

Ergodic theory of Kusuoka measures

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Abstract. In the analysis on self-similar fractal sets, the Kusuoka measure plays an important role (cf. [13], [7], and [2]). Here we investigate the Kusuoka measure from an ergodic theoretic viewpoint, seen as an invariant measure on a symbolic space. Our investigation shows that the Kusuoka measure generalizes Bernoulli measures and their properties to higher dimensions of an underlying finite dimensional vector space. Our main result is that the transfer operator on functions has a spectral gap when restricted to a certain Banach space that contains the Hölder continuous functions, as well as the highly discontinuous g -function associated to the Kusuoka measure. As a consequence, we obtain exponential decay of correlations. In addition, we provide some explicit rates of convergence for a family of generalized Sierpiński gaskets.

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1. Introduction

1.1. Background and problems. The Kusuoka measure has recently attracted some attention, since it gives rise to a well-working Laplacian on fractal sets (see, e.g., [15]). The Laplacian is usually defined weakly with respect to a measure on the fractal set. A standard way of accomplishing this is to first define a Dirichlet energy form $\mathcal{E}(f, f)$ on the fractal K , in analogy with $\int |\nabla f|^2 d\mu(x)$, and then to define the Laplacian by equating the corresponding bilinear form $\mathcal{E}(u, v)$ with $-\int (\Delta_\mu u)v d\mu$, for functions v vanishing on the boundary. It is well-known that with respect to the normalized Hausdorff measure, the domain of the Laplacian is not even closed under multiplication. By contrast, the Kusuoka measure is well-behaved in this sense and in some other more subtler ways, e.g., for the Laplacian it provides Gaussian heat kernel estimates with respect to the effective resistance metric and can be regarded as a second order differential operator [10].

Recently, Strichartz and his collaborators ([2] and [16]) have proved some basic properties of the Kusuoka measure. Here we provide an investigation of the Kusuoka measure from the point of view of ergodic theory on symbolic shift spaces. For instance, we provide exponential mixing results as a consequence of the quasi-compactness of a transfer operator as it acts on a Banach space which contains functions that may have a dense set of discontinuity points, but which can be regarded as “smooth” when they are integrated with respect to the Kusuoka measure. In fact, the associated transfer operator is given by a simple multiplication when acting on a certain space of matrix-valued processes. However, when restricting the transfer operator to ordinary functions, the corresponding transition probability function has a dense set of discontinuity points, which presents difficulties.

Our abstract way of treating the Kusuoka measure is rather similar to the one in the original work by Kusuoka ([13]) and covers in fact a general class of measures that can be defined by products of matrices. We point out that the Kusuoka measure is really a family of measures that generalizes the Bernoulli measures to higher dimensions. We also note that the theory of matrix product state representations of quantum Potts models (see e.g. [14]) seems to be quite related, although we have not used any particular result from this theory.

We believe that our analysis opens the door to interesting further research. For example, it should now be possible to compute the entropy of the measure explicitly. In view of our exponential mixing results, it should then be possible to provide a multifractal formalism for the Kusuoka measure. A major challenge would be to generalise the type of results we provide here for matrices (as our restriction maps) to infinite dimensional operators. Using infinite dimensional

operators, one could hope to be able treat the Kusuoka measure on fractal sets with infinite boundaries, such as that of the Sierpiński carpet. However, it is not immediately clear how one should define the Kusuoka measure even in the case of the Vicsek set, which has a countably infinite boundary.

Other challenges in the fractal realm would include, e.g., the problem of relating our results to the Cartesian product of a Sierpiński gasket with itself, or if one glues together the boundary points of two such copies, producing a “fractafold.”

1.2. Summary of the main results. We prove quasicompactness of a transfer operator defined on a Banach space, with a norm that is an integrated Hölder norm in terms the variations of functions on cylinder sets of a symbolic space. In some sense, we are studying the transfer operator of a space of “Besov type,” since the moral is that we look at a “smooth” space that may have many discontinuities (since we integrate), and this is necessary in order to handle the dense set of discontinuity points of the g -function that defines the transfer operator.

To be more precise: Let S be a finite set and let \mathcal{X} denote the symbolic space $\mathcal{X} = S^{\mathbb{Z}_+}$ ($\mathbb{Z}_+ = \{0, 1, 2, \dots\}$) of functions $x: \mathbb{Z}_+ \rightarrow S$. The (point) shift map $T: \mathcal{X} \rightarrow \mathcal{X}$ is defined as $(Tx)(n) = x(n + 1)$. In our abstract setting, the Kusuoka measure ν ([13]) is a shift-invariant measure on the space \mathcal{X} . The transfer operator L is the dual of the shift operator $Tf = f \circ T$ on the Hilbert-space of functions $L^2(\mathcal{X}, \nu)$. It has the form

$$Lf(x) = \sum_{s \in S} g(sx) f(sx)$$

where the g -function can be defined as

$$g(x) = \lim_{n \rightarrow \infty} \frac{\nu([x]_n)}{\nu([Tx]_{n-1})},$$

where $[x]_n$ denote the cylinder of length n containing x .

Given a real number $\gamma, 0 < \gamma < 1$, we define for $f \in L^2(\mathcal{X}, \nu)$ a Banach space $L^2_\gamma \subset L^2$ with norm $\|f\|_{L^2_\gamma}$ by

$$\|f\|_{L^2_\gamma} = \sum_{n=0}^{\infty} \gamma^{-n} \|f^{(n)}\|_{L^2}.$$

Here $(f^{(n)})_{n=0}^\infty$ is the martingale difference sequence for f given by $f^{(0)} = f_0$ and $f^{(n)}(x) = f_n(x) - f_{n-1}(x)$ for $n \geq 1$, and where $x \mapsto f_n(x) := f([x]_n)$ is the

orthogonal projection of f onto the finite dimensional subspace L_n^2 of $L^2(\mathcal{X}, \nu)$ of \mathcal{F}_n -measurable functions, where \mathcal{F}_n is the σ -algebra generated by cylinder sets with length n .

Quasicompactness of L on our ‘‘Besov space’’ L_γ^2 , means that there exists $0 < \rho < 1$ such that for any $f \in L_\gamma^2$, where γ is sufficiently close to one, we have

$$\left\| L^m f - \int f d\mu \right\|_{L_\gamma^2} \leq C\rho^m \|f\|_{L_\gamma^2} \quad (1)$$

for a uniform constant C .

We prove (1) by representing L as a *dilation* of a transfer operator \mathcal{L} defined on a larger graded Hilbert-space $\mathbb{V} = \overline{\lim} \mathbb{V}_n$ consisting of matrix-valued processes. The graded Hilbert space $L^2 = \overline{\lim} L_n^2$ is isometrically embedded into \mathbb{V} . It is fairly straightforward to show that quasi-compactness holds for \mathcal{L} on \mathbb{V}_γ and, since $L = Q \circ \mathcal{L}$ where $Q: \mathbb{V} \rightarrow L^2$ is the orthogonal projection. This result carries over to L on L_γ^2 for those γ such that Q is continuous as an operator from \mathbb{V}_γ to L_γ^2 .

From the quasicompactness result (1), exponential decay of correlations (mixing at an exponential rate) follows automatically: if $f \in L^2(\mathcal{X}, \nu)$ and $g \in L_\gamma^2$ then for some $0 < \rho < 1$ and some uniform constant C , we have

$$\left| \int f(g \circ T^n) d\nu - \int f d\nu \int g d\nu \right| \leq C\rho^n.$$

We note that our quasicompactness results depend on the general symbolic formulation, where we use the ultrametric on the symbolic space \mathcal{X} and not some underlying geometric distance. Hence, the quasicompactness on our ‘‘Besov space’’ will not immediately translate into quasicompactness on a Besov space defined on the metric space of an underlying fractal, such as those of Jonsson [6] and Grigor’yan [4].

1.3. More results and the structure of the paper. In Section 2, we present the Kusuoka measure from an abstract point of view, namely on cylinder sets, which corresponds to the products of matrices that act on a finite-dimensional vector space that corresponds to a space of harmonic functions. We give two special examples. The first shows that the Kusuoka measure in one dimension reduces to the class of Bernoulli measures. We can thus view abstract the Kusuoka measure as a natural generalisation of the Bernoulli measure, the difference being that we ‘‘multiply matrices instead of numbers.’’ The second example is a brief discussion of a well-studied case, that of the Sierpiński gasket, extensively studied in [1], [2], [15], [16], and in many other works.

In Section 2, we state the main results: quasicompactness of the transfer operator on the space L_γ^2 , as well as exponential decay of correlations. We also consider special results for the Sierpiński gasket and the family SG_n , defined in Subsection 3.2. In Theorem 4, we obtain precise mixing rates of convergence in a simplified case, when we shift cylinders of a fixed length. We have only stated this result for the Sierpiński gasket, but we have made some calculations for the mixing rates for SG_n , $n = 3, 4, 5$; see Example 3.

In Section 4, we introduce a Hilbert space \mathbb{V} on which a transfer operator that acts on matrix-valued operators is easily analysed in terms of the matrix operator M , defined in (6). Here we obtain a simple expression for the transfer operator as the dual of the shift map T , so here the “higher-dimension” generalisation of Bernoulli measures is exploited. The proof of Theorem 6 that states the quasicompactness on a space \mathbb{V}_γ , equipped with a certain “smooth” norm $\|\cdot\|_\gamma$, relies essentially on the contraction of the matrix-operator M and the contraction ratio $\theta_1 < 1$ remains the same. In Subsection 4.5, we obtain a strict contraction of M acting on symmetric matrices and this is used to obtain more precise rates of convergence (Theorem 4) in the case of the Sierpiński gasket.

