, c)$-semistable Lévy process on $\mathbb{R}$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) = \begin{cases} \dim_H B \cdot \max(\alpha, 1) & \textit{if } \alpha \dim_H B \leq 1, \\ 1 + \dim_H B - \dfrac{1}{\alpha} & \textit{if } \alpha \dim_H B > 1. \end{cases}$$

Let $X = (X(t))_{t\geq 0}$ be a (c^E, c)-operator semistable Lévy process on $\mathbb{R}^d$ and let $\alpha_1 > \cdots > \alpha_p$ denote the reciprocals of the real parts of the eigenvalues of E as defined in Section 2.1. We want to calculate the Hausdorff dimension of the graph $\mathrm{Gr}_X(B)$ of X for an arbitrary Borel set $B \subseteq \mathbb{R}_+$. Therefore, we define the process $Z = (Z(t))_{t\geq 0}$ as $Z(t) = (t, X(t))$ for all $t \geq 0$. This gives us $\dim_H Z(B) = \dim_H \mathrm{Gr}_X(B)$. One can easily see that Z is also a Lévy process and fulfills the scaling property of a (c^F, c)-operator semistable process where

$$F = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}.$$

Nevertheless, the process Z itself is not operator semistable in the sense of the definition given in the Introduction as the distribution of $Z(1)$ is obviously not full.

As mentioned in the Introduction, the Hausdorff dimension $\dim_H X(B)$ of the range of an operator semistable Lévy process X has already been calculated in [8] as

$$\dim_{\mathrm{H}} X(B) = \begin{cases} \alpha_1 \dim_{\mathrm{H}} B & \text{if } \alpha_1 \dim_{\mathrm{H}} B \leq d_1, \\ 1 + \alpha_2 \Big(\dim_{\mathrm{H}} B - \dfrac{1}{\alpha_1} \Big) & \text{if } \alpha_1 \dim_{\mathrm{H}} B > d_1, \end{cases} \tag{3.1}$$

almost surely for $d \geq 2$. Hence, for the reasons mentioned above, we are now able to use the parts of the result (3.1) and the corresponding proofs where fullness of the process was not required. All other parts, however, have to be calculated anew.

The proof of Theorem 3.1 is split into two parts. First we will obtain the upper bounds for $\dim_H \mathrm{Gr}_X(B)$ by choosing a suitable sequence of coverings. This method goes back to Pruitt and Taylor [17] and Hendricks [3]. Afterwards we will use standard capacity arguments in order to prove the lower bounds.

3.1. Upper bounds. For a Lévy process $\{X(t)\}_{t\geq 0}$ let

$$T_X(a, s) = \int_0^s 1_{B(0,a)}(X(t))dt \tag{3.2}$$

be the sojourn time in the closed ball $B(0, a)$ with radius a centered at the origin up to time $s > 0$.

The following covering lemma is due to Pruitt and Taylor (see Lemma 6.1 in [17])

Lemma 3.3. *Let $Z = \{Z(t)\}_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{d+1}$ and let $\Lambda(a)$ be a fixed K_1-nested family of cubes in $\mathbb{R}^{d+1}$ of side a with $0 < a \leq 1$. For any $u \geq 0$ let $M_u(a, s)$ be the number of cubes in $\Lambda(a)$ hit by $Z(t)$ at some time $t \in [u, u+s]$. Then*

$$\mathbb{E}\left[M_u(a, s)\right] \leq 2\, K_1 s \cdot \Big(\mathbb{E}\Big[T_Z\Big(\frac{a}{3}, s\Big)\Big]\Big)^{-1}.$$

In order to prove the upper bounds of Theorem 3.1 we now need to calculate sharp lower bounds of the expected sojourn times $\mathbb{E}[T_Z(a,s)]$ of the graph $Z = \{(t, X(t)), t \geq 0\}$ of an operator semistable Lévy process on $\mathbb{R}^d$.

In their paper Kern and Wedrich calculated in Theorem 2.6 in [8] upper and lower bounds for the expected sojourn times $\mathbb{E}[T_X(a,s)]$ of an operator semistable Lévy process:

Theorem 3.4. *Let $X = \{X(t)\}_{t\geq 0}$ be as in Theorem 3.1. For any $0 < \alpha_2'' < \alpha_2 < \alpha_2' < \alpha_1'' < \alpha_1 < \alpha_1'$ there exist positive and finite constants $K_6, \dots, K_9$ such that*

(i) *if $\alpha_1 \leq d_1$, then for all $0 < a \leq 1$ and $a^{\alpha_1} \leq s \leq 1$ we have*

$$K_6 a^{\alpha_1'} \leq \mathbb{E}[T_X(a,s)] \leq K_7 a^{\alpha_1''};$$

(ii) *if $\alpha_1 > d_1 = 1$, for all $0 < a \leq a_0$ with $a_0 > 0$ sufficiently small, and all $a^{\alpha_2} \leq s \leq 1$ we have*

$$K_8 a^{\rho'} \leq \mathbb{E}[T_X(a,s)] \leq K_9 a^{\rho''},$$

where

$$\rho'' = 1 + \alpha_2''\Big(1 - \frac{1}{\alpha_1}\Big) \quad \text{and} \quad \rho' = 1 + \alpha_2'\Big(1 - \frac{1}{\alpha_1}\Big).$$

Looking at the proof of the lower bounds of Theorem 3.4 (i) (i.e. Theorem 2.6 (i) in [8]), we find that the condition $\alpha_1 \leq d_1$ is not needed here. Hence, the same proof additionally gives us the following corollary:

Corollary 3.5. *Let $X = \{X(t)\}_{t\geq 0}$ be as in Theorem 3.1. For any $0 < \alpha_1 < \alpha_1'$ there exists a positive and finite constant $\widetilde{K}_6$ such that for all $0 < a \leq 1$ and $a^{\alpha_1} \leq s \leq 1$ we have*

$$\widetilde{K}_6 a^{\alpha_1'} \leq \mathbb{E}[T_X(a,s)].$$

Similarly to the results above we will now calculate lower bounds for the expected sojourn times $\mathbb{E}[T_Z(a,s)]$ of the graph $Z = \{(t, X(t)), t \geq 0\}$ of an operator semistable Lévy process on $\mathbb{R}^d$. The upper bounds can also be calculated but are not stated here as they are not needed to determine the Hausdorff dimension.

