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Attractors of iterated function systems with uncountably many maps and infinite sums of Cantor sets

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Abstract. We study the topological properties of attractors of iterated function systems (IFS) on the real line, consisting of affine maps of homogeneous contraction ratio. These maps define what we call a *second generation* IFS: they are uncountably many and the set of their fixed points is a Cantor set. We prove that when this latter either is the attractor of a finite, non–singular, hyperbolic, IFS (of *first generation*), or it possesses a particular *dissection* property, the attractor of the second generation IFS is the union of a finite number of closed intervals. We also prove a theorem that generalizes this result to certain infinite sums of compact sets, in the sense of Minkowski and under the Hausdorff metric.

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Contents

1. Introduction and discussion of the main results

This paper addresses a problem that belongs to the important study of the topology of fractals: to characterize the attractor of iterated function systems (IFS) that consist of an uncountable infinity of similarities in one dimension.

The recent surveys [7, 8] offer a good perspective of attractors of finite IFS. On the other hand, attractors of IFS with a countable infinity of maps have also been considered, *e.g.* in [9, 14, 16, 25, 28]. Much less studied is the case, discussed herein, when such infinity is uncountable, so that [16, 18, 20, 22, 23, 24, 26, 32] is an almost complete list of references.

Our result, somehow surprising in its simplicity – that, in a large family of cases, the attractor is a finite union of closed intervals, also describes certain infinite sums of Cantor sets. As such, it has relevance to the problem of Minkowski sums and more generally to the study of algebraic/Boolean operations with Cantor sets, a very partial list of related literature being [1, 2, 11, 15, 30, 31]. Further motivation to solve this problem is to control the support of IFS invariant measures that model multiple pursuit games [22] and fractal inverse problems [23].

1.1. Problem formulation. We consider $\mathcal{K}$, the set of non–empty compact subsets of a compact interval I in $\mathbb{R}$, endowed with the Hausdorff metric, defined in the standard way as follows. Let $d(x, y)$ be the Euclidean distance of two points x and y in $\mathbb{R}$, and let $d(x, A) = \min\{d(x, a), a \in A\}$ be the distance of the point x from the non–empty compact subset A of I. The Hausdorff distance d_H between two compact sets A and B in $\mathcal{K}$ is then defined as

$$d_H(A, B) = \max\{\max\{d(a, B), a \in A\}, \max\{d(b, A), b \in B\}\}. \tag{1.1}$$

Under this distance $\mathcal{K}$ is a compact space [33].

Let Ψ be a non-empty set of contractive transformations of a non-empty compact interval I in $\mathbb{R}$, such that there exists $r < 1$ for which every $\psi \in \Psi$ is r-Lipschitz. Also, let the operator $\mathbf{U}_\Psi$ be defined by [17, 5, 4]

$$\mathbf{U}_\Psi(A) = \overline{\bigcup_{\psi \in \Psi} \psi(A)} \tag{1.2}$$

for every $A \in \mathcal{K}$, where the bar denotes topological closure.

In this setting, $\mathbf{U}_\Psi$ is a contractive operator on $\mathcal{K}$ and $K = K_\Psi$ is the unique element of $\mathcal{K}$ that solves the equation

$$K = \mathbf{U}_\Psi(K). \tag{1.3}$$

Following Hutchinson and Barnsley–Demko K_Ψ is called the *invariant set*, or the *attractor* of the iterated functions system Ψ [17, 5], a construction that generalizes the first insights of Moran and Mandelbrot [29, 21]. For an IFS composed of finitely many affine maps, the attractor may take different topological forms, as diverse as an interval, a countable union of intervals, a Cantor set, a finite number of points (in the singular case of maps with zero contraction ratios).

As mentioned, IFS with a countable infinity of maps have been considered *e.g.* in [9, 14, 16, 25, 28]. Their study requires the introduction of the closure of the right hand side of eq. (1.2), at difference with the case of finitely many transformations.

In this paper we will study the topological properties of the attractor in a more general class of IFS, composed of an *uncountable* set of maps, but we will restrict ourselves in two ways. First, following Elton and Yan [13] we will consider *homogeneous* affine maps. Secondly, as in [22, 23, 24] these maps will be structured as a *second generation IFS*.

Precisely, we start from a *first generation IFS* Ψ, that we require to be *finite, hyperbolic* and *non–singular*. This is defined by the following conditions:

i) Ψ consists of a finite number M of real maps (*i.e.* maps from $\mathbb{R}$ to itself),

$$\Psi = \{\psi_i : i = 1, \dots, M\}; \tag{1.4}$$

ii) there exists a closed interval I such that $\psi_i(I) \subseteq I$ for any $i = 1, \dots, M$;

iii) every map ψ_i is C^2 on I;

iv) there exist constants δ and σ such that for any $i = 1, \dots, M$

$$0 < \sigma \le |\psi_i'(x)| \le \delta < 1, \tag{1.5}$$

for all x in I, and at least two maps in Ψ have different fixed points.

A paradigmatic example of this situation is offered by non–linear IFS generating real Julia sets (see *e.g.* [6, 10]).

Next, consider a new set of affine maps, of equal contraction ratio $0 < \alpha < 1$,

$$\phi(\beta; x) = \alpha(x - \beta) + \beta = \alpha x + (1 - \alpha)\beta, \tag{1.6}$$

where $\beta \in \mathbb{R}$ is the fixed point of $\phi(\beta; \cdot)$. A *second generation IFS* Φ consists of all maps of the form (1.6), whose fixed points β belong to the attractor K_Ψ of the first generation IFS Ψ:

$$\Phi = \{\phi(\beta; \cdot),\ \beta \in K_\Psi\}. \tag{1.7}$$

Note that $x \mapsto \phi(\beta, x)$ maps I into I for every $\beta \in K_\Psi$. We use again eq. (1.2) to define the operator $\mathbf{U}_\Phi$, replacing the set of maps Ψ by Φ. Since K_Ψ is a compact set, the closure at right hand side of eq. (1.2) is here redundant.

Let therefore K_Φ denote the fixed point of $\mathbf{U}_\Phi$: $K_\Phi = \mathbf{U}_\Phi(K_\Phi)$. This set is the attractor of the second generation IFS Φ derived from Ψ and α, the properties of which are the object of this paper.

1.2. Main results and their discussion. Our main result shows that K_Φ has a very simple structure, under very general conditions on the first–generation attractor K_Ψ upon which it is constructed.

Theorem 1.1. *For any finite, non–singular, hyperbolic IFS Ψ and for any $0<\alpha<1$ the attractor K_Φ of the second generation IFS Φ derived from Ψ and α consists of a finite union of closed intervals. The same is true when K_Ψ in definition* (1.7) *is replaced by a Cantor set K admitting a construction of uniformly lower bounded dissection.*

As stated, the theorem holds also for a class of Cantor sets, defined via constructions of uniformly lower bounded dissection: these latter will be described in the course of the paper.

The first part of this theorem has been conjectured in [24], Conjecture 1, for disconnected, affine IFS. In the same work, a weaker result was found in a specific case: namely, it was proven (Theorem 1 in [24]) that *when Ψ is a two–maps, disconnected, affine IFS* (but the proof can be extended to any finite number of maps) *the attractor K_Φ contains an interval.* Theorem 1.1 solves the problem completely and in wider generality, under the hypotheses above. The following consequences of this theorem are to be noted.

Firstly, it can be used in conjunction with a localization analysis of the set K_Φ. Formulae somehow simplify when the convex hull of K_Ψ is the interval $[-1, 1]$: by a suitable rescaling we can always put ourselves in this situation. Then, the following proposition was proven in [22, 23] (see also Lemma 1 in [24]).

Proposition 1.2. *Let* $\mathrm{Conv}(K_\Psi) = [-1, 1]$. *Then*

$$K_\Psi \subseteq K_\Phi \subseteq [-\alpha, \alpha] + (1-\alpha)K_\Psi \subseteq B_{2\alpha}(K_\Psi),$$

where $B_{2\alpha}(K_\Psi)$ is the closed 2α–neighborhood of K_Ψ. The last two sets in the chain of inclusions consist of a finite number of closed intervals.

We shall give a different proof of Proposition 1.2 later on, in Section 2. A first consequence of it is that in the limit case of $\alpha = 0$ K_Φ is equal to K_Ψ. Also observe that this proposition yields in a rather simple way a cover of K_Φ by intervals. The difficult step achieved by Theorem 1.1 is to prove that intervals are contained in K_Φ and indeed that it exactly consists of a finite number of them.

Finally, Theorem 1.1 implies that algorithm A2 in [24] terminates in a finite number of steps, hence it provides an efficient means of computation of the intervals composing the set K_Φ.

Remark 1.3. Observe that when B is a countable dense subset of the compact set K_Ψ and when $\widetilde{\Phi} = \{\phi(b;\cdot),\ b \in B\}$, the induced operator is the same as that of the complete IFS: $U_{\widetilde{\Phi}} = U_\Phi$. Hence, one can construct a countable IFS with the same attractor as the uncountable one – however, closure in eq. (1.2) becomes essential. This is a particular case of the general Proposition 9 in [20].

While the previous remark might seem to downplay the importance of uncountable IFS, it must be observed that their distinctive role is fully appreciated when considering *balanced measures* on their supports, as done in [13, 26]. In the case of the second generation IFS considered in this paper these measures have been studied in [22, 23]: they are always of pure type and can be either absolutely continuous or singular continuous with respect to the Lebesgue measure. Discriminating between the two cases appears to be an interesting and challenging problem [22]. Theorem 1.1 helps in this endeavor by settling the problem on the nature of the support of these measures.

From a more general perspective, the results of this paper also belong to the study of the topological properties of sums of Cantor sets. A key ingredient of our proof is a result (Theorem 7.1 below) by Cabrelli *et al.* [11] on finite sums of Cantor sets. Recall the notion of Minkowski sum of two non–empty sets A and B in $\mathbb{R}$ (see [33]): this sum is the set

$$A + B = \{a + b,\ a \in A, b \in B\}. \tag{1.8}$$

In this context, we first prove a result on sums of Cantor sets of the kind $K_n = \gamma_1 K + \cdots + \gamma_n K$, Proposition 8.1, where K is a Cantor (see below for hypotheses) and γ_j are arbitrary positive numbers.

Next, we show that the attractor K_Φ defined above and characterized via Theorem 1.1 can be written as a geometric series of Cantor sets (see eq. (25) in [24] and Section 9 below):

$$K_\Phi = (1-\alpha) \sum_{j=0}^{\infty} \alpha^j K_\Psi. \tag{1.9}$$

In the above, convergence of the series is to be understood in the sense of Hausdorff metric. We can study more general series than (1.9): this results in the following theorem.

Theorem 1.4. *Let $K = K_\Psi$ be the attractor of a finite, non–singular hyperbolic IFS Ψ, or a Cantor set K admitting a construction of uniformly lower bounded dissection. Let $\sum_j \alpha_j < \infty$ be a convergent series of positive real entries. Then, the infinite series*

$$\sum_{j=0}^{\infty} \alpha_j K \tag{1.10}$$

is convergent in the Hausdorff metric to a non–empty compact set, which is a finite union of closed, disjoint intervals.

Clearly, because of eq. (1.9), Theorem 1.4 is a generalization of Theorem 1.1.

1.3. Organization of the paper. In the next section we review some basic properties of IFS and their attractors. These properties are well known (see *e.g.* Section 3 in [17]) and we reproduce them here solely for convenience and as a way to introduce notations. Partially new is the proof of Proposition 1.2, presented at the end of this section. The successive Section 3 explains a standard way to describe Cantor sets in the real line. We mainly follow Section 2 in [11] and we extend it by proving a few results needed in the remainder of the paper. The fundamental property of *uniformly lower bounded dissection* (*ulbd*) of Cantor sets is also defined in this section. In Section 4 we prove that this property holds for Cantor sets generated as attractors of two–maps, non–singular, hyperbolic IFS. We then derive some useful lemmas on the relation of ulbd property with certain operations on sets: Section 5 contains an explicit construction by which it is proven that the union of two separated, ulbd Cantor sets is also ulbd, while Section 6 proves that a ulbd Cantor set with prescribed properties can be found in the sum of a finite collection of ulbd Cantor sets.

We then move to the core of the problem: in Section 7 we recall Cabrelli *et al.* result on finite sums of ulbd Cantor sets [11], to which we add two useful Lemmas. This leads us to Section 8 where we prove a proposition about the finite truncations of the series in eqs. (1.9) and (1.10) and we prove Theorem 1.1. The theory of infinite series of compact sets of the kind (1.10), in the sense of Minkowski and under the Hausdorff metric, is briefly developed in the final Section 9, where we prove Theorem 1.4.

2. Basic properties of IFS maps and attractors

In this section we let Ψ be a set of contractive transformations, more precisely of r-Lipschitz transformations on a compact interval I, where r is a suitable number with $0 < r < 1$. Such a set Ψ, according to standard terminology, is called a *hyperbolic IFS*. When the cardinality of Ψ is finite we will use the notation $\Psi := \{\psi_i : i = 1, \dots, M\}$ and assume there exist at least two maps with different fixed points. Then, $m_\Psi = \min K_\Psi$ is strictly smaller than $M_\Psi = \max K_\Psi$. For all $n \geq 1$ let Ψ^n be the IFS consisting of the n-fold composition of the maps in Ψ:

$$\Psi^n := \{\psi_{i_1,\dots,i_n} : i_1, \dots, i_n = 1, \dots, M\}, \quad \psi_{i_1,\dots,i_n} := \psi_{i_1} \circ \dots \circ \psi_{i_n}. \tag{2.1}$$

Clearly, $\mathbf{U}_\Psi^n = \mathbf{U}_{\Psi^n}$ on $\mathcal{K}$. Acting with $\mathbf{U}_\Psi$ n times in eq. (1.3) yields $K_\Psi = \mathbf{U}_\Psi^n(K_\Psi) = \mathbf{U}_{\Psi^n}(K_\Psi)$ and this implies that, for all $n \geq 1$,

$$K_{\Psi^n} = K_\Psi. \tag{2.2}$$

Lemma 2.1. *If Ψ and Ψ' are hyperbolic I.F.S on a compact interval I, the following hold*:

i) *if $A \in \mathcal{K}$ and $\mathbf{U}_\Psi(A) \supseteq A$, then $K_\Psi \supseteq A$;*

ii) *if $B \in \mathcal{K}$ and $\mathbf{U}_\Psi(B) \subseteq B$, then $K_\Psi \subseteq B$;*

iii) *if $\Psi \subseteq \Psi'$ then $K_\Psi \subseteq K_{\Psi'}$.*

Proof. i) The set $X := \{D \in \mathcal{K} : \mathbf{U}_\Psi(D) \supseteq A\}$ is a closed nonempty subset of $\mathcal{K}$ (X is nonempty since $A \in X$ by the hypothesis), thus it is a complete metric space with respect to the Hausdorff metric. The map $\mathbf{U}_\Psi$ is a contraction from X into itself (since $D \in X$ implies $\mathbf{U}_\Psi^2(D) \supseteq \mathbf{U}_\Psi(A) \supseteq A$), thus it has a fixed point C: $\mathbf{U}_\Psi(C) = C$. Because of uniqueness, this latter is the same as the IFS attractor: $C = K_\Psi$. Moreover, since $K_\Psi = C \in X$, then $K_\Psi = \mathbf{U}_\Psi(K_\Psi) \supseteq A$.

ii) Same as i), with $X := \{D \in \mathcal{K} : \mathbf{U}_\Psi(D) \subseteq B\}$.

iii) We have $\mathbf{U}_{\Psi'}(K_\Psi) \supseteq \mathbf{U}_\Psi(K_\Psi) = K_\Psi$. Now, iii) follows from i) with $A = K_\Psi$. □

Remark 2.2. The above lemma is folklore: we included its proof for completeness. Indeed, it is a consequence of order/monotonicity properties of Ψ. A thorough analysis of its implications can be found in [19].