In Section 5, we prove that the quasicompactness result in Section 4, for the matrix-valued space \mathbb{V}_γ , may be retrieved for *functions* in L_γ^2 , by means of a projection; see Lemma 9 and its proof in Subsection 5.2, the most technical and difficult part of the paper. In Lemma 9 a new contraction factor $\theta_2 < 1$ is introduced and the final contraction ratio ρ expressed in terms of the quasicompactness of Theorem 1 must be strictly larger than both θ_1 and θ_2 . It remains an open problem, even in the case of the Sierpiński gasket, whether $\theta_2 = \theta_1$. In Subsection 5.3, we restrict our attention to the Sierpiński gasket and prove Theorem 4.

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2. The Kusuoka measure

2.1. Cylinders and cylinder sets. An elementary *cylinder* is a function

$$\alpha: [a, b) \longrightarrow S$$

defined on some integer interval $[a, b) = \{a, a + 1, \dots, b - 1\}$. The *length* of the cylinder is $\ell(\alpha) = b - a$. The corresponding *cylinder set* $\alpha \subset \mathcal{X}$ is the set of $x \in \mathcal{X}$ that coincides with α on $[a, b)$. (Notice that we make no notational distinction between a cylinder and the equivalent cylinder set.)

A cylinder is an *initial cylinder* if the domain is $[a, b) = [0, n)$ for some n and we write S^n instead of $S^{[0, n)}$, and also S^* for the set $\cup_n S^n$ of initial cylinders. The set S^0 consists of the empty cylinder \emptyset . Let $[x]_n$ denote the initial cylinder obtained by restricting x to the interval $[0, n)$. Let \mathcal{F}_n be the algebra generated by the cylinder sets $[x]_n$, $x \in \mathcal{X}$ and let \mathcal{F} be the limit σ -algebra as $n \rightarrow \infty$.

For a cylinder $\alpha \in S^{[a, b)}$ and a symbol $s \in S$, an expression of the form αs is understood as the concatenation of the cylinder with the symbol to the right, so that αs is a cylinder in $S^{[a, b+1)}$ with $(\alpha s)(b) = s$. If $a > 0$ then $s\alpha$ is the corresponding concatenation to the left, but, if $\alpha \in S^n$ is an initial cylinder then $s\alpha \in S^{n+1}$ with $(s\alpha)(0) = s$ and $(s\alpha)(k) = \alpha(k - 1)$, $k = 1, \dots, n + 1$. The expression sx refers in the same way to the concatenated and shifted sequence $sx \in \mathcal{X}$, where $(sx)(0) = s$ and $sx(n) = x(n - 1)$, $n \geq 1$.

2.2. Construction of an abstract Kusuoka measure. In order to define the Kusuoka measure, we consider a fixed finite dimensional Hilbert space H having scalar product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B} = \mathcal{B}(H)$ denote the space of bounded operators on H . For any cylinder $\alpha \in S^{[a, b)}$, we associate the compound “restriction map”

$$A(\alpha) = A_{\alpha(a)} \dots A_{\alpha(b-1)},$$

where $A_s \in \mathcal{B}$, $s \in S$ are operators with certain properties specified later. We define the *Kusuoka measure* ν on the cylinder set $\alpha = \{x: x|_{[a, b)} = \alpha\} \subset \mathcal{X}$ as the trace

$$\nu(\alpha) = \text{Tr}(A(\alpha)^* \mathcal{E} A(\alpha)), \quad (2)$$

where \mathcal{E} is a positive definite symmetric operator $H \rightarrow H$ such that $\text{Tr}(\mathcal{E}) = 1$.

The definition (2) defines a consistent probability measure on the measurable space $(\mathcal{X}, \mathcal{F})$ if and only if the system $\{A_s: s \in S\}$ of maps satisfies the following two conditions

$$\sum_s A_s^* \mathcal{E} A_s = \mathcal{E} \quad (3)$$

and

$$\sum_s A_s A_s^* = I. \quad (4)$$

Consistency of definition of ν follows: e.g. (4) gives that

$$\sum_s \nu(s\alpha) = \sum_s \text{Tr}(A_s^* A(\alpha)^* \mathcal{E} A(\alpha) A_s) = \text{Tr}(A(\alpha)^* \mathcal{E} A(\alpha) I) = \nu(\alpha),$$

so ν is consistent with extensions to the left. Similarly, (3) shows that $\sum_s \nu(\alpha s) = \nu(\alpha)$. It is also clear that ν will be a shift invariant measure on $(\mathcal{X}, \mathcal{F})$, since $\nu(\alpha)$ is determined by the *word* corresponding to the cylinder $\alpha \in S^{[a,b]}$.

As is shown in [13], the Kusuoka measure is moreover *ergodic* if one assumes that the system is *irreducible* in the sense that

the linear maps $A_s, s \in S$, have no common nontrivial invariant subspace W . (5)

That is, there exists no subspace $W, (0) \subsetneq W \subsetneq H$, such that $A_s(W) \subset W$ for all $s \in S$.

We will consider the space $\mathcal{B} = \mathcal{B}(H)$ of operators on H . Note that, if we define the operators $M: \mathcal{B} \rightarrow \mathcal{B}$ and $M^*: \mathcal{B} \rightarrow \mathcal{B}$ by

$$M(B) = \sum_s A_s B A_s^*, \quad \text{and} \quad M^*(B) = \sum_s A_s^* B A_s \quad (6)$$

then (3) and (4) can be expressed as a statement of fixed points, i.e. that $M(I) = I$ and $M^*(\mathcal{E}) = \mathcal{E}$. The operator $M^*(B)$ is the adjoint of M on \mathcal{B} with respect to the Hilbert–Schmidt scalar product $\langle A, B \rangle_{HS} = \text{Tr}(B^* A)$.

We will often use the the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ with associated norm $\|A\|_{\mathcal{E}} = \langle A, A \rangle_{\mathcal{E}}^{1/2}$ given by

$$\langle A, B \rangle_{\mathcal{E}} = \text{Tr}(\mathcal{E} A B^*) = \text{Tr}(B^* \mathcal{E} A), \quad A, B \in \mathcal{B}. \quad (7)$$

Notice that $\nu(\alpha) = \|A(\alpha)\|_{\mathcal{E}}^2$ and that (3) and (4) are equivalent to the statement that the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is “bi-invariant” in the sense that

$$\begin{aligned} \langle X, Y \rangle_{\mathcal{E}} &= \sum_{\alpha \in S^k} \langle A(\alpha) X, A(\alpha) Y \rangle_{\mathcal{E}} \\ &= \sum_{\alpha \in S^k} \langle X A(\alpha), Y A(\alpha) \rangle_{\mathcal{E}}, \quad \text{for all } X, Y \in \mathcal{B}, \end{aligned} \quad (8)$$

for all $k \geq 0$.

For our main results, we use an irreducibility condition, implying (5), stating that for some $k > 1$

$$c_k = \inf_F \sum_{\alpha \in S^k} \langle A(\alpha)F, A(\alpha) \rangle_{\mathcal{E}}^2 > 0 \quad (9)$$

where the infimum (minimum) is over the compact set of all symmetric operators $F \in \mathcal{B}$ such that $\|F\|_{\mathcal{E}} = 1$ and $\langle F, I \rangle_{\mathcal{E}} = 0$. Notice that $c_k < 1$, since, by Cauchy–Schwarz, we have

$$\sum_{\alpha \in S^k} \langle A(\alpha)F, A(\alpha) \rangle_{\mathcal{E}}^2 \leq \sum_{\alpha} \|A(\alpha)F\|_{\mathcal{E}}^2 \cdot \|A(\alpha)\|_{\mathcal{E}}^2 = \sum_{\alpha} \|A(\alpha)F\|_{\mathcal{E}}^2 \cdot \nu(\alpha),$$

where we conclude from the irreducibility condition (5) that $\nu(\alpha) < 1$. Moreover, (8) implies that $\sum_{\alpha \in S^k} \|A(\alpha)F\|_{\mathcal{E}}^2$ equals $\|F\|_{\mathcal{E}}^2 = 1$.

It is not clear to us in what circumstances the condition (9) is a consequence of the irreducibility condition (5). Note that the stronger irreducibility condition (9) follows if the maps $B \mapsto A_s B A_s^*$ have no non-trivial common invariant subspace of \mathcal{B} .

2.3. Examples. The Kusuoka measure can usefully be viewed as a general construction for a large class of shift invariant measures.

Example 1 (Bernoulli measure). The product form of Kusuoka measure shows that it is a natural generalisation of the Bernoulli measure. Indeed, in the special case when $H = \mathbb{R}$ and A_s is $v \mapsto q_s v$, where (4) states that $q_1^2 + \dots + q_k^2 = 1$. In this case, ν is the Bernoulli measure associated to the distribution $p(s) = q_s^2$ on S . The energy operator is here the identity operator, which clearly has trace 1. Notice that the irreducibility condition (9) is trivially satisfied in this case.

Example 2 (classical Kusuoka measure on SG). The terminology we use comes from applications in the context of harmonic analysis on certain fractals: the space H is the finite dimensional space of harmonic functions modulo constants on a self-similar fractal K with a prescribed finite “boundary.”

The *restriction map* A_s for a symbol $s \in S$ represents the restriction of harmonic functions to one of the $|S|$ sub-fractals K_s , $s \in S$. The quadratic form \mathcal{E} on H is an *energy form* which the harmonic functions in H are minimising. By self-similarity we have an isomorphism $K_s \cong K$ and, by using this isomorphism and a suitable scaling, we can represent the restriction of harmonic functions to K_s as a map $A_s: H \rightarrow H$. The invariance relation (3) follows since the energy on the whole fractal is the sum of the energies on the sub-fractals. There is

also a unique dual invariant form \mathfrak{R} on H^* , but we identify \mathfrak{R} with a given inner product on the Hilbert space H . Hence we obtain the relation (4).