Theorem 3.6. *Let* $Z = \{(t, X(t)), t \geq 0\}$, *where* $X = \{X(t), t \geq 0\}$ *is as in Theorem 3.1.*

(i) *If* $\alpha_1 \geq 1$, *there exists a positive and finite constant* K_2 *such that for all* $0 < a \leq 1$ *and* $a^{\alpha_1} \leq s \leq 1$ *and any* $\alpha_1 < \alpha_1'$

$$\mathbb{E}[T_Z(a,s)] \geq K_2 a^{\alpha_1'}.$$

(ii) *If* $\alpha_1 < 1$, *there exists a positive and finite constant* K_3 *such that for all* $0 < a \leq 1$ *and* $a \leq s \leq 1$ *and any* $\epsilon > 0$

$$\mathbb{E}[T_Z(a,s)] \geq K_3 a^{1+\epsilon}.$$

(iii) *If* $\alpha_1 > d_1 = 1$ *and* $\alpha_2 \geq 1$, *there exists a positive and finite constant* K_4 *such that for any* $0 < \alpha_2 < \alpha_2' < \alpha_1$ *and all* $a > 0$ *small enough, say* $0 < a \leq a_0$, *and all* $a^{\alpha_2} \leq s \leq 1$

$$\mathbb{E}[T_Z(a,s)] \geq K_4 a^{1+\alpha_2'(1-\frac{1}{\alpha_1})}.$$

(iv) *If* $\alpha_1 > d_1 = 1$ *and* $\alpha_2 < 1$, *there exists a positive and finite constant* K_5 *such that for all* $a > 0$ *small enough, say* $0 < a \leq a_0$, *and all* $\frac{a}{\sqrt{p+1}} \leq s \leq 1$

$$\mathbb{E}[T_Z(a,s)] \geq K_5 a^{2-\frac{1}{\alpha_1}}.$$

Proof. (i) & (ii) Let $\alpha_1' > \alpha_1$. Looking at the proof of Corollary 3.5 (i.e. Theorem 2.6 part (i) in [8]) one realizes that the fullness is not needed there. Hence we can use this result to prove part (i) and (ii) of the present theorem. In order to do so we need to further examine the exponent

$$F = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

of the process Z. Analogously to Section 2.1 denote by $\tilde{\alpha}_1 > \cdots > \tilde{\alpha}_q$ the reciprocals of the real parts of the eigenvalues of F and by $\tilde{d}_1$ the dimension of the F_1 invariant subspace of $\mathbb{R}^{d+1}$, where F_1 is (analogously to E_1) the blockmatrix, whose eigenvalues have real part equal to $\tilde{\alpha}_1^{-1}$. Furthermore, let $\tilde{\alpha}_1'$ be such that $\tilde{\alpha}_1' = \tilde{\alpha}_1 + \alpha_1' - \alpha_1$.

In part (i) we have that $\alpha_1 \geq 1$. Then $\tilde{\alpha_1} = \alpha_1$ and $\tilde{d_1} \geq d_1$. By Corollary 3.5 there now exists a positive constant K_2 such that

$$\mathbb{E}[T_Z(a,s)] \geq K_2 a^{\tilde{\alpha}_1'} = K_2 a^{\alpha_1'}$$

for all $0 < a \leq 1$ and $a^{\alpha_1} \leq s \leq 1$.

On the other hand in part (ii) we have $\alpha_1 < 1$. Then $\tilde{\alpha}_1 = 1$ and $\tilde{d}_1 = 1$. For any $\epsilon > 0$, by Corollary 3.5 there now exists a positive constant K_3 such that

$$\mathbb{E}[T_Z(a,s)] \geq K_3 a^{\tilde{\alpha}_1+\epsilon} = K_3 a^{1+\epsilon}$$

for all $0 < a \leq 1$ and $a \leq s \leq 1$.

(iii) Let $0 < \alpha_j < \alpha_j' < \alpha_{j-1}$ for all $j = 2, \dots, p$. Choose $i_0, i_1 \in \mathbb{N}_0$ such that $c^{-i_0} < a \leq c^{-i_0+1}$ and $c^{-i_1} < c^{-i_0\alpha_2'} \leq c^{-i_1+1}$. For $t \in (0,1]$ we can write $t = mc^{-i}$ with $m \in [1,c)$ and $i \in \mathbb{N}_0$. By Lemma 2.1 we then have

$$\|X^{(j)}(t)\| \stackrel{\mathrm{d}}{=} \|c^{-iE_j} X^{(j)}(m)\| \leq \|c^{-iE_j}\| \, \|X^{(j)}(m)\| \leq K\, c^{-i/\alpha_j'} \|X^{(j)}(c^i t)\| \tag{3.3}$$

for all $j = 1, \dots, p$. Note that, since $d_1 = 1$, for $j = 1$ in (3.3) we can choose $K = 1$ and $\alpha_1' = \alpha_1$. Furthermore, since $\alpha_2' > 1$ there exists a constant $a_0 > 0$ such that for all $0 < a \leq a_0$ we have $a^{\alpha_2'} \leq \frac{a}{\sqrt{p+1}}$. Altogether, for $0 < a \leq a_0$ this gives us