Remark 2.3. The previous lemma can be used to construct monotonic sequences of compact sets converging to the attractor K_Ψ. Take A as in i) and define $A_n = \mathbf{U}_\Psi^n(A)$, B as in ii) and $B_n = \mathbf{U}_\Psi^n(B)$. Then,

$$A \subseteq \dots \subseteq A_n \subseteq A_{n+1} \dots \subseteq K_\Psi \subseteq \dots \subseteq B_{n+1} \subseteq B_n \subseteq \dots \subseteq B.$$

The ascending sequence has been called the construction from the inside; the descending, construction from the outside; A the nucleus and B the absorbing set [3, 7]. Typical choices for A and B are a finite set of fixed points, and $[m_\Psi, M_\Psi]$, the convex hull of K_Ψ, respectively. (Notice that an alternative proof of i) and ii) can be obtained via the previous equation.)

A corollary of this result is the following

Lemma 2.4. *Let $\Psi := \{\psi_i : i = 1, \ldots, M\}$ be a hyperbolic IFS on a compact interval I. Let β be the fixed point of $\psi_{i_1,\ldots,i_n}$, with $i_1, \ldots, i_n = 1, \ldots, M$. Then $\beta \in K_\Psi$. Also $\psi_{i'_1,\ldots,i'_{k'}}(\beta) \in K_\Psi$ for all $i'_1, \ldots, i'_{k'} = 1, \ldots, M$.*

Proof. By definition, $\mathbf{U}_{\Psi^n}(\{\beta\}) = \mathbf{U}^n_\Psi(\{\beta\}) \supseteq \psi_{i_1,\ldots,i_n}(\{\beta\}) = \{\beta\}$. Thus, by Lemma 2.1 i) $K_\Psi = K_{\Psi^n} \supseteq \{\beta\}$. Next,

$$\psi_{i'_1,\ldots,i'_{k'}}(\beta) \in \psi_{i'_1,\ldots,i'_{k'}}(K_\Psi) \subseteq U_{\Psi^{k'}}(K_\Psi) = K_\Psi. \qquad \square$$

Observe that the above also holds for infinite cardinality IFS.

Lemma 2.5. *Suppose that the hyperbolic IFS Ψ is composed of finitely many maps. Then, there exists $\Psi' \subseteq \Psi$ such that Ψ' has precisely two elements and $m_{\Psi'} = m_\Psi$, $M_{\Psi'} = M_\Psi$.*

Proof. Firstly, $m_\Psi \in K_\Psi = \mathbf{U}_\Psi(K_\Psi)$. There exists $\psi_1 \in \Psi$ such that

$$m_\Psi \in \psi_1(K_\Psi).$$

Thus, there exists $x \in K_\Psi$ such that $m_\Psi = \psi_1(x)$. We have either $x = m_\Psi$ or $x = M_\Psi$. In fact, in the opposite case there exist $x_1, x_2 \in K_\Psi$ such that $x_1 < x < x_2$. Thus $\psi_1(x_1), \psi_1(x_2) \in \psi_1(K_\Psi) \subseteq \mathbf{U}_\Psi(K_\Psi) = K_\Psi$ and one of the numbers $\psi_1(x_1)$, $\psi_1(x_2)$ is less than m_Ψ, a contradiction. Similarly, we can prove that there exist $\psi_2 \in \Psi$ and $y \in \{m_\Psi, M_\Psi\}$ such that $\psi_2(y) = M_\Psi$. Let $\Psi' := \{\psi_1, \psi_2\}$. We claim that Ψ' satisfies the Lemma. Note that, since $K_{\Psi'} \subseteq K_\Psi \subseteq [m_\Psi, M_\Psi]$, it suffices to prove that

$$m_\Psi, M_\Psi \in K_{\Psi'} \tag{2.3}$$

We distinguish four different cases.

Case 1. $x = m_\Psi$, $y = M_\Psi$. Here, m_Ψ is the fixed point of ψ_1 and M_Ψ is the fixed point of ψ_2. Now, eq. (2.3) follows from Lemma 2.4.

Case 2. $x = y = m_\Psi$. Then m_Ψ is the fixed point of ψ_1, and $M_\Psi = \psi_2(m_\Psi)$, and eq. (2.3) follows again from Lemma 2.4.

CASE 3. $x = y = M_\Psi$. We proceed as in Case 2.

CASE 4. $x = M_\Psi$, $y = m_\Psi$. We have $\psi_1 \circ \psi_2(m_\Psi) = m_\Psi$, $\psi_2 \circ \psi_1(M_\Psi) = M_\Psi$, and eq. (2.3) follows again from Lemma 2.4. □

To complete this section we prove the localization result described in the Introduction.

Proof of Proposition 1.2. Since $U_\Phi(K_\Psi) \supseteq K_\Psi$ by the definition of Φ, the first inclusion follows from by Lemma 2.1, i). Next, observe that Conv K_Φ = Conv K_Ψ: the maps $\phi(\beta; x)$ in eq. (1.6) are contractive, the set of their fixed points is K_Ψ and since α is positive $\mathbf{U}_\Phi(\text{Conv } K_\Psi) \subseteq \text{Conv } K_\Psi$ which implies that $K_\Phi \subseteq \text{Conv } K_\Psi$ so that Conv $K_\Phi \subseteq$ Conv K_Ψ. On the other hand, since $K_\Psi \subseteq K_\Phi$ the reverse inclusion also holds. By hypothesis we so have that Conv K_Φ = Conv $K_\Psi = [-1, 1]$. Write explicitly $K_\Phi = \mathbf{U}_\Phi K_\Phi$ using the Minkowski sum of sets in the form

$$\begin{aligned} K_\Phi &= \bigcup_{\beta \in K_\Psi} [\alpha K_\Phi + (1-\alpha)\beta] \\ &= \alpha K_\Phi + (1-\alpha)K_\Psi \subseteq [-\alpha, \alpha] + (1-\alpha)K_\Psi \\ &:= \widehat{K}_\Phi, \end{aligned} \tag{2.4}$$

where the set $\widehat{K}_\Phi$ defined in the above is a cover of K_Φ. Observe that it can also be written as a union of intervals of fixed length 2α

$$\widehat{K}_\Phi = \bigcup_{\beta \in K_\Psi} [(1-\alpha)\beta - \alpha, (1-\alpha)\beta + \alpha].$$

It follows that $\widehat{K}_\Phi$ is closed and included in $[-1, 1]$, hence compact. In addition, since the intervals in the union have fixed length 2α it can be written as a finite union of (different) intervals, each of length larger than, or equal to, 2α.

Furthermore, let $\beta \in K_\Psi$, so that also $|\beta| \leq 1$, which shows that

$$[-\alpha + (1-\alpha)\beta, \alpha + (1-\alpha)\beta] \subseteq [\beta - 2\alpha, \beta + 2\alpha].$$

Taking the union over all $\beta \in K_\Psi$ proves the third inclusion of the thesis. The same argument used above shows that also this last set, the closed 2α neighborhood of K_Ψ, consists of a finite number of closed intervals. □

Remark 2.6. Suppose that

$$d(x, (1-\alpha)K_\Psi) > \alpha + \epsilon. \tag{2.5}$$

It follows from the above proof that $\overline{B_\epsilon}(x) \bigcap K_\Phi = \varnothing$, so that the closed ball of radius ϵ at x is contained in the complement of K_Φ. The set N_ϵ introduced in [24] (that we here write for the case when $\mathrm{Conv}(K_\Psi) = [-1, 1]$):

$$N_\epsilon = \{x \in [-1,1] \colon [x-\alpha-\epsilon, x+\alpha+\epsilon] \cap (1-\alpha)K_\Psi = \varnothing\} \tag{2.6}$$

is the collection of the centers of these balls, or equivalently of the points $x \in [-1, 1]$ which satisfy the inequality (2.5). For any $\epsilon \geq 0$, N_ϵ is a finite collection of open intervals contained in the complement of K_Φ. When $\epsilon = 0$ N_ϵ is the complement of $\widehat{K}_\Phi$ in $[-1, 1]$.

3. Construction of Cantor sets via dissections

Recall that a Cantor set is a compact, totally disconnected, nonempty subset of $\mathbb{R}$, with no isolated points. Iterated function systems yield a convenient construction of families of Cantor sets on the real line. We now use a different description, of general scope, that has been employed also in [11]. The Hausdorff dimension of sets constructed in this fashion has been studied in [12]. The main idea behind this construction is that the complement of a real Cantor set is a countable union of open intervals. How to organize this countable set is the core of the description, which requires symbolic coding, as follows.

Let W be the set of finite binary words, with $\varnothing$ being the empty word:

$$W := \varnothing \cup \{w_1, \ldots, w_r \colon r = 1, 2, 3, \ldots, w_i \in \{0, 1\}\}.$$

Define the wordlength function $|\cdot|$ via $|\varnothing| = 0$, $|w_1, \ldots, w_r| = r$. If $w, w' \in W$, let ww' be the concatenation of w and w', $w\varnothing = \varnothing w = w$. Let us now associate a closed interval on the real line to each word in W: that is to say, we define a map $\mathcal{I} \colon W \to \{\text{closed intervals } \subset \mathbb{R}\}$, such that

$$\mathcal{I} \colon w \longrightarrow I_w := [a_w, b_w], \tag{3.1}$$

with a_w and b_w denoting the end points of I_w (clearly, we always require $a_w < b_w$). This map is defined iteratively. The initial seed is an arbitrary interval $I = [a, b]$ that is associated to the empty word. That is, $I_\varnothing = I$, $a_\varnothing = a$, $b_\varnothing = b$. The iteration rule is then the following: given $I_w = [a_w, b_w]$ with $a_w < b_w$, choose two points c_w and d_w, so that $a_w < c_w < d_w < b_w$ holds with strict inequalities and define the new intervals $I_{w0} = [a_{w0}, b_{w0}]$ and $I_{w1} = [a_{w1}, b_{w1}]$ via

$$a_{w0} = a_w, b_{w0} = c_w, a_{w1} = d_w, b_{w1} = b_w.$$

In simpler terms, the interval corresponding to a word w, of length $|w|$ generates two intervals, corresponding to the words $w0$ and $w1$ of length $|w| + 1$. It is convenient to define the *ratios of dissection* $r(\mathfrak{I})_{w0}$ and $r(\mathfrak{I})_{w1}$ associated to these intervals, as

$$r(\mathfrak{I})_{wj} = \frac{b_{wj} - a_{wj}}{b_w - a_w} = \frac{d(I_{wj})}{d(I_w)}, \quad j = 0, 1 \tag{3.2}$$

where here and in the following we use the notation $d(A) = \text{diam}\,(A)$ for the diameter of the set A. Clearly, $d(A) = \max A - \min A$, for every non–empty compact subset A of $\mathbb{R}$. Note that, since $d(I_{w0}) + d(I_{w1}) < d(I_w)$, we have

$$r(\mathfrak{I})_{w0} + r(\mathfrak{I})_{w1} < 1. \tag{3.3}$$

The ratio of dissection $r(\mathfrak{I})_v$ is so defined for any word $v \in W \backslash \{\varnothing\}$ and it measures the ratio of the diameters of I_v and of its immediate ancestor $I_{v'}$ (associated with the word v', obtained from v by deleting the last binary digit).

To complete the construction, take the union of the intervals I_w of fixed length $|w|$, and then intersect these latter sets:

$$C_n(\mathfrak{I}) := \bigcup_{|w|=n} I_w, \quad C(\mathfrak{I}) := \bigcap_{n=0}^{\infty} C_n(\mathfrak{I}). \tag{3.4}$$

Note that $I_{wj} \subseteq I_w$, thus in particular $C_{r+1}(\mathfrak{I}) \subseteq C_r(\mathfrak{I})$ and $C(\mathfrak{I})$ is not empty. We say that $C(\mathfrak{I})$ *is the quasi-Cantor set constructed on* $\mathfrak{I}$, or that $\mathfrak{I}$ *constructs* $C(\mathfrak{I})$.

Consider the following specific case: for $j = 0, 1$, let j^n denote the word composed of the letter j repeated n times, with $n \in \mathbb{N}$, where $j^0 = \varnothing$. These words are labels of the extreme intervals in $C_n(\mathfrak{I})$, so that

$$\min I_{0^n} = \min I, \qquad \max I_{1^n} = \max I,$$

and

$$\min C(\mathfrak{I}) = \min I, \quad \max C(\mathfrak{I}) = \max I.$$

The correspondence $\mathfrak{I} \to C(\mathfrak{I})$ is not one-to-one: the same Cantor set may be constructed on different $\mathfrak{I}$'s. This is clearly seen by considering the complement of a quasi-Cantor set $C(\mathfrak{I})$: define the *gaps* of $C(\mathfrak{I})$ as the bounded connected components of the complement of $C(\mathfrak{I})$. Their countable union is precisely $I_\varnothing \setminus C(\mathfrak{I})$.