A well-studied example is the Sierpiński gasket, SG, which is the unique nonempty compact set satisfying

$$\text{SG} = \bigcup_{i=0}^2 F_i \text{SG},$$

where $F_i = \frac{1}{2}(x + q_i)$, and where $\{q_i\}_{i=0}^2$ are the vertices of an equilateral triangle. These three points are also the boundary points of SG. We obtain the Kusuoka measure on SG (see, e.g., [1], [2], [15], and [16]) in the special case $S = \{0, 1, 2\}$ and corresponding matrices $A_s = R^{-s}DR^s$, where R is the rotation by $2\pi/3$, and where

$$D = \begin{pmatrix} \frac{3}{\sqrt{15}} & 0 \\ 0 & \frac{1}{\sqrt{15}} \end{pmatrix}.$$

For a non-zero harmonic function h , the *energy measure* ν_h is defined on an elementary cylinder $[w] = \{x: [x]_k = w\}$ by

$$\nu_h([w]) = \mathcal{E}(A_w h, A_w h),$$

where we have lifted the restriction of a harmonic function on

$$F_w \text{SG} = F_{w_1} \dots F_{w_n} \text{SG}$$

to an element $A_w h$, which is also a harmonic function on SG. We have

$$\mathcal{E}(h, h) = \sum_{w \in S^k} \mathcal{E}(A_w h, A_w h),$$

where one should observe that the usual normalisation constants are built into the restriction maps A_s , and also that

$$A_w = A_{w_k} A_{w_{k-1}} \dots A_{w_1},$$

if $w = w_1 w_2 w_3 \dots w_k$. We can for instance choose the basis of two harmonic functions (see [16]), $h_1 = \frac{\sqrt{2}}{3}(1, -\frac{1}{2}, -\frac{1}{2})$ and $h_2 = \frac{1}{\sqrt{6}}(0, 1, -1)$. We can obtain the Kusuoka measure on SG as the sum $\nu = \nu_{h_1} + \nu_{h_2}$ of energy measures for the orthonormal basis of H . In this case the restriction maps are symmetric matrices, whence $M = M^*$ and $\mathcal{E} = (1/2)I$. We obtain by direct computation that the action of M on the subspace of symmetric matrices in \mathcal{B} is given by

$$M \left(\begin{bmatrix} a & b \\ b & -a \end{bmatrix} + cI \right) = \frac{4}{5} \cdot \begin{bmatrix} a & b \\ b & -a \end{bmatrix} + cI. \quad (10)$$

Similarly, it contracts with a factor $4/5$ on the space of anti-symmetric matrices. It follows that M acts as a contraction on the space of trace-less matrices with the contraction constant $\theta_1 = 4/5$, which is one of the constants that will be important to us in the sequel in order to describe mixing rates. From this it follows (Corollary 5 below) that if $A \in \mathcal{F}_k$ (measurable with respect to cylinders sets of length k) and $B \in \mathcal{F}$ (any Borel set), we have

$$|\nu(T^{-(n+k)} A \cap B) - \nu(A)\nu(B)| \leq 2\left(\frac{4}{5}\right)^n.$$

3. Results

3.1. A spectral gap for the transfer operator on the associated Banach space.

A standard approach to studying ergodic properties of T -invariant measures on \mathcal{X} is to use transfer operators defined on spaces of functions. In particular, if we consider the real Hilbert space $L^2(\mathcal{X}, \nu)$ with the scalar product $\langle f, g \rangle = \int fg d\nu$ and norm $\|f\| = \langle f, f \rangle^{1/2}$ then we can define the *transfer operator* $L: L^2(\mathcal{X}, \nu) \rightarrow L^2(\mathcal{X}, \nu)$ as the dual of the shift map, i.e.,

$$\langle Lf, g \rangle = \langle f, g \circ T \rangle \tag{11}$$

for $f, g \in L^2(\mathcal{X}, \nu)$. It is easy to see that that the operator norm of L is one and that it has a maximum modulus eigenvalue with the constant function 1 as the normalised eigenvector.

The operator L takes the explicit form

$$Lf(x) = \sum_s g(sx) f(sx)$$

where the *g-function*, $g: X \rightarrow [0, 1]$, can be defined as

$$g(x) = \lim_{n \rightarrow \infty} \frac{\nu([x]_n)}{\nu([Tx]_{n-1})}.$$

The *g-function* exists, on account of the martingale convergence theorem, ν -almost everywhere. Bell, Ho and Strichartz [2] showed that the *g-function* associated to the Kusuoka measure for the Sierpiński gasket has a dense countable family of discontinuities. In particular, the Kusuoka measure is not a Gibbs measure and therefore not amenable to the classical thermodynamic ideas.

We say that an operator L on a Banach space has a *spectral gap* if it has a unique eigenvalue λ of maximum modulus and if all other elements of the

spectrum of L has modulus less than some $\rho < |\lambda|$. In order to prove that there is a spectral gap for the operator L , one usually needs to restrict it to a more regular class of functions which is considerably smaller than L^2 . For the Kusuoka measure, because of the discontinuities in the g -function, it is not appropriate to consider, say, Hölder continuous functions. Instead we consider functions where the martingale sequence converges in L^2 -norm quickly enough.

Any element $f \in L^2(\mathcal{X}, \nu)$ can be uniquely represented by the corresponding *martingale process*

$$f(\alpha) = \mathbb{E}[f \mid \alpha] = \nu(\alpha)^{-1} \int_{\alpha} f \, d\nu, \quad \alpha \in S^*.$$

We will usually refer to the martingale process $f(\alpha)$ by the same name as the element $f(x)$ in $L^2(\mathcal{X}, \nu)$. The function $x \mapsto f_n(x) := f([x]_n)$ is the orthogonal projection onto the finite dimensional subspace $m\mathcal{F}_n$ of $L^2(\mathcal{X}, \nu)$ of \mathcal{F}_n -measurable functions and by the martingale convergence theorem, we have $\lim_n f_n(x) = f(x)$, ν -almost everywhere. The *martingale difference sequence* $(f^{(n)})_{n=0}^\infty$ of f is given by $f^{(0)} = f_0$ and $f^{(n)}(x) = f_n(x) - f_{n-1}(x)$ for $n \geq 1$.

Given a real number γ , $0 < \gamma < 1$, we define for $f \in L^2(\mathcal{X}, \nu)$ a norm $\|f\|_\gamma$ by

$$\|f\|_\gamma = \sum_{n=0}^\infty \gamma^{-n} \|f^{(n)}\|. \tag{12}$$

The space of functions $f: X \rightarrow \mathbb{R}$ such that the γ -norm $\|f\|_\gamma$ is finite is denoted L^2_γ . We observe that L^2_γ is a Banach space which is dense in $L^2(\mathcal{X}, \nu)$. One can perhaps think of it as a type of Besov space.

Note also that if $f: X \rightarrow \mathbb{R}$ is α -Hölder continuous in the sense that

$$\text{var}_n f = \sup_{[x]_n=[y]_n} |f(x) - f(y)| = O(2^{-\alpha n})$$

then it belongs to L^2_γ for $\gamma > 2^{-\alpha}$.

Our main result involves proving a spectral gap for the transfer operator L if we restrict L to the spaces L^2_γ . Let c_k , $0 \leq c_k < 1$ be as in the irreducibility condition (9).

Theorem 1. *Assume that the irreducibility condition (9) holds and that γ is as above. Define*

$$\theta_2 := \inf_k (1 - c_k)^{1/k} < 1.$$

Then, for $\gamma > \theta_2$, the transfer operator L restricts to a continuous operator $L: L^2_\gamma \rightarrow L^2_\gamma$ having a spectral gap.

Remark 1. The bound $\theta_2 = (1 - c_k)^{1/k}$ is not meant to be optimal; it is based on an argument where we use a pointwise estimate of a convergence rate $Q_n G$ of projections.

As a simple consequence of our results we have the following result, which expresses how quickly the Kusuoka measure can be approximated.

Corollary 2. *There exists $0 < \rho < 1$ such that for any $f \in L^2_\gamma$*

$$\left\| L^m f - \int f d\mu \right\|_{L^2_\gamma} \leq C \rho^m \|f\|_{L^2_\gamma}$$

where C is a uniform constant.

The rate of convergence, ρ , depends on both θ_2 and a contraction constant θ_1 in (29), or, equivalently, (27). From Theorem 6 it follows that we may choose $\rho > \max\{\theta_1, \gamma\}$, where $\gamma > \theta_2$, as above.

As a consequence, we have exponential decay of correlations.

Corollary 3. *If $f \in L^2(\mathcal{X}, \nu)$ and $g \in L^2_\gamma$ (e.g., an α -Hölder continuous function, if $\gamma = 2^{-\alpha}$), then for some $0 < \rho < 1$ and some uniform constant C , we have*

$$\left| \int f(g \circ T^n) d\nu - \int f d\nu \int g d\nu \right| \leq C \rho^n.$$

3.2. Specialisation to Sierpiński gaskets. We now specialize to some explicit estimates of rates of convergence for the cases that correspond to the Sierpiński gasket SG and the family SG_n , $n = 2, 3, \dots$ ($SG = SG_2$), which are realized in \mathbb{R}^2 and constructed by $n(n+1)/2$ contraction mappings $F_j(x) = x/n + b_{j,n}$ for suitable choices of $b_{j,n}$, so that SG_n is the unique nonempty compact set that satisfies

$$SG_n = \bigcup_{j=1}^{\frac{1}{2}n(n+1)} F_j(SG_n).$$

By a direct computation of θ_1 , we obtain the following result for SG. In Example 3 have included the corresponding rates for SG_n , $n = 3, 4, 5$. These explicit approximation results depend on the fact that for SG_n we have symmetric restriction maps A_s .