$$\begin{aligned}
\mathbb{E}[T_Z(a,s)] &= \int_0^s \mathbb{P}\left(\|Z(t)\| < a\right) dt \\
&= \int_0^s \mathbb{P}\left(\|(t, X(t))\| < a\right) dt \\
&\geq \int_0^s \mathbb{P}\Big(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, \\
&\qquad\qquad 2 \leq j \leq p, |t| < \frac{a}{\sqrt{p+1}}\Big) dt \\
&\geq \int_0^{a^{\alpha_2'}} \mathbb{P}\Big(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, \\
&\qquad\qquad 2 \leq j \leq p\Big) dt \\
&\geq \int_0^{c^{-i_1}} \mathbb{P}\Big(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, \\
&\qquad\qquad 2 \leq j \leq p\Big) dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=i_1+1}^{\infty} \int_{c^{-i}}^{c^{-i+1}} \mathbb{P}\Big(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \\
&\qquad\qquad \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, 2 \le j \le p\Big) dt \\
&\ge \sum_{i=i_1+1}^{\infty} \int_{c^{-i}}^{c^{-i+1}} \mathbb{P}\Big(|X^{(1)}(c^i t)| < \frac{c^{\frac{i}{\alpha_1}-i_0}}{\sqrt{p+1}}, \\
&\qquad\qquad \|X^{(j)}(c^i t)\| < K^{-1} \frac{c^{\frac{i}{\alpha_j'}-i_0}}{\sqrt{p+1}}, 2 \le j \le p\Big) dt \\
&\ge \sum_{i=i_1+1}^{\infty} c^{-i} \int_1^c \mathbb{P}\Big(|X^{(1)}(m)| < \frac{c^{\frac{i}{\alpha_1}-i_0}}{\sqrt{p+1}}, \\
&\qquad\qquad \|X^{(j)}(m)\| < K^{-1} \frac{c^{\frac{i}{\alpha_j'}-i_0}}{\sqrt{p+1}}, 2 \le j \le p\Big) dm,
\end{aligned}
$$

where the penultimate inequality follows from (3.3). By Lemma 2.4 choose $K_{10} > 0$, $r > 0$ and uniformly bounded Borel sets $J_m \subseteq \mathbb{R}^{d-1}$ with Lebesgue measure $0 < K_9 \le \lambda^{d-1}(J_m) < \infty$ for every $m \in [1, c)$ such that the bounded continuous density $g_m(x_1, \dots, x_p)$ of $X(m) = X^{(1)}(m) + \dots + X^{(p)}(m)$ fulfills

$$
g_m(x_1, \dots, x_p) \ge K_{10} > 0 \quad \text{for all } (x_1, \dots, x_p) \in [-r, r] \times J_m
$$

and for every $m \in [1, c)$. Since $\{J_m\}_{m \in [1,c)}$ is uniformly bounded by Lemma 2.4 we are able to choose $0 < \delta \le c^{-3} < 1$ such that

$$
\bigcup_{m \in [1,c)} J_m \subseteq \Big\{ \|x_j\| \le \frac{K^{-1} c^{\frac{-\alpha_1}{\alpha_p}}}{\delta \sqrt{p+1}}, 2 \le j \le p \Big\}.
$$

Let $\eta = c^{\frac{2}{\alpha_p}} / (r\sqrt{p+1})$.

Since $\alpha_1 > \alpha_2' > 1$ there exists a constant $a_0 \in (0, 1]$ such that $(\eta a)^{\alpha_1} < (\delta a)^{\alpha_2'}$ for all $0 < a \le a_0$. Now, choose $i_2, i_3 \in \mathbb{N}_0$ such that

$$
c^{-i_2} < (\delta c^{-i_0+1})^{\alpha_2'} \le c^{-i_2+1} \quad \text{and} \quad c^{-i_3} < (\eta c^{-i_0})^{\alpha_1} \le c^{-i_3+1}.
$$

Note that

$$
c^{-i_3} < (\eta c^{-i_0})^{\alpha_1} < (\delta a)^{\alpha_2'} \le (\delta c^{-i_0+1})^{\alpha_2'} \le c^{-i_2+1}
$$

and

$$
\begin{aligned}
c^{-(i_1+1)} &\ge c^{-2} \cdot c^{-i_0 \alpha_2'} \ge (c^{-2} \cdot c^{-i_0})^{\alpha_2'} = (c^{-3} \cdot c^{-i_0+1})^{\alpha_2'} \\
&\ge (\delta c^{-i_0+1})^{\alpha_2'} > c^{-i_2},
\end{aligned}
$$

hence $i_3 \geq i_2 - 1$ and $i_1 + 1 \leq i_2$. We further have for all $i = i_2, \ldots, i_3 + 1$ and every $j = 2, \ldots, p$

$$\frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \leq \frac{c^{(i_3+1)/\alpha_1 - i_0}}{\sqrt{p+1}} \leq \frac{c^{2/\alpha_1}(\eta c^{-i_0})^{-1} c^{-i_0}}{\sqrt{p+1}} = \frac{c^{2/\alpha_1}}{\eta\sqrt{p+1}} < r \tag{3.4}$$

and, since $\alpha_2' \geq \alpha_j'$ for $j = 2, \ldots, p$,

$$\begin{aligned} \frac{c^{i/\alpha_j' - i_0}}{\sqrt{p+1}} \geq \frac{c^{i_2/\alpha_j' - i_0}}{\sqrt{p+1}} &\geq \frac{(\delta c^{-i_0+1})^{-\alpha_2'/\alpha_j'} c^{-i_0}}{\sqrt{p+1}} \\ &= \frac{(\delta^{-1} c^{i_0 - 1})^{\alpha_2'/\alpha_j'} c^{-i_0}}{\sqrt{p+1}} \geq \frac{c^{-\alpha_2'/\alpha_j'}}{\delta\sqrt{p+1}} \geq \frac{c^{-\alpha_1/\alpha_p}}{\delta\sqrt{p+1}}. \end{aligned} \tag{3.5}$$