Lemma 3.1. *The gaps of $C(\mathcal{I})$ are the sets*

$$I_w \setminus (I_{w0} \cup I_{w1})\colon w \in W. \tag{3.5}$$

Proof. The sets in (6) are clearly bounded components of the complement of $C(\mathcal{I})$. Conversely, suppose A is a bounded component of the complement of $C(\mathcal{I})$ and take $x \in A$. Then $x \in I_\varnothing$. In fact, in the opposite case, either $x < a$ or $x > b$. In the former case $x \in]-\infty, a[$, in the latter $x \in]b, +\infty[$, so that A being the connected component of x contains either $]-\infty, a[$ or $]b, +\infty[$, thus is unbounded, a contradiction. Now recall that $I_\varnothing = C_0(\mathcal{I})$, so that $x \in C_0(\mathcal{I}) \setminus (\bigcap_{r=0}^{+\infty} C_r(\mathcal{I}))$ and there exists $r \in \mathbb{N}$ such that $x \in C_r(\mathcal{I}) \setminus C_{r+1}(\mathcal{I})$, and also $w \in W$ such that $|w| = r$ and $x \in I_w \setminus (I_{w0} \cup I_{w1})$. Therefore, $A = I_w \setminus (I_{w0} \cup I_{w1})$. □

Note that $I_w \setminus (I_{w0} \cup I_{w1}) =]c_w, d_w[$. Therefore, the above construction of $C(\mathcal{I})$, defined by $\mathcal{I}$, can also be seen as a construction of its complementary in $[a, b]$, described by a function $\mathcal{G}$ from W to the set of open intervals. Keeping fixed the image of this map, *i.e.* the gaps, any map $\mathcal{G}$, that respects a simple prescription (gaps appear in interlacing sequence) yields the same Cantor set. We will use this freedom later in the paper. We will also use a specific symbol for the diameter of the gaps:

$$\gamma(\mathcal{I})_w := d_w - c_w = \min I_{w1} - \max I_{w0}, \quad \gamma(C) = \sup_{w \in W} \gamma(\mathcal{I})_w. \tag{3.6}$$

We now give a condition for $C(\mathcal{I})$ being a Cantor set. Let us start with a symbolic coding of all points in $C(\mathcal{I})$. Denote by $\widetilde{W}$ the set of infinite strings of 0 and 1, *i.e.* $\widetilde{W} = \{0, 1\}^{\mathbb{N} \setminus \{0\}}$, and for $\tilde{w} \in \widetilde{W}$, write $\tilde{w} = i_1 i_2 i_3 \ldots$. Also, let $\tilde{w}_n$ be the finite string of length n obtained by truncation of $\tilde{w}$: $\tilde{w}_n = i_1 \ldots i_n$. With this vocabulary, a point $x \in \mathbb{R}$ belongs to $C(\mathcal{I})$ if and only if there exists $\tilde{w} \in \widetilde{W}$ such that $x \in \bigcap_{n=1}^{\infty} I_{\tilde{w}_n}$. Indeed, when the set is Cantor, this intersection is the singleton $\{x\}$, as the following standard lemma shows:

Lemma 3.2. *The set $C(\mathcal{I})$ is a Cantor set if and only if for every $\tilde{w} \in \widetilde{W}$*

$$d(I_{\tilde{w}_n}) \xrightarrow[n \to +\infty]{} 0.$$

Proof. In fact, being the sets $I_{\tilde{w}_n}$ serially enclosed, the sequence of their diameters is monotonic, and if for a certain $\tilde{w} \in \widetilde{W}$ it does not converge to zero, then it has a strictly positive limit: $d(I_{\tilde{w}_n}) \xrightarrow[n \to +\infty]{} c > 0$. In this case, $C(\mathcal{I}) \supseteq \bigcap_{n=1}^{\infty} I_{\tilde{w}_n} \supseteq [\alpha, \beta]$ with $\alpha < \beta$. Thus $C(\mathcal{I})$ is not totally disconnected.

Conversely, suppose that for every $\widetilde{w} \in \widetilde{W}$ we have $d(I_{\widetilde{w}_n}) \xrightarrow[n\to+\infty]{} 0$. Let $x \in C(\mathcal{I})$ and take $\widetilde{w} \in \widetilde{W}$ such that $x \in \bigcap_{n=1}^{\infty} I_{\widetilde{w}_n}$. Put $I_{\widetilde{w}_n} = [a_n, b_n]$. Since $a_n \leq x \leq b_n$ and $b_n - a_n \xrightarrow[n\to+\infty]{} 0$, for every U neighborhood of x there exists n such that $[a_n, b_n] \subseteq U$. If $\widetilde{w}' \in \widetilde{W}$ and $\widetilde{w}'_n = \widetilde{w}_n$ but $\widetilde{w}' \neq \widetilde{w}$ and $x' \in \bigcap_{n=1}^{\infty} I_{\widetilde{w}'_n}$, then $x' \in C(\mathcal{I}) \cap [a_n, b_n]$, thus $x' \in U$, but $x' \neq x$, so that $C(\mathcal{I})$ has no isolated points. Also, since $C_n(\mathcal{I})$ is the *disjoint* union of closed intervals including $[a_n, b_n]$, then the component of x in $C(\mathcal{I})$ is contained in $[a_n, b_n]$, for every n, thus in $\bigcap_{n=1}^{\infty} I_{\widetilde{w}_n}$, which, since $b_n - a_n \xrightarrow[n\to+\infty]{} 0$ amounts to $\{x\}$, hence $C(\mathcal{I})$ is totally disconnected. □

We can use the previous lemma in conjunction with the following:

Lemma 3.3. *A sufficient condition for $C(\mathcal{I})$ being a Cantor set is that there exists a positive constant a, such that all dissection ratios are larger than, or equal to a.*

Proof. Equation (3.3) implies that $r(\mathcal{I})_{wj} < 1 - r(\mathcal{I})_{w(1-j)} \leq 1 - a < 1$. Hence, since $d(I_{wj}) = r(\mathcal{I})_{wj} d(I_w) \leq (1-a) d(I_w)$, it follows that for every $\widetilde{w} \in \widetilde{W}$ $d(I_{\widetilde{w}_n}) \leq (1-a)^n d(I)$, thus $d(I_{\widetilde{w}_n}) \xrightarrow[n\to+\infty]{} 0$. □

Clearly, this condition is not so much intended to exclude that dissection ratios get too small – which could still be compatible with having a Cantor set, and hence the condition is not necessary – rather, because of the inequality (3.3) it implies that dissection ratios cannot tend to one.

Given the role that this condition will play in the following, we find it convenient to embody it into a formal definition:

Definition 3.4. A construction $\mathcal{I}$ that satisfies the condition in Lemma 3.3 will be defined to be of *uniformly lower bounded dissection* (*ulbd*), and a Cantor set admitting one such construction will also be said to possess the ulbd property.

Remark 3.5. Note that the proof of Lemma 3.3 shows that if $\mathcal{I}$ is ulbd, then $d(I_w) \leq (1-a)^n d(I)$ whenever $w \in W$, $|w| = n$, and a is as given in Lemma 3.3.

We end this section by defining a further element in the algebra of quasi-Cantor sets. Observe that each interval in the above construction can be thought of as the starting interval in the construction of a Cantor set, subset of the former. In fact, let the mapping $\mathcal{I}$ be fixed, and let us focus on a finite word $w \in W$ and on its associated interval $I_w = \mathcal{I}(w)$. Define a new mapping $\mathcal{I}_w$ by the formula (compare with eq. (3.1)):

$$\mathcal{I}_w : w' \to \mathcal{I}_w(w') := \mathcal{I}(ww') = I_{ww'}. \tag{3.7}$$

Denote by $C(\mathcal{J}_w)$ the quasi Cantor set constructed in the set I_w by this mapping. Clearly

$$C(\mathcal{J}_w) = I_w \cap C(\mathcal{J}). \tag{3.8}$$

Moreover, the set of gaps of $\mathcal{J}_w$ is contained in the set of gaps of $\mathcal{J}$, and the set of ratios of $\mathcal{J}_w$ is contained in the set of ratios of $\mathcal{J}$. Namely, $\gamma(\mathcal{J}_w)_{w'} = \gamma(\mathcal{J})_{ww'}$ and $r(\mathcal{J}_w)_{w'} = r(\mathcal{J})_{ww'}$. Of course, it also holds true that

$$\max C(\mathcal{J}_w) = \max I_w, \quad \min C(\mathcal{J}_w) = \min I_w.$$

4. Ulbd property of IFS attractors

In this section we prove that Cantor sets constructed via IFS with two maps satisfying (1.5) possess the uniformly lower bounded dissection property. Prior to that, we need a technical lemma, that will also be useful elsewhere.

Let V be the set of finite words in the M letters $\{1, 2, \ldots, M\}$ (*i.e.*, the labels of the IFS maps), that obviously also include the empty word. This is a trivial generalization of the two letters case. For each $v \in V$, let $\psi_v = \psi_{v_1} \circ \cdots \circ \psi_{v_n}$, as in eq. (2.1).

Lemma 4.1. *Let Ψ be a set of real maps, as in eq.* (1.4)*, that satisfy conditions* i–iv *in Section* 1 *on an interval $I \subset \mathbb{R}$. Then, there exists $c > 0$ such that, for any interval $J \subseteq I$, any $v \in V$, any $i = 1, \ldots, M$, we have*

$$r_{vi} := \frac{d(\psi_{vi}(J))}{d(\psi_v(J))} \geq c. \tag{4.1}$$

Proof. We need to estimate the ratios r_{vi}. Let us first consider the numerator in eq. (4.1): this is the length of an interval, that can be evaluated as

$$d(\psi_{vi}(J)) = |\psi'_{vi}(\eta)| d(J),$$

where η is a point in J. Similarly, $d(\psi_v(J)) = |\psi'_v(\zeta)| d(J)$, $\zeta \in J$. Let n be the length of v. The chain rule for the derivative of these composed functions leads us to define two sequences of points η_k, ζ_k, for $k = 1, \ldots, n$, as follows:

$$\eta_k = (\psi_{v_{k+1}} \circ \cdots \circ \psi_{v_n} \circ \psi_i)(\eta), \quad k = 1, \ldots, n-1,$$
$$\zeta_k = (\psi_{v_{k+1}} \circ \cdots \circ \psi_{v_n})(\zeta), \qquad k = 1, \ldots, n-1,$$

and $\eta_n = \psi_i(\eta)$, $\zeta_n = \zeta$. With these notations, the derivative of the composed function can be factored as

$$\psi'_{vi}(\eta) = \psi'_{v_1}(\eta_1) \cdot \cdots \cdot \psi'_{v_n}(\eta_n) \cdot \psi'_i(\eta),$$
$$\psi'_v(\zeta) = \psi'_{v_1}(\zeta_1) \cdot \cdots \cdot \psi'_{v_n}(\zeta_n).$$

Because of contractivity of the maps, the points η_k, ζ_k approach each other geometrically, when n grows. In fact, $\eta_n, \zeta_n \in I$ and

$$\eta_k, \zeta_k \in (\psi_{v_{k+1}} \circ \cdots \circ \psi_{v_n})(I), \quad k = 1, \ldots, n-1.$$

Using eq. (1.5) we obtain

$$d((\psi_{v_{k+1}} \circ \cdots \circ \psi_{v_n})(I)) \leq \delta^{n-k} d(I), \quad k = 1, \ldots, n-1.$$

The above information permits to compute the logarithm of the inverse of the ratio r_{vi}: we call it l_{vi} and we show that it is bounded from above. In fact,

$$l_{vi} := -\log(r_{vi}) = -\log(|\psi'_i(\eta)|) + \sum_{k=1}^{n} \log(|\psi'_{v_k}(\zeta_k)|) - \log(|\psi'_{v_k}(\eta_k)|). \quad (4.2)$$

Therefore,

$$l_{vi} \leq \log(1/\sigma) + \sum_{k=1}^{n} |\log(|\psi'_{v_k}(\zeta_k)|) - \log(|\psi'_{v_k}(\eta_k)|)|. \quad (4.3)$$

Consider now the functions $g_i(x) = \log(|\psi'_i(x)|)$, where $i = 1, \ldots, M$. Because of eq. (1.5) $g_i(x)$ is differentiable and

$$|g'_i(x)| = \frac{|\psi''_i(x)|}{|\psi'_i(x)|},$$

so that each term in the summation at right hand side of eq. (4.3) can be estimated as

$$|\log(|\psi'_{v_k}(\zeta_k)|) - \log(|\psi'_{v_k}(\eta_k)|)| = \frac{|\psi''_{v_k}(\theta_k)|}{|\psi'_{v_k}(\theta_k)|} |\zeta_k - \eta_k| \leq \frac{B}{\sigma} \delta^{n-k} d(I), \quad (4.4)$$

with θ_k an intermediate point between η_k and ζ_k and where B is the maximum of the absolute value of the second derivative of all ψ_i's over I. In conclusion, we have

$$l_{vi} \leq \log(1/\sigma) + \frac{B}{\sigma} d(I) \sum_{k=1}^{n} \delta^{n-k} \leq \log(1/\sigma) + \frac{B}{\sigma(1-\delta)} d(I). \quad (4.5)$$

The term at right hand side is a finite quantity C, independent of w and i, and this proves the lemma: $r_{vi} \geq e^{-C}$ for all $v \in V$, $i = 1, \ldots, M$. □

The second lemma of this section is now the following.

Lemma 4.2. *Let $\Psi = \{\psi_0, \psi_1\}$ be a set of two real maps, that satisfy conditions* i–iv *in Section* 1 *on an interval $I \subset \mathbb{R}$. Then, the attractor K_Ψ is either an interval, or a Cantor set that admits a construction $\mathfrak{I}$ that is of ulbd.*

Proof. Remark that the attractor of a two–maps IFS with different fixed points is either a full interval or a Cantor set, as it is easy to see. In fact, let $I_\varnothing = \mathrm{Conv}(K_\Psi) = [a, b]$, the convex hull of K_Ψ. Let now $J_i = \psi_i([a, b])$, $i = 0, 1$. The extreme point a must belong to one of these two intervals, and b to the other: in fact, they belong to K_Ψ and therefore also to $U_\Psi(K_\Psi)$. If these two intervals are not disjoint, we have that $U_\Psi([a, b]) = [a, b]$ and therefore the attractor is the full convex hull, $K_\Psi = [a, b]$. In the opposite case, J_0 is either strictly to the left of J_1, or to its right. In the first case we assign a permutation g of $\{0, 1\}$, such that $g(0) = 0$ and $g(1) = 1$ (the identity). In the second case we invert indices: $g(0) = 1$, $g(1) = 0$, so that in both cases we define $I_i = [a_i, b_i] = \psi_{g(i)}(I_\varnothing)$ and we have $a_0 = a$, $b_1 = b$. Disconnectedness of the two intervals imply that $b_0 < a_1$, thereby completing the first step in the construction of the Cantor set.

We then proceed by induction: consider the words $w \in W$ of length $n-1$, the maps ψ_w (as in eq. (2.1)) and the permutation g of the set of $n-1$ letter words that defines the lexicographically ordered intervals $I_w = [a_w, b_w] = \psi_{g(w)}(I_\varnothing)$. Define the intervals $J_{wi} = (\psi_{g(w)} \circ \psi_i)(I_\varnothing)$, for $i = 0, 1$. Clearly, $J_{wi} \subset I_w$, and these two intervals are disjoint. Extend the permutation g to the set of n-letter words as follows: $g(w0) = g(w)0$, $g(w1) = g(w)1$ if J_{w0} is to the left of J_{w1}, or $g(w0) = g(w)1$, $g(w1) = g(w)0$ if otherwise. This implies that $I_{wi} = [a_{wi}, b_{wi}] = \psi_{g(wi)}(I_\varnothing)$, and $a_w = a_{w0} < b_{w0} < a_{w1} < b_{w1} = b_w$. This proves that the map $\mathfrak{I}$ so defined yields a Cantor set.