Theorem 4. For any function f which is \mathcal{F}_k -measurable (i.e., measurable with respect to the finite algebra generated by cylinders of length k), we have

$$\left\| L^{n+k} f(x) - \int f \, d\nu \right\|_{\infty} \leq 2 \left(\frac{4}{5} \right)^n \|f\|_1, \quad (13)$$

where $\|\cdot\|_{\infty}$ denotes the essential supremum norm.

Corollary 5. If ν is the Kusuoka measure on three symbols related to the SG, we have for $A \in \mathcal{F}_k$ and $B \in \mathcal{F}$ that

$$|\nu(T^{-(n+k)} A \cap B) - \nu(A)\nu(B)| \leq 2 \left(\frac{4}{5} \right)^n.$$

Remark 2. In this simplified case, the rate of convergence can be expressed in terms of $\theta_1 = \frac{4}{5}$ only. In Theorem 1 we also need to consider the constant θ_2 from Lemma 9 below in order to obtain the uniform rate of convergence expressed, e.g., in Corollary 2. Notice that in Theorem 4 we use members of \mathcal{F}_k as test functions and we need to start the convergence at this level k , whereas in Corollary 2 we may use any $f \in L^2_{\nu}$ and we do not relate the number of iterates to the (lack of) regularity of f . Nevertheless, Theorem 4 may give some insight about the rate of convergence from a practical point of view. We have given an argument for this special case only for SG, just in order to simplify matters, but a similar argument for this type of result for convergence in the $\|\cdot\|_{\infty}$ -norm may be devised in the general Kusuoka measure case. The constant 2 in front of the $(\frac{4}{5})^n$ can be interpreted as the dimension of the space of harmonic functions modulo constants, i.e., the number of boundary points minus one. That we do not have a “general” uniform, but unknown, constant C in front of the $(\frac{4}{5})^n$ is due to a strict contraction result, Lemma 8, which we have obtained for symmetric restriction maps, and which is thus valid for all SG_n .

Remark 3. If we have the probability weights $p_j(x) = \frac{1}{15} + \frac{12}{15} \frac{dv_j}{dv}$, $j = 0, 1, 2$, for the iterated function system $\{F_j\}_{j=0}^2$ that defines the Sierpiński gasket (as in Bell, Ho and Strichartz [2]), where ν_j are the energy measures so that the Kusuoka measure $\nu = \sum_{j=0}^2 \nu_j$, then a standard conjugation between symbolic space and the fractal SG gives the same rate of convergence (namely $(4/5)^n$ for SG) with respect to the essential supremum norm for an associated transfer operator defined on Hölder continuous functions f on SG as $Lf(x) = \sum_{j=0}^2 p_j(x) f(F_j(x))$. Notice that we can view the natural extension of the full left shift on the symbol space as an iterated function system, where the probability $p_j(x)$ of choosing the symbol j to go from the state $x = (x_0, x_1, \dots)$ to (j, x_0, x_1, \dots) is given by $g(jx)$.

Example 3. We have explicitly computed the rate of convergence in the case of SG_n . For SG_3 , the level 3 Sierpiński gasket, is generated by the iterated function system $F_j(x) = \frac{1}{3}x + \frac{2}{3}v_j$, $j = 1, 2, 3, 4, 5, 6$, where v_1, v_2, v_3 are the vertices of an equilateral triangle and where $v_4 = \frac{v_2+v_3}{2}$, $v_5 = \frac{v_1+v_3}{2}$, $v_6 = \frac{v_1+v_2}{2}$. We approximate SG_3 with a graph sequence, and we use the same two initial orthonormal harmonic functions as in the case of SG (the three boundary points are the same). We obtain two families of matrices (the restriction maps) A_s with three in each family being rotations by 120° of each other. That is, we have one family of three matrices that restricts values to the three triangles with one vertex at the original vertex points v_1, v_2, v_3 and another family of three matrices that restricts values to the three other triangles. There are similar and obvious ways to describe the other fractals in the family SG_n .

In these cases we get $\theta_1 = \frac{5}{7}$ for SG_3 , $\theta_1 = \frac{2822}{4223}$ for SG_4 and $\theta_1 = \frac{209527}{327611}$ for SG_5 .

4. The transfer operator on the space \mathbb{V}

Instead of working with the L^2 -space of regular functions on \mathcal{X} , the idea is to work with a Hilbert space \mathbb{V} consisting of “operator valued process limits,” where the shift operator \mathbb{T} is defined. The action of the corresponding transfer operator has a simple explicit description. The Hilbert space $L^2(\mathcal{X}, \nu)$ has a representation as a subspace \mathbb{L} of \mathbb{V} . The space \mathbb{L} itself is not invariant under the action of \mathcal{L} , but the transfer operator L on $L^2(\mathcal{X}, \nu)$ can be recovered as a dilation such that $L^k = Q \circ \mathcal{L}^k$, where Q is the orthogonal projection onto \mathbb{L} . We show in Subsection 5.2 that Q is continuous with respect to the γ -norm, for suitable γ , and, hence, that the spectral properties of \mathcal{L} on \mathbb{V} carry over to results on the action of L on the spaces $L^2(\mathcal{X}, \nu)$.

4.1. Construction of a graded Hilbert space of process limits. We will use a general construction of a certain graded Hilbert space of process limits on \mathcal{X} under a given system of “restriction operators” ψ_s , $s \in S$. The Hilbert space is modeled by the martingale representation of functions in $L^2(\mathcal{X}, \mu)$, but with the difference that they do not necessarily converge to functions. The construction can be generalised to non-self similar systems using a systematic approach based on direct and inverse limits.

4.1.1. E -valued processes and finite degree process limits. Let E denote a finite-dimensional linear space. An E -valued process is a function $f: S^* \rightarrow E$,

where S^* is the set of initial cylinders. A process $f(\alpha)$, $\alpha \in S^*$, can be identified with the sequence $f_n(x)$ of \mathcal{F}_n -measurable functions $f_n(x) = f([x]_n)$.

Let \mathbb{E}_0 denote the direct sum $\bigoplus_{\alpha} E_{\alpha}$, where for all initial cylinders $\alpha \in S^*$, E_{α} is a copy of E . We interpret \mathbb{E}_0 as the space of all processes $f(\alpha)$ such that there is a smallest integer $\text{deg}(f) \geq 0$ where $f(\alpha) = 0$ for all cylinders α of length $\ell(\alpha) > \text{deg}(f)$. We refer to $\text{deg}(f)$ as the *degree* of $f \in \mathbb{E}_0$.

We will need a construction which, more formally (and more generally), involves taking direct and inverse limits. Given a set $\psi = \{\psi_s : s \in S\}$ (a *self similar* system of restriction maps) of linear maps $\psi_s : E \rightarrow E$, let $\Psi : \mathbb{E}_0 \rightarrow \mathbb{E}_0$ be the map $f \mapsto \Psi f$ given by

$$(\Psi f)(\alpha s) = \psi_s f(\alpha), \quad (\Psi f)(\emptyset) = f(\emptyset).$$

We assume that $\Psi : \mathbb{E}_0 \rightarrow \mathbb{E}_0$ is an injective map.

Let \mathbb{E}_* be the space $\mathbb{E}_0 = \bigoplus_{\alpha} E_{\alpha}$ modulo the subspace $\text{Ker}(I - \Psi)$. The space \mathbb{E}_* is the space of limit orbits for the map Ψ . Elements f in \mathbb{E}_* can be represented by E -valued processes which are “eventually constant” in the following sense: there is a smallest number $\text{deg}(f) \geq 0$ where $f(\alpha s) = \psi_s f(\alpha)$ for $\ell(\alpha) > \text{deg}(f)$. Two processes, $f(\alpha)$ and $g(\alpha)$ of degree at most n are identified if $f(\alpha) = g(\alpha)$ for all $\alpha \in S^n$. The finite dimensional space \mathbb{E}_n of elements $f \in \mathbb{E}_*$ of degree $\text{deg}(f) \leq n$, consists of those elements $f \in \mathbb{E}_*$ that can be written on the form $f = \tilde{f} + \text{Ker}(I - \Psi)$ where $\tilde{f} \in \mathbb{E}_0$ has degree less than or equal to n .

4.1.2. Invariant bilinear forms. A bilinear form \mathcal{F} on \mathbb{E}_* is *local* if it has the form

$$\mathcal{F}(f, g) = \lim_{n \rightarrow \infty} \sum_{\alpha \in S^n} \mathcal{F}(\alpha)(f(\alpha), g(\alpha)), \quad \text{for all } f, g \in \mathbb{E}_*, \quad (14)$$

where, for each α , $\mathcal{F}(\alpha)$ is a given form on E . The local form \mathcal{F} on \mathbb{E}_* is well defined if, for all $\alpha \in S^*$, we have the invariance condition

$$\mathcal{F}(\alpha)(u, v) = \sum_s \mathcal{F}(\alpha s)(\psi_s u, \psi_s v), \quad \text{for all } u, v \in E. \quad (15)$$

In this case the limit in (14) is the limit of an eventually constant sequence.

