J. Fractal Geom. 4 (2017), 73–103
DOI 10.4171/JFG/45

Journal of Fractal Geometry

Graph-directed sprays and their tube volumes via functional equations

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Abstract. The notion of sprays introduced by Lapidus and his co-workers has proved useful in the context of fractal tube formulas. In the present note, we define a notion of a graph-directed spray, associated with a weighted directed graph. Using a simple functional equation satisfied by the volume of the inner ε-neighborhood of such a graph-directed spray, we establish a tube formula for them, where we allow the generators of the spray to be pluriphase. We give also an example to illustrate the application of this notion to the computation of the tube volume of graph-directed fractals.

Mathematics Subject Classification (2010). Primary: 28A80, 28A75; Secondary: 52A38.

Keywords. Tube formulas, graph-directed sprays, functional equations, Mellin transform.

Contents

1. Introduction

Tube volumes for fractals has been an interesting topic of research in the last decade. M. Lapidus and his coworkers established several tube formulas as series in terms of residues of some associated zeta functions [7]. Recently, we have proposed a simple alternative approach to tube formulas for self-similar sprays via functional equations ([4]). The so-called sprays introduced by Lapidus and Pomerance in [10] (to be defined below) are a convenient notion in dealing with fractal tube volumes.

Beyond the self-similar fractals, the next family of interest is the class of graph-directed fractals. For tube volumes of graph-directed fractals there are formulas (see [2]), which are analogous to the tube formulas of Lapidus and Pearse ([8]). So far as we know, the notion of spray has not been extended to the graph-directed setting. In the present note, we want to propose the notion of graph-directed sprays and establish a tube formula for them via functional equations.

We now first recall the notion of a spray in Euclidean space. Let $G \subseteq \mathbb{R}^n$ be a non-empty bounded open set. A spray generated by G is a collection $\mathcal{S} = (G_i)_{i\in\mathbb{N}}$ of pairwise disjoint open sets $G_i \subseteq \mathbb{R}^n$ such that G_i is a scaled copy of G by some $\lambda_i > 0$; in other words, G_i is congruent (i.e. isometric) to $\lambda_i G$. The sequence $(\lambda_i)_{i\in\mathbb{N}}$ is called the associated scaling sequence of the spray. In applications one has typically $\lambda_i < 1$ and often $\lambda_0 = 1$ so that G_0 is equal (or isometric) to G. Furthermore, it is meaningful to assume $\sum_{i\in\mathbb{N}} \lambda_i^n < \infty$ to make the volume of $\bigcup_{i\in\mathbb{N}} G_i$ finite, as one deals with the inner tube of $\bigcup_{i\in\mathbb{N}} G_i$ (the inner ε-tube of an open set $A \subseteq \mathbb{R}^n$ is the set of points of A within a distance less than ε to the boundary of A).

The most important class of sprays is that of self-similar sprays, for which the scaling sequence is of a very special type.

Let $\{r_1, r_2, \ldots, r_J\}$ be a so-called ratio list (i.e. $0 < r_j < 1$ for $j = 1, 2, \ldots, J$) and consider the formal expression

$$\frac{1}{1-(r_1+r_2+\cdots+r_J)} = 1 + \sum_{k=1}^{\infty} \sum_{i_1,i_2,\ldots,i_k\in\{1,\ldots,J\}} r_{i_1} r_{i_2} \ldots r_{i_k}. \tag{1}$$

If the scaling sequence $(\lambda_i)_{i\in\mathbb{N}}$ of a spray is given by the terms of the series on the right-hand side of (1) for an appropriate ratio list, then the spray is called a self-similar spray.

To give a flavour of tube formulas, we note the following theorem ([8], [9], and [3]).

Theorem 1.1. *Let $(G_i)_{i\in\mathbb{N}}$ be a self-similar spray generated by $G \subseteq \mathbb{R}^n$ with a scaling sequence associated with a ratio list $\{r_1, r_2, \ldots, r_J\}$. Assume $G \subseteq \mathbb{R}^n$ to be monophase, i.e. let its inner ε-tube volume function $V_G(\varepsilon)$ be given by*

$$V_G(\varepsilon) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} \kappa_i \varepsilon^{n-i} & \text{if } 0 \le \varepsilon \le g, \\ \mathrm{Vol}(G) & \text{if } \varepsilon \ge g, \end{cases}$$

where g is the inradius of G (i.e. the supremum of the radii of the balls contained in G). Then the volume $V_{\cup G_i}(\varepsilon)$ of the inner ε-tube of $\cup G_i$ is given by the formula

$$V_{\cup G_i}(\varepsilon) = \sum_{\omega \in \mathfrak{D} \cup \{0,1,2,\dots,n-1\}} \mathrm{res}(\zeta(s)\, \varepsilon^{n-s}; \omega) \quad \text{for } \varepsilon < g,$$

where

$$\zeta(s) = \frac{1}{1 - (r_1^s + \cdots + r_J^s)} \sum_{i=0}^{n} \frac{g^{s-i}}{s-i} \kappa_i \quad (\text{with } \kappa_n = -\mathrm{Vol}(G))$$

and $\mathfrak{D}$ is the set of complex roots of the equation $r_1^s + \cdots + r_J^s = 1$.

For a far-reaching generalization of this theorem to general fractal sprays and arbitrary generators see [9].

The motivation behind the notion of the self-similar spray is that they naturally emerge as the "hollow spaces" in self-similar fractals. For example, if you start with an interval, and construct a Cantor set by deleting successively the open middle thirds of the intervals, the collection of deleted open intervals constitute a 1-dimensional self-similar spray with a scaling sequence associated with the ratio list $\{\frac{1}{3}, \frac{1}{3}\}$, the generator being the first deleted middle third. Likewise, if you start with a triangle and successively delete the open middle fourths of the triangles to obtain in the end a Sierpinski Gasket, then the collection of the deleted open triangles constitute a 2-dimensional self-similar spray with a scaling sequence associated with the ratio list $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$, the generator being again the first deleted middle fourth.

One can in principle allow any scaling sequence for a spray, but to obtain manageable tube formulas some sort of restrictions seem (as of yet) to be necessary. To be associated with a ratio list is, for example, such a condition. A weaker condition (called "subshift of finite-type") was formulated to handle the hollow spaces of graph-directed fractals in [3], but a more natural approach demands a consideration of "graph-directed" sprays, since the hollow spaces of graph-directed fractals are composed of copies of several generators, each associated with a node of the graph and each scaled with a different scaling sequence (see Figures 3-5). This is in contrast with self-similar sprays with several generators, where all generators are scaled with the same scaling sequence, as for example in the pentagasket tiling (see [8]). Another point is that, in the definition of classical sprays connectivity of generators are not explicitly required, so that, for example, the six generators of the pentagasket could also be viewed as a single disconnected generator. In the graph-directed setting however, irrespective of the connectivity of generators, the presence of different scaling sequences distinguish them from the classical self-similar sprays.

In Section 2 we define this more general concept of graph-directed sprays. In Section 3 we consider a natural functional equation for inner tube volumes of graph-directed sprays where we allow pluriphase generators, formulate a multi-dimensional renewal lemma to handle it and establish an inner tube formula for graph-directed sprays as our main result (Theorem 3.7). In Section 4 we give the proof, taking into account the additional difficulties arising from the presence of the Mauldin–Williams matrix. We give also an explicit sufficient region for ε on which the main theorem holds.

In the appendix we give a more detailed discussion of the relationship between sprays and graph-directed sprays.

2. Graph-directed sprays

Let $\mathcal{G} = (V, E, r)$ be a finite weighted directed graph with weights $r\colon E \to (0, 1)$. For an edge $e \in E$, we denote the initial vertex of e by $i(e)$ and the terminal vertex by $t(e)$. For vertices $u, v \in V$, we denote the set of edges from u to v by E_{uv} and the set of edges starting from u by E_u. If $E_u \neq \emptyset$ for all $u \in V$, such a graph is called a Mauldin–Williams graph. If any two vertices u and v can be joined by a (directed) path, then the graph is said to be strongly connected. We will generally assume that the Mauldin–Williams graphs be strongly connected.

We define the weight of a path $\alpha = e_1 e_2 \dots e_k$ by $r(\alpha) = r(e_1)\cdot r(e_2)\cdot \dots \cdot r(e_k)$. α is called a path from the vertex u to v if $i(e_1) = u$ and $t(e_k) = v$. We also write $i(\alpha) = u$ and $t(\alpha) = v$. We assign an empty path ϕ_u to every vertex u with weight $r(\phi_u) = 1$.

Now we define graph-directed sprays.

Definition 2.1. Let $\mathcal{G} = (V, E, r)$ be a Mauldin–Williams graph and G_u $(u \in V)$ be bounded open sets in $\mathbb{R}^n$. A graph-directed spray $\mathcal{S}$ associated with $\mathcal{G}$ and generated by the open sets G_u $(u \in V)$ is a collection of pairwise disjoint open sets G_α in $\mathbb{R}^n$ (where α is a path in the graph), such that G_α is a scaled isometric copy of $G_{t(\alpha)}$ with scaling ratio $r(\alpha)$.

Remark 2.2. Note that G_u, with $u \in V$, is a generator and G_α, where α denotes a path, is a copy of a generator. If $\alpha = \phi_u$ then G_{ϕ_u} is a scaled isometric copy of $G_{t(\phi_u)} = G_u$ with scaling ratio $r(\phi_u) = 1$, i.e. G_{ϕ_u} is an isometric copy of G_u.

Remark 2.3. Note that if $\mathcal{G}$ has only one node this notion reduces to the ordinary notion of a self-similar spray generated by a single (possibly non-connected) open set with a scaling sequence associated with the ratio list consisting of the weights of the loops around the single vertex.

The spray $\mathcal{S}$ can naturally be decomposed into subcollections

$$\mathcal{S}_u = \{G_\alpha \mid i(\alpha) = u\}.$$

Notice that the subcollection $\mathcal{S}_u$ is also composed of pairwise disjoint scaled copies of all generators G_v with scaling ratios $r(\alpha)$ for paths α starting at u. This decomposition of $\mathcal{S}$ into the subcollections $\mathcal{S}_u$ will prove useful in establishing tube formulas.