Let

$$I_m = \Big(-\frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}}, \frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \Big) \times J_m.$$

Then together with the calculations above, we get using (3.4) and (3.5)

$$\begin{aligned} \mathbb{E}[T(a,s)] &\geq \sum_{i=i_2}^{i_3+1} c^{-i} \int_1^c P\Big(|X^{(1)}(m)| < \frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \\ &\qquad\qquad \|X^{(j)}(m)\| \leq K^{-1} \frac{c^{i/\alpha_j' - i_0}}{\sqrt{p+1}}, 2 \leq j \leq p \Big) dm \\ &\geq \sum_{i=i_2}^{i_3+1} c^{-i} \int_1^c \int_{I_m} g_m(x)\, dx\, dm \\ &\geq \sum_{i=i_2}^{i_3+1} c^{-i} (c-1)\, 2\, \frac{c^{i/\alpha_1 - i_0}}{\sqrt{p+1}} \cdot K_{10} \cdot K_9 \\ &= K c^{-i_0} \sum_{i=i_2}^{i_3+1} (c^{-i})^{1-\frac{1}{\alpha_1}} \\ &= K c^{-i_0} \Big(\frac{1 - (c^{-(i_3+2)})^{1-\frac{1}{\alpha_1}}}{1 - c^{\frac{1}{\alpha_1} - 1}} - \frac{1 - (c^{-i_2})^{1-\frac{1}{\alpha_1}}}{1 - c^{\frac{1}{\alpha_1} - 1}} \Big) \\ &= K c^{-i_0} ((c^{-i_2})^{1-\frac{1}{\alpha_1}} - (c^{-(i_3+2)})^{1-\frac{1}{\alpha_1}}) \\ &\geq K_{41} (c^{-i_0})^{1+\alpha_2'\left(1-\frac{1}{\alpha_1}\right)} - K_{42} (c^{-i_0})^{\alpha_1}. \end{aligned} \tag{3.6}$$

Since $1+\alpha_2'\big(1-\frac{1}{\alpha_1}\big) < 1+\alpha_1\big(1-\frac{1}{\alpha_1}\big) = \alpha_1$ we have $(c^{-i_0})^{\alpha_1-\left(1+\alpha_2'\left(1-\frac{1}{\alpha_1}\right)\right)} \to 0$ if $a \to 0$, i.e. $i_0 \to \infty$. Hence we can further choose a_0 sufficiently small, such that

$$\mathbb{E}[T_Z(a,s)] \geq K_4 a^{1+\alpha_2'(1-\frac{1}{\alpha_1})}$$

for all $0 < a \leq a_0$.

(iv) Let $0 < \alpha_j < \alpha_j' < \alpha_{j-1}$ for all $j = 2,\dots,p$, and additionally, let $\alpha_2 < \alpha_2' < 1$. Now choose $i_0, i_1 \in \mathbb{N}_0$ such that $c^{-i_0} < a \leq c^{-i_0+1}$ and $c^{-i_1} < \frac{a}{\sqrt{p+1}} \leq c^{-i_1+1}$. For $t \in (0,1]$ we can write $t = mc^{-i}$ with $m \in [1,c)$ and $i \in \mathbb{N}_0$. By (3.3) we then have

$$\|X^{(j)}(t)\| \stackrel{\mathrm{d}}{=} \|c^{-iE_j} X^{(j)}(m)\| \leq K\, c^{-i/\alpha_j'} \|X^{(j)}(c^i t)\| \tag{3.7}$$

for all $j = 1,\dots,p$. Note that, since $d_1 = 1$, for $j = 1$ in (3.7) we can choose $K = 1$ and $\alpha_1' = \alpha_1$. Similarly to the proof of part (iii), this gives us

$$\begin{aligned}
\mathbb{E}[T_Z(a,s)] &\geq \int_0^{\frac{a}{\sqrt{p+1}}} \mathbb{P}\Big(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, \\
&\qquad\qquad 2 \leq j \leq p\Big) dt \\
&\geq \int_0^{c^{-i_1}} \mathbb{P}\Big(|X^{(1)}(t)| < \frac{a}{\sqrt{p+1}}, \|X^{(j)}(t)\| < \frac{a}{\sqrt{p+1}}, \\
&\qquad\qquad 2 \leq j \leq p\Big) dt \\
&\geq \sum_{i=i_1+1}^{\infty} \int_{c^{-i}}^{c^{-i+1}} \mathbb{P}\Big(|X^{(1)}(c^i t)| < \frac{c^{\frac{i}{\alpha_1}-i_0}}{\sqrt{p+1}}, \\
&\qquad\qquad \|X^{(j)}(c^i t)\| < K^{-1} \frac{c^{\frac{i}{\alpha_j'}-i_0}}{\sqrt{p+1}}, 2 \leq j \leq p\Big) dt \\
&\geq \sum_{i=i_1+1}^{\infty} c^{-i} \int_1^c \mathbb{P}\Big(|X^{(1)}(m)| < \frac{c^{\frac{i}{\alpha_1}-i_0}}{\sqrt{p+1}}, \\
&\qquad\qquad \|X^{(j)}(m)\| < K^{-1} \frac{c^{\frac{i}{\alpha_j'}-i_0}}{\sqrt{p+1}}, 2 \leq j \leq p\Big) dm,
\end{aligned}$$

where the penultimate inequality follows from (3.7). As in the proof of part (iii), by Lemma 2.4 choose $K_{10} > 0$, $r > 0$ and uniformly bounded Borel sets $J_m \subseteq \mathbb{R}^{d-1}$ with Lebesgue measure $0 < K_9 \leq \lambda^{d-1}(J_m) < \infty$ for every