Let us now prove that this construction is of *ulbd*. In fact, contraction ratios are defined by eq. (3.2): in this case, they are given by

$$r(\mathfrak{I})_{wj} = \frac{d(I_{wj})}{d(I_w)} = \frac{d(\psi_{g(wj)}(I_\varnothing))}{d(\psi_{g(w)}(I_\varnothing))}, \tag{4.6}$$

with $j = 0, 1$. Since $g(wj) = g(w)h_w(j)$, where h_w is a permutation of a last letter (that depends on w, but this is not an issue), we can apply Lemma 4.1 to prove that these ratios are uniformly bounded from below. □

5. Union of ulbd Cantor sets

This section explains how to organize the union of two ulbd Cantor sets, into a single construction, $\mathcal{I}'$, that is also ulbd, perhaps with a smaller lower bound. To do this, we shall exploit the non uniqueness of the construction.

Lemma 5.1. *Let $C^{(1)}$ and $C^{(2)}$ be two Cantor sets of ulbd, separated so that* $\max C^{(1)} < \min C^{(2)}$. *Then, $C := C^{(1)} \cup C^{(2)}$ is a Cantor set, that admits a construction $\mathcal{I}$ that is also of ulbd with dissection ratios larger than, or equal to a'. This value can be estimated as follows. Let $a^{(1)}$ and $a^{(2)}$ be the (strictly positive) lower bounds to the dissection ratios of $C^{(1)}$ and $C^{(2)}$. Let also*

$$a = \min\left\{\frac{\max C^{(1)} - \min C^{(1)}}{\min C^{(2)} - \min C^{(1)}}, \frac{\max C^{(2)} - \min C^{(2)}}{\max C^{(2)} - \max C^{(1)}}, a^{(1)}, a^{(2)}\right\}. \tag{5.1}$$

Then, $a' = \frac{a^2}{a+1}$.

Finally,

$$\gamma(C) = \max\{\gamma(C^{(1)}), \gamma(C^{(2)}), \min C^{(2)} - \max C^{(1)}\}. \tag{5.2}$$

Proof. First, it is clear that $C := C^{(1)} \cup C^{(2)}$ is a Cantor set, because of the separation condition $\max C^{(1)} < \min C^{(2)}$. Therefore, it can be constructed on a map $\mathcal{I}$, although not in a unique way. Since the set of gaps do not depend on the construction $\mathcal{I}$, the gaps of C are the union of those of $C^{(1)}$ and $C^{(2)}$ and the open interval $]\max C^{(1)}, \min C^{(2)}[$, eq. (5.2) follows. We now need to prove that such a construction exists, that is of uniformly lower bounded dissection. We denote by $\mathcal{I}$ this construction.

Suppose $C^{(1)}$ and $C^{(2)}$ are constructed on $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$, with ratios of dissection $r(\mathcal{I}^{(1)})_w$, $r(\mathcal{I}^{(2)})_w$, which by hypothesis are all larger than, or equal to a. Without loss of generality, assume that $I^{(1)}$ is wider than $I^{(2)}$:

$$d(I^{(1)}) \geq d(I^{(2)}). \tag{5.3}$$

The hypothesis and eq. (3.3) imply that $r(\mathcal{I}^{(j)})_w < 1 - a$ for every $w \in W \setminus \{\varnothing\}$ and $j = 1, 2$. Then, $d(I^{(j)}_{wi}) = r^{(j)}_{wi}\, d(I^{(j)}_w) \leq (1-a) d(I^{(j)}_w)$ and we conclude that

$$d(I^{(j)}_w) \leq (1-a)^{|w|} d(I^{(j)})$$

for each $w \in W$ and $i, j = 1, 2$.

Then, because of (5.3), there exists $\bar{n} \in \mathbb{N}$ such that

$$d(I^{(1)}_{1^n}) \geq d(I^{(2)}) \quad \text{for } n \leq \bar{n}, \tag{5.4a}$$

$$d(I^{(1)}_{1^n}) < d(I^{(2)}) \quad \text{for } n > \bar{n}. \tag{5.4b}$$

In view of this result, let us define a construction $\mathcal{I}$ as follows: for any $w \in W$ define I_w by

$$I_w = \begin{cases} [\min I^{(1)}_w, \max I^{(2)}] & \text{if } w = 1^n,\ 0 \leq n \leq \bar{n} \quad \text{(first case)}, \\ I^{(1)}_{1^{\bar{n}}w'} & \text{if } w = 1^{\bar{n}}0w' \quad \text{(second case)}, \\ I^{(2)}_{w'} & \text{if } w = 1^{\bar{n}}1w' \quad \text{(third case)}, \\ I^{(1)}_w & \text{otherwise} \quad \text{(fourth case)}. \end{cases} \tag{5.5}$$

We first prove that $C^{(1)} \cup C^{(2)}$ is constructed on $\mathcal{I}$. That is, $\mathcal{I}$ constructs a quasi Cantor set and, putting $C_n := \bigcup_{|w|=n} I_w$, we have

$$C^{(1)} \cup C^{(2)} = \bigcap_{n=0}^{\infty} C_n. \tag{5.6}$$

We will use systematically the following evident remark: Since $\max I^{(1)} = \max C^{(1)} < \min C^{(2)} = \min I^{(2)}$ and the intervals $I^{(j)}_w$ are all contained in $I^{(j)}$ $(j = 1, 2)$, then any element of $I^{(1)}_w$ is strictly less than any element of $I^{(2)}_{w'}$ for every $w, w' \in W$.

Note that by hypothesis $\min I^{(1)} < \max I^{(1)} < \min I^{(2)} < \max I^{(2)}$ and in our construction $\mathcal{I}$, eq. (5.5), $I = I_\varnothing = [\min I^{(1)}, \max I^{(2)}]$. Let again $I_w = [a_w, b_w]$. Note that by construction, in any case $a_w < b_w$. If $w = 1^n$, $n \leq \bar{n}$, this follows from the above remark, since $\min I^{(1)}_w \leq \max I^{(1)}$. In the other cases the intervals considered are of the form $I^{(j)}_v$ with $j = 1, 2$, which by hypothesis satisfy the inequality $a_v < b_v$.

Next, we have to prove that $a_w = a_{w0} < b_{w0} < a_{w1} < b_w = b_{w1}$, *i.e.*

$$\min I_{w0} = \min I_w, \tag{5.7}$$

$$\max I_{w0} < \min I_{w1}, \tag{5.8}$$

$$\max I_{w1} = \max I_w. \tag{5.9}$$

In first case of eq. (5.5) we have $I_w = [\min I^{(1)}_w, \max I^{(2)}]$ and either one of the two possibilities holds:

$$n < \bar{n}, \quad I_{w0} = I^{(1)}_{w0}, \qquad I_{w1} = [\min I^{(1)}_{w1}, \max I^{(2)}], \tag{5.10}$$

$$n = \bar{n}, \quad I_{w0} = I^{(1)}_{1^{\bar{n}}} = I^{(1)}_w, \quad I_{w1} = I^{(2)}. \tag{5.11}$$

We see that (5.7) holds in both cases. Moreover, (5.8) holds trivially if $n < \bar{n}$, while if $n = \bar{n}$ we have $\max I_{w0} = \max I^{(1)}_w \le \max I^{(1)} < \min I^{(2)} = \min I_{w1}$.

Finally (5.9) is trivial in both subcases. In second case, $w0 = 1^{\bar{n}}0w'0$ and $w1 = 1^{\bar{n}}0w'1$. Thus,

$$I_w = I^{(1)}_{1^{\bar{n}}w'}, \quad I_{w0} = I^{(1)}_{1^{\bar{n}}w'0}, \quad I_{w1} = I^{(1)}_{1^{\bar{n}}w'1}, \tag{5.12}$$

so that (5.7), (5.8) and (5.9) follow from the corresponding properties of $I^{(1)}_w$. In the third case, then $w0 = 1^{\bar{n}}1w'0$ and $w1 = 1^{\bar{n}}1w'1$. Thus,

$$I_w = I^{(2)}_{w'}, \quad I_{w0} = I^{(2)}_{w'0}, \quad I_{w1} = I^{(2)}_{w'1}, \tag{5.13}$$

and we proceed as above. In the fourth case we have

$$I_w = I^{(1)}_{w}, \quad I_{w0} = I^{(1)}_{w0}, \quad I_{w1} = I^{(1)}_{w1}, \tag{5.14}$$

and we proceed similarly. Thus, we have proven that in fact $\mathcal{I}$ constructs a quasi Cantor set. We now prove that $C^{(1)} \cup C^{(2)}$ is constructed on $\mathcal{I}$, that is eq. (5.6). Note that for every $n > \bar{n}$ we have

$$C^{(1)}_n \cup C^{(2)}_n \subseteq C_n \subseteq C^{(1)}_{n-\bar{n}-1} \cup C^{(2)}_{n-\bar{n}-1}. \tag{5.15}$$

To prove (5.15), suppose first $x \in C^{(1)}_n \cup C^{(2)}_n$. Then, either $x \in C^{(1)}_n$ or $x \in C^{(2)}_n$. In the former case, there exists $w \in W$ with $|w| = n$ such that $x \in I^{(1)}_w$, and either w can be written as $w = 1^{\bar{n}}w'$, in which case $x \in I_{1^{\bar{n}}0w'} \subseteq C_{n+1} \subseteq C_n$, or w is not of the form $w = 1^{\bar{n}}w'$, in which case $x \in I_w \subseteq C_n$. If instead $x \in C^{(2)}_n$, then $x \in I^{(2)}_w$ for some $w \in W$ with $|w| = n$. Then, $x \in I_{1^{\bar{n}}1w} \subseteq C_{n+\bar{n}+1} \subseteq C_n$. The first inclusion in (5.15) is so proven. Let us prove the second. Suppose that $x \in C_n$, thus $x \in I_w$ for some $w \in W$ with $|w| = n$. Then by definition, since $n > \bar{n}$ we are not in the first case in definition of I_w. If the second case holds, then $x \in I^{(1)}_v$ with $|v| = n - 1$; in the third case one has $x \in I^{(2)}_v$ with $|v| = n - 1 - \bar{n}$, and in the fourth case $x \in I^{(1)}_v$ with $|v| = n$. In any case, since the sequences of sets $C^{(j)}_n$ are decreasing we have that $x \in C^{(1)}_{n-\bar{n}-1} \cup C^{(2)}_{n-\bar{n}-1}$ and (5.15) is proven.

At this point, from (5.15), since $C^{(j)}_n \subseteq C^{(j)}_0 = I^{(j)}$ and $I^{(1)} \cap I^{(2)} = \varnothing$, and $\bigcap_{n=0}^{\infty} C^{(j)}_n = C^{(j)}$, eq. (5.6) easily follows. Thus, we have proven that $C^{(1)} \cup C^{(2)}$ is constructed on $\mathcal{I}$.

Finally, we have to prove the fundamental part of the lemma, that is, there exists a positive constant a' such that $r(\mathcal{I})_{wj} \ge a'$ for all $w \in W$, $j = 0, 1$. Note that by the hypothesis (5.1) we have

$$\max I_2 - \min I_2 \ge a(\max I_2 - \min I_2 + \min I_2 - \max I_1)$$

hence, using also (5.4),

$$\max I^{(1)}_{1^n} - \min I^{(1)}_{1^n} \geq \max I_2 - \min I_2 \geq \frac{a}{1-a}(\min I_2 - \max I_1) \tag{5.16}$$

for every $n \leq \bar{n}$. Similarly, since eq. (5.1) implies that

$$\max I^{(1)} - \min I^{(1)} \geq a(\min I^{(2)} - \max I^{(1)} + \max I^{(1)} - \min I^{(1)})$$

we have

$$\max I^{(1)} - \min I^{(1)} \geq \frac{a}{1-a}(\min I^{(2)} - \max I^{(1)}). \tag{5.17}$$

Moreover, by (5.4),

$$d(I^{(2)}) > d(I^{(1)}_{1^{\bar{n}+1}}) = r(\mathcal{J}^{(1)})_{1^{\bar{n}+1}}\, d(I^{(1)}_{1^{\bar{n}}}) \geq a d(I^{(1)}_{1^{\bar{n}}}).$$

Hence,

$$d(I^{(2)}) \leq d(I^{(1)}_{1^{\bar{n}}}) \leq \frac{1}{a} d(I^{(2)}). \tag{5.18}$$

Following these inequalities, we can evaluate $r(\mathcal{J})_{wj}$, with $w \in W$.

When (5.10) holds, we first estimate $r(\mathcal{J})_{w0}$:

$$r(\mathcal{J})_{w0} = \frac{\max I_{w0} - \min I_{w0}}{\max I_w - \min I_w} = \frac{\max I^{(1)}_{w0} - \min I^{(1)}_{w0}}{\max I^{(2)} - \min I^{(1)}_w}. \tag{5.19}$$

Now, since $w = 1^n$, $n \leq \bar{n}$, by (5.16) we have

$$\begin{aligned}
&\max I^{(2)} - \min I^{(1)}_w \\
&\quad = \max I^{(2)} - \min I^{(2)} + \min I^{(2)} - \max I^{(1)}_{1^n} + \max I^{(1)}_{1^n} - \min I^{(1)}_{1^n} \\
&\quad = \max I^{(2)} - \min I^{(2)} + \min I^{(2)} - \max I^{(1)} + \max I^{(1)}_{1^n} - \min I^{(1)}_{1^n} \\
&\quad \leq 2(\max I^{(1)}_{1^n} - \min I^{(1)}_{1^n}) + \min I^{(2)} - \max I^{(1)} \\
&\quad \leq \Big(2 + \frac{1-a}{a}\Big)(\max I^{(1)}_{1^n} - \min I^{(1)}_{1^n}) \qquad (5.20) \\
&\quad = \Big(1 + \frac{1}{a}\Big)(\max I^{(1)}_w - \min I^{(1)}_w) \\
&\quad = \frac{1}{r(\mathcal{J}^{(1)})_{w0}} \frac{a+1}{a}(\max I^{(1)}_{w0} - \min I^{(1)}_{w0}) \\
&\quad \leq \frac{a+1}{a^2}(\max I^{(1)}_{w0} - \min I^{(1)}_{w0}).
\end{aligned}$$

Thus, also in view of (5.19), we have

$$r(\mathfrak{I})_{w0} \geq \frac{a^2}{a+1}. \tag{5.21}$$

Let us now consider $r(\mathfrak{I})_{w1}$, still when (5.10) holds. We have

$$r(\mathfrak{I})_{w1} = \frac{\max I_{w1} - \min I_{w1}}{\max I_w - \min I_w} = \frac{\max I^{(2)} - \min I^{(1)}_{w1}}{\max I^{(2)} - \min I^{(1)}_w}.$$

Now,

$$\begin{aligned}\max I^{(1)}_w - \min I^{(1)}_{w1} &= \max I^{(1)}_{w1} - \min I^{(1)}_{w1}\\ &= r(\mathfrak{I}^{(1)})_{w1}(\max I^{(1)}_w - \min I^{(1)}_w)\\ &\geq a(\max I^{(1)}_w - \min I^{(1)}_w).\end{aligned}$$