The motivation for the definition of graph-directed sprays comes, in analogy to the motivation of self-similar sprays, from hollow spaces of graph-directed fractals. Let us very briefly recall the notion of graph-directed fractals.

Let $\mathcal{G} = (V, E, r)$ be a Mauldin–Williams graph, $(A_u)_{u \in V}$ be a list of complete subsets of $\mathbb{R}^n$ and let $f_e\colon A_{t(e)} \to A_{i(e)}$ be similarities with similarity ratios $r(e)$. Such an assignment is called a realization of the graph $\mathcal{G}$ in $\mathbb{R}^n$. Given such a realization, there is a unique list $(K_u)_{u \in V}$ of nonempty compact sets with $K_u \subset A_u$ $(u \in V)$ satisfying

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} f_e(K_v)$$

for all $u \in V$ (see [5]).

In favorable cases the maps f_e can be restricted to the convex hull of the graph-directed "attractors" $K_{t(e)}$ and these attractors can be imagined to be formed by deleting successively pieces of the convex hull analogous to the construction of the Cantor set or the Sierpinski Gasket by deleting successively pieces of an interval or a triangle. The collection of deleted open pieces will constitute a graph-directed spray in the above defined sense, with so many generators as there are nodes of the graph. Before making this idea precise, it will be best to study an example.

Example 2.4. Consider the Mauldin–Williams graph with $V = \{1, 2\}$, with 9 edges and the corresponding weights as shown in Figure 1.

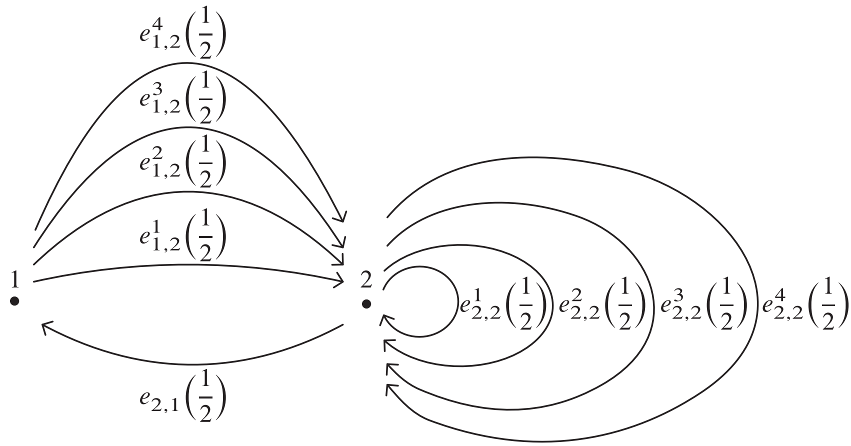

Figure 1. A Mauldin–Williams graph with 2 nodes and 9 edges (the weights are shown in parenthesis).

Let A_1 and A_2 be the square and the triangle in $\mathbb{R}^2$ as shown in Figure 2 (a). The similarities associated with the edges are indicated in Figure 2 (b). The graph-directed fractals K_1 and K_2 of the system are shown in Figure 3. $(A_1 \setminus K_1) \cup (A_2 \setminus K_2)$ is a collection of connected open sets which constitute a graph-directed spray with generators G_1 and G_2 satisfying Definition 2.1 (see Figures 4-5). The generators G_1 and G_2 thereby are defined by $G_u = A_u^\circ \setminus \bigcup_{e \in E_u} f_e(A_{t(e)})$ for $u = 1, 2$.

Note that the subcollections $\mathcal{S}_1 = \{G_\alpha \,|\, i(\alpha) = 1\}$ and $\mathcal{S}_2 = \{G_\alpha \,|\, i(\alpha) = 2\}$ contain copies of both of G_1 and G_2 so that we can not view either of them as a spray in the classical sense.

To illustrate the formation of G_α for a path α, we give several examples in Figure 6.

From the point of view of our present concern to establish inner tube formulas for graph-directed sprays, the special positions of the scaled copies of the generators are not important as long as they are pairwise disjoint. But if one wishes to compute tube volumes of fractals, one should be careful in relating the tube of the fractal to the inner tube of an associated spray. In the above example the ε-tube volume of K_1 can be expressed as the sum of the inner ε-tube volume of the collection $\mathcal{S}_1$ and the outer ε-tube of the square A_1 (likewise for K_2) (see Figure 7). But this convenient relationship does not hold always as the following example shows.

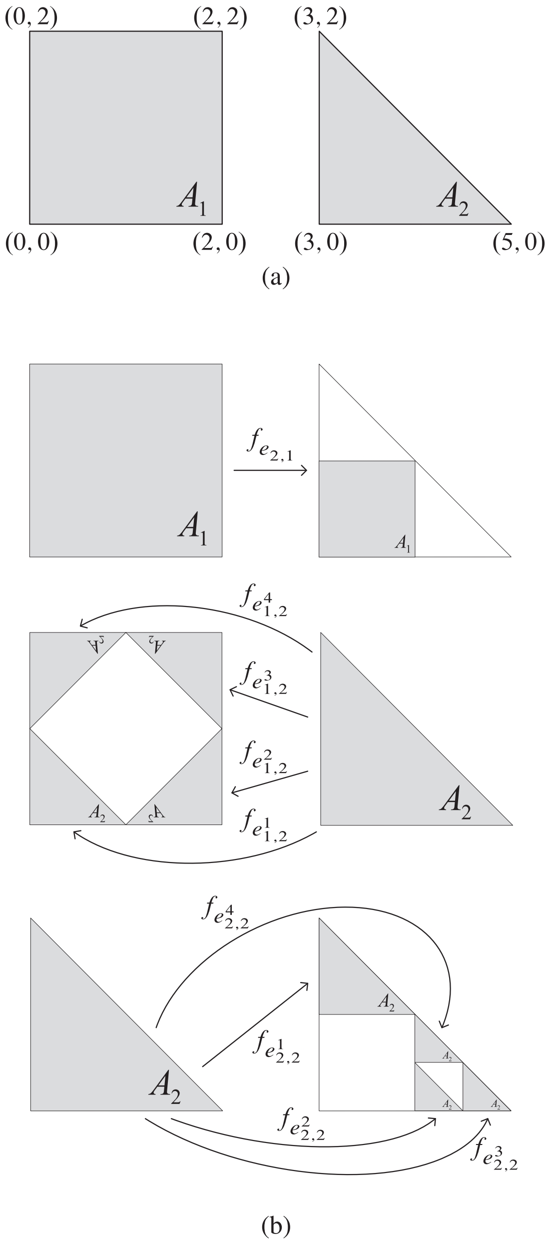

Figure 2. A realization of the Mauldin–Williams graph of Example 2.4 (shown in Figure 1). (a) The complete spaces associated with the 2 nodes. (b) The similarities associated with the edges.

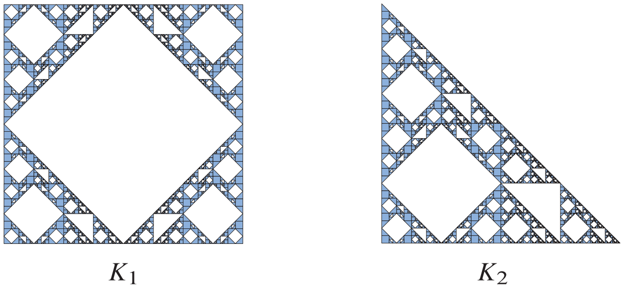

Figure 3. The attractors of the realization shown in Figure 2 of the Mauldin–Williams graph of Example 2.4.

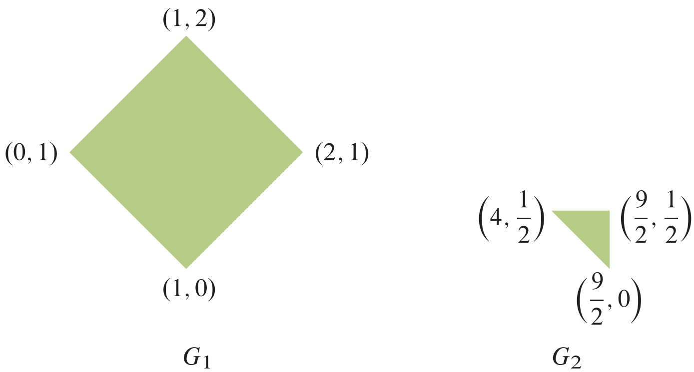

Figure 4. The generators of the realization shown in Figure 2 of the Mauldin–Williams graph of Example 2.4.

Figure 5. The hollow spaces of the realization shown in Figure 2 of the Mauldin–Williams graph of Example 2.4 ($\mathcal{S}_1$ left, $\mathcal{S}_2$ right).