$m \in [1,c)$ such that the bounded continuous density $g_m(x_1,\dots,x_p)$ of $X(m) = X^{(1)}(m)+\dots+X^{(p)}(m)$ fulfills

$$g_m(x_1,\dots,x_p) \ge K_{10} > 0 \quad \text{for all } (x_1,\dots,x_p) \in [-r,r]\times J_m$$

and for every $m \in [1,c)$. Since $\{J_m\}_{m\in[1,c)}$ is uniformly bounded by Lemma 2.4 we are now able to choose $0<\delta \le (\sqrt{p+1}\cdot c^3)^{-1} < 1$ such that

$$\bigcup_{m\in[1,c)} J_m \subseteq \Big\{ \|x_j\| \le \frac{K^{-1}c^{\frac{-\alpha_1}{\alpha_p}}}{\delta\sqrt{p+1}}, 2\le j\le p \Big\}.$$

Let $\eta = c^{\frac{2}{\alpha_p}}/(r\sqrt{p+1})$.

Since $\alpha_1>1$ there exists a constant $0<a_0\le 1$ such that we have $(\eta a)^{\alpha_1} < \delta a$ for all $0<a\le a_0$. Now, choose $i_2,i_3\in\mathbb{N}_0$ such that $c^{-i_2} < \delta c^{-i_0+1} \le c^{-i_2+1}$ and $c^{-i_3} < (\eta c^{-i_0})^{\alpha_1} \le c^{-i_3+1}$. Note that

$$c^{-i_3} < (\eta c^{-i_0})^{\alpha_1} < (\eta a)^{\alpha_1} < \delta a \le \delta c^{-i_0+1} \le c^{-i_2+1}$$

and, since $\delta \le \frac{1}{\sqrt{p+1}}\cdot c^{-3}$,

$$c^{-(i_1+1)} \ge c^{-2}\cdot\frac{a}{\sqrt{p+1}} > \frac{c^{-3}}{\sqrt{p+1}}\cdot c^{-i_0+1} \ge \delta c^{-i_0+1} > c^{-i_2}.$$

Hence, we also get $i_2-1\le i_3$ and $i_1+1\le i_2$. As in (3.4), we further have for all $i=i_2,\dots,i_3+1$ that

$$\frac{c^{i/\alpha_1-i_0}}{\sqrt{p+1}} \le r \tag{3.8}$$

and, since $\alpha_j' < 1$ for all $j=2,\dots,p$,

$$\begin{aligned}\frac{c^{i/\alpha_j'-i_0}}{\sqrt{p+1}} &\ge \frac{c^{i_2/\alpha_j'-i_0}}{\sqrt{p+1}} \ge \frac{(\delta c^{-i_0+1})^{-1/\alpha_j'}c^{-i_0}}{\sqrt{p+1}} = \frac{(\delta^{-1}c^{i_0-1})^{1/\alpha_j'}c^{-i_0}}{\sqrt{p+1}}\\ &\ge \frac{c^{-1/\alpha_j'}}{\delta\sqrt{p+1}} \ge \frac{c^{-1/\alpha_p}}{\delta\sqrt{p+1}} \ge \frac{c^{-\alpha_1/\alpha_p}}{\delta\sqrt{p+1}}.\end{aligned} \tag{3.9}$$

Define the subsets $\{I_m : m\in[1,c)\}\subseteq\mathbb{R}^d$ as above. Similarly to the calculations in (3.6), using (3.8) and (3.9) we arrive at

$$\mathbb{E}[T_Z(a,s)] \ge Kc^{-i_0}\big((c^{-i_2})^{1-\frac{1}{\alpha_1}} - (c^{-(i_3+2)})^{1-\frac{1}{\alpha_1}}\big) \tag{3.10}$$

Altogether, we get

$$\begin{aligned}\mathbb{E}[T_Z(a,s)] &\geq K_{51}c^{-i_0}(c^{-i_0})^{1-\frac{1}{\alpha_1}} - K_{52}c^{-i_0}(c^{-i_0})^{\alpha_1-1}\\ &= K_{51}(c^{-i_0})^{2-\frac{1}{\alpha_1}} - K_{52}(c^{-i_0})^{\alpha_1}.\end{aligned}$$

Since $\alpha_1 > 1$ and therefore $2 - 1/\alpha_1 = 1 + (1 - 1/\alpha_1) < 1 + \alpha_1(1 - 1/\alpha_1) = \alpha_1$, we can choose a_0 sufficiently small, such that

$$\mathbb{E}[T_Z(a,s)] \geq K_5 a^{2-\frac{1}{\alpha_1}}.$$

for all $0 < a \leq a_0$. □

Similarly to the proof of Lemma 3.4 in [8], we can now find a suitable covering of $Z(B)$ and prove the desired upper bounds.

Lemma 3.7. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator semistable Lévy process on $\mathbb{R}^d$ with $d \geq 2$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) \leq \begin{cases} \alpha_1 \dim_H B & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 \geq 1, & \text{(i)}\\ \dim_H B & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 < 1, & \text{(ii)}\\ 1 + \alpha_2\left(\dim_H B - \frac{1}{\alpha_1}\right) & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > \alpha_2 \geq 1, & \text{(iii)}\\ 1 + \dim_H B - \frac{1}{\alpha_1} & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > 1 > \alpha_2. & \text{(iv)}\end{cases}$$

Proof. (i) Assume $\alpha_1 \dim_H B \leq d_1$ and $\alpha_1 \geq 1$. Analogously to the proof of Lemma 3.4 in [8] for the case $\alpha_1 \dim_H B \leq 1$, it follows by Lemma 3.3 and Theorem 3.6 (i) that $\dim_H Z(B) \leq \alpha_1 \dim_H B$ almost surely.