In particular, a given fixed form \mathcal{E} on E gives a *constant* invariant form on \mathbb{E}_* if and only if

$$\mathcal{E}(u, v) = \sum_s \mathcal{E}(\psi_s u, \psi_s v), \quad \text{for all } u, v \in E. \quad (16)$$

4.1.3. The Hilbert space of E -valued processes and the orthogonal decomposition. Given a restriction system $\psi = \{\psi_s\}$ and an local invariant positive definite form \mathcal{F} satisfying (15), we obtain a non-degenerate inner product $\langle f, g \rangle_{\mathbb{E}}$ on \mathbb{E}_* as the limit in (14). We let $\mathbb{E} = \mathbb{E}(\psi, \mathcal{F})$ be the Hilbert space obtained as the completion of \mathbb{E}_* with respect to the norm

$$\|f\|_{\mathbb{E}} = (\langle f, f \rangle_{\mathbb{E}})^{1/2}.$$

The spaces \mathbb{E}_n of processes of degree less than n are closed subspaces of \mathbb{E} . We can define $\mathbb{E}^{(n)} := \mathbb{E}_n \ominus \mathbb{E}_{n-1}$ as the orthogonal complement of \mathbb{E}_{n-1} inside \mathbb{E}_n . This gives us an orthogonal decomposition of \mathbb{E} , so that any $f \in \mathbb{E}$ has a unique expression $f = \sum_{n=0}^{\infty} f^{(n)}$, where $f^{(n)} \in \mathbb{E}^{(n)}$ and $\|f\|_{\mathbb{E}}^2 = \sum_{n=0}^{\infty} \|f^{(n)}\|_{\mathbb{E}}^2$. This orthogonal decomposition lets us express a process limit f by a *unique* representative process

$$f(\alpha) = f^{(0)}(\alpha) + \cdots + f^{(n)}(\alpha), \quad \text{for } \alpha \in S^n.$$

For a number $\gamma \in (0, 1)$, we define the Banach space \mathbb{E}_γ with the norm $\|x\|_\gamma$

$$\|f\|_{\mathbb{E}_\gamma} = \sum_{i=0}^{\infty} \gamma^{-i} \|f^{(i)}\|_{\mathbb{E}}.$$

4.1.4. The martingale representation of $L^2(\mathcal{X}, \mu)$. Note that for any probability measure μ on \mathcal{X} , we can construct the martingale representation of the space $L^2(\mathcal{X}, \mu)$ as a graded Hilbert space according to the scheme above as follows: let E be the space \mathbb{R} of real numbers and let the restriction system $\{\psi_s\}$ be given by $\psi_s(x) = x$. This gives the usual restriction of functions and the corresponding limits in \mathbb{E}_* are locally constant functions: the spaces \mathbb{E}_n , $n \geq 0$, will correspond to the spaces of \mathcal{F}_n -measurable functions. We use a non-constant local invariant form $\mathcal{F}(\alpha)$ given by $\mathcal{F}(\alpha)(x, y) = \mu(\alpha)xy$ and the invariance condition (15) holds since

$$\sum_s \mu(\alpha s)xy = \mu(\alpha)xy.$$

Since, for $f, g \in \mathbb{E}_n$, the limit $\langle f, g \rangle_{\mathbb{E}}$ in (14), gives

$$\langle f, g \rangle_{\mathbb{E}} = \sum_{\alpha} f(\alpha)g(\alpha)\mu(\alpha) = \int f(x)g(x) d\mu(x).$$

The closure \mathbb{E} of \mathbb{E}_* will hence give an isometric copy of the space $L^2(\mathcal{X}, \mu)$.

4.2. The operator valued Hilbert space \mathbb{V} . We show (by copying some arguments given by Kusuoka in [13]) that, starting from a system $\{A_s\}$ of restriction maps $A_s: H \rightarrow H$ on a finite dimensional space H , we can define a Hilbert space $\mathbb{V} = \mathbb{V}(\{A_s\})$ of process limits taking values in the space $\mathcal{B}(E)$ of linear operators on E . The inner product $\langle \cdot | \cdot \rangle$ on \mathbb{V} is induced by a constant bi-invariant (see (19)) positive definite bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ on \mathcal{B} . The space \mathbb{V} also allow us to define the shift operator $\top f, f \in \mathbb{V}$, as an isometric injective map.

Kusuoka’s paper [13] starts with a self-similar system $\{A_s\}$ of injective maps on a finite dimensional space H which is irreducible in the sense (5). It is proved that, modulo a re-scaling (i.e. we replace A_s with λA_s for some $\lambda > 0$), there exists a unique invariant positive definite form \mathcal{E} on H . In addition, there is a corresponding *dual invariant* positive definite form \mathcal{R} defined on H^* , such that

$$\mathcal{R}(z, w) = \sum_s \mathcal{R}(A_s^* z, A_s^* w), \quad \text{for all } z, w \in H^*. \tag{17}$$

From the pair \mathcal{E} and \mathcal{R} , we define the inner product $\langle A, B \rangle_{\mathcal{E}}$ on the space of operators $\mathcal{B}(E) \cong H \otimes H^*$ by setting

$$\langle u \otimes v^*, f \otimes g^* \rangle_{\mathcal{E}} = \mathcal{E}(u, f) \mathcal{R}(v^*, g^*), \quad u, f \in H, v^*, g^* \in H^*, \tag{18}$$

for rank one operators and then extend it by bilinearity. If we have a Hilbert space structure on H (and H^*) so that \mathcal{E} and \mathcal{R} are represented as symmetric operators in \mathcal{B} then the inner product is given by $\langle A, B \rangle_{\mathcal{E}} = \text{Tr}(B^* \mathcal{E} A \mathcal{R})$. In particular, if we assume that the inner product on H is the form \mathcal{R} — so that \mathcal{R} is represented by the identity operator — then we see that (16) and (17) take the forms (3) and (4), which were our starting points.

The system $\{A_s\}$ acts on $F \in \mathcal{B}(E)$ both from the left, $A_s F = A_s \circ F$, and from the right, $F A_s = F \circ A_s$. The form $\langle F, G \rangle_{\mathcal{E}}$ is then both left and right invariant, i.e.

$$\langle F, G \rangle_{\mathcal{E}} = \sum_s \langle A_s F, A_s G \rangle_{\mathcal{E}} = \sum_s \langle F A_s, G A_s \rangle_{\mathcal{E}}, \tag{19}$$

on account of \mathcal{E} satisfying (16) and \mathcal{R} satisfying (17). In particular, we can consistently define the corresponding Kusuoka measure on $(\mathcal{X}, \mathcal{F})$ by taking $\nu(\alpha) = \langle A(\alpha), A(\alpha) \rangle_{\mathcal{E}}$, where $A(\alpha)$ denotes the composition $A_{\alpha_1} \circ \dots \circ A_{\alpha_n}$.

The left invariance of $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ in (19), states that the form $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ on the finite dimensional space \mathcal{B} is invariant with respect to the restriction system $\psi_s(F) = A_s F$. We obtain, by the general construction above, a graded Hilbert space \mathbb{V} with a scalar product

$$(F | G) = \lim_{n \rightarrow \infty} \sum_{\alpha \in S^n} \langle F(\alpha), G(\alpha) \rangle_{\mathcal{E}}. \tag{20}$$

The grading means that \mathbb{V} has the orthogonal decomposition $\bigoplus_{n \geq 0} \mathbb{V}^{(n)}$ so that

$$(F | G) = \sum_{n \geq 0} (F^{(n)} | G^{(n)}).$$

An element $G(\alpha s) \in \mathbb{V}_n$, $\alpha s \in S^n$, belongs to $\mathbb{V}^{(n)}$, $n \geq 1$, if and only if the relation

$$\sum_s A_s^* \mathcal{E} G(\alpha s) = \mathbf{0} \quad (21)$$

holds for all $\alpha \in S^{n-1}$. This follows since if $F(\alpha) \in \mathbb{V}_{n-1}$ then

$$(F | G) = \sum_{\alpha} \sum_s \langle G(\alpha s), A_s F(\alpha) \rangle_{\mathcal{E}} = \sum_{\alpha} \text{Tr} \left(\left(\sum_s A_s^* \mathcal{E} G(\alpha s) \right) F(\alpha) \right).$$

This can be zero for arbitrary F only if G satisfies (21).

A process $F \in \mathbb{V}$ belongs to $\mathbb{V}^{(0)}$ if and only if $F(\alpha) = A(\alpha) F_0$ for some constant operator $F_0 = F(\emptyset) \in \mathcal{B}$. The process $A(\alpha)$ denotes the ‘‘identity process’’ $A \in \mathbb{V}^{(0)}$ with $A(\emptyset) = I$.

4.3. The shift operator and the transfer operator on \mathbb{V} and a spectral gap.

Because of the right invariance in (19), we can furthermore consistently define the left shift operator \mathbb{T} as the injective map $\mathbb{T}: \mathbb{V} \rightarrow \mathbb{V}$ given by

$$\mathbb{T} F(s\alpha) = F(\alpha) A_s. \quad (22)$$

It is an isometric embedding in the sense that $(F | G) = (\mathbb{T} F | \mathbb{T} G)$. Note also that if $F \in \mathbb{V}_*$ has finite degree, then $\deg(\mathbb{T} F) = \deg(F) + 1$ and that the space $\mathbb{V}^{(k)}$, $k \geq 0$, is embedded by \mathbb{T} into the space $\mathbb{V}^{(k+1)}$, since \mathbb{T} manifestly preserves the orthogonality condition (21).