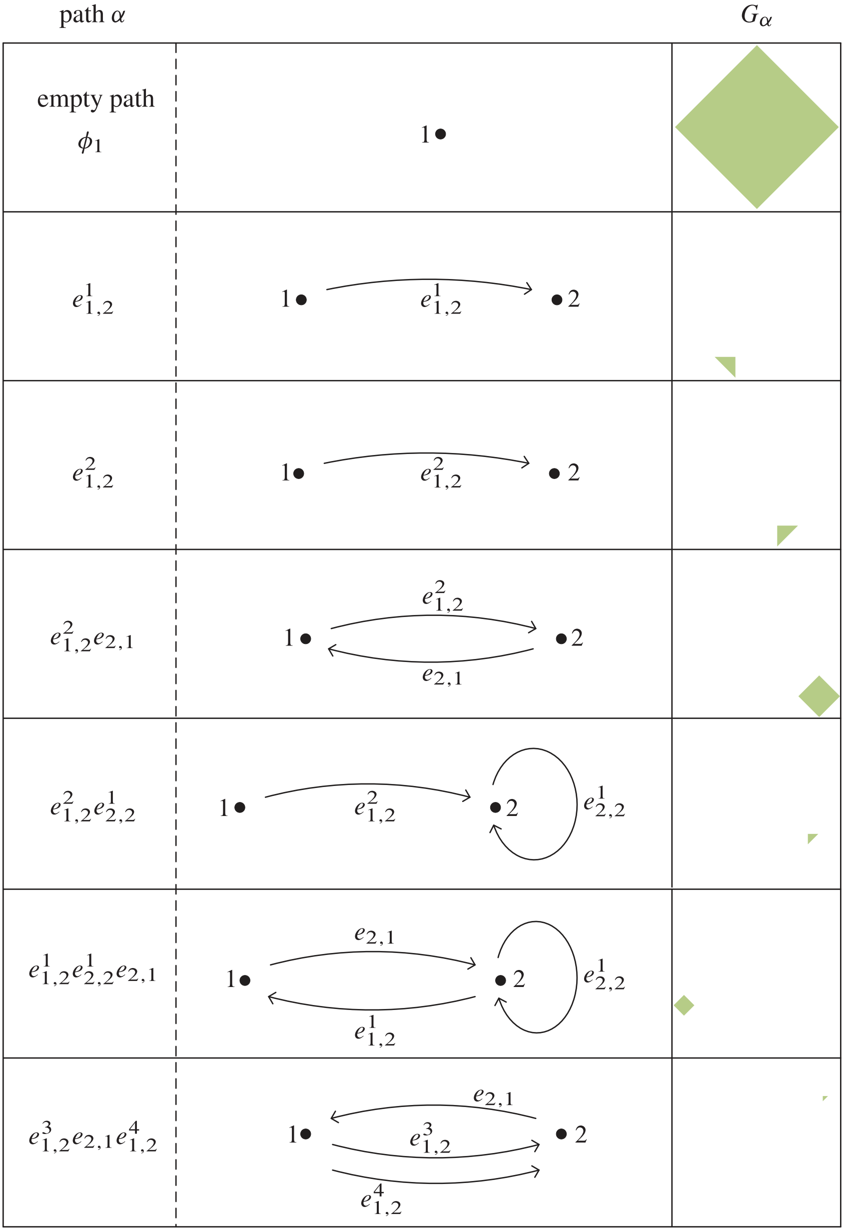

Figure 6. A few examples of actions of paths on generators for the realization shown in Figure 2 of the Mauldin–Williams graph of Example 2.4.

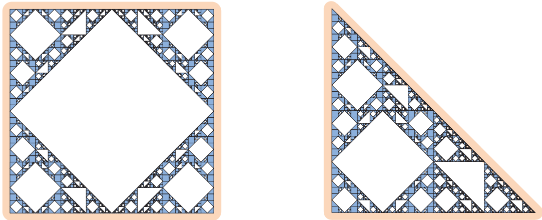

Figure 7. The ε-tubes of the attractors of Example 2.4 as the union of the inner ε-tube of the hollow spaces and the outer ε-tube of the convex hull of the attractor.

Example 2.5. As the Mauldin–Williams graph we choose the same graph in Figure 1 with the only difference that we delete the edge e_{22}^4. We choose the same realization of this graph in $\mathbb{R}^2$ discarding the map corresponding to e_{22}^4. The emerging graph-directed fractals L_1 and L_2 and the corresponding graph-directed spray with generators H_1 and H_2 satisfying Definition 2.1 are shown in Figures 8 and 9.

In this example the ε-tube volume of the graph-directed fractals can not be meaningfully related to the inner ε-tube volume of the graph-directed spray. Two types of problematic boundaries of generator copies are indicated in Figure 9 by dotted lines.

The simple relationship between the fractal tube volume and the inner spray volume observed in Example 2.4 still remains true for a more general class of graph-directed systems, if the following assumptions hold.

i) $\dim(C_u) = n$, where C_u is the convex hull $[K_u]$ of K_u.

ii) Tileset condition. The open set condition should be satisfied with $O_u = C_u^\circ$. We recall that (see [5]) the graph-directed system satisfies the open set condition if there exists a list $(O_u)_{u\in V}$ of open sets $O_u \subset \mathbb{R}^n$ such that,

(a) for any $e \in E_{uv}$, $f_e(O_v) \subset O_u$,

(b) for any two distinct $e_1, e_2 \in E_u$, $f_{e_1}(O_{t(e_1)}) \cap f_{e_2}(O_{t(e_2)}) = \emptyset$.

iii) Nontriviality condition. $C_u^\circ \nsubseteq \displaystyle\bigcup_{e\in E_u} f_e(C_{t(e)})$.

iv) Pearse–Winter condition, [12]. $\partial C_u \subset K_u$.

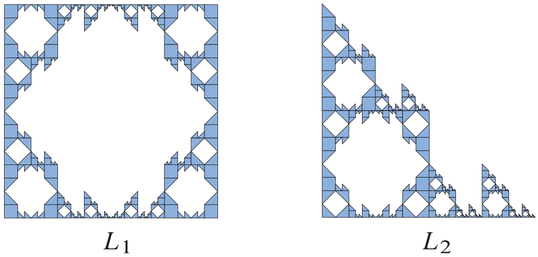

Figure 8. Attractors of Example 2.5.

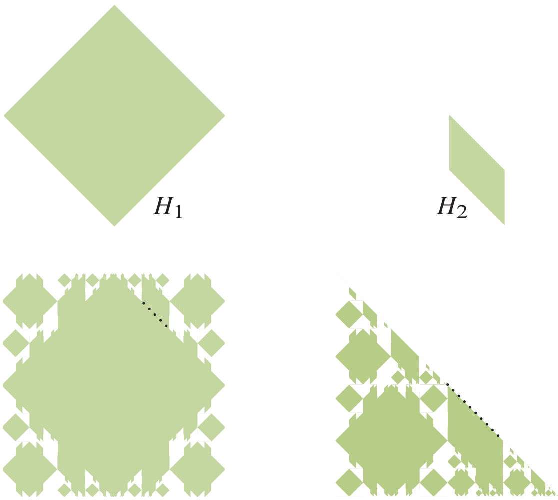

Figure 9. Generators and the hollow spaces of Example 2.5 showing that the ε-tube of the attractors needn't be the union of the inner ε-tube of the hollow space and the outer ε-tube of the convex hull of the attractor.

Now, if we define

$$G_u = C_u^\circ \setminus \bigcup_{e \in E_u} f_e(C_{t(e)})$$

then we get a graph-directed spray $\mathcal{S}$ generated by the open sets $(G_u)_{u \in V}$ with $G_\alpha = G_{e_1 e_2 \ldots e_k} = f_{e_1} f_{e_2} \ldots f_{e_k}(G_{t(e_k)})$ for a path α in the graph $\mathcal{G}$.

Under the above conditions one can compute the ε-tube volume of the graph-directed fractals with the help of the inner tube volume of the graph-directed spray as in Example 2.4. So we now consider the inner tube volumes for graph-directed sprays in the next section.

3. Inner tube volumes of graphdirected sprays via functional equations

Let $\mathcal{G} = (V, E, r)$ be a Mauldin–Williams graph, $(G_u)_{u\in V}$ bounded open sets in $\mathbb{R}^n$ and $\mathcal{S}$ be a graph-directed spray associated with $\mathcal{G}$ and generated by the open sets $(G_u)_{u\in V}$. Let $\mathcal{S}_u$ be the subcollection of the spray $\mathcal{S}$ corresponding to the paths with initial vertex $u \in V$.

The volume of the inner ε-tube of the collection $\mathcal{S}_u$ satisfies the following functional equation (for all $u \in V$) as one can easily verify:

$$V_{\mathcal{S}_u}(\varepsilon) = \sum_{v\in V} \sum_{e\in E_{uv}} r_e^n \, V_{\mathcal{S}_v}\left(\frac{\varepsilon}{r_e}\right) + V_{G_u}(\varepsilon). \tag{2}$$

Our strategy will be, as in the self-similar case ([4]), to apply the Mellin transform to this functional equation and then try to recover the volume function by applying the inverse Mellin transform. To apply the Mellin transform we need an estimate of $V_{\mathcal{S}_u}(\varepsilon)$ as $\varepsilon \to 0$. We now formulate a multi-dimensional renewal lemma which will enable us to find such an estimate (For the one-dimensional renewal lemma see [11]).

We recall that for a strongly connected Mauldin–Williams graph the spectral radius of the matrix

$$A(s) = [a_{uv}(s)]_{u,v\in V} \quad \text{with } a_{uv}(s) = \sum_{e\in E_{uv}} r_e^s$$

(and $a_{uv}(s) = 0$ if $E_{uv} = \emptyset$) takes the value 1 for a unique $s_0 \geq 0$, which is called the sim-value of the graph and which we denote by D below (see [5]). We will always assume $D < n$.

Remark 3.1. The assumption $D < n$ is in fact equivalent to the condition that the total volume of the graph-directed spray is finite. To see this, one can easily verify that the volumes of the subcollections $\mathcal{S}_u$ can be expressed as follows:

$$[\mathrm{Vol}(\mathcal{S}_u)]_{u\in V} = (I + A(n) + A^2(n) + \dots)\, [\mathrm{Vol}(G_u)]_{u\in V},$$

where $[\mathrm{Vol}(\mathcal{S}_u)]_{u\in V}$ is a column vector. Note that, the matrix power $A^k(n)$ codes the contribution of paths of length k to the total spray volume.

We first note that the spectral radius of $A(s)$ is a strictly decreasing function for $s \geq 0$ and the spectral radius of $A(D) = 1$ ([5]). Now, if $D < n$, the spectral radius of $A(n)$ is strictly less than 1 and hence the matrix power series $\sum_{k=0}^{\infty} A^k(n)$ converges ([6, Theorem 5.6.15]), so that the volume of the spray is finite. On the other hand if $D \geq n$, then the spectral radius of $A(n)$ is greater than or equal to 1, and hence at least one entry of the matrix sequence $A^k(n)$ does not tend to zero [6, Theorem 5.6.12], so that the series $\sum_{k=0}^{\infty} A^k(n)$ diverges and the total volume of the spray is infinite.