(ii) Assume $\alpha_1 \dim_H B \leq d_1$ and $\alpha_1 < 1 \leq d_1$. For $\gamma > \dim_H B$, choose $\beta > 1$ such that $\gamma' = 1 - \beta + \gamma > \dim_H B$. For $\varepsilon \in (0, 1]$, by definition of the Hausdorff dimension, there exists a sequence $\{I_i\}_{i\in\mathbb{N}}$ of intervals in $\mathbb{R}_+$ of length $|I_i| < \varepsilon$ such that

$$B \subseteq \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} |I_i|^{\gamma'} < 1.$$

Let $s_i = b_i := |I_i|$; then $b_i/3 < s_i$. It follows by Lemma 3.3 and Theorem 3.6 (ii) that $Z(I_i)$ can be covered by M_i cubes $C_{ij} \in \Lambda(b_i)$ of side b_i such that for every $i \in \mathbb{N}$ we have

$$\mathbb{E}[M_i] \leq 2K_1 s_i \Big(\mathbb{E}\Big[T_Z\Big(\frac{b_i}{3}, s_i\Big)\Big]\Big)^{-1} \leq 2K_1 s_i K_3^{-1} \Big(\frac{b_i}{3}\Big)^{-\beta} = K\, s_i b_i^{-\beta} = K\,|I_i|^{1-\beta}.$$

Note that $Z(B) \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{M_i} C_{ij}$, where $b_i\sqrt{d+1}$ is the diameter of C_{ij}. In other words, $\{C_{ij}\}$ is a $(\varepsilon\sqrt{d+1})$-covering of $X(B)$. By monotone convergence we have

$$\mathbb{E}\Big[\sum_{i=1}^{\infty} M_i b_i^{\gamma}\Big] = \sum_{i=1}^{\infty} \mathbb{E}[M_i b_i^{\gamma}] \leq \sum_{i=1}^{\infty} K\,|I_i|^{1-\beta}\,|I_i|^{\gamma} = K \sum_{i=1}^{\infty} |I_i|^{\gamma'} \leq K.$$

Letting $\varepsilon \to 0$, i.e $b_i \to 0$ and applying Fatou's lemma, we get

$$\mathbb{E}[\mathcal{H}^{\gamma}(X(B))] \leq \mathbb{E}\Big[\liminf_{\varepsilon\to 0} \sum_{i=1}^{\infty}\sum_{j=1}^{M_i} (b_i\sqrt{d+1})^{\gamma}\Big] \leq \liminf_{\varepsilon\to 0} \sqrt{d+1}^{\,\gamma}\, \mathbb{E}\Big[\sum_{i=1}^{\infty} M_i b_i^{\gamma}\Big] \leq \sqrt{d+1}^{\,\gamma} K < \infty,$$

which shows that $\dim_H Z(B) \leq \gamma$ almost surely. And since $\gamma > \dim_H B$ is arbitrary, we get $\dim_H Z(B) \leq \dim_H B$ almost surely.

(iii) Assume $\alpha_1 \dim_H B > d_1$ and $\alpha_2 \geq 1$. Since $\dim_H B \leq 1$, we have $\alpha_1 > d_1 = 1$. For $\gamma > \dim_H B$ choose $\alpha_2' > \alpha_2$ such that $\gamma' = 1 - \frac{\alpha_2'}{\alpha_2} + \frac{\alpha_2'}{\alpha_2}\gamma > \dim_H B$. For $\varepsilon \in (0,1]$ define $\{I_i\}_{i\in\mathbb{N}}$ as in part (ii) and let $s_i := |I_i|$ and $b_i := |I_i|^{\frac{1}{\alpha_2}}$. Then $(b_i/3)^{\alpha_2} < s_i$. Again, by Lemma 3.3 and Theorem 3.6 (iii) it follows that $Z(I_i)$ can be covered by M_i cubes $C_{ij} \in \Lambda(b_i)$ of side b_i such that for every $i \in \mathbb{N}$ we have

$$\mathbb{E}[M_i] \leq 2K_1 s_i \Big(\mathbb{E}\Big[T_Z\Big(\frac{b_i}{3}, s_i\Big)\Big]\Big)^{-1} \leq 2K_1 s_i K_4^{-1} \Big(\frac{b_i}{3}\Big)^{-1-\alpha_2'\left(1-\frac{1}{\alpha_1}\right)} = K\, s_i b_i^{-1-\alpha_2'\left(1-\frac{1}{\alpha_1}\right)} = K\,|I_i|^{1-\frac{1}{\alpha_2}-\frac{\alpha_2'}{\alpha_2}\cdot\left(1-\frac{1}{\alpha_1}\right)}.$$

By monotone convergence we have

$$\mathbb{E}\Big[\sum_{i=1}^{\infty} M_i b_i^{1+\alpha_2'\left(\gamma-\frac{1}{\alpha_1}\right)}\Big] \le \sum_{i=1}^{\infty} K\,|I_i|^{1-\frac{1}{\alpha_2}-\frac{\alpha_2'}{\alpha_2}\cdot\left(1-\frac{1}{\alpha_1}\right)}\,|I_i|^{\frac{1}{\alpha_2}+\frac{\alpha_2'}{\alpha_2}\left(\gamma-\frac{1}{\alpha_1}\right)}$$

$$= K\sum_{i=1}^{\infty}|I_i|^{\gamma'} \le K.$$

Since $\gamma > \dim_H B$ and $\alpha_2' > \alpha_2$ are arbitrary, with the same arguments as in part (ii) we get $\dim_H Z(B) \le 1 + \alpha_2(\dim_H B - \frac{1}{\alpha_1})$ almost surely.