Since $\max I^{(2)} > \max I^{(1)}_w$ and $a < 1$,

$$\begin{aligned}\max I^{(2)} - \min I^{(1)}_{w1} &= \max I^{(2)} - \max I^{(1)}_w + \max I^{(1)}_w - \min I^{(1)}_{w1}\\ &\geq \max I^{(2)} - \max I^{(1)}_w + a(\max I^{(1)}_w - \min I^{(1)}_w)\\ &\geq a(\max I^{(2)} - \max I^{(1)}_w) + a(\max I^{(1)}_w - \min I^{(1)}_w)\\ &= a(\max I^{(2)} - \min I^{(1)}_w),\end{aligned}$$

hence

$$r(\mathfrak{I})_{w1} \geq a. \tag{5.22}$$

We next evaluate $r(\mathfrak{I})_{w0}$ and $r(\mathfrak{I})_{w1}$ when (5.11) holds. We have $w = 1^{\bar{n}}$ and $\max I^{(1)}_{1^{\bar{n}}} = \max I^{(1)}$. Then,

$$\begin{aligned}&\max I^{(2)} - \min I^{(1)}_w\\ &= \max I^{(2)} - \min I^{(2)} + \min I^{(2)} - \max I^{(1)} + \max I^{(1)}_{1^{\bar{n}}} - \min I^{(1)}_{1^{\bar{n}}}\\ &\leq \max I^{(1)}_{1^{\bar{n}}} - \min I^{(1)}_{1^{\bar{n}}} + \frac{1-a}{a}(\max I^{(1)}_{1^{\bar{n}}} - \min I^{(1)}_{1^{\bar{n}}}) + (\max I^{(1)}_{1^{\bar{n}}} - \min I^{(1)}_{1^{\bar{n}}})\\ &= \frac{a+1}{a}(\max I^{(1)}_{1^{\bar{n}}} - \min I^{(1)}_{1^{\bar{n}}})\end{aligned}$$

where the inequalities follow from (5.16) and (5.18). Hence

$$r(\mathfrak{I})_{w0} = \frac{\max I_{w0} - \min I_{w0}}{\max I_w - \min I_w} = \frac{\max I^{(1)}_{1^{\bar{n}}} - \min I^{(1)}_{1^{\bar{n}}}}{\max I^{(2)} - \min I^{(1)}_w} \geq \frac{a}{a+1}. \tag{5.23}$$

Similarly,

$$\begin{aligned}
&\max I^{(2)} - \min I^{(1)}_w \\
&\quad = \max I^{(2)} - \min I^{(2)} + \min I^{(2)} - \max I^{(1)} + \max I^{(1)}_{1^{\bar n}} - \min I^{(1)}_{1^{\bar n}} \\
&\quad \le \max I^{(2)} - \min I^{(2)} + \frac{1-a}{a}(\max I^{(2)} - \min I^{(2)}) \\
&\qquad + \frac{1}{a}(\max I^{(2)} - \min I^{(2)}) \\
&\quad = \frac{2}{a}(\max I^{(2)} - \min I^{(2)}).
\end{aligned}$$

Hence

$$r(\mathcal{J})_{w1} = \frac{\max I_{w1} - \min I_{w1}}{\max I_w - \min I_w} = \frac{\max I^{(2)} - \min I^{(2)}}{\max I^{(2)} - \min I^{(1)}_w} \ge \frac{a}{2} \ge \frac{a^2}{a+1}. \tag{5.24}$$

Finally, we easily see that if (5.12), (5.13), or (5.14) holds, then we have respectively $r(\mathcal{J})_{wj} = r(\mathcal{J}^{(1)})_{1^{\bar n} w' j}$, $r(\mathcal{J})_{wj} = r(\mathcal{J}^{(2)})_{w' j}$, $r(\mathcal{J})_{wj} = r(\mathcal{J}^{(1)})_{wj}$, and by hypothesis such numbers are all larger than, or equal to $a \ge \frac{a^2}{a+1}$.

To sum up, in view of (5.21), (5.22), (5.23), and (5.24), we have $r(\mathcal{J})_{wj} \ge a'$ where a' is given by $\frac{a^2}{a+1}$ and the Lemma is completely proven. □

6. Finite sums of ulbd Cantor Sets

This section contains a single Lemma, in which we prove that any finite sum of ulbd Cantor sets contains a Cantor set of uniformly lower bounded dissections, with the same convex hull as the full sum, and maximum gap size not larger than those of the individual Cantor sets.

Lemma 6.1. *Let $C^{(1)}, \dots, C^{(m)}$ be Cantor sets constructed on $I^{(1)}, \dots, I^{(m)}$ with ratios of dissection larger than, or equal to, a. Then there exists a Cantor set $C \subseteq C^{(1)} + \cdots + C^{(m)}$ constructed with all ratios of dissection larger than a_m, where $a_m > 0$ depends only on a and m, such that*

$$\gamma(C) \le \max_{s=1,\dots,m} \gamma(C^{(s)}), \tag{6.1}$$

$$\min C = \sum_{s=1}^{m} \min C^{(s)}, \quad \max C = \sum_{s=1}^{m} \max C^{(s)}. \tag{6.2}$$

Proof. The proof of this Lemma is rather long and technical. For better clarity, it is organized in successive steps.

STEP 1. The lemma holds trivially for $m = 1$. We first show by induction that if it holds for $m = 2$, then it holds for any m. It will then be sufficient to prove the Lemma for $m = 2$. In fact, suppose that the Lemma holds for $m = 2$, and that it also holds for a generic value $m \geq 1$. This implies that it holds for $m + 1$, as the following argument shows. Let $C^{(1)}, \dots, C^{(m)}, C^{(m+1)}$ be Cantor sets constructed with all ratios of dissection at least a. Then by hypothesis there exists a Cantor set $C' \subseteq C^{(1)} + \cdots + C^{(m)}$ which can be constructed with all ratios of dissection at least a_m, such that $\gamma(C') \leq \max_{s=1,\dots,m} \gamma(C^{(s)})$ and $\min C' = \sum_{s=1}^{m} \min C^{(s)}$, $\max C' = \sum_{s=1}^{m} \max C^{(s)}$. Put now $a' = \min\{a, a_m\}$ and let $a_{m+1} = a'$. Then, by the Lemma for $m = 2$ applied to the pair $C', C^{(m+1)}$ there exists a Cantor set

$$C \subseteq C' + C^{(m+1)} \subseteq C^{(1)} + \cdots + C^{(m)} + C^{(m+1)}$$

with all dissection ratios at least a_{m+1} such that

$$\gamma(C) \leq \max\{\gamma(C'), \gamma(C^{(m+1)})\} \leq \max_{s=1,\dots,m,m+1} \gamma(C^{(s)}),$$

$$\min C = \min C' + \min C^{(m+1)} = \sum_{s=1}^{m+1} \min C^{(s)},$$

$$\max C = \max C' + \max C^{(m+1)} = \sum_{s=1}^{m+1} \max C^{(s)},$$

and the Lemma for $m + 1$ holds. In the next steps we will prove the Lemma for $m = 2$.

STEP 2. Let $C^{(1)}, C^{(2)}$ be Cantor sets constructed on $\mathcal{J}^{(1)}, \mathcal{J}^{(2)}$ with all ratios of dissection at least a. Based on these latter, we will define a construction $\mathcal{J}$ of a new Cantor set C. Prior to do that, we need to study auxiliary sets A_n. For any $n \in \mathbb{N}$ (including obviously $n = 0$), put

$$A_n = A_n^{(1)} \cup A_n^{(2)},$$

where the terms in the union are defined as follows. Suppose that the following condition holds:

$$\gamma(\mathcal{J}^{(1)})_{0^n} \leq \gamma(\mathcal{J}^{(2)})_{0^n}, \tag{6.3}$$

γ being the gap size defined in eq. (3.6). In this case, let

$$A_n^{(1)} := C^{(2)}_{0^n 1} + \max C^{(1)}_{0^{n+1}}, \quad A_n^{(2)} := C^{(1)}_{0^n 1} + \max C^{(2)}_{0^n 1}. \tag{6.4}$$

In the opposite case the indices (1) and (2) at right hand side of eq. (6.4) are exchanged:

$$A_n^{(1)} := C_{0^n1}^{(1)} + \max C_{0^{n+1}}^{(2)},$$
$$A_n^{(2)} := C_{0^n1}^{(2)} + \max C_{0^n1}^{(1)}.$$

Therefore, let us consider the case in eqs. (6.3,6.4), the other giving results that can be obtained by exchanging superscripts. Clearly, $A_n^{(1)}$ and $A_n^{(2)}$ are two Cantor sets with gaps not larger than $\max\{\gamma(C^{(1)}), \gamma(C^{(2)})\}$, constructed with ratios of dissections at least a. We have

$$\begin{aligned}
\min A_n^{(2)} - \max A_n^{(1)} &= \min C_{0^n1}^{(1)} + \max C_{0^n1}^{(2)} - (\max C_{0^n1}^{(2)} + \max C_{0^{n+1}}^{(1)}) \\
&= \min I_{0^n1}^{(1)} + \max I_{0^n1}^{(2)} - \max I_{0^n1}^{(2)} - \max I_{0^{n+1}}^{(1)} \\
&= \min I_{0^n1}^{(1)} - \max I_{0^{n+1}}^{(1)} \\
&= \gamma(\mathcal{J}^{(1)})_{0^n}.
\end{aligned}$$

Since the last quantity is positive, this also proves that $\max A_n^{(1)} < \min A_n^{(2)}$. Following the same kind of computation, we also have that

$$\begin{aligned}
\max A_n^{(2)} - \max A_n^{(1)} &= \max C_{0^n1}^{(1)} - \max C_{0^n0}^{(1)} \\
&= \max I_{0^n1}^{(1)} - \max I_{0^n0}^{(1)} \\
&\leq \max I_{0^n1}^{(1)} - \min I_{0^n0}^{(1)} \\
&= \max I_{0^n}^{(1)} - \min I_{0^n}^{(1)} \\
&= \frac{1}{r(\mathcal{J}^{(1)})_{0^n1}} (\max I_{0^n1}^{(1)} - \min I_{0^n1}^{(1)}) \\
&= \frac{1}{r(\mathcal{J}^{(1)})_{0^n1}} (\max A_n^{(2)} - \min A_n^{(2)}) \\
&\leq \frac{1}{a} (\max A_n^{(2)} - \min A_n^{(2)}).
\end{aligned}$$

Therefore,

$$\frac{\max A_n^{(2)} - \min A_n^{(2)}}{\max A_n^{(2)} - \max A_n^{(1)}} \geq a. \tag{6.5}$$

Moreover

$$\begin{aligned}
\min A_n^{(2)} - \min A_n^{(1)} &= \min A_n^{(2)} - \max A_n^{(1)} + \max A_n^{(1)} - \min A_n^{(1)} \\
&= \gamma(\mathcal{J}^{(1)})_{0^n} + \max C^{(2)}_{0^n 1} - \min C^{(2)}_{0^n 1} \\
&\leq \gamma(\mathcal{J}^{(2)})_{0^n} + \max I^{(2)}_{0^n 1} - \min I^{(2)}_{0^n 1} \\
&= \min I^{(2)}_{0^n 1} - \max I^{(2)}_{0^{n+1}} + \max I^{(2)}_{0^n 1} - \min I^{(2)}_{0^n 1} \\
&\leq \max I^{(2)}_{0^n 1} - \min I^{(2)}_{0^{n+1}} \\
&= \max I^{(2)}_{0^n} - \min I^{(2)}_{0^n} \\
&= \frac{1}{r(\mathcal{J}^{(2)})_{0^n 1}} (\max I^{(2)}_{0^n 1} - \min I^{(2)}_{0^n 1}) \\
&= \frac{1}{r(\mathcal{J}^{(2)})_{0^n 1}} (\max A_n^{(1)} - \min A_n^{(1)}) \\
&\leq \frac{1}{a} (\max A_n^{(1)} - \min A_n^{(1)}).
\end{aligned}$$

As a consequence,

$$\frac{\max A_n^{(1)} - \min A_n^{(1)}}{\min A_n^{(2)} - \min A_n^{(1)}} \geq a. \tag{6.6}$$

Thus, by (6.5) and (6.6) $A_n^{(1)}$ and $A_n^{(2)}$ satisfy (5.1). We can so use Lemma 5.1, that implies that A_n is a Cantor set and can be constructed on a map with all ratios of dissection larger than, or equal to, a'. Let us denote this map with $\bar{\mathcal{J}}(n)$ and its image intervals by $\bar{I}(n)_w$: recall that a different map is defined for any value of n, including zero. Moreover, by construction

$$\begin{aligned}
\gamma(A_n) &\leq \max\{\gamma(A_n^{(1)}), \gamma(A_n^{(2)}), \min A_n^{(2)} - \max A_n^{(1)}\} \\
&\leq \max\{\gamma(C_1), \gamma(C_2), \gamma(\mathcal{J}^{(1)}_{0^n})\} \\
&\leq \max\{\gamma(C_1), \gamma(C_2)\}.
\end{aligned}$$

STEP 3. We can now introduce the new map $\mathcal{J}$ that constructs the Cantor set C in the thesis of this Lemma. Recall the notation that associates an interval to any finite word, eq (3.1): $\mathcal{J}(w) = I_w$. Let us define all such intervals, parting the set of finite binary words W according to the number of leading zeros. In fact, let

$$\begin{cases} I_{0^n} = I^{(1)}_{0^n} + I^{(2)}_{0^n}, \\ I_{0^n 1 w'} = \bar{I}(n)_{w'}. \end{cases} \tag{6.7}$$

In the above, w' is any word, $n \in \mathbb{N}$ can take the value 0, and the intervals $\bar{I}(n)_{w'}$, $I^{(1)}_{0^n}$ and $I^{(2)}_{0^n}$ have been defined in the previous step. As before, 0^0 is to be intended as the empty set.