Lemma 3.2. *Let $\mathcal{G} = (V, E, r)$ be a strongly connected Mauldin–Williams graph and*

$$h_u(x) = \sum_{v \in V} \sum_{e \in E_{uv}} r_e^D\, h_v\Big(x - \log\Big(\frac{1}{r_e}\Big)\Big) + \psi_u(x) \quad (u \in V), \tag{3}$$

be a system of renewal equations on $\mathbb{R}$, where D is the sim-value of the graph. Assume $\psi_u(x) = O(e^{-\tau|x|})$ for some $\tau > 0$. Let $(h_u)_{u \in V}$ be a solution of this system of renewal equations. If $h_u(x)$ tends to 0 for $x \to -\infty$ for all $u \in V$, then h_u is bounded (for all $u \in V$).

Proof. Let $\gamma = \min_{e \in E}\{\log 1/r_e\}$.

Since the Mauldin–Williams graph $\mathcal{G}$ is strongly-connected, the corresponding Mauldin–Williams matrix $A(s)$ is irreducible and by the Perron-Frobenius theorem, for $s = D$ the spectral radius 1 is also an eigenvalue with a positive eigenvector $p = (p_u)_{u \in V}$ (with $p_u > 0$) so that we have

$$p_u = \sum_{v \in V} a_{uv}(D)\, p_v = \sum_{v \in V} \sum_{e \in E_{uv}} r_e^D\, p_v. \tag{4}$$

Since the functions h_u $(u \in V)$ tend to zero for $x \to -\infty$, one can choose $x_0 \in \mathbb{R}$ such that $|h_u(x)| \leq p_u$ for $x \in (-\infty, x_0]$.

Let $x \in [x_0, x_0 + \gamma]$. From (3) and (4),

$$\begin{aligned} |h_u(x)| &\leq \sum_{v \in V} \sum_{e \in E_{uv}} r_e^D\, p_v + \sup_{x \in [x_0, x_0+\gamma]} |\psi_u(x)| \\ &\leq p_u + \sup_{x \in [x_0, x_0+\gamma]} |\psi_u(x)|. \end{aligned}$$

By the assumption on ψ_u, we can find an M such that $|\psi_u(x)| \leq p_u\, M\, e^{-\tau|x|}$ (for all $u \in V$). Hence for $x \in [x_0, x_0 + \gamma]$

$$|h_u(x)| \leq p_u(1 + M \sup_{x \in [x_0, x_0+\gamma]} e^{-\tau|x|}). \tag{5}$$

Since $|h_u(x)| \leq p_u$ for $x \in (-\infty, x_0]$, the inequality (5) holds for all $x \in (-\infty, x_0 + \gamma]$.

Now, let $x \in [x_0 + \gamma, x_0 + 2\gamma]$. As above,

$$\begin{aligned}|h_u(x)| &\le \sum_{v\in V}\sum_{e\in E_{uv}} r_e^D p_v(1 + M \sup_{x\in[x_0,x_0+\gamma]} e^{-\tau|x|}) + \sup_{x\in[x_0+\gamma,x_0+2\gamma]} |\psi_u(x)| \\ &\le p_u(1 + M \sup_{x\in[x_0,x_0+\gamma]} e^{-\tau|x|}) + p_u M \sup_{x\in[x_0+\gamma,x_0+2\gamma]} e^{-\tau|x|} \\ &= p_u(1 + M \sup_{x\in[x_0,x_0+\gamma]} e^{-\tau|x|} + M \sup_{x\in[x_0+\gamma,x_0+2\gamma]} e^{-\tau|x|}).\end{aligned}$$

The above inequality clearly holds for all $x \in (-\infty, x_0 + 2\gamma]$. Repeating the above argument, we see that for all $x \in \mathbb{R}$,

$$|h_u(x)| \le p_u\Big(1 + M \sum_{k=0}^{\infty} \sup_{x\in[x_0+k\gamma,x_0+(k+1)\gamma]} e^{-\tau|x|}\Big).$$

This shows that h_u is bounded on $\mathbb{R}$. □

Now we can derive an estimate for $V_{\mathcal{S}_u}(\varepsilon)$ as $\varepsilon \to 0$.

Lemma 3.3. *Assume, there exists an $\alpha > 0$ such that $V_{G_u}(\varepsilon) = O(\varepsilon^\alpha)$ for $\varepsilon \to 0$. We further assume that $n - \alpha < D < n$, where D is the sim-value of the graph. Then, it holds $V_{\mathcal{S}_u}(\varepsilon) = O(\varepsilon^{n-D})$ as $\varepsilon \to 0$.*

Remark 3.4. If the generator G_u is monophase or pluriphase (i.e. the volume of the inner ε-tube of G_u is piecewise polynomial), then $\alpha \ge 1$ and the assumption $n - \alpha < D$ reduces to $n - 1 < D$. But if G_u is a complicated set of a fractal nature, then α could be less than 1.

Proof of Lemma 3.3. Let us define

$$W_{\mathcal{S}_u}(\varepsilon) = \frac{V_{\mathcal{S}_u}(\varepsilon)}{\varepsilon^{n-D}}$$

to obtain

$$W_{\mathcal{S}_u}(\varepsilon) = \sum_{v\in V}\sum_{e\in E_{uv}} r_e^D\, W_{\mathcal{S}_v}\Big(\frac{\varepsilon}{r_e}\Big) + \frac{V_{G_u}(\varepsilon)}{\varepsilon^{n-D}} \tag{6}$$

from the equation (2). Let us now apply the change of variable $\varepsilon = e^{-x}$ in the equation (6). We obtain the following system of renewal equations on $\mathbb{R}$

$$h_u(x) = \sum_{v\in V}\sum_{e\in E_{uv}} r_e^D\, h_v\Big(x - \log\frac{1}{r_e}\Big) + \psi_u(x),$$

where

$$h_u(x) = W_{\mathcal{S}_u}(e^{-x}) \quad \text{and} \quad \psi_u(x) = \frac{V_{G_u}(e^{-x})}{e^{-x(n-D)}}.$$

Now we have to verify the assumptions of Lemma 3.2.

We notice that

$$h_u(x) = h_u(-\log \varepsilon) = \frac{V_{\mathcal{S}_u}(\varepsilon)}{\varepsilon^{n-D}} \longrightarrow 0$$

as $x \to -\infty$, i.e. $\varepsilon \to \infty$, since $D < n$ and the volume of the spray is finite.

To check the assumption on ψ_u, choose $\tau = \min\{n - D, D - n + \alpha\}$.

By Lemma 3.2, $h_u(x) = V_{\mathcal{S}_u}(\varepsilon)/\varepsilon^{n-D}$ is bounded, so that $V_{\mathcal{S}_u}(\varepsilon) = O(\varepsilon^{n-D})$. $\square$

We will now apply the Mellin transform to the equation (2) and to this end, it will be convenient to define the auxiliary functions

$$f_u(\varepsilon) = \frac{V_{\mathcal{S}_u}(\varepsilon)}{\varepsilon^n}, \quad \text{for } u \in V$$

(these functions can be viewed as a kind of "normed" tube volumes). The system (2) of functional equations transforms into the following system:

$$f_u(\varepsilon) = \sum_{v \in V} \sum_{e \in E_{uv}} f_v\left(\frac{\varepsilon}{r_e}\right) + \frac{V_{G_u}(\varepsilon)}{\varepsilon^n}. \tag{7}$$

Recall that the Mellin transform of a function $f\colon (0, \infty) \to \mathbb{R}$ is given by

$$\mathcal{M}(f)(s) \equiv \tilde{f}(s) = \int_0^\infty f(x)\, x^{s-1} dx.$$

If this integral exists for some $c \in \mathbb{R}$ and if the function f is continuous at $x \in (0, \infty)$ and of bounded variation in a neighborhood of x, then $f(x)$ can be recovered by the inverse Mellin transform ([13])

$$\frac{1}{2\pi \mathbf{i}} \lim_{T \to \infty} \int_{c-\mathbf{i}T}^{c+\mathbf{i}T} \tilde{f}(s)\, x^{-s} ds.$$

The function f_u is continuous and $f_u(\varepsilon) = O(\varepsilon^{-n})$ as $\varepsilon \to \infty$. If for some $\alpha > 0$, $V_{G_u}(\varepsilon) = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ and $n - \alpha < D < n$, then by Lemma 3.3, $f_u(\varepsilon) = O(\varepsilon^{-D})$ as $\varepsilon \to 0$. So the integral

$$\int_0^\infty f_u(\varepsilon)\, \varepsilon^{s-1}\, d\varepsilon$$

exists for any s with $D < \mathrm{Re}(s) < n$. Likewise, the integral

$$\int_0^\infty \frac{V_{G_u}(\varepsilon)}{\varepsilon^n}\, \varepsilon^{s-1}\, d\varepsilon$$

exists for $n - \alpha < \mathrm{Re}(s) < n$. We can then take the Mellin transform of (7) to obtain

$$\widetilde{f_u}(s) = \sum_{v\in V} \Big(\sum_{e\in E_{uv}} r_e^s \Big) \widetilde{f_v}(s) + \mathcal{M}\Big(\frac{V_{G_u}(\varepsilon)}{\varepsilon^n}\Big)(s)$$

for $D < \mathrm{Re}(s) < n$. This system can also be written as a matrix equation

$$F(s) = A(s)F(s) + \Phi(s) \quad (D < \mathrm{Re}(s) < n)\,,$$

where $F(s)$ is the column vector $[\widetilde{f_u}(s)]_{u\in V}$,

$$\Phi(s) = \Big[\mathcal{M}\Big(\frac{V_{G_u}(\varepsilon)}{\varepsilon^n}\Big)(s)\Big]_{u\in V},$$

and $A(s)$ is the Mauldin–Williams matrix.

Lemma 3.5. *For* $\mathrm{Re}(s) > D$, *the matrix* $I - A(s)$ *is invertible, so that it holds*

$$F(s) = [I - A(s)]^{-1}\, \Phi(s) \tag{8}$$

for $D < \mathrm{Re}(s) < n$.