The *transfer operator* $\mathcal{L}: \mathbb{V} \rightarrow \mathbb{V}$ is defined as the dual of \mathbb{T} , i.e. $\mathcal{L}: \mathbb{V} \rightarrow \mathbb{V}$ satisfies $(\mathcal{L} F | G) = (F | \mathbb{T} G)$ for all $F, G \in \mathbb{V}$. For $F, G \in \mathbb{V}_*$, we have

$$\begin{aligned} (F | \mathbb{T} G) &= \sum_{\alpha} \sum_s \text{Tr}(A_s^* G(\alpha)^* \mathcal{E} F(s\alpha)) \\ &= \sum_{\alpha} \text{Tr} \left(G(\alpha)^* \mathcal{E} \left(\sum_s F(s\alpha) A_s^* \right) \right) \end{aligned}$$

provided $\ell(\alpha)$ is large enough. It follows that the operator $\mathcal{L}: \mathbb{V} \rightarrow \mathbb{V}$ is explicitly given by the expression

$$(\mathcal{L} F)(\alpha) = \sum_{s \in S} F(s\alpha) A_s^*, \quad (23)$$

which is a simple multiplicative operator, analogous to the transfer operator for a Bernoulli measure.

Since it is dual to the isometric embedding T , it must be that \mathcal{L} is a contraction, i.e. $\|\mathcal{L} F\| \leq \|F\|$. From (4), it is also clear that the process $A(\alpha)$ is an eigenvector to \mathcal{L} corresponding to eigenvalue $\lambda = 1$, since $\sum_s A(\alpha) A_s A_s^* = A(\alpha)$. The operator \mathcal{L} acts as a reverse shift operator on the spaces $\mathbb{T}^k(\mathbb{V})$, $k > 0$. Its essential spectral radius is therefore 1.

Note also that \mathcal{L} preserves the space of constants $\mathbb{V}^{(0)}$: from (23) it follows that if $G(\alpha) = A(\alpha)G_0$, $\alpha \in S^*$, then

$$\mathcal{L} G(\alpha) = A(\alpha) M(G_0), \tag{24}$$

where, $M: \mathcal{B} \rightarrow \mathcal{B}$ is the operator defined in (6).

For $0 < \gamma \leq 1$, we define the Banach space \mathbb{V}_γ by the norm

$$\|F\|_\gamma = \sum_{n=0}^{\infty} \gamma^{-n} \|F^{(n)}\|. \tag{25}$$

We want to show that the operator $\mathcal{L}_\gamma = \mathcal{L}|_{\mathbb{V}_\gamma}$, i.e. \mathcal{L} restricted to \mathbb{V}_γ , has a spectral gap. Recall that the spectrum $\sigma(\mathcal{L})$ of an operator $\mathcal{L}: \mathbb{V} \rightarrow \mathbb{V}$ is the set of complex numbers z such that the operator $zI - \mathcal{L}$ is not invertible.

Theorem 6. *Assume $0 < \gamma < 1$ and consider the operator \mathcal{L}_γ of \mathcal{L} restricted to \mathbb{V}_γ . Then \mathcal{L}_γ has the eigenvalue $\lambda = 1$ corresponding to the unique eigenvector A . There is also a constant $\theta_1 < 1$, such that $\sigma(\mathcal{L}_\gamma) \setminus \{1\}$ is contained in the disc of radius $\rho = \max\{\gamma, \theta_1\}$.*

Note that the essential spectral radius of \mathcal{L}_γ is γ .

4.4. Proof of Theorem 6. We prove the following two bounds. Firstly that

$$\|\mathcal{L} X\|_\gamma = \gamma^{-k+1} \|(\mathcal{L} X)^{(k-1)}\| \leq \gamma \|X\|_\gamma, \quad \text{for all } X \in \mathbb{V}^{(k)}, k \geq 1. \tag{26}$$

Secondly, we show that for some $\theta_1 < 1$ and some $C \geq 1$, we have that

$$\|\mathcal{L}^k X\| \leq C \theta_1^k \|X\|, \quad \text{for all } X \in \mathbb{V}^{(0)}, (X|A) = 0. \tag{27}$$

For $\rho = \max\{\theta_1, \gamma\}$, it follows from (26) and (27) that for

$$Y = Y^{(0)} + Y^{(1)} + \dots,$$

such that $(Y | A) = (Y^{(0)} | A) = 0$, we have

$$\begin{aligned} \|\mathcal{L}^k Y\|_\gamma &\leq \sum_{j=0}^{\infty} \|\mathcal{L}^k Y^{(j)}\|_\gamma \\ &\leq \sum_{j=0}^k \gamma^j \cdot C\theta_1^{k-j} \|Y^{(j)}\|_\gamma + \sum_{j=k+1}^{\infty} \gamma^k \cdot \|Y^{(j)}\|_\gamma. \end{aligned}$$

Here we use that $(\mathcal{L}^j Y^{(j)} | A) = 0$ which follows from the observation that

$$(\mathcal{L} X | A) = (X | T A) = (X | A).$$

Hence,

$$\|\mathcal{L}^k Y\|_\gamma \leq C\rho^k \cdot \sum_{j=0}^{\infty} \|Y^{(j)}\|_\gamma = C\rho^k \cdot \|Y\|_\gamma.$$

This shows that the spectral radius of \mathcal{L} restricted to the space of the elements Y in \mathbb{V}_γ such that $(Y | A) = 0$ is less than ρ .

4.4.1. Proof of the bound (26). The bound (26) is a consequence of \mathcal{L} being a contraction, that is

$$\|\mathcal{L} X\| \leq \|X\|.$$

Since \mathbb{T} is an isometric embedding any ON-basis $\{E_i\}$ of \mathbb{V} is transported to an ON-basis $\{\mathbb{T} E_i\}$ of $\mathbb{T}(\mathbb{V})$ and, by Parseval's identity, the squared norm satisfies

$$\|\mathcal{L} X\|^2 = \sum_i (\mathcal{L} X | E_i)^2 = \sum_i (X | \mathbb{T} E_i)^2 \leq \|X\|^2. \quad (28)$$

If $X \in \mathbb{V}^{(k)}$, $k \geq 0$, then we find that $\mathcal{L} X \in \mathbb{V}^{(k-1)}$, since the dual operator, \mathbb{T} , restricts to an isometric embedding of $\mathbb{V}^{(k-1)}$ into $\mathbb{V}^{(k)}$. Taking an ON-basis $E_l^{(j)}$ of $\mathbb{V}^{(j)}$ and using (28) shows that

$$\|\mathcal{L} X\| = \|(\mathcal{L} X)^{(k-1)}\| = \|\mathcal{L} \circ \text{proj}_W X\| \leq \|X\|,$$

where proj_W is the orthogonal projection onto the subspace $W = \mathbb{T}(\mathbb{V}^{(k-1)})$.

4.4.2. Proof of the bound (27). For convenience, we extend the setting to complex matrices in order to include the case of anti-symmetric matrices. Let \mathcal{D} be the space of Hermitian (self-adjoint) operators B such that $\langle B, I \rangle_{\mathcal{E}} = \text{Tr}(\mathcal{E}B) = 0$. We extend the operator M to complex matrices and \mathcal{D} is then an M -invariant subspace, since

$$\text{Tr}(\mathcal{E} M(B)I) = \text{Tr}(M^*(\mathcal{E})B) = \text{Tr}(\mathcal{E}B)$$

on account of (3).

Lemma 7. *Assume (5) holds. There are constants $\theta_1 < 1$ such that for any $B \in \mathcal{D}$ we have*

$$\|M^k(B)\| \leq C_0 \theta_1^k \|B\|_{\mathcal{E}}, \quad (29)$$

for some $C_0 > 0$.

In order to show (27) it is enough, by (24), to consider the operator M in (6) acting on \mathcal{B} . Since any element in \mathcal{B} uniquely can be represented as an orthogonal sum of an symmetric and anti-symmetric operator, it suffices to analyse the action of M restricted to \mathcal{D} , since the map $B \mapsto i \cdot B$ is an isomorphism between the space \mathcal{D} and the M -invariant space of anti-Hermitian matrices.

4.4.3. Proof of Lemma 7. We can take θ_1 as the maximum eigenvalue of the operator M restricted to \mathcal{D} .

For a Hermitian operator $B \in \mathcal{D}$, let $\sigma(B) = \{\lambda_i\} \subset \mathbb{R}$ denote its spectrum. Let $\sigma(M(B)) = \{\gamma_j\} \subset \mathbb{R}$ denote the spectrum of $M(B) \in \mathcal{D}$. We have the spectral decompositions

$$B = \sum_i \lambda_i P_i \quad \text{and} \quad M(B) = \sum_i \lambda_i M(P_i) = \sum_j \gamma_j Q_j, \quad (30)$$

where P_i and Q_j refers to systems of orthogonal projections such that $\sum_i P_i = \sum_j Q_j = I$. Let, for the moment, $\langle A, B \rangle := \text{Tr}(AB^*)$ denote the Hilbert–Schmidt scalar product on $\mathcal{B} = \mathcal{B}(H)$. Then $\{P_i\}$ and $\{Q_j\}$ are orthogonal sets under $\langle \cdot, \cdot \rangle$. (We have $P_i P_{i'} = 0$ if $i \neq i'$.) Taking the orthogonal projection in the direction of Q_j of both sides in (30) gives, for each j , the equation

$$\left(\sum_i c_{ij} \lambda_i \right) Q_j = \gamma_j Q_j \quad (31)$$

where

$$c_{ij} = \frac{\langle M(P_i), Q_j \rangle}{\langle Q_j, Q_j \rangle} \quad \text{and} \quad \sum_i c_{ij} = \frac{\langle I, Q_j \rangle}{\langle Q_j, Q_j \rangle} = 1, \quad (32)$$

since $\sum_i M(P_i) = M(I) = I$ by (4). The coefficients c_{ij} cannot be negative, since

$$\langle M(P_i), Q_j \rangle = \text{Tr}(M(P_i)Q_j) = \text{Tr}(Q_j M(P_i) Q_j) \geq 0. \tag{33}$$

This is a consequence of the positivity of M , i.e. that M preserves the cone of positive semidefinite matrices so that $M(P_i)$, and hence $Q_j M(P_i) Q_j$, both are positive semi-definite.