It is instructive to write down explicitly the first few formulae: let $n = 0$, to obtain $I_\varnothing = I^{(1)}_\varnothing + I^{(2)}_\varnothing$. This is the convex hull of C and it is clearly made by an interval composed of the arithmetic sums of any pair of numbers, one in $I^{(1)}$ and one in $I^{(2)}$. Therefore, it is also the convex hull of $C^{(1)} + C^{(2)}$. Consider next I_1. It can be obtained from the second formula in (6.7): $I_1 = \bar{I}(0)_\varnothing$, that is, the convex hull of A_0. All intervals corresponding to words starting with 1 are then constructed by the map $\bar{\mathfrak{I}}(0)$: in fact, eq. (6.7) yields $I_{1w} = \bar{I}(0)_w$; as remarked above these intervals construct the Cantor set $A(0)$. Observe that the maximum of this Cantor set is equal to the maximum of $I^{(1)} + I^{(2)}$ and therefore to the maximum of $C^{(1)} + C^{(2)}$. Let us also describe the case $n = 1$. This permits to write the interval I_0 as $I^{(1)}_0 + I^{(2)}_0$. Constructing A_1 via the map $\bar{\mathfrak{I}}(1)$ then enables us to define all intervals $I_{01w} = \bar{I}(1)_w$, *et cetera.*

STEP 4. We now prove formally that eq. (6.7) is a consistent construction of a quasi Cantor set. Clearly, I_w is an interval for every $w \in W$, so that we just need to prove that for every $w \in W$ we have

$$\min I_w = \min I_{w0}, \tag{6.8}$$

$$\max I_{w0} < \min I_{w1}, \tag{6.9}$$

$$\max I_{w1} = \max I_w. \tag{6.10}$$

If $w = 0^n$, we have

$$\begin{aligned}
\min I_w &= \min(I^{(1)}_{0^n} + I^{(2)}_{0^n}) \\
&= \min I^{(1)}_{0^n} + \min I^{(2)}_{0^n} \\
&= \min I^{(1)}_{0^{n+1}} + \min I^{(2)}_{0^{n+1}} \\
&= \min(I^{(1)}_{0^{n+1}} + I^{(2)}_{0^{n+1}}) \\
&= \min I_{0^{n+1}} \\
&= \min I_{0^n 0},
\end{aligned}$$

and (6.8) holds. Next, if (6.3) holds, then

$$\begin{aligned}
\min I_{w1} - \max I_{w0} &= \min(C^{(2)}_{0^n 1} + \max C^{(1)}_{0^{n+1}}) - \max(I^{(1)}_{0^{n+1}} + I^{(2)}_{0^{n+1}}) \\
&= \min I^{(2)}_{0^n 1} + \max I^{(1)}_{0^{n+1}} - \max I^{(1)}_{0^{n+1}} - \max I^{(2)}_{0^{n+1}}
\end{aligned}$$

$$= \min I^{(2)}_{0^n 1} - \max I^{(2)}_{0^{n+1}}$$
$$= \gamma(\mathcal{J}^{(2)})_{0^n}$$

and the last quantity is larger than zero. On the contrary, if (6.3) does not hold, we have $\min I_{w1} - \max I_{w0} = \gamma(\mathcal{J}^{(1)})_{0^n} > 0$, so that in both cases (6.9) holds. To sum up, for any word 0^n there is $j \in \{1, 2\}$ so that

$$\min I_{0^n 1} - \max I_{0^n 0} = \gamma(\mathcal{J}^{(j)})_{0^n}. \tag{6.11}$$

We now prove (6.10). We have

$$\begin{aligned}
\max I_w &= \max(I^{(1)}_{0^n} + I^{(2)}_{0^n}) \\
&= \max I^{(1)}_{0^n} + \max I^{(2)}_{0^n} \\
&= \max I^{(1)}_{0^n 1} + \max I^{(2)}_{0^n 1} \\
&= \max C^{(1)}_{0^n 1} + \max C^{(2)}_{0^n 1} \\
&= \max A^{(2)}_n \\
&= \max A_n \\
&= \max \bar{I}(n) \\
&= \max I_{0^n 1} \\
&= \max I_{w1}
\end{aligned}$$

and (6.10) is proven.

Suppose now $w = 0^n 1 w'$. In this case (6.8), (6.9) and (6.10) follow immediately from the corresponding properties of $\bar{\mathcal{J}}(n)$. In conclusion, the above proves that $\mathcal{J}$ in fact constructs a quasi Cantor set, which we denote by C.

Step 5. Equation (6.11) and a straightforward argument when $w = 0^n 1 w'$, imply that eq. (6.1) holds:

$$\gamma(C) \le \max\{\gamma(C_1), \gamma(C_2)\}. \tag{6.12}$$

Let us now prove eq. (6.2). We have that

$$\begin{aligned}
\min C &= \min I \\
&= \min I_{0^0} \\
&= \min(I^{(1)}_{0^0} + I^{(2)}_{0^0}) \\
&= \min I^{(1)} + \min I^{(2)} \\
&= \min C^{(1)} + \min C^{(2)}
\end{aligned}$$

and similarly

$$\max C = \max C^{(1)} + \max C^{(2)}.$$

STEP 6. We now prove that $C \subseteq C^{(1)} + C^{(2)}$. Take $x \in C$. Then, there exists an infinite string $\widetilde{w} = i_1 i_2 i_3 \dots$ such that $x \in I_{i_1 \dots i_n}$ for all n. We distinguish two cases. If $i_s = 0$ for all s, then for all n,

$$\begin{aligned} x \in I_{0^n} &= I^{(1)}_{0^n} + I^{(2)}_{0^n} \\ &= [\min I^{(1)}_{0^n}, \max I^{(1)}_{0^n}] + [\min I^{(2)}_{0^n}, \max I^{(2)}_{0^n}] \\ &= [\min I^{(1)}, \min I^{(1)} + \max I^{(1)}_{0^n} - \min I^{(1)}_{0^n}] \\ &\quad + [\min I^{(2)}, \min I^{(2)} + \max I^{(2)}_{0^n} - \min I^{(2)}_{0^n}] \\ &= [\min I^{(1)} + \min I^{(2)}, \min I^{(1)} + \min I^{(2)} + d_n] \end{aligned}$$

where

$$d_n := \max I^{(1)}_{0^n} - \min I^{(1)}_{0^n} + \max I^{(2)}_{0^n} - \min I^{(2)}_{0^n} \xrightarrow[n \to +\infty]{} 0,$$

since $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ construct two Cantor sets. Hence, $x = \min I^{(1)} + \min I^{(2)} = \min C^{(1)} + \min C^{(2)} \in C^{(1)} + C^{(2)}$. Suppose instead, there exists $\bar{s}$ such that $i_{\bar{s}} = 1$, and we can and do assume $i_s = 0$ for every $s < \bar{s}$ (in other words, $\bar{s}$ is the first occurrence of 1 in the symbolic sequence $\widetilde{w}$). Putting $n = \bar{s} - 1$ and $\widetilde{w}' = i_{\bar{s}+1} i_{\bar{s}+2} \dots$, we have $\widetilde{w}_{\bar{s}+m} = 0^n 1 \widetilde{w}'_m$ for all $m > 0$. Thus for every $m > 0$,

$$x \in I_{0^n 1 \widetilde{w}'_m} = \bar{I}(n)_{\widetilde{w}'_m} \implies x \in A_n = A^{(1)}_n \cup A^{(2)}_n \subseteq C^{(1)} + C^{(2)}.$$

Thus, $C \subseteq C^{(1)} + C^{(2)}$ is proven.

STEP 7. It finally remains to prove that $r(\mathcal{I})_w \geq a'$ for all $w \in W$ (as usual this also proves that C is not only quasi Cantor but also Cantor). Let us start by considering the word $w = 0^n$. Recalling that $r(\mathcal{I}^{(j)})_v \geq a$, for $j = 1, 2$ and $v \in W \setminus \{\varnothing\}$, we have the following estimates. The diameter of the interval I_w is

$$\begin{aligned} d(I_w) &= d(I_{0^n}) \\ &= \max(I^{(1)}_{0^n} + I^{(2)}_{0^n}) - \min(I^{(1)}_{0^n} + I^{(2)}_{0^n}) \\ &= \max I^{(1)}_{0^n} - \min I^{(1)}_{0^n} + \max I^{(2)}_{0^n} - \min I^{(2)}_{0^n} \\ &= d(I^{(1)}_w) + d(I^{(2)}_w). \end{aligned}$$

The diameter of the interval I_{w0} is

$$\begin{aligned} d(I_{w0}) &= d(I_{0^{n+1}}) \\ &= d(I_{w0}^{(1)}) + d(I_{w0}^{(2)}) \\ &= r(\mathcal{J}^{(1)})_{w0} d(I_w^{(1)}) + r(\mathcal{J}^{(2)})_{w0} d(I_w^{(2)}) \\ &\geq a d(I_w^{(1)}) + a d(I_w^{(2)}) \\ &= a d(I_w). \end{aligned}$$

The diameter of the interval I_{w1} is

$$\begin{aligned} d(I_{w1}) &= d(I_{0^n 1}) \\ &= d(\bar{I}(n)) \\ &= \max \bar{I}(n) - \min \bar{I}(n) \\ &= \max A_n - \min A_n \\ &= \max A_n^{(2)} - \min A_n^{(1)} \\ &= \max C_{0^n 1}^{(1)} + \max C_{0^n 1}^{(2)} - \min C_{0^n 1}^{(2)} - \max C_{0^{n+1}}^{(1)}. \end{aligned}$$

If (6.3) holds, we can continue as follows:

$$\begin{aligned} d(I_{w1}) &= \max I_{0^n 1}^{(2)} - \min I_{0^n 1}^{(2)} + \max I_{0^n 1}^{(1)} - \max I_{0^{n+1}}^{(1)} \\ &\geq \max I_{0^n 1}^{(2)} - \min I_{0^n 1}^{(2)} + \max I_{0^n 1}^{(1)} - \min I_{0^n 1}^{(1)} \\ &= d(I_{0^n 1}^{(2)}) + d(I_{0^n 1}^{(1)}) \\ &= d(I_{w1}^{(2)}) + d(I_{w1}^{(1)}) \\ &= r(\mathcal{J}^{(2)})_{w1} d(I_w^{(2)}) + r(\mathcal{J}^{(1)})_{w1} d(I_w^{(1)}) \\ &\geq a d(I_w^{(2)}) + a d(I_w^{(1)}) \\ &= a d(I_w). \end{aligned}$$

Hence,

$$r(\mathcal{J})_{wj} = \frac{d(I_{wj})}{d(I_w)} \geq a \geq a', \quad j = 0, 1.$$

Note that for $j = 1$ we have used eq. (6.3); when it does not hold we exchange the indices 1 and 2.

Let now consider the words $w = 0^n 1 w'$, with w' any finite word. Recalling that $r(\bar{\mathcal{I}}(n))_v \geq a'$, for $n \in \mathbb{N}$ and $v \in W \setminus \{\varnothing\}$, we have

$$r(\mathcal{I})_{wj} = \frac{d(\bar{I}_{wj})}{d(\bar{I}_w)} = \frac{d(\bar{I}(n)_{w'j})}{d(\bar{I}(n)_{w'})} = r(\bar{\mathcal{I}}(n))_{w'j} \geq a'$$

and the Lemma is completely proven. □

7. More on sums of ulbd Cantor sets

In this section, we first recall a result from the literature on the Minkowski sum of ulbd Cantor sets, and we then derive a couple of further Lemmas.

Theorem 7.1 (Theorem 3.2 in [11]). *Let $C^{(1)}, \dots, C^{(m)}$ be Cantor sets constructed on $I^{(1)}, \dots, I^{(m)}$ with constructions $\mathcal{I}^{(1)}, \dots, \mathcal{I}^{(m)}$ of uniformly lower bounded dissections, larger than $a > 0$. In addition, suppose that $a \leq \frac{1}{3}$ and that m is such that*

$$(m-1)\frac{a^2}{(1-a)^3} + \frac{a}{1-a} \geq 1.$$

Finally suppose that no translate of any of these Cantor sets is contained in a gap of another. Then, the sum of these Cantor sets is a closed interval:

$$C^{(1)} + \cdots + C^{(m)} = \Big[\sum_{i=1}^{m} \min C^{(i)}, \sum_{i=1}^{m} \max C^{(i)}\Big].$$

The result of this theorem can be easily extended to any sets containing the Cantor sets $C^{(i)}$ and enclosed in the intervals $I^{(i)}$.

Lemma 7.2. *Let $C^{(i)}$ be Cantor sets constructed on $I^{(i)} = \mathrm{Conv}(C^{(i)})$, for $i = 1, \dots, m$ and suppose that $C^{(1)} + \cdots + C^{(m)}$ is an interval. Then, for any sets $D^{(i)}$ such that*

$$C^{(i)} \subseteq D^{(i)} \subseteq I^{(i)}, \quad i = 1, \dots, m, \tag{7.1}$$

and

$$\min C^{(i)} = \min D^{(i)}, \max C^{(i)} = \max D^{(i)}, \quad i = 1, \dots, m, \tag{7.2}$$

we have that

$$D^{(1)} + \dots + D^{(m)} = \Big[\sum_{i=1}^{m} \min C^{(i)}, \sum_{i=1}^{m} \max C^{(i)}\Big]. \tag{7.3}$$

Proof. Observe that, by eq. (7.1),

$$C^{(1)} + \dots . + C^{(m)} \subseteq D^{(1)} + \dots + D^{(m)} \subseteq I^{(1)} + \dots + I^{(m)}.$$

Moreover, we trivially have

$$I_1 + \dots + I_m = \Big[\sum_{i=1}^{m} \min I^{(i)}, \sum_{i=1}^{m} \max I^{(i)}\Big] = \Big[\sum_{i=1}^{m} \min C^{(i)}, \sum_{i=1}^{m} \max C^{(i)}\Big]$$

and, since $C^{(1)} + \dots + C^{(m)}$ is an interval, it is clearly

$$C^{(1)} + \dots + C^{(m)} = \Big[\sum_{i=1}^{m} \min C^{(i)}, \sum_{i=1}^{m} \max C^{(i)}\Big]. \qquad \square$$

We now use Theorem 7.1 to prove the following important lemma:

Lemma 7.3. *Suppose that $C^{(l)}$, $l = 1, 2, \dots$ are Cantor sets of ulbd larger than $a > 0$, such that $d(C^{(l)}) \in [A_1, A_2]$ for any l, with $0 < A_1 < A_2 < \infty$. Then, there exist $n \in \mathbb{N} \setminus \{0\}$, that depends only on A_1, A_2 and a, such that*

$$C^{(1)} + C^{(2)} + \dots + C^{(n)} = \Big[\sum_{l=1}^{n} \min C^{(l)}, \sum_{l=1}^{n} \max C^{(l)}\Big].$$

Proof. Take the smallest $m \in \mathbb{N}$ such that $mA_1 > A_2$. Let $h \in \mathbb{N}$ and consider the finite sums $S^{(h)} = C^{(hm+1)} + \dots + C^{(hm+m)}$. Because of Lemma 6.1, for every $h \in \mathbb{N}$ there exist a Cantor set $D^{(h)}$, constructed with ratios of dissection larger than, or equal to a_m (where a_m do not depend on h and where, of course, we can take $a_m \in]0, \frac{1}{3}]$) such that the following relations hold:

$$D^{(h)} \subseteq C^{(hm+1)} + \dots + C^{(hm+m)}, \tag{7.4}$$

$$\min D^{(h)} = \sum_{l=hm+1}^{hm+m} \min C^{(l)}, \tag{7.5}$$

$$\max D^{(h)} = \sum_{l=hm+1}^{hm+m} \max C^{(l)}, \tag{7.6}$$

$$\gamma(D^{(h)}) \le \max_{l=hm+1,\dots,hm+m} \gamma(C^{(l)}) \le A_2. \tag{7.7}$$

By (7.5) and (7.6) we have

$$d(D^{(h)}) = \max D^{(h)} - \min D^{(h)} = \sum_{l=hm+1}^{hm+m} d(C^{(l)}) \geq mA_1 > A_2,$$

so that, in view of (7.7) $d(D^{(h)}) > \gamma(D^{(h')})$ for any $h, h' \in \mathbb{N}$, *i.e.* no translate of $D^{(h)}$ is contained in a gap of $D^{(h')}$. Now, by Theorem 7.1, if $H \in \mathbb{N}$ is large enough, then

$$D^{(0)} + \cdots + D^{(H)} = \Big[\sum_{h=0}^{H} \min D^{(h)}, \sum_{h=0}^{H} \max D^{(h)} \Big],$$

so that on one hand

$$C^{(1)} + \cdots + C^{(Hm+m)} \subseteq \Big[\sum_{i=1}^{Hm+m} \min C^{(i)}, \sum_{i=1}^{Hm+m} \max C^{(i)} \Big].$$

On the other hand, by (7.4) and (7.5)

$$\begin{aligned} C^{(1)} + \cdots + C^{(Hm+m)} &\supseteq D^{(0)} + \cdots + D^{(H)} \\ &= \Big[\sum_{h=0}^{H} \min D^{(h)}, \sum_{h=0}^{H} \max D^{(h)} \Big] \\ &= \Big[\sum_{i=1}^{Hm+m} \min C^{(i)}, \sum_{i=1}^{Hm+m} \max C^{(i)} \Big] \end{aligned}$$

and the Lemma is proven for $n = Hm + m$. □

8. Proof of Theorem 1.1 and related results

In this section we prove Theorem 1.1, via an additional proposition, interesting in its own right, that describes a situation in which the sum of Cantor sets is an interval. We shall present it in a form that will also be useful in Section 9. Such proposition is the following.