Proof. This is a consequence of some well-known results from matrix algebra. For $s \in \mathbb{R}$, $s > D$, the spectral radius $\rho(A(s))$ is less than 1 ([5]). Then by [6, Theorem 5.6.12] $\lim_{k\to\infty} A^k(s) = 0$ entry-wise. For arbitrary $s \in \mathbb{C}$ with $\mathrm{Re}(s) > D$, we have

$$|a_{uv}(s)| = \Big| \sum_{e\in E_{uv}} r_e^s \Big| \le \sum_{e\in E_{uv}} r_e^{\mathrm{Re}(s)} = a_{uv}(\mathrm{Re}(s)).$$

This holds for the entries of $A^k(s)$ and $A^k(\mathrm{Re}(s))$ also, giving $\lim_{k\to\infty} A^k(s) = 0$ entry-wise. We then have by [6, Theorem 5.6.12], $\rho(A(s)) < 1$, and thus $I - A(s)$ is invertible. □

We can write the matrix equation (8) also as follows:

$$[\widetilde{f}_u(s)]_{u\in V} = \frac{1}{\det(I - A(s))}[\mathrm{adj}(I - A(s))]_{uv}\Big[\mathcal{M}\Big(\frac{V_{G_u}(\varepsilon)}{\varepsilon^n}\Big)(s)\Big]_{v\in V},$$

or

$$\widetilde{f}_u(s) = \frac{1}{\det(I - A(s))}\sum_{v\in V}\mathrm{adj}(I - A(s))_{uv}\,\mathcal{M}\Big(\frac{V_{G_v}(\varepsilon)}{\varepsilon^n}\Big)(s) \quad \text{for all } u \in V, \tag{9}$$

where "adj" means the adjugate matrix.

We now apply the inverse Mellin transform to the equation (9). For $D < c < n$,

$$\begin{aligned} f_u(\varepsilon) &= \frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty}\widetilde{f}_u(s)\,\varepsilon^{-s}ds \\ &= \frac{1}{2\pi\mathbf{i}}\sum_{v\in V}\Big(\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty}\frac{\mathrm{adj}(I - A(s))_{uv}}{\det(I - A(s))}\,\mathcal{M}\Big(\frac{V_{G_v}(\varepsilon)}{\varepsilon^n}\Big)(s)\,\varepsilon^{-s}\,ds\Big). \end{aligned}$$

Definition 3.6. Let $\mathcal{G} = (V, E, r)$ be a Mauldin–Williams graph, G_u ($u \in V$) bounded open sets in $\mathbb{R}^n$ and $\mathcal{S}$ a graph-directed spray associated with $\mathcal{G}$ and generated by the open sets G_u. Let $A(s) = [a_{uv}(s)]_{u,v\in V}$ with $a_{uv}(s) = \sum_{e\in E_{uv}} r_e^s$ be the Mauldin–Williams matrix of the graph. We define the geometric zeta function of the graph-directed spray with respect to the node $u \in V$ as follows:

$$\zeta_u(s) = \sum_{v\in V}\frac{\mathrm{adj}(I - A(s))_{uv}}{\det(I - A(s))}\,\mathcal{M}\Big(\frac{V_{G_v}(\varepsilon)}{\varepsilon^n}\Big)(s),$$

for $D < \mathrm{Re}(s) < n$ where D is the sim-value of the Mauldin–Williams graph.

$f_u(\varepsilon)$ can now be expressed as

$$f_u(\varepsilon) = \frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty}\zeta_u(s)\,\varepsilon^{-s}ds.$$

At this point, we need some assumptions about the inner tube volumes of the generators to manipulate this expression further. We assume that the generators are monophase or pluriphase. Note that in this case the geometric zeta function which is analytic for $D < \mathrm{Re}(s) < n$ can be extended meromorphically to the whole plane $\mathbb{C}$, as can be seen from the explicit expressions for the Mellin transforms given in the Remarks 3.8-3.9 below.

We can now express our main result as follows.

Theorem 3.7. *Let $\mathcal{G} = (V, E, r)$ be a Mauldin–Williams graph, $G_u\,(u \in V)$ bounded open sets in $\mathbb{R}^n$ and $\mathcal{S}$ a graph-directed spray associated with $\mathcal{G}$ and generated by the open sets G_u. Let $A(s)$ be the matrix of the Mauldin–Williams graph.*

We assume the generators G_u to be monophase or pluriphase. We furthermore assume that the sim-value D of the Mauldin–Williams graph satisfies $n-1 < D < n$.

Then for small ε, the volume of the inner ε-tube of the graph-directed spray $\mathcal{S}$ can be expressed pointwise as the following residue formula:

$$V_{\mathcal{S}}(\varepsilon) = \sum_{u \in V} \sum_{\omega \in \mathfrak{D} \cup \{0,1,2,\ldots,n-1\}} \operatorname{res}(\zeta_u(s)\,\varepsilon^{n-s}; \omega),$$

where $\mathfrak{D}$ is the set of zeros of $\det(I - A(s))$*, which we call the complex dimensions of the graph-directed spray* (*For an exact bound for ε, see the last paragraph of the proof*).

Remark 3.8. If the generator G_u is monophase with tube formula

$$V_{G_u}(\varepsilon) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} \kappa_i^u\,\varepsilon^{n-i} & \text{for } 0 \le \varepsilon \le g_u, \\ \operatorname{Vol}(G_u) & \text{for } \varepsilon \ge g_u, \end{cases}$$

then

$$\mathcal{M}\Big(\frac{V_{G_u}(\varepsilon)}{\varepsilon^n}\Big)(s) = \sum_{i=0}^{n} \kappa_i^u\,\frac{g_u^{s-i}}{s-i},$$

where $\kappa_n^u = -\operatorname{Vol}(G_u)$.

Remark 3.9. If the generator G_u is pluriphase, let us assume that it has the tube formula

$$V_{G_u}(\varepsilon) = \begin{cases} \displaystyle\sum_{i=0}^{n} \kappa_i^{m,u}\,\varepsilon^{n-i} & \text{for } g_{m-1,u} \le \varepsilon \le g_{m,u}\,,\ m = 1,2,\ldots,M_u, \\ \operatorname{Vol}(G_u) & \text{for } \varepsilon \ge g_u, \end{cases}$$

where $g_{0,u} = 0$, $g_{M_u,u} = g_u$ (g_u the inradius of G_u) and $\kappa_n^{1,u} = 0$. It will be more convenient to write the above formula as

$$V_{G_u}(\varepsilon) = \sum_{i=0}^{n} \kappa_i^{m,u}\,\varepsilon^{n-i} \ \text{ for } g_{m-1,u} \le \varepsilon \le g_{m,u}, \quad m = 1,2,\ldots,M_u+1,$$

where we set $\kappa_i^{M_u+1,u} = 0$ for $i = 0, 1, \ldots, n-1$, $\kappa_n^{M_u+1,u} = \mathrm{Vol}(G_u)$ and $g_{M_u+1,u} = \infty$. Then

$$\mathcal{M}\Big(\frac{V_{G_u}(\varepsilon)}{\varepsilon^n}\Big)(s) = \sum_{m=1}^{M_u}\sum_{i=0}^{n}(\kappa_i^{m,u} - \kappa_i^{m+1,u})\frac{g_{m,u}^{s-i}}{s-i}.$$

Example 3.10 (Example 2.4 continued). The volumes of the inner ε-neighborhood of G_1 and G_2 are given by the following functions:

$$V_{G_1}(\varepsilon) = \begin{cases} 4\sqrt{2}\varepsilon - 4\varepsilon^2 & \text{if } \varepsilon \le \dfrac{\sqrt{2}}{2}, \\[2ex] 2 & \text{if } \varepsilon \ge \dfrac{\sqrt{2}}{2}, \end{cases}$$

$$V_{G_2}(\varepsilon) = \begin{cases} \left(\dfrac{2+\sqrt{2}}{2}\right)\varepsilon - (3+2\sqrt{2})\varepsilon^2 & \text{if } \varepsilon \le \dfrac{2-\sqrt{2}}{4}, \\[2ex] \dfrac{1}{8} & \text{if } \varepsilon \ge \dfrac{2-\sqrt{2}}{4}. \end{cases}$$

The corresponding Mauldin–Williams matrix of the graph is

$$A(s) = \begin{pmatrix} 0 & 4\,\dfrac{1}{2^s} \\[2ex] \dfrac{1}{2^s} & \dfrac{1}{2^s} + 3\,\dfrac{1}{4^s} \end{pmatrix}$$

and the sim-value of the graph is

$$D = \log_2\Big(\frac{\sqrt{29}+1}{2}\Big).$$

The complex dimensions of the graph-directed spray are given by

$$\Big\{\log_2\Big(\frac{\sqrt{29}+1}{2}\Big) + \mathbf{i}kp \;\Big|\; k \in \mathbb{Z}\Big\} \cup \Big\{\log_2\Big(\frac{\sqrt{29}-1}{2}\Big) + \mathbf{i}\Big(k + \frac{1}{2}\Big)p \;\Big|\; k \in \mathbb{Z}\Big\},$$

where $p = 2\pi/\ln 2$. Using Theorem 3.7, we obtain the volume of the ε-neighborhood of $\mathcal{S}$ as

$$V_{\mathcal{S}}(\varepsilon) = \frac{4}{7}\varepsilon^2 - \frac{28}{5}\sqrt{2}\varepsilon + \Sigma_1 + \Sigma_2 + \frac{2}{7}(3+2\sqrt{2})\varepsilon^2 - \frac{3}{5}(2+\sqrt{2})\varepsilon + \Sigma_3 + \Sigma_4,$$

where

$$\Sigma_1 = \frac{\varepsilon^{2-(D+\mathbf{i}kp)}}{\sqrt{29}\ln 2}\Big[\frac{2\sqrt{29}-2}{7}\Big(-4\frac{\big(\frac{\sqrt{2}}{2}\big)^{D+\mathbf{i}kp}}{D+\mathbf{i}kp} + 4\sqrt{2}\frac{\big(\frac{\sqrt{2}}{2}\big)^{D-1+\mathbf{i}kp}}{D-1+\mathbf{i}kp} - 2\frac{\big(\frac{\sqrt{2}}{2}\big)^{D-2+\mathbf{i}kp}}{D-2+\mathbf{i}kp}\Big) + 4\Big(-(3+2\sqrt{2})\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D+\mathbf{i}kp}}{D+\mathbf{i}kp} + \frac{2+\sqrt{2}}{2}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D-1+\mathbf{i}kp}}{D-1+\mathbf{i}kp} - \frac{1}{8}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D-2+\mathbf{i}kp}}{D-2+\mathbf{i}kp}\Big)\Big],$$