Thus (31) expresses each eigenvalue γ_j of $M(B)$ as a convex combination of the eigenvalues $\{\lambda_j\}$. In particular, if we order them so that $\gamma_1 > \gamma_2 > \dots$ and $\lambda_1 > \lambda_2 > \dots$ then

$$\gamma_1 = \sum_i c_{i1} \lambda_i \leq \lambda_1.$$

Since M is an operator on finite dimensional space, there is some eigenvector $B \in \mathcal{D}$ corresponding to $\theta \in \sigma(M)$ of maximum modulus. If we assume that B is an eigenvector of M then $\gamma_j = \theta \lambda_j$ and we can assume that $Q_j = P_j$.

If we assume that $|\theta| = 1$, then it must hold that $c_{11} = 1$ and $c_{i1} = 0$ for $i = 2, \dots$. But that is equivalent to the equalities

$$\langle M(P_1), P_1 \rangle = \langle P_1, P_1 \rangle \quad \text{and} \quad \langle M(P_1), I - P_1 \rangle = 0. \tag{34}$$

A consequence of positivity is that $P_i M(I - P_i) P_i \geq 0$ and thus, since $M(I) = I$, that $0 \leq P_i M(P_i) P_i \leq P_i$. Hence, it follows from (34) that, in fact, $M(P_1) = P_1$.

Moreover, since each term in the sum

$$M(P_1) = \sum_s A_s P_1 A_s^* = P_1$$

is positive definite, it follows that $A_s^*(W) \subset W$ for all s , where W is the range of P_1 . This contradicts the irreducibility condition (5) unless $P_1 = I$. However, that would imply that B is a scalar multiple of I , which contradicts the condition $\langle B, I \rangle_{\mathcal{E}} = 0$.

4.5. Strict contraction in Schatten norms. If the operators $\{A_s\}$ are symmetric, then the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is proportional to $\langle \cdot, \cdot \rangle$. In other words $\mathcal{E} = \frac{1}{d}I$, where d is the dimension of H and, in particular, we have the identity

$$\text{Tr}(M(B)) = \langle M(B), I \rangle = \langle B, I \rangle = \text{Tr}(B), \tag{35}$$

since this holds for $\langle \cdot, \cdot \rangle_{\mathcal{E}}$.

In this case, we obtain strict contractivity of M for all Schatten norms $\|\cdot\|_p$ on \mathcal{D} , which we define for $p \geq 1$ as

$$\|A\|_p = \text{Tr}(|A|^p)^{1/p}$$

where $|A|$ denotes the positive part of the operator A , i.e. the unique positive definite operator $|A|$ such that $A = |A|R$ for some orthonormal operator R . For any Hermitian matrix A we can write $|A| = \sum_i |\lambda_i| P_i$, where $A = \sum_i \lambda_i P_i$ is the spectral decomposition of A . The norm $\|A\|_p$ can then be expressed as

$$\|A\|_p = \left(\sum_i |\lambda_i|^p \langle P_i, P_i \rangle \right)^{1/p}.$$

Lemma 8. *Assume that the restriction maps A_s are all symmetric. For all $p \geq 1$, there is a constant $\theta_{1,p}$, $0 < \theta_{1,p} < 1$, such that*

$$\|M(B)\|_p \leq \theta_{1,p} \|B\|_p, \tag{36}$$

for all $B \in \mathcal{D}$.

From the explicit action of M given in (10) it is clear that $\theta_{1,p} = 4/5$, for all p , if we restrict to the particular case of the Sierpiński gasket.

Proof of Lemma 8. We use the notation from the argument showing Lemma 7. From the fact that γ_j can be expressed as a convex combination of the λ_i s, we obtain

$$\|M(B)\|_p^p = \sum_j \left| \sum_i c_{ij} \lambda_i \right|^p \langle Q_j, Q_j \rangle \leq \sum_j \sum_i c_{ij} |\lambda_i|^p \langle Q_j, Q_j \rangle$$

using Jensen’s inequality.

The definition $c_{ij} = \langle M(P_i), Q_j \rangle / \langle Q_j, Q_j \rangle$ of c_{ij} shows that

$$\begin{aligned} \|M(B)\|_p^p &\leq \sum_i |\lambda_i|^p \left(\sum_j \langle M(P_i), Q_j \rangle \right) \\ &= \sum_i |\lambda_i|^p \langle M(P_i), I \rangle \\ &= \sum_i |\lambda_i|^p \langle P_i, P_i \rangle \\ &= \|B\|_p^p \end{aligned}$$

on account of (35).

The irreducibility condition implies that Jensen’s inequality must be strict for all B . Compactness leads us to deduce that M is a strict contraction on \mathcal{D} in the norm $\|\cdot\|_p$. \square

5. Proofs of the main results

5.1. Proof of Theorem 1. Consider the construction of $L^2(\mathcal{X}, \mu)$ given in Section 4.1.4 above. In the case when the measure considered is the Kusuoka measure ν for the system $\{A_s\}$, we can represent a function $f(x) \in L^2(\mathcal{X}, \nu)$ with an operator valued process limit in \mathbb{V} by

$$\Phi(f)(\alpha) = f(\alpha) A(\alpha), \tag{37}$$

where $f(\alpha)$ is the martingale process corresponding to f . This becomes an *isometry*, since we have

$$\mathcal{F}(\alpha)(f(\alpha), g(\alpha)) = \nu(\alpha) f(\alpha)g(\alpha) = \langle f(\alpha) A(\alpha), g(\alpha) A(\alpha) \rangle_{\mathcal{E}}. \tag{38}$$

Hence, Φ gives an isometric representation of $\mathbb{L} = L^2(\mathcal{X}, \nu)$ as a closed subspace in \mathbb{V} .

Notice that Φ preserves the grading, so that $\text{deg}(\Phi(f)) = \text{deg}(f)$ for $f \in \mathbb{L}_*$ and $\Phi(f) \in \mathbb{V}_*$. Hence the representations of $f \in \mathbb{L}_\gamma$ as $\Phi(f) \in \mathbb{V}_\gamma$ are isometric as well. It is also clear that Φ commutes with the shift operator, i.e.

$$\Phi(\mathbb{T} f)(s\alpha) = A(s\alpha) f(\alpha) = A_s \Phi(f)(\alpha) = \mathbb{T}(\Phi(f)(s\alpha)).$$

From now on we view the space \mathbb{L} as a closed subspace of \mathbb{V} and drop the explicit use of Φ .

For the qualitative results of this paper, it is inessential if we replace (9) by the stronger condition that

$$c := \inf_F \sum_s \langle A_s F A_s^*, I \rangle_{\mathcal{E}}^2 > 0, \tag{39}$$

and then take $\theta_2 = (1 - c)^{1/2}$. If (9) holds for a certain $k > 1$, we can then use an “amalgamated” symbolic space based on the symbols $S' = S^k$ and the maps by $A'_{s'} = A(s')$, $s' \in S'$. Moreover, using $\gamma' = \gamma^k$ will relate the γ -norms on the original and the amalgamated system.

Let Q be the orthogonal projection of \mathbb{V} onto \mathbb{L} . The transfer operator L on \mathbb{L} , is defined by the duality (11). By the (implied) isometry Φ , we have

$$\langle Lf, g \rangle_{\mathbb{L}} = \langle f, \mathbb{T}g \rangle_{\mathbb{L}} = (f | \mathbb{T}g) = (\mathcal{L} f | g) = \langle Q \circ \mathcal{L} f, g \rangle_{\mathbb{L}}. \tag{40}$$

The operator $L: \mathbb{L} \rightarrow \mathbb{L}$ can hence be expressed as the composition $L = Q \circ \mathcal{L}$.

From using \mathcal{L}^k and \mathbb{T}^k instead of \mathcal{L} and \mathbb{T} in (40), we deduce furthermore that

$$L^k = Q \circ \mathcal{L}^k, \quad \text{for } k \geq 1.$$

It follows that $R(s) = (L - s1)^{-1}$ is given by $Q \circ \mathcal{R}(s)$, where $\mathcal{R}(s) = (\mathcal{L} - s1)^{-1}$. If Q is a continuous map on \mathbb{V}_γ then $\|R(s)\|_\gamma \leq C \|\mathcal{R}(s)\|$. The spectrum $\sigma(L)$ of L is therefore contained in the spectrum $\sigma(\mathcal{L})$ of \mathcal{L} . Since Theorem 6 states that $\mathcal{L}|_{\mathbb{V}_\gamma}$ has a spectral gap for all $\gamma < 1$, we deduce that Theorem 1 holds for those γ such that the operator Q is a well defined and continuous on \mathbb{V}_γ .