(iv) Assume $\alpha_1 \dim_H B > d_1$ and $\alpha_2 < 1$. Since $\dim_H B \le 1$, we have $\alpha_1 > d_1 = 1$. Let $\gamma = \gamma' > \dim_H B$. For $\varepsilon \in (0, 1]$ define $\{I_i\}_{i\in\mathbb{N}}$ as in part (ii) and let $s_i := |I_i|$ and $b_i := |I_i|$. Then $b_i/(3\sqrt{p+1}) < s_i$. Again, by Lemma 3.3 and Theorem 3.6 (iv) it follows that $Z(I_i)$ can be covered by M_i cubes $C_{ij} \in \Lambda(b_i)$ of side b_i such that for every $i \in \mathbb{N}$ we have

$$\begin{aligned}
\mathbb{E}[M_i] &\le 2K_1 s_i\Big(\mathbb{E}\Big[T_Z\Big(\frac{b_i}{3}, s_i\Big)\Big]\Big)^{-1}\\
&\le 2K_1 s_i K_5^{-1}\Big(\frac{b_i}{3}\Big)^{-2+\frac{1}{\alpha_1}}\\
&= K\, s_i b_i^{-2+\frac{1}{\alpha_1}}\\
&= K\,|I_i|^{-1+\frac{1}{\alpha_1}}.
\end{aligned}$$

By monotone convergence we have

$$\begin{aligned}
\mathbb{E}\Big[\sum_{i=1}^{\infty} M_i b_i^{1+\gamma-\frac{1}{\alpha_1}}\Big] &\le \sum_{i=1}^{\infty} K\,|I_i|^{-1+\frac{1}{\alpha_1}}\,|I_i|^{1+\gamma-\frac{1}{\alpha_1}}\\
&= K\sum_{i=1}^{\infty}|I_i|^{\gamma}\\
&= K\sum_{i=1}^{\infty}|I_i|^{\gamma'}\\
&\le K.
\end{aligned}$$

Since $\gamma > \dim_H B$ is arbitrary, we get $\dim_H Z(B) \le 1 + \dim_H B - \frac{1}{\alpha_1}$ almost surely. □

3.2. Lower bounds. In order to obtain the lower bounds of $\dim_H \mathrm{Gr}_X(B)$ we apply Frostman's Lemma and Theorem and use the relationship between the Hausdorff dimension and the capacitary dimension (see [2, 14] for details).

Lemma 3.8. *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator semistable Lévy process on $\mathbb{R}^d$ with $d \geq 2$. Then for any Borel set $B \subseteq \mathbb{R}_+$ we have almost surely*

$$\dim_H \mathrm{Gr}_X(B) \geq \begin{cases} \alpha_1 \dim_H B & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 \geq 1, & \text{(i)} \\ \dim_H B & \text{if } \alpha_1 \dim_H B \leq d_1, \alpha_1 < 1, & \text{(ii)} \\ 1 + \alpha_2\Big(\dim_H B - \dfrac{1}{\alpha_1}\Big) & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > \alpha_2 \geq 1, & \text{(iii)} \\ 1 + \dim_H B - \dfrac{1}{\alpha_1} & \text{if } \alpha_1 \dim_H B > d_1, \alpha_1 > 1 > \alpha_2. & \text{(iv)} \end{cases}$$

Proof. (i)+(iii) Since projections are Lipschitz continuous, we have

$$\dim_H \mathrm{Gr}_X(B) \geq \dim_H X(B).$$

Hence, the desired lower bounds in these two parts can be deduced from the dimension result (3.1) for the range of an operator semistable process.

(ii) Choose $0 < \gamma < \dim_H B \leq 1$. Then by Frostman's lemma there exists a probability measure σ on B such that

$$\int_B \int_B \frac{\sigma(ds)\sigma(dt)}{|s-t|^\gamma} < \infty. \tag{3.13}$$

In order to prove $\dim_H Gr_X(B) = \dim_H Z(B) \geq \gamma$ almost surely, by Frostman's theorem [7, 14] it suffices to show that

$$\int_B \int_B \mathbb{E}\left[\|Z(s) - Z(t)\|^{-\gamma}\right] \sigma(ds)\,\sigma(dt) < \infty. \tag{3.14}$$

Let $s, t \in B \subseteq \mathbb{R}_+$. Then

$$\mathbb{E}\left[\left\|\begin{pmatrix} t \\ X(t)\end{pmatrix} - \begin{pmatrix} s \\ X(s)\end{pmatrix}\right\|^{-\gamma}\right] \leq \mathbb{E}[|s-t|^{-\gamma}] = |s-t|^{-\gamma}.$$

Hence, (3.14) follows directly from (3.13).

(iv) Assume $\alpha_1 \dim_H B > d_1$ then $\alpha_1 > d_1 = 1$. Choose

$$1 < \gamma < 1 + \dim_H B - \frac{1}{\alpha_1},$$

then

$$\rho = \gamma - 1 + \frac{1}{\alpha_1} < \dim_H B.$$

By Frostman's lemma, there exists again a probability measure σ on B such that

$$\int_B \int_B \frac{\sigma(ds)\sigma(dt)}{|s-t|^\rho} < \infty.$$

Again, in order to verify (3.14) we split the domain of integration into two parts.