$$\Sigma_2 = \frac{\varepsilon^{2-(D'+\mathbf{i}(k+\frac{1}{2})p)}}{\sqrt{29}\ln 2}\Big[\frac{2\sqrt{29}+2}{7}\Big(-4\frac{\big(\frac{\sqrt{2}}{2}\big)^{D'+\mathbf{i}(k+\frac{1}{2})p}}{D'+\mathbf{i}\big(k+\frac{1}{2}\big)p} + 4\sqrt{2}\frac{\big(\frac{\sqrt{2}}{2}\big)^{D'-1+\mathbf{i}(k+\frac{1}{2})p}}{D'-1+\mathbf{i}\big(k+\frac{1}{2}\big)p} - 2\frac{\big(\frac{\sqrt{2}}{2}\big)^{D'-2+\mathbf{i}(k+\frac{1}{2})p}}{D'-2+\mathbf{i}\big(k+\frac{1}{2}\big)p}\Big) - 4\Big(-(3+2\sqrt{2})\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D'+\mathbf{i}(k+\frac{1}{2})p}}{D'+\mathbf{i}\big(k+\frac{1}{2}\big)p} + \frac{2+\sqrt{2}}{2}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D'-1+\mathbf{i}(k+\frac{1}{2})p}}{D'-1+\mathbf{i}\big(k+\frac{1}{2}\big)p} - \frac{1}{8}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D'-2+\mathbf{i}(k+\frac{1}{2})p}}{D'-2+\mathbf{i}\big(k+\frac{1}{2}\big)p}\Big)\Big],$$

$$\Sigma_3 = \frac{\varepsilon^{2-(D+\mathbf{i}kp)}}{\sqrt{29}\ln 2}\Big[\Big(-4\frac{\big(\frac{\sqrt{2}}{2}\big)^{D+\mathbf{i}kp}}{D+\mathbf{i}kp} + 4\sqrt{2}\frac{\big(\frac{\sqrt{2}}{2}\big)^{D-1+\mathbf{i}kp}}{D-1+\mathbf{i}kp} - 2\frac{\big(\frac{\sqrt{2}}{2}\big)^{D-2+\mathbf{i}kp}}{D-2+\mathbf{i}kp}\Big)$$
$$+ \frac{\sqrt{29}+1}{2}\Big(-(3+2\sqrt{2})\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D+\mathbf{i}kp}}{D+\mathbf{i}kp}$$
$$+ \frac{2+\sqrt{2}}{2}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D-1+\mathbf{i}kp}}{D-1+\mathbf{i}kp}$$
$$- \frac{1}{8}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D-2+\mathbf{i}kp}}{D-2+\mathbf{i}kp}\Big)\Big],$$

$$\Sigma_4 = -\frac{\varepsilon^{2-(D'+\mathbf{i}(k+\frac{1}{2})p)}}{\sqrt{29}\ln 2}\Big[\Big(-4\frac{\big(\frac{\sqrt{2}}{2}\big)^{D'+\mathbf{i}(k+\frac{1}{2})p}}{D'+\mathbf{i}\big(k+\frac{1}{2}\big)p}$$
$$+ 4\sqrt{2}\frac{\big(\frac{\sqrt{2}}{2}\big)^{D'-1+\mathbf{i}(k+\frac{1}{2})p}}{D'-1+\mathbf{i}\big(k+\frac{1}{2}\big)p}$$
$$- 2\frac{\big(\frac{\sqrt{2}}{2}\big)^{D'-2+\mathbf{i}(k+\frac{1}{2})p}}{D'-2+\mathbf{i}\big(k+\frac{1}{2}\big)p}\Big)$$
$$- \frac{\sqrt{29}+1}{2}\Big(-(3+2\sqrt{2})\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D'+\mathbf{i}(k+\frac{1}{2})p}}{D'+\mathbf{i}\big(k+\frac{1}{2}\big)p}$$
$$+ \frac{2+\sqrt{2}}{2}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D'-1+\mathbf{i}(k+\frac{1}{2})p}}{D'-1+\mathbf{i}\big(k+\frac{1}{2}\big)p}$$
$$- \frac{1}{8}\frac{\big(\frac{2-\sqrt{2}}{4}\big)^{D'-2+\mathbf{i}(k+\frac{1}{2})p}}{D'-2+\mathbf{i}\big(k+\frac{1}{2}\big)p}\Big)\Big].$$

4. Proof of the main theorem (Theorem 3.7)

Since

$$V_S(\varepsilon) = \varepsilon^n \sum_{u \in V} f_u(\varepsilon)$$

and

$$\begin{aligned} f_u(\varepsilon) &= \frac{1}{2\pi \mathbf{i}} \sum_{v \in V} \left(\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \mathcal{M}\Big(\frac{V_{G_v}(\varepsilon)}{\varepsilon^n}\Big)(s)\ \varepsilon^{-s}\, ds \right) \\ &= \frac{1}{2\pi \mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \zeta_u(s)\, \varepsilon^{-s} ds, \end{aligned}$$

we have to evaluate the integral on the right hand side. As is well-known there is a general procedure to evaluate this integral by applying the residue theorem. In the present case of graph-directed sprays however, the $\det(I - A(s))$ in the denominator calls for a more cautious treatment in order to be able to give an explicit validity range of the tube formula for small ε.

The above sum consists of integrals of the type

$$\frac{1}{2\pi \mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds \quad (i = 0, 1, \ldots, n),$$

for $n - 1 < D < c < n$. Recall that $\det(I - A(s))$ is non-zero for $\operatorname{Re}(s) > D$.

We first note that there exists a vertical strip containing all the zeros of $\det(I - A(s))$. To see this, notice that $\det(I - A(s))$ can be expressed as a sum

$$1 + \sum_{\alpha} p_\alpha^s - \sum_{\beta} q_\beta^s$$

with $0 < p_\alpha, q_\beta < 1$ since the entries of the matrix $A(s)$ are of the form $\sum r_e^s$ for $0 < r_e < 1$. As $\operatorname{Re}(s) \to -\infty$, the smallest of p_α, q_β will dominate and avoid $\det(I - A(s))$ to vanish. We choose a $c_l < 0$ such that $|\det(I - A(s))| > \delta$ for some $\delta > 0$ and for all $\operatorname{Re}(s) \le c_l$.

We will choose a sequence $(\tau_j)_{j \in \mathbb{N}} \to \infty$ such that $|\det(I - A(s))|$ will be uniformly away from zero on the line segments $c_l \le \operatorname{Re}(s) \le c$, $\operatorname{Im}(s) = \pm\tau_j$.

Lemma 4.1. *There exists an increasing sequence $(\tau_j)_{j \in \mathbb{N}}$, $\tau_j \to \infty$ and a $K > 0$ such that $|\det(I - A(s))| > K$ for $c_l \le \operatorname{Re}(s) \le c$ and $\operatorname{Im}(s) = \pm\tau_j$.*

Proof. Being an entire function, $\det(I - A(s))$ has isolated zeros and we can choose $\tau_1 > 0$ such that there are no zeros on the segment $[c_l, c] \times \{\tau_1\}$. Let $2K$ be the minimum of $|\det(I - A(s))|$ on this segment.

To construct τ_2, we need the following lemma.

Lemma 4.2 (Dirichlet lemma, [1]). *Let $M, N \in \mathbb{N}$ and $T > 0$. Let $a_1, a_2, \dots, a_N$ be real numbers. There exists a real number $h \in [T, T M^N]$ such that*

$$\|a_i h\| \leq \frac{1}{M} \quad (1 \leq i \leq N).$$

Here $\|\cdot\|$ denotes "the distance to the nearest integer" function on $\mathbb{R}$.

The idea for choosing $\tau_2 = \tau_1 + h$ will be the following. We want to arrange h such that

$$|\det(I - A(s+ih)) - \det(I - A(s))| < K$$

for $c_l \leq \mathrm{Re}(s) \leq c$. Then the minimum of $|\det(I - A(s+ih))|$ will be greater than K. Since

$$\det(I - A(s)) = 1 + \sum_\alpha p_\alpha^s - \sum_\beta q_\beta^s,$$

we have

$$\begin{aligned}
|\det(I - A(s+ih)) - \det(I - A(s))| &= \left| \sum_\alpha (p_\alpha^{s+ih} - p_\alpha^s) - \sum_\beta (q_\beta^{s+ih} - q_\beta^s) \right| \\
&\leq \sum_\alpha |p_\alpha^{s+ih} - p_\alpha^s| + \sum_\beta |q_\beta^{s+ih} - q_\beta^s| \\
&= \sum_\alpha p_\alpha^{\mathrm{Re}(s)} |p_\alpha^{ih} - 1| + \sum_\beta q_\beta^{\mathrm{Re}(s)} |q_\beta^{ih} - 1|.
\end{aligned}$$

Since the number of terms and $p_\alpha^{\mathrm{Re}(s)}, q_\beta^{\mathrm{Re}(s)}$ are bounded, it will be enough to make the factors $|p_\alpha^{ih} - 1|$, $|q_\beta^{ih} - 1|$ small enough. To realize this, we can apply the Dirichlet Lemma (Lemma 4.2) to make $p_\alpha^{ih} = e^{ih \ln p_\alpha}$ and $q_\beta^{ih} = e^{ih \ln q_\beta}$ close enough to 1, by choosing $\|h \ln p_\alpha/(2\pi)\|$ and $\|h \ln q_\beta/(2\pi)\|$ small enough.