Proposition 8.1. *Let $\{\gamma_j\}_{j=1}^{\infty}$ be a sequence of real positive numbers with $\gamma_j > 1$ for all j, and let K be either a ulbd Cantor set of dissection ratios larger than $a > 0$, or the attractor of a finite, hyperbolic, non–singular IFS Ψ. Then, there exists $n \in \mathbb{N} \setminus \{0\}$ that depends only on the set K and not on the specific choice of the numbers γ_j, such that $K_n = \gamma_1 K + \cdots + \gamma_n K$ is the disjoint union of finitely many closed intervals.*

Proof. Let first K be a ulbd Cantor set, $K = C(\mathcal{I})$, and let $A = d(I_\varnothing) = d(K)$. Consider the following condition: for a given $l \in \mathbb{N} \setminus \{0\}$ require that $w \in W$ is such that

$$d(I_w) \geq A/\gamma_l, \quad d(I_{wj}) < A/\gamma_l \tag{8.1}$$

for at least one value of $j \in \{0, 1\}$. Clearly, this condition may or may not be verified by a finite word w. Let $\mathcal{B}_l$ be the set of words that pass the test, for a given value $l \in \mathbb{N} \setminus \{0\}$:

$$\mathcal{B}_l = \{w \in W : (8.1) \text{ holds}\}. \tag{8.2}$$

Recall that $\mathcal{I}_w$ is the induced construction on I_w defined in eq. (3.7) yielding the Cantor set $C(\mathcal{I}_w)$ in eq. (3.8). For all $l \in \mathbb{N} \setminus \{0\}$, K may be written as the union of a finite number of Cantor sets:

$$K = \bigcup_{w \in \mathcal{B}_l} C(\mathcal{I}_w). \tag{8.3}$$

In fact, because of eq. (3.4), for any $x \in K$, there exists $\tilde{w} \in \widetilde{W}$ such that $x \in I_{\tilde{w}_n}$ for every natural $n \in \mathbb{N}$ and $x \in C(\mathcal{I}_{\tilde{w}_n})$. When $n = 0$, $\tilde{w}_0 = \varnothing$ and $d(I_\varnothing) = A \geq A/\gamma_l$, since $\gamma_l > 1$ for any l. If either $d(I_0)$ or $d(I_1)$ is smaller than A/γ_l, then condition (8.1) holds for $w = \varnothing$. Otherwise $d(I_{\tilde{w}_1}) \geq A/\gamma_l$ and the same argument can be repeated for $\tilde{w}_1 0$ and $\tilde{w}_1 1$. Clearly, since $d(I_{\tilde{w}_n})$ tends to zero when n tends to infinity, there exists a value n such that (8.1) holds for $\tilde{w}_n$. This proves that K is enclosed in the right hand side (8.3). The other inclusion is trivial, since $C(\mathcal{I}_w) \subseteq K$ for any $w \in W$. By remark 3.5, only a finite number of words can verify the first inequality in (8.1), hence the cardinality of $\mathcal{B}_l$ is finite, for any $l \in \mathbb{N} \setminus \{0\}$.

Now, take $w \in \mathcal{B}_l$. Combining the first part of (8.1), that is $d(I_w) \geq A/\gamma_l$, with the second: there exists $j \in \{0, 1\}$ such that $A/\gamma_l > d(I_{wj}) = r(\mathcal{I})_{wj} d(I_w) \geq a d(I_w)$, where we have used the ulbd property, we obtain

$$\frac{A}{\gamma_l} \leq d(C(\mathcal{I}_w)) = d(I_w) < \frac{A}{a\gamma_l}. \tag{8.4}$$

Next, multiply all terms in eq. (8.3) by γ_l, to prove that for any value $l \in \mathbb{N} \setminus \{0\}$ the set $\gamma_l K$ can be written as a finite union of Cantor sets $\gamma_l C(\mathcal{I}_w)$, $w \in \mathcal{B}_l$, each of which has the following properties: it has uniformly lower bounded dissection larger than $a > 0$, and its diameter lies in the interval $[A, A/a]$.

Consider now the set $K_n = \gamma_1 K + \cdots + \gamma_n K$ in the thesis of the Lemma. Using eq. (8.3), it can be written as follows:

$$K_n = \bigcup_{w_1 \in \mathcal{B}_1, \ldots, w_n \in \mathcal{B}_n} \gamma_1 C(\mathcal{I}_{w_1}) + \cdots + \gamma_n C(\mathcal{I}_{w_n}). \tag{8.5}$$

We can now apply Lemma 7.3 to prove that there exists an integer n that depends only on a and A (and hence, *not* on the numbers γ_j) such that each term in the above union, $\gamma_1 C(\mathcal{I}_{w_1}) + \cdots + \gamma_n C(\mathcal{I}_{w_n})$, is an interval. Since the cardinality of each $\mathcal{B}_l$ is finite, it follows at once that for such value n, K_n is the union of a finite number of disjoint, closed intervals.

More complicated is the case when $K = K_\Psi$ is the attractor of a finite, non–singular, hyperbolic IFS. Denote again by $I_\varnothing$ the convex hull of K, $A = d(K) = d(I_\varnothing)$. Let δ and σ be as in eq. (1.5). As in section 4 we need to consider finite words in M letters, denoted by $v \in V$. Also recall that ψ_v denotes the composite map defined in eq. (2.1), with $v = i_1, \ldots, i_n$. Consider the diameters of the sets $\psi_v(K)$. Clearly, $d(\psi_v(K)) = d(\psi_v(I_\varnothing))$: to these latter we can apply Lemma 4.1, which proves that letting $c = \exp(-C)$, $C > 0$, computed as in eq. (4.5), we have

$$d(\psi_{vi}(K)) \geq c d(\psi_v(K)) \tag{8.6}$$

for any $v \in V$, $i = 1, \ldots, M$, denoting again by vi the composed word. We now replace condition (8.1) by the following: for $l \in \mathbb{N} \setminus \{0\}$ require that $v \in V$ satisfies

$$cA/\gamma_l < d(\psi_v(K)) \leq A/\gamma_l, \tag{8.7}$$

and define accordingly

$$\mathcal{B}_l = \{v \in V : \text{eq. (8.7) holds}\}. \tag{8.8}$$

The analogue of eq. (8.3) is now

$$K = \bigcup_{v \in \mathcal{B}_l} \psi_v(K). \tag{8.9}$$

To prove eq. (8.9) observe that for any $x \in K$ there exists $i_1 \in \{1, \ldots, M\}$ such that $x \in \psi_{i_1}(K)$, then there exists $i_2 \in \{1, \ldots, M\}$ such that $x \in \psi_{i_1,i_2}(K)$ and so proceeding: there exist $i_1, i_2, i_3, i_4, \cdots \in \{1, \ldots, M\}$ such that for every s we have $x \in \psi_{i_1,\ldots,i_s}(K)$. For $s = 0$ condition (8.7) is not verified: $\psi_\varnothing(K) = K$ and

$d(\psi_\varnothing(K)) = A > A/\gamma_l$. Choose the first index $s > 0$ such that $d(\psi_{i_1,\dots,i_s}(K)) \leq d(K)/\gamma_l$, which surely exists, since $d(\psi_{i_1,\dots,i_s}(K)) \leq \delta^s d(K)$. At the same time, since $d(\psi_{i_1,\dots,i_{s-1}}(K)) > d(K)/\gamma_l$, eq. (8.6) implies that

$$d(\psi_{i_1,\dots,i_s}(K)) \geq c d(\psi_{i_1,\dots,i_{s-1}}(K)) > c d(K)/\gamma_l$$

so that (8.7) holds for $v = i_1,\dots,i_s$ and therefore K is a subset of the union at right hand side of (8.9). The other inclusion is obvious from $K = U^n_\Psi(K)$. We can again prove that $\mathcal{B}_l$ has only finitely many elements: this follows easily from the inequality $d(\psi_{i_1,\dots,i_s}(K)) \leq \delta^s d(K)$ and from the first inequality in condition (8.7).

Equation (8.9) yields the analogue of eq. (8.5):

$$K_n = \bigcup_{v_1\in\mathcal{B}_1,\dots,v_n\in\mathcal{B}_n} \gamma_1\psi_{v_1}(K) + \cdots + \gamma_n\psi_{v_n}(K). \tag{8.10}$$

We need to analyze this union. At difference with the first part of the proof, we cannot apply Lemma 7.3 directly, because we have control of the diameter of the sets $\gamma_l\psi_{v_l}(K)$ (that are contained in the interval $[cA, A]$) but not of their nature: we do not know whether they are ulbd Cantor sets. Therefore, we continue as follows.

Let Ψ' be the two-maps IFS related to Ψ as in Lemma 2.5. Recall that Ψ' is obtained by selecting two maps out of the full set Ψ, in such a way to conserve the convex hull of the attractor. Let $K' := K_{\Psi'}$ be the attractor of Ψ'. We have $K' \subseteq K$ (Lemma 2.1, iii). By Lemma 4.2 K' is either a closed interval, or a Cantor set, with the same convex hull as K, $I_\varnothing$.

In the former case $K = K' = I_\varnothing$, and therefore $\gamma_1 K + \cdots + \gamma_n K$ is an interval, and the thesis of this Lemma follows easily.

The second case is more interesting: Ψ' is a two–maps, non–singular IFS, whose attractor K' is a Cantor set. Lemma 4.2 establishes that K' has a ulbd construction. Consider now the images $K'_v := \psi_v(K')$, with $v \in V$. Each of these is a Cantor set. We can easily prove that they too are of ulbd. In fact, a construction $\mathcal{I}^v$ for each of them can be obtained from $\mathcal{I}$ in Lemma 4.2, in analogy with eq. (3.7) as

$$I^v_{h(w)} = (\psi_v \circ \psi_w)(I_\varnothing) = \psi_{vw}(I_\varnothing),$$

where w is a finite word in the labels of the two maps that compose Ψ' and h is a permutation of the finite word w, constructed along the same lines of lemma 4.2. Lemma 4.1, which we have also used above, proves that the dissection ratios of K'_v are uniformly lower bounded by the value $c = \exp(-C) > 0$, eq. (4.5), computed over the full set of maps composing Ψ.

Let us now replace K by K' at right hand side in eq. (8.10). Since K and K' have the same convex hull, conditions (8.7) and (8.8) imply that the diameters of $\gamma_l \psi_{v_l}(K')$ are all contained in $[cA, A]$. Lemma 7.3 can now be applied, to prove that for sufficiently large n (that depends only on c, δ and A) the finite sum $\gamma_1 \psi_{v_1}(K') + \cdots + \gamma_n \psi_{v_n}(K')$ is an interval, for any choice of $v_1, \ldots, v_n$ in the respective sets $\mathcal{B}_1, \ldots, \mathcal{B}_n$. We must now resort to Lemma 7.2: setting $C^{(i)} = \gamma_i \psi_{v_i}(K')$, $D^{(i)} = \gamma_i \psi_{v_i}(K)$, $I_\varnothing$ the convex hull of K and $I^{(i)} = \gamma_i \psi_{v_i}(I_\varnothing)$, we are in the conditions of the Lemma, and thus every term in the union in the right hand side of (8.10) is also a closed interval. Then the Lemma follows again by the finite cardinality of each $\mathcal{B}_l$. □

Remark 8.2. Notice that in the previous proposition, one can multiply the real, positive numbers $\gamma_1, \ldots, \gamma_n$ by any strictly positive real constant, without changing the nature of the set K_n, so that the requirement $\gamma_j > 1$ in the hypothesis can be relaxed to $\gamma_j > 0$.

We can now prove the first main result of this paper, Theorem 1.1.

Let again K_Φ be the attractor of the IFS composed of the maps Φ in eqs. (1.7), which is the unique solution in $\mathcal{K}$ of the equation $\mathbf{U}_\Phi(K_\Phi) = K_\Phi$. Note that of course $\mathbf{U}_\Phi^n(K_\Phi) = K_\Phi$, where the map $\mathbf{U}_\Phi^n$ is defined by

$$\mathbf{U}_\Phi^n(A) = \bigcup_{\beta_n, \ldots, \beta_1 \in K} \phi_{\beta_n} \circ \cdots \circ \phi_{\beta_1}(A) \quad \text{for all } A \in \mathcal{K}. \tag{8.11}$$

Equation (8.11) easily follows by induction on n. We now need an algebraic formula.