First assume that $|s-t| = mc^{-i} \leq 1$ with $m \in [1, c)$ and $i \in \mathbb{N}_0$. Since $d_1 = 1$ we get

$$\begin{aligned}
&\mathbb{E}\left[\left\| \begin{pmatrix} t \\ X(t) \end{pmatrix} - \begin{pmatrix} s \\ X(s) \end{pmatrix} \right\|^{-\gamma}\right] \\
&\leq \mathbb{E}[(c^{-i\frac{2}{\alpha_1}} \cdot |X^{(1)}(m)|^2 + |s-t|^2)^{-\frac{\gamma}{2}}] \\
&\leq K \int_{\mathbb{R}} \frac{1}{c^{-i\frac{\gamma}{\alpha_1}} \cdot |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&= K \int_{\mathbb{R}} \frac{1}{m^{-\frac{\gamma}{\alpha_1}} \left(mc^{-i}\right)^{\frac{\gamma}{\alpha_1}} \cdot |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&\leq K \int_{\mathbb{R}} \frac{1}{c^{-\frac{\gamma}{\alpha_1}} \cdot |s-t|^{\frac{\gamma}{\alpha_1}} |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&\leq K \int_{\mathbb{R}} \frac{1}{|s-t|^{\frac{\gamma}{\alpha_1}} |x_1|^\gamma + |s-t|^\gamma} \cdot g_m(x_1) dx_1 \\
&= K \cdot |s-t|^{-\frac{\gamma}{\alpha_1}} \int_{\mathbb{R}} \frac{1}{|x_1|^\gamma + |s-t|^{\gamma\left(1-\frac{1}{\alpha_1}\right)}} \cdot g_m(x_1) dx_1 =: K \cdot |s-t|^{-\frac{\gamma}{\alpha_1}} \cdot I_m,
\end{aligned}$$

where $g_m(x_1)$ is the density function of $X^{(1)}(m)$. Let

$$F_m(r_1) = \mathbb{P}(|X^{(1)}(m)| \leq r_1) = \int_{|x_1| \leq r_1} g_m(x_1) dx_1$$

and note that by Lemma 2.2

$$\sup_{m \in [1,c)} \sup_{x_1 \in \mathbb{R}} |g_m(x_1)| \leq K_8 < \infty.$$

This leads to

$$F_m(r_1) \le 1 \wedge 2K_8 \cdot r_1 \quad \text{for all } r_1 \ge 0 \text{ and } m \in [1, c).$$

We denote $z = |s-t|^{1-\frac{1}{\alpha_1}}$. By using integration by parts, we deduce

$$\begin{aligned}
I_m &= \int_0^\infty \frac{1}{r_1^\gamma + z^\gamma}\, F_m(dr_1) \\
&= \Big[\frac{1}{r_1^\gamma + z^\gamma}\, F_m(r_1)\Big]_0^\infty + \int_0^\infty \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + z^\gamma)^2}\, F_m(r_1) dr_1 \\
&\le K \int_0^\infty \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + z^\gamma)^2}\, r_1 dr_1 \\
&= K \int_0^\infty \frac{\gamma r_1^{\gamma}}{(r_1^\gamma + z^\gamma)^2}\, dr_1 \\
&= K \int_0^\infty \frac{z\gamma \cdot (z s_1)^{\gamma}}{((z s_1)^\gamma + z^\gamma)^2}\, ds_1 \\
&= K z^{-(\gamma-1)} \cdot \int_0^\infty \frac{\gamma s_1^{\gamma}}{(s_1^\gamma + 1)^2}\, ds_1 \\
&\le K z^{-(\gamma-1)} = K\, |s-t|^{-(\gamma-1)\left(1-\frac{1}{\alpha_1}\right)},
\end{aligned}$$

where the last integral is finite since $\gamma > 1$. Together we get for $|s-t| \le 1$

$$\mathbb{E}\left[\left\|\begin{pmatrix} t \\ X(t)\end{pmatrix} - \begin{pmatrix} s \\ X(s)\end{pmatrix}\right\|^{-\gamma}\right] \le K\, |s-t|^{-\gamma+1-\frac{1}{\alpha_1}} = K\, |s-t|^{-\rho}.$$

For $|s-t| \ge 1$ we have

$$\begin{aligned}
\sup_{|s-t|\ge 1} \mathbb{E}\left[\left\|\begin{pmatrix} t \\ X(t)\end{pmatrix} - \begin{pmatrix} s \\ X(s)\end{pmatrix}\right\|^{-\gamma}\right] &\le \sup_{|s-t|\ge 1} \mathbb{E}[|s-t|^{-\gamma}] \\
&= \sup_{|s-t|\ge 1} |s-t|^{-\gamma} \le 1.
\end{aligned}$$

Therefore it follows from the calculations above that

$$\int_B \int_B \mathbb{E}\left[\left\|\begin{pmatrix} t \\ X(t)\end{pmatrix} - \begin{pmatrix} s \\ X(s)\end{pmatrix}\right\|^{-\gamma}\right] \sigma(ds)\sigma(dt) < \infty.$$

Using Frostman's theorem we get

$$\dim_H Gr_X(E) \ge \gamma.$$

Since $\gamma < 1 + \dim_H B - \frac{1}{\alpha_1}$ was arbitrary this concludes the proof. □

3.3. Proof of main results. Theorem 3.1 now follows directly from Lemma 3.7 and Lemma 3.8. It remains to prove the corresponding dimension result for the one-dimensional case as stated in Theorem 3.2. For $\alpha \dim_H B \leq 1$ Lemma 3.7 and 3.8 are still valid for $d = 1$ with $\alpha_1 := \alpha$. In case $\alpha \dim_H B > 1 = d$ the proof runs analogously to Lemma 3.7 part (iv) and Lemma 3.8 part (iv).

Remark 3.9. For $B = [0, 1]$, an alternative way to calculate $\dim_H \mathrm{Gr}_X(B)$ can be to examine the index introduced by Khoshnevisan et al. in [10], which depends on the asymptotic behavior of the Lévy exponent of the process X. As this is subject of current research, it is not addressed in the present paper.

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Lina Wedrich, Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, Universitätsstr. 1, D-40225 Düsseldorf, Germany

e-mail: lina.wedrich@uni-duesseldorf.de