As we have control on choosing h on any range, we can repeat this procedure to get a sequence of segments $[c_l, c] \times \{\tau_j\}$ on all of which $|\det(I - A(s))|$ is bounded below by K. $\square$

Let us now consider the rectangles $R_j = [c_l, c] \times [-\tau_j, \tau_j]$ and denote its oriented edges by $L_{1,j}, L_{2,j}, L_{3,j}, L_{4,j}$ as shown in Figure 10. We will show that for small enough ε the integrals on $L_{2,j}, L_{3,j}$ and $L_{4,j}$ will tend to zero as $j \to \infty$ so that by residue theorem we will get the integral on the vertical line at c as a series of residues on the strip $c_l < \mathrm{Re}(s) < c$:

$$\begin{aligned}
&\frac{1}{2\pi\mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \frac{\mathrm{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds \\
&\quad = \sum_{\omega \in \mathfrak{D} \cup \{i\}} \mathrm{res}\left(\frac{\mathrm{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i}; \omega \right)
\end{aligned}$$

for $i = 0, 1, \ldots, n-1$. For $i = n$ the same formula holds with the only difference that the residues are taken on $\mathfrak{D}$.

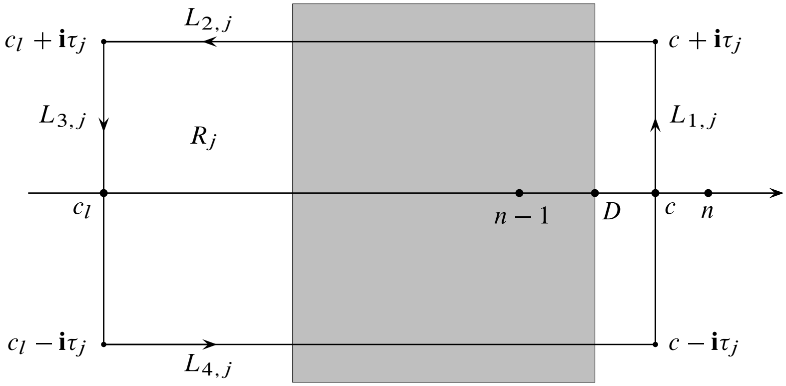

Figure 10. The strip containing the poles of the geometric zeta function $\zeta_u(s)$ and the rectangle R_j with the oriented boundary segments $L_{1,j}, L_{2,j}, L_{3,j}, L_{4,j}$ used in the proof.

First consider the integral on $L_{2,j} = t + \mathbf{i}\tau_j$, $c_l \le t \le c$.

$$\begin{aligned}\left| \int_{L_{2,j}} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds \right| \\ \le \int_{c_l}^{c} \left| \frac{\operatorname{adj}(I - A(t+\mathbf{i}\tau_j))_{uv}}{\det(I - A(t+\mathbf{i}\tau_j))} \right| \frac{g^{t-i}}{|t + \mathbf{i}\tau_j - i|} \varepsilon^{-t}\, dt\end{aligned}$$

$\operatorname{adj}(I - A(s))_{uv}$ is of the form

$$\tilde{1} + \sum_{\alpha} p_\alpha^s - \sum_{\beta} q_\beta^s$$

($\tilde{1}$ indicates that 1 might be present or absent) and therefore is bounded for $c_l \le \operatorname{Re}(s) \le c$, so that we can write

$$\left| \int_{L_{2,j}} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds \right| \le C \int_{c_l}^{c} \frac{dt}{\tau_j}$$

(C being a constant not depending on j), which tends to zero for $\tau_j \to \infty$.

Similarly the integral on $L_{4,j} \to 0$ for $\tau_j \to \infty$.

Finally we consider the integral on $L_{3,j}$:

$$\int_{L_{3,j}} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds = \int_{C_j} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds,$$

where $C_j = c_l + \tau_j e^{\mathbf{i}t}$, $\pi/2 \le t \le 3\pi/2$ (see Figure 11). $\operatorname{adj}(I - A(s))_{uv}$ is of the form $\tilde{1} + \sum_\alpha p_\alpha^s - \sum_\beta q_\beta^s$. In any term, p_α, q_β there can appear at most $(N-1)$-th power of the smallest weight $r_{\min}$ of the graph where N is the number of the nodes of the graph. We can thus dominate $|\operatorname{adj}(I - A(s))_{uv}|$ by $C'\, r_{\min}^{(N-1)\operatorname{Re}(s)}$. When $s \in C_j$, we have $\operatorname{Re}(s) \le c_l < 0$, therefore $|\det(I - A(s))| > \delta$ by the choice of c_l.

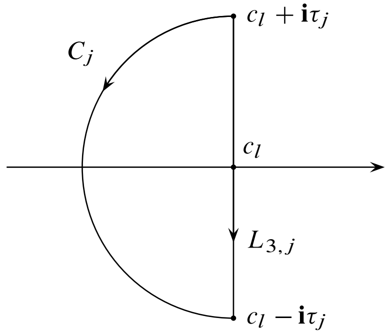

Figure 11. The semi-circle C_j used to evaluate the integral on the segment $L_{3,j}$.

We can now write

$$\begin{aligned}
&\left| \int_{C_j} \frac{\operatorname{adj}(I - A(s))_{uv}}{\det(I - A(s))} \frac{g^{s-i}}{s-i} \varepsilon^{-s}\, ds \right| \\
&\quad \le \int_{C_j} \frac{C'\, r_{\min}^{(N-1)\operatorname{Re}(s)}}{\delta} \frac{g^{\operatorname{Re}(s)-i}}{|s-i|} \varepsilon^{-\operatorname{Re}(s)}\, |ds| \\
&\quad \le C'' \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{r_{\min}^{N-1}\, g}{\varepsilon}\right)^{\tau_j \cos t} \frac{\tau_j}{|c_l + \tau_j e^{\mathbf{i}t} - i|} dt \\
&\quad \le C'' \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{r_{\min}^{N-1}\, g}{\varepsilon}\right)^{\tau_j \cos t} dt
\end{aligned}$$

since $\tau_j \le |c_l + \tau_j e^{\mathbf{i}t} - i|$. By the Jordan Lemma (which states that

$$\lim_{n\to\infty} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} a^{n \cos t}\, dt = 0,$$

for any fixed $a > 1$) this integral tends to zero if

$$\frac{r_{\min}^{N-1}\, g}{\varepsilon} > 1.$$

So we come to the conclusion that our tube formula is valid pointwise for $\varepsilon < r_{\min}^{N-1} \min_{u \in V} \{g_u\}$ for monophase generators. In the pluriphase case, one should take $\varepsilon < r_{\min}^{N-1} \min_{u \in V} \{g_{1,u}\}$. Needless to say, these are only sufficient bounds.

Remark 4.3. There is an unfortunate misprint in [4, p. 159, line 12], where $\varepsilon < g$ should be $\varepsilon < g_1$.

5. Appendix

In this appendix we want to discuss the relationship between the sprays as defined in the work of Lapidus, Pearse and Winter in [9] and graph-directed sprays as defined in the present paper, in Section 2.

According to [9], a spray (or fractal spray) is essentially a collection of disjoint bounded open sets $\{U_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$, where each U_i is a copy of a fixed bounded open set $U \subseteq \mathbb{R}^n$ under a similarity transformation ψ_i on $\mathbb{R}^n$ with scaling ratio $\lambda_i > 0$, i.e. $U_i = \psi_i(U)$.

The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is called a fractal string in [9] and it is assumed to be a non-increasing sequence (of positive real numbers) satisfying $\lim_{i \to \infty} \lambda_i = 0$. (We call this sequence also a "scaling sequence" and assume that $\sum_{i=1}^{\infty} \lambda_i^n < \infty$, which are secondary to the matter.)

The set U is not assumed to be connected and its components, which are assumed to be finitely many, are called the generators of the spray. From the point of view of tube formulas, it can be assumed that there is a single generator $G = U$, i.e. U is connected. This is also secondary to the matter and the main point about sprays is that, there is a single basic open set $U \subseteq \mathbb{R}^n$ and there is a single fractal string (scaling sequence) encoding the similarity scalings of the set U.

In graph-directed sprays however, there is a finite collection $G_u (u \in V)$ of open bounded basic sets in $\mathbb{R}^n$ and a derived collection $\{G_\alpha\}_\alpha$ of pairwise disjoint open sets in $\mathbb{R}^n$, each of which is a scaled copy of one of these basic sets. There is not a single fractal string (scaling sequence) governing these scalings of the basic open sets G_u. Instead, the scalings are obtained from a weighted directed graph as follows.

Let $\mathcal{G} = (V, E, r)$ be a weighted directed graph which we assume to be strongly connected in the sense that given any two vertices $u, v \in V$, there exists a (directed) path α starting at u and terminating at v. If we denote the initial vertex of an edge $e \in E$ by $i(e)$ and the terminal vertex by $t(e)$, then such a path α is a sequence $\alpha = e_1 e_2 \ldots e_k$ with $i(e_1) = u$, $t(e_k) = v$ and $t(e_j) = i(e_{j+1})$ for $j = 1, \ldots, k-1$.

For such a path α we define the scaling ratio $r(\alpha)$ by $r(\alpha) = r(e_1)\dots r(e_k)$. The "scaling system" of the graph-directed spray is given by the collection $\{r(\alpha)\}_\alpha$ of all scaling ratios of paths α on the weighted directed graph $\mathcal{G}$. We can now define a graph-directed spray $\mathcal{S}$ with basic sets $G_u (u \in V)$ and associated with the graph $\mathcal{G}$ as a collection $\{G_\alpha\}_\alpha$ of pairwise disjoint open sets G_α in $\mathbb{R}^n$ (where α is a path in the graph $\mathcal{G}$), such that G_α is an isometric copy of $G_{t(\alpha)}$ with scaling ratio $r(\alpha)$.

The main differences between (fractal) sprays and graph-directed sprays are the following ones.

1) In sprays there is a single (connected or disconnected) basic set, but in graph-directed sprays there are finitely many basic sets;

2) In sprays, there is a single sequence $\{\lambda_i\}_{i=1}^\infty$ of scalings (called the fractal string or scaling sequence), which governs the scalings of the basic set. In graph directed sprays however, there is a system $\{r(\alpha)\}_\alpha$ of scalings derived from a weighted directed graph, with a rule specifying how they are related to the basic sets.

