

Non-spectral fractal measures with Fourier frames

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Abstract. We generalize the compatible tower condition given by Strichartz to the almost-Parseval-frame tower and show that non-trivial examples of almost-Parseval-frame tower exist. By doing so, we demonstrate the first singular fractal measure which has only finitely many mutually orthogonal exponentials (and hence it does not admit any exponential orthonormal bases), but it still admits Fourier frames.

Mathematics Subject Classification (2010). Primary 28A25, 42A85, 42B05.

Keywords. Fourier frame, frame spectral measures, fractal measures, almost Parseval frames, spectra.

1. Introduction

Let μ be a compactly supported Borel probability measure on \mathbb{R}^d . We say that μ is a *frame spectral measure* if there exists a collection of exponential functions $\{e^{2\pi i \langle \lambda, x \rangle}\}_{\lambda \in \Lambda}$ such that there exists $0 < A \leq B < \infty$ with

$$A \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} \left| \int f(x) e^{-2\pi i \langle \lambda, x \rangle} d\mu(x) \right|^2 \leq B \|f\|_2^2, \quad \text{for all } f \in L^2(\mu).$$

Whenever such Λ exists, $\{e^{2\pi i \langle \lambda, x \rangle}\}_{\lambda \in \Lambda}$ is called a *Fourier frame* for $L^2(\mu)$ and Λ is a *frame spectrum* for μ . When μ admits an exponential orthonormal basis, we say that μ is a *spectral measure* and the corresponding frequency set Λ is called a *spectrum* for μ .