For a process $G(\alpha) \in \mathbb{V}$, the projected process $Q G$ has explicitly the form

$$(Q G)(\alpha) = \nu(\alpha)^{-1} \langle G(\alpha), A(\alpha) \rangle_\mathcal{E} A(\alpha). \tag{41}$$

In other words, the value $Q G(\alpha)$ is, locally for each α , the $\langle \cdot, \cdot \rangle_\mathcal{E}$ -orthogonal projection of the value $G(\alpha) \in \mathbb{B}$ onto the line in \mathbb{B} spanned by $A(\alpha)$. The explicit form (41) follows since an orthonormal basis for $\mathbb{L}_n = \mathbb{V}_n \cap \mathbb{L}$ is given by

$$\{\nu^{-1/2}(\beta) A 1_\beta : \beta \in S^n\},$$

where 1_β denotes the real-valued process

$$1_\beta(\alpha) = \begin{cases} 1 & \text{if } \alpha = \beta\gamma \text{ for some } \gamma \in S^*, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\langle A(\alpha), A(\alpha) \rangle_\mathcal{E} = \nu(\alpha)$ so, from (41), we deduce that the squared norm $\|Q_m F\|^2$ of a projection can be expressed as

$$\|Q_m F\|^2 = \sum_{\beta \in S^m} \langle F(\beta), A(\beta) \rangle_\mathcal{E}^2. \tag{42}$$

That the projection Q is continuous as a projection operator between \mathbb{V}_γ and \mathbb{L}_γ is a nontrivial result since the projection does not preserve the grading: the image of $\mathbb{V}^{(k)}$ under Q spreads out on the spaces $\mathbb{L}^{(j)} = \mathbb{L} \cap \mathbb{V}^{(j)}$, for $j \geq k$. Hence, the norm $\|Q F\|_{\mathbb{V}_\gamma}$, $F \in \mathbb{V}_\gamma$, is not necessarily bounded in terms of $\|F\|_{\mathbb{V}_\gamma}$. We state and prove it as a lemma.

5.2. The continuity of the projection

Lemma 9 (Continuity of Q). *Let $\theta_2 = \sqrt{1-c}$ where c is the constant in the irreducibility condition (39). For any fixed k and any $G \in \mathbb{V}^{(k)}$*

$$\|(Q G)^{(j)}\| \leq \theta_2^{j-k} \|G\|.$$

In particular, we get, for any $G \in \mathbb{V}_\gamma$, that

$$\|QG\|_\gamma \leq \frac{1}{1 - (\theta_2/\gamma)} \cdot \|G\|_\gamma,$$

provided $\gamma > \theta_2$. Let Q_n denote the orthogonal projection onto the closed subspace \mathbb{L}_n . For G in $\mathbb{V}^{(k)}$, we have $Q_n G = 0$ for $n < k$. Let

$$Z^{[n]} = G - Q_n G, \quad n \geq k$$

so that $QZ^{[n]} = \sum_{m>n} (QG)^{(m)}$ and $(QG)^{(n+1)} = Q_{n+1} Z^{[n]}$.

Proof of Lemma 9. It is enough to show that, with c as in the irreducibility condition (9), we have, for all $n > k$, that

$$\|Q_{n+1} Z^{[n]}\|^2 \geq c \|QZ^{[n]}\|^2. \quad (43)$$

By induction and orthogonality of $Q_{n+1} Z^{[n]}$ and $QZ^{[n+1]}$, we obtain

$$\|QZ^{[n+1]}\|^2 = \|QZ^{[n]}\|^2 - \|Q_{n+1} Z^{[n]}\|^2 \leq \theta_2^2 \|QZ^{[n]}\|^2$$

and the sought after statement in Lemma 9 follows by induction, since

$$\|QZ^{[k]}\| \leq \|G\|.$$

Any process F of degree $\deg(F) \leq n$, has the orthogonal decompositions

$$F = \sum_{\alpha \in S^n} F1_\alpha$$

of “localised” processes. Furthermore, the projections Q_m , $m \geq 1$, respect this localisation, i.e. $Q_m F$ is the orthogonal sum $\sum_\alpha Q_m(F1_\alpha)$. It follows that it is enough to show that

$$\|Q_{n+1} Z\|^2 \geq c \|QZ\|, \quad (44)$$

for a part $Z = Z^{[n]}1_\alpha$ of Z , where $\alpha \in S^n$ is fixed.

Furthermore, for $\beta = \alpha\gamma \in S^m$, where $m \geq n$, we have

$$\langle Z(\beta), A(\beta) \rangle_\mathcal{E} = \text{Tr}(\mathcal{E}Z(\beta)A(\alpha)^*A(\gamma)^*) = \langle Z(\alpha\gamma)A(\alpha)^*, A(\gamma) \rangle_\mathcal{E}. \quad (45)$$

It follows that

$$\|Q_m Z\| = \|Q_{m-n} X\|, \quad \text{for } m \geq n,$$

where $X(\gamma) = A(\gamma)X_0 \in \mathbb{V}^{(0)}$ is a constant process with $X_0 = Z(\alpha)A(\alpha)^*$. Moreover, it follows from (45) that

$$Q_n X = Q_n \tilde{X},$$

where

$$\tilde{X}(\alpha) = A(\alpha) \frac{1}{2}(X_0 + X_0^*),$$

i.e the process based on the symmetric part of X_0 : if $W \in \mathcal{B}$ is anti-symmetric and $Y = A W$ then $Q Y = 0$ since

$$\langle A(\alpha)W, A(\alpha) \rangle_{\mathcal{E}} = \text{Tr}(\mathcal{E} A(\alpha)W A(\alpha)^*) = 0.$$

It follows that to show (44) is equivalent to showing that

$$\|Q_1 X\|^2 = \sum_s \langle A_s X_0 A_s^*, I \rangle_{\mathcal{E}}^2 \geq c \|Q X\|^2, \tag{46}$$

where $X = A X_0 \in \mathbb{V}^{(0)}$ and X_0 is the symmetric part of $Z^{[n]}(\alpha) A(\alpha)^*$. Since $Z^{[n]} = G - Q_n G$, we moreover have that

$$\langle X_0, I \rangle_{\mathcal{E}} = \langle Z^{[n]}(\alpha), A(\alpha) \rangle_{\mathcal{E}} = 0.$$

But, since Q is a projection, we have

$$\|Q X\|^2 \leq \|X\|^2 = \|X_0\|_{\mathcal{E}}^2,$$

and thus (46) is a direct consequence of the strong irreducibility condition (9). \square

5.3. Proof of Theorem 4. In the case of the of the Sierpiński gasket and its generalizations, the family SG_n , the restriction maps A_s , $s \in S$, are symmetric. It follows that the $\|A\|_{\mathcal{E}}$ is $1/d$ times the Hilbert–Schmidt norm. Moreover, as is shown in Section 4.5, the symmetry of A_s also implies that M contracts strictly in the Schatten-norms.

It follows directly from (3) that

$$H_n(x) = \frac{A_n(x)^* \mathcal{E} A_n(x)}{\text{Tr}(A_n(x)^* \mathcal{E} A_n(x))}$$

is a positive semi-definite and bounded matrix-valued ν -martingale process that, by the Martingale Convergence Theorem, converges ν -almost everywhere to a limit $H(x)$ such that

$$\text{Tr}(H(x)) = 1. \tag{47}$$

We can write

$$\nu(\alpha \mid [x]_n) = \frac{\text{Tr}(\mathcal{E} A_n(x) A(\alpha) A(\alpha)^* A_n(x))}{\text{Tr}(A_n(x)^* \mathcal{E} A_n(x))} = \text{Tr}(H_n(x) A(\alpha) A(\alpha)^*).$$

If we take the limit $n \rightarrow \infty$ we obtain

$$\nu(\alpha|x) = \text{Tr} \left(H(x) A(\alpha) A(\alpha)^* \right) \quad (48)$$

ν -almost everywhere.

Assume now that f is \mathcal{F}_k -measurable function, $f(x) = f(\alpha)$ for $\alpha \in S^{[0,k]}$. By linearity of trace,

$$\begin{aligned} L^{m+k} f(x) &= \sum_{\substack{\alpha \in S^k \\ \beta \in S^{[k,m+k]}}} \nu(\alpha\beta|x) f(\alpha\beta x) \\ &= \sum_{\alpha} \text{Tr} \left(H(x) \left(\sum_{\beta} A(\beta) A(\alpha) A(\alpha)^* A(\beta)^* \right) \right) f(\alpha) \\ &= \sum_{\alpha} \text{Tr} \left(H(x) M^m (A(\alpha) A(\alpha)^*) \right) f(\alpha). \end{aligned}$$

Since $M^m(I) = I$ and $\text{Tr}(H(x)) = 1$, we obtain that

$$L^{m+k} f(x) - \int f d\nu = \sum_{\alpha} \text{Tr}(H(x) M^m (A(\alpha) A(\alpha)^* - \nu(\alpha)I)) f(\alpha).$$

For the proof of Theorem 4, it thus remains to show that for some constant $0 < \theta_1 < 1$

$$|\text{Tr}(H(x) M^m (A(\alpha) A(\alpha)^* - \nu(\alpha)I))| \leq d \cdot \theta_1^m \cdot \nu(\alpha), \quad (49)$$

uniformly for all x

Let B_{α} denote the symmetric zero-trace matrix $B_{\alpha} = A(\alpha) A(\alpha)^* - \nu(\alpha)I$. By Hölder's inequality for the Schatten norms we have

$$|\text{Tr}(H(x) M^m (B_{\alpha}))| \leq \|H(x)\|_{\infty} \cdot \|M^m (B_{\alpha})\|_1. \quad (50)$$

We have $\|H(x)\|_{\infty} \leq \|H(x)\|_1 = 1$, by (47), and (36) implies that

$$\text{Tr}(|M^m (B_{\alpha})|) \leq \theta_1^m \text{Tr}(|B_{\alpha}|) = d \cdot \theta_1^m \text{Tr}(\mathcal{E}|B_{\alpha}|),$$

where $\theta_1 = \theta_{1,1}$. We then see that (49) follows from the estimate

$$\text{Tr}(\mathcal{E}|B_{\alpha}|) = \text{Tr}(\mathcal{E} \cdot |A(\alpha) A(\alpha)^* - \nu(\alpha)I|) \leq \text{Tr}(\mathcal{E} A(\alpha) A(\alpha)^*) = \nu(\alpha).$$

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