We illustrate these schemes in Figures 12 and 13.

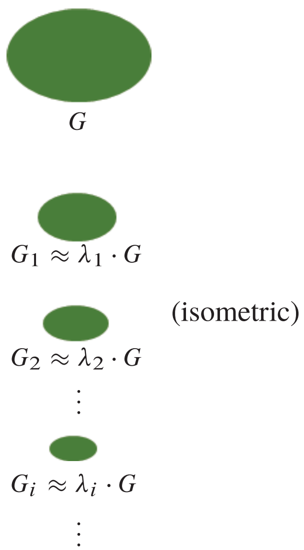

Figure 12. A spray generated by a basic set (or generator) G with a scaling sequence $\{\lambda_i\}_{i=1}^\infty$.

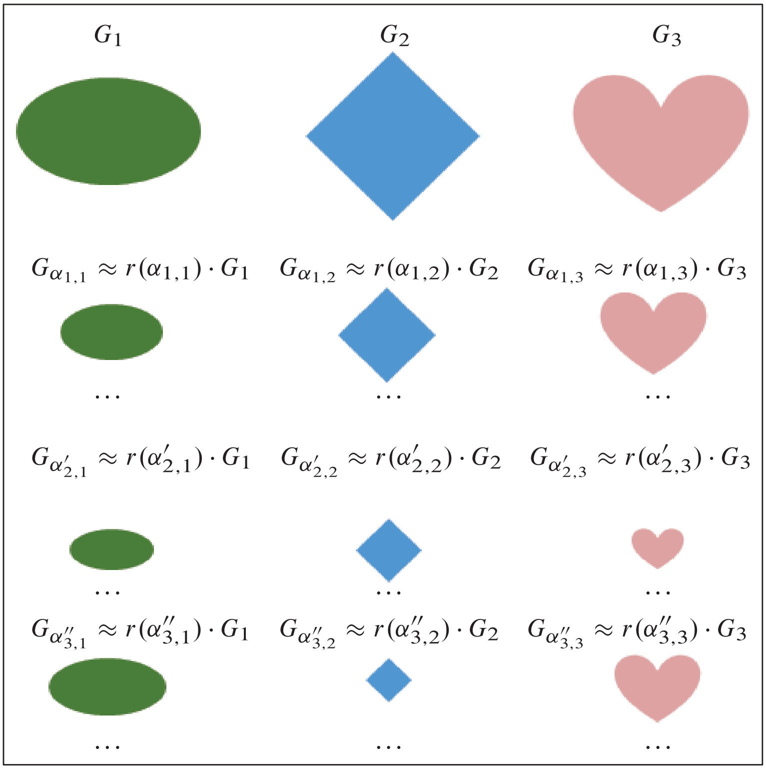

Figure 13. A graph directed spray with (three) basic sets (or generators) G_1, G_2, G_3 with a scaling system $\{r(\alpha_{uv})\}$ derived from a weighted directed graph $\mathcal{G}$ (with $V = \{1, 2, 3\}$). (α_{uv}, or α'_{uv},α''_{uv},... are paths on $\mathcal{G}$ from vertex u to vertex v.) $r(\alpha_{uv})$ is the scaling ratio associated to the path α_{uv}. The collection of sets $\mathcal{S}^u$ in each column constitutes a spray (in the sense of [9]) by itself, but their volumes of inner ε-tubes are unrelated.

A graph-directed spray is obviously not a (fractal) spray, since the collection $\{G_\alpha\}_\alpha$ is not given by scaled copies of a single basic set with the help of a single scaling sequence.

This collection can however be decomposed into the subcollections $\mathcal{S}^u = \{G_\alpha \mid t(\alpha) = u\}$ and these subcollections are indeed (fractal) sprays with basic set G_u and scaling ratios $\{r(\alpha) \mid t(\alpha) = u\}$, which can be ordered as a sequence. From the point of view of tube formulas, it is technically difficult to manage these sequences. There is however another decomposition of the graph-directed spray $\{G_\alpha\}_\alpha$ into subcollections $\mathcal{S}_u$, given by $\mathcal{S}_u = \{G_\alpha \mid i(\alpha) = u\}$ as shown in Figure 14. These subcollections are definitely no longer sprays in the sense of [9], but they are extremely convenient subcollections for computations of tube volumes, since there is a very natural functional equation relating the tube volumes of $\mathcal{S}_u$ as explained and used efficiently in the present paper.

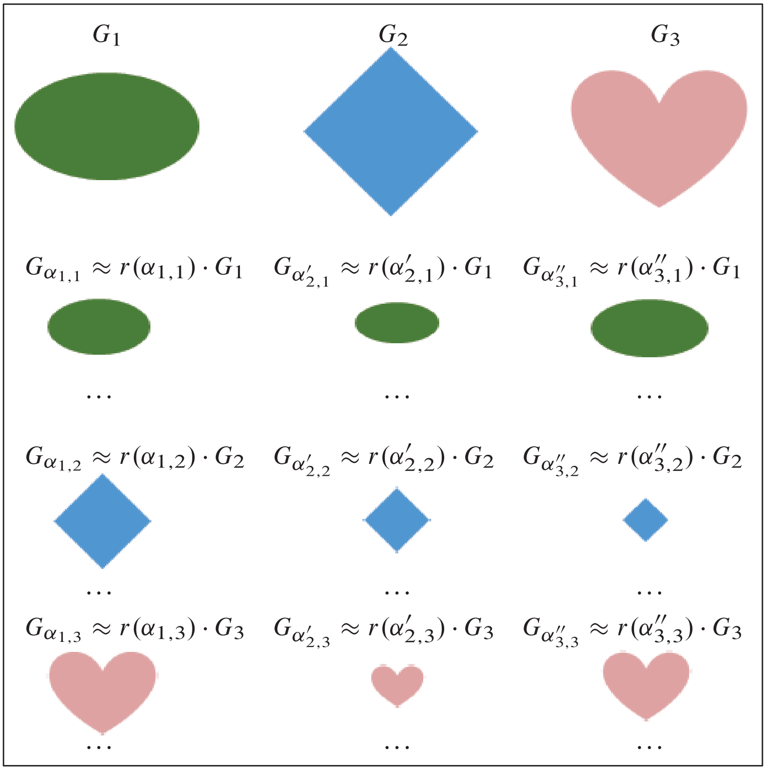

Figure 14. The same graph directed spray as in Figure 13, but with another display. The collection of sets $\mathcal{S}_u$ in each column does not constitute a spray (in the sense of [9]), but the inner ε-tube volumes of $\mathcal{S}_u$ satisfy a natural functional equation (2).

To summarize, a graph-directed spray $\{G_\alpha\}$ is not a spray in the sense of [9]; but there are two decompositions of $\{G_\alpha\}$, one of which gives a collection of sprays in the sense of [9]. The other decomposition does not give sprays in this sense, but it is very convenient for the volume computations.

On the other hand, a spray in the sense of [9] which has a self-similar fractal string, is naturally a special case of graph directed sprays, with a graph consisting of a single node. If a spray in the sense of [9] is not self-similar, then it is not a special instance of a graph directed spray.

References

[1] P. T. Bateman and H. G. Diamond, *Analytic number theory.* An introductory course. Monographs in Number Theory, 1. Hackensack, N.J., World Scientific, 2004. MR 2111739 Zbl 1074.11001

[2] B. Demir, A. Deniz, Ş. Koçak, and A. E. Üreyen, Tube formulas for graph-directed fractals. *Fractals* **18** (2010), no. 3, 349–361. MR 2683935 Zbl 1206.28010

[3] A. Deniz, Ş. Koçak, Y. Özdemir, and A. E. Üreyen, Tube formulas for self-similar and graph-directed fractals. *Math. Intelligencer* **35** (2013), no. 3, 36–49. MR 3111970 Zbl 1283.28005

[4] A. Deniz, Ş. Koçak, Y. Özdemir, and A. E. Üreyen, Tube volumes via functional equations. *J. Geom.* **106** (2015), no. 1, 153–162. MR 3320885 Zbl 1318.28021

[5] G. Edgar, *Measure, topology and fractal geometry.* Second edition. Undergraduate Texts in Mathematics. Springer, New York, 2008. MR 2356043 Zbl 1152.28008

[6] R. A. Horn and C. R. Johnson, *Matrix analysis.* Cambridge University Press, Cambridge, 2013. MR 2978290 Zbl 1267.15001

[7] M. L. Lapidus and M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions.* Geometry and spectra of fractal strings. Springer Monographs in Mathematics. Springer, New York, 2006; second ed., 2012. MR 2245559 MR 2977849 (second ed.) Zbl 1119.28005 Zbl 1261.28011 (second ed.)

[8] M. L. Lapidus and E. P. J. Pearse, Tube formulas and complex dimensions of self-similar tilings. *Acta Appl. Math.* **112** (2010), no. 1, 91–136. MR 2684976 Zbl 1244.28013

[9] M. L. Lapidus, E. P. J. Pearse, and S. Winter, Pointwise tube formulas for fractal sprays and self-similar tilings with arbitrary generators. *Adv. Math.* **227** (2011), no. 4, 1349–1398. MR 2799798 Zbl 274.28016

[10] M. L. Lapidus and C. Pomerance, Counterexamples to the modified Weyl–Berry conjecture on fractal drums. *Math. Proc. Cambridge Philos. Soc.* **119** (1996), no. 1, 167–178. MR 1356166 Zbl 0858.58052

[11] M. Levitin and D. Vassiliev, Spectral asymptotics, renewal theorem, and the Berry conjecture for a class of fractals. *Proc. London Math. Soc.* (3) **72** (1996), no. 1, 188–214. MR 1357092 Zbl 0837.35105

[12] E. P. J. Pearse and S. Winter, Geometry of canonical self-similar tilings. *Rocky Mountain J. Math.* **42** (2012), no. 4, 1327–1357. MR 2981047 Zbl 1266.28006

[13] E. C. Titchmarsh, Introduction to the theory of Fourier integrals. Clarendon Press, Oxford, 1948.

Received August 18, 2015

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