Lemma 8.3. *The n-fold map composition in eq.* (8.11) *takes the following form: for every* $x \in \mathbb{R}$

$$\phi_{\beta_n} \circ \cdots \circ \phi_{\beta_1}(x) = \alpha^n x + \alpha^n (1 - \alpha) \sum_{i=1}^{n} \frac{\beta_i}{\alpha^i}. \tag{8.12}$$

Proof. We proceed by induction. For $n = 0$ and $n = 1$ the result is trivial. Suppose it holds for n. For $n + 1$ we have

$$\begin{aligned}\phi_{\beta_{n+1}} \circ \phi_{\beta_n} \circ \cdots \circ \phi_{\beta_1}(x) &= \phi_{\beta_{n+1}}(\phi_{\beta_n} \circ \cdots \circ \phi_{\beta_1}(x)) \\ &= \phi_{\beta_{n+1}}\Big(\alpha^n x + \alpha^n(1-\alpha)\sum_{i=1}^{n} \frac{\beta_i}{\alpha^i}\Big)\end{aligned}$$

$$= \alpha\Big(\alpha^n x + \alpha^n(1-\alpha)\sum_{i=1}^{n}\frac{\beta_i}{\alpha^i}\Big) + (1-\alpha)\beta_{n+1}$$

$$= \alpha^{n+1}x + \alpha^{n+1}(1-\alpha)\sum_{i=1}^{n+1}\frac{\beta_i}{\alpha^i}.$$ □

Proof of Theorem 1.1. Because of eq. (8.11), for any $n \in \mathbb{N}$

$$K_\Phi = \bigcup_{\beta_n,\dots,\beta_1\in K} \phi_{\beta_n}\circ\cdots\circ\phi_{\beta_1}(K_\Phi) = \bigcup_{x\in K_\Phi}\ \bigcup_{\beta_n,\dots,\beta_1\in K}\{\phi_{\beta_n}\circ\cdots\circ\phi_{\beta_1}(x)\}. \tag{8.13}$$

Use now eq. (8.12) to get

$$\bigcup_{\beta_n,\dots,\beta_1\in K}\{\phi_{\beta_n}\circ\cdots\circ\phi_{\beta_1}(x)\} = \alpha^n x + \alpha^n(1-\alpha)\sum_{i=1}^{n}\frac{K}{\alpha^i}. \tag{8.14}$$

Because of Proposition 8.1 we can choose n in such a way that $\alpha^n(1-\alpha)\sum_{i=1}^{n}\frac{K}{\alpha^i}$ is a finite union of closed intervals, call it $\mathcal{J}$. The term $\alpha^n x$ in eq. (8.14) merely shifts these intervals, so that eq. (8.13) becomes

$$K_\Phi = \bigcup_{x\in K_\Phi}\alpha^n x + \mathcal{J}. \tag{8.15}$$

Since K_Φ is bounded and closed and since eq. (8.15) shows that K_Φ is the union of shifted intervals in $\mathcal{J}$, K_Φ is itself a finite union of closed intervals. □

9. Series of Cantor sets

In this section we generalize the results of the previous section to series of compact sets of the form (1.10), $\sum_{j=0}^{\infty}\alpha_j K$, where K is either the attractor of an IFS or a Cantor set of ulbd. Recall that we use the Minkowski sum of sets, eq. (1.8), in combination with the Hausdorff metric in $\mathcal{K}$, eq. (1.1).

A simple lemma is useful:

Lemma 9.1. *Let K and T be two non–empty compact subsets of $\mathbb{R}$. Then,*

$$d_H(K, K+T) \le \max\{|t|, t\in T\}. \tag{9.1}$$

Proof. We estimate the two quantities at right hand side of (1.1) separately. The first is

$$\max\{d(k, K+T), k \in K\} = \max_{k\in K} \min_{k'\in K, t\in T} d(k, k'+t) \leq \max_{k\in K} \min_{t\in T} d(k, k+t) = \min_{t\in T} |t|.$$

The second is

$$\max_{k'\in K, t\in T} \min_{k\in K} d(k'+t, k) \leq \max_{k'\in K, t\in T} d(k', k'+t) = \max_{k'\in K, t\in T} |t| = \max_{t\in T} |t|. \qquad \square$$

The following lemma characterizes a sort of absolute convergence of infinite series of the kind (1.10).

Lemma 9.2. *Let $\sum_{j=0}^{\infty} \alpha_j$ be a convergent series of real positive entries and let K be a non–empty compact set in $\mathbb{R}$. Then, the series $\sum_{j=0}^{\infty} \alpha_j K$ is also convergent. Any permutation of its terms yields the same value for the sum of the series.*

Proof. Let $S_n = \sum_{j=0}^{n} \alpha_j K$ be the n-th partial sum of the above series. Clearly, if s is the sum of the series of real numbers, S_n is a subset of the compact set $[-s\bar{k}, s\bar{k}]$, where $\bar{k} = \max\{|k|, k \in K\}$. Let us prove that S_n compose a Cauchy sequence in $\mathcal{K}$, the set of compact non–empty subsets of $[-s\bar{k}, s\bar{k}]$. In fact, let $m > n \in \mathbb{N}$:

$$d_H(S_n, S_m) \leq \sum_{j=n}^{m-1} d_H(S_j, S_{j+1}) \leq \bar{k} \sum_{j=n}^{m-1} \alpha_{j+1}, \tag{9.2}$$

where we have used Lemma 9.1, since $S_{j+1} = S_j + \alpha_{j+1} K$. At right hand side of the above equation we find the difference of two partial summations of the real sequence, which is itself Cauchy and therefore can be made arbitrarily small if n is sufficiently large: this proves that S_n is a Cauchy sequence. Since $\mathcal{K}$ is complete, the sequence of sets S_n is convergent and we call S its sum.

Let now Ξ be any permutation of the set of natural numbers, Ξ_n the set of its first $n+1$ elements, *i.e* $\Xi_n = \Xi(\{0, 1, \ldots, n\})$ and $S_n(\Xi) = \sum_{j\in\Xi_n} \alpha_j K$. With these notations the previous partial sum S_n coincides with $S_n(\text{Id})$, being Id the identity permutation. What proven above implies that the limit $S(\Xi) = \lim_{n\to\infty} S_n(\Xi)$ exists for any permutation Ξ. We want to prove that it coincides with $S = S(\text{Id})$, the sum of the original series. In fact, for any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $\bar{k} \sum_{j=m+1}^{\infty} \alpha_j < \epsilon$, since the series of positive real numbers α_j

is convergent. Moreover, there exists $n' \geq m$ such that $\mathrm{Id}_m = \{0, 1, \dots, m\} \subseteq \Xi_n$ for all $n \geq n'$ and clearly also $\mathrm{Id}_m \subseteq \mathrm{Id}_n$. This yields

$$S_n(\Xi) = \sum_{j=0}^{m} \alpha_j K + \sum_{j \in \Xi_n,\, j>m} \alpha_j K = S_m + \sum_{j \in \Xi_n,\, j>m} \alpha_j K.$$

Using Lemma 9.1 this means that

$$d_H(S_m, S_n(\Xi)) \leq \max \Big\{ |t|, t \in \sum_{j \in \Xi_n,\, j>m} \alpha_j K \Big\} \leq \bar{k} \sum_{j=m+1}^{\infty} \alpha_j < \epsilon.$$

The same calculation proves that $d_H(S_m, S_n) \leq \bar{k} \sum_{j=m+1}^{\infty} \alpha_j < \epsilon$, so that

$$d_H(S_n(\Xi), S_n) \leq 2\epsilon$$

for any $n \geq n'$. Since ϵ is arbitrary this proves that $S(\Xi) = S$. □

We can now prove the second theorem of this paper.

Proof of Theorem 1.4. Because of lemma 9.2 we can rearrange terms in the series (1.10) – without changing its sum – so that α_j becomes a monotonic sequence. Next, write the finite summation

$$\sum_{j=0}^{n} \alpha_j K = \frac{\alpha_n}{\eta} \Big(\eta K + \frac{\eta \alpha_{n-1}}{\alpha_n} K + \frac{\eta \alpha_{n-2}}{\alpha_n} K + \cdots + K \frac{\eta \alpha_0}{\alpha_n} \Big), \tag{9.3}$$

where $\eta > 1$. Letting $\gamma_j = \eta \frac{\alpha_{n-j+1}}{\alpha_n}$ yields a monotonic, non decreasing sequence of positive numbers, with $\gamma_1 = \eta > 1$. The set in brackets at right hand side of eq. (9.3) can then be written as $K_n = \gamma_1 K + \cdots + \gamma_{n+1} K$. To this set we can apply the previous Proposition 8.1, to show that there exists $n \in \mathbb{N} \setminus \{0\}$ such that K_n is a finite union of disjoint intervals, and so is obviously also $\frac{\alpha_n}{\eta} K_n$, that can therefore be written as $\bigcup_{s=1}^{S} F_s$, F_s being closed intervals. As a consequence, we obtain

$$\sum_{j=0}^{\infty} \alpha_j K = \sum_{j=0}^{n} \alpha_j K + \sum_{j=n+1}^{\infty} \alpha_j K = \Big(\bigcup_{s=1}^{S} F_s \Big) + G,$$

where $G \in \mathcal{K}$ is the compact set, sum of the convergent series $\sum_{j=n+1}^{\infty} \alpha_j K$ (that exists because of Lemma 9.2). Indicating by g the elements of the set G, we finally write

$$\sum_{j=0}^{\infty} \alpha_j K = \bigcup_{g \in G} \bigcup_{s=1}^{S} (F_s + g).$$

The left hand side of the above is a compact set, again because of Lemma 9.2. The right hand side is the union of shifted intervals of positive minimal length $l = \min_s d(F_s)$. Hence, the left hand side is also a finite union of closed, disjoint intervals. □

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References

[1] R. Anisca and C. Chlebovec, On the structure of arithmetic sums of Cantor sets with constant ratios of dissection. *Nonlinearity* **22** (2009), no. 9, 2127–2140. MR 2534296 Zbl 1185.28011

[2] W. C. Abram and J. C. Lagarias, Intersections of multiplicative translates of 3-adic Cantor sets. *J. Fractal Geom.* **1** (2014), no. 4, 349–390. MR 3299817 Zbl 1332.11074

[3] C. Bandt, Self-similar sets. I: Topological Markov chains and mixed self-similar sets. *Math. Nachr.* **142** (1989), 107–123. MR 1017373 Zbl 0707.28004

[4] M. F. Barnsley, *Fractals everywhere*. Academic Press, Inc., Boston, MA, 1988. MR 0977274 Zbl 0691.58001

[5] M. F. Barnsley and S. G. Demko, Iterated function systems and the global construction of fractals. *Proc. Roy. Soc. London Ser. A* **399** (1985), no. 1817, 243–275. MR 0799111 Zbl 0588.28002

[6] M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, Almost periodic Jacobi matrices associated with Julia sets for polynomials. *Comm. Math. Phys.* **99** (1985), no. 3, 303–317. MR 0795106 Zbl 0574.58016

[7] M. F. Barnsley, D. C. Wilson, and K. Leśniak, Some recent progress concerning topology of fractals. In K. P. Hart, J. van Mill, and P. Simon (eds.), *Recent progress in general topology*. III. Atlantis Press, Paris, 2014, 69–92. MR 3205482 Zbl 1322.54002

[8] M. F. Barnsley and A. Vince, Developments in fractal geometry. *Bull. Math. Sci.* **3** (2013), no. 2, 299–348. MR 3084010 Zbl 1275.28006

[9] M. F. Barrozo and U. Molter, Countable contraction mappings in metric spaces: invariant set and measure. *Cent. Eur. J. Math.* **12** (2014), no. 4, 593–602. MR 3152173

[10] D. Bessis and G. Mantica, Orthogonal polynomials associated to almost periodic Schrödinger operators. *J. Comput. Appl. Math.* **48** (1993), no. 1-2, 17–32. MR 1246849 Zbl 0793.35023

[11] C. Cabrelli, K. Hare, and U. Molter, Sums of Cantor sets yielding an interval. *J. Aust. Math. Soc.* **73** (2002), no. 3, 405–418. MR 1936262 Zbl 1019.28002

[12] C. Cabrelli, F. Mendivil, U. M. Molter, and R. Shonkwiler, On the Hausdorff h-measure of Cantor sets. *Pacific J. Math.* **217** (2004), no. 1, 45–59. MR 2105765 Zbl 1063.28006

[13] J. H. Elton and Z. Yan, Approximation of measures by Markov processes and homogeneous affine iterated function systems. *Constr. Approx.* **5** (1989), no. 1, 69–87. MR 0982725 Zbl 0658.60093

[14] H. Fernau, Infinite iterated function systems. *Math. Nachr.* **170** (1994), 79–91. MR 1302367 Zbl 0817.28006

[15] K. H. Hare and T. C. O'Neil, N-fold sums of Cantor sets. *Mathematika* **47** (2000), no. 1-2, 243–250. MR 1924501 Zbl 1024.28006

[16] M. Hille, Remarks on limit sets of infinite iterated function systems. *Monatsh. Math.* **168** (2012), no. 2, 215–237. MR 2984148 Zbl 1272.28006

[17] J. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747. MR 0625600 Zbl 0598.28011

[18] H. Kunze, D. La Torre, F. Mendivil, and E. R. Vrscay, *Fractal-based methods in analysis.* Springer, Berlin etc., 2012. MR 3014680 Zbl 1237.28002

[19] K. Leśniak, Extremal sets as fractals. *Nonlinear Anal. Forum* **7** (2002), no. 2, 199–208. MR 1959879 Zbl 1161.54301

[20] K. Leśniak, Random iteration for infinite nonexpansive iterated function systems. *Chaos* **25** (2015), no. 8, article id. 083117, 5 pp. MR 3390405

[21] B. B. Mandelbrot, *Fractals*: *forms, chance and dimension.* Translated from the French. Revised edition. W. H. Freeman and Co., San Francisco, CA, 1977. MR 0471493 Zbl 0376.28020

[22] G. Mantica, Dynamical systems and numerical analysis: the study of measures generated by uncountable I.F.S. *Numer. Algorithms* **55** (2010), no. 2-3, 321–335. MR 2720635 Zbl 1200.65037

[23] G. Mantica, Direct and inverse computation of Jacobi matrices of infinite IFS. *Numer. Math.* **125** (2013), no. 4, 705–731. MR 3127328 Zbl 1282.65050

[24] G. Mantica, On the attractor of one–dimensional infinite iterated function systems. *Int. J. Appl. Nonlinear Science* **1** (2013), 87–99. Zbl 1291.28010

[25] D. Mauldin and M. Urbański, Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc.* (3) **73** (1996), no. 1, 105–154. MR 1387085 Zbl 0852.28005

[26] F. Mendivil, A generalization of IFS with probabilities to infinitely many maps. *Rocky Mountain J. Math.* **28** (1998), no. 3, 1043–1051. MR 1657044 Zbl 0995.37010

[27] A. Mihail and R. Miculescu, The shift space for an infinite iterated function system. *Math. Rep.* (*Bucur.*) **11(61)** (2009), no. 1, 21–32. MR 2506506 Zbl 1199.28030

[28] M. Moran, Hausdorff measure of infinitely generated self-similar sets. *Monatsh. Math.* **122** (1996), no. 4, 387–399. MR 1418125 Zbl 0862.28008

[29] P. A. P. Moran, Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.* **42** (1946). 15–23. MR 0014397 Zbl 0063.04088

[30] J. Palis and J. C. Yoccoz, On the arithmetic sum of regular Cantor sets. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), no. 4, 439–456. MR 1464530 Zbl 0895.58020

[31] S. Pedersen and J. D. Phillips, On intersections of Cantor sets: self-similarity. *Commun. Math. Anal.* **16** (2014), no. 1, 1–30. MR 3161733 Zbl 1298.28020

[32] R. H. Riedi and B. B. Mandelbrot, Multifractal formalism for infinite multinomial measures. *Adv. in Appl. Math.* **16** (1995), no. 2, 132–150. MR 1334146 Zbl 0866.28005

[33] R. Schneider, *Convex bodies*: *the Brunn–Minkowski theory*. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1993. MR 1216521 Zbl 0798.52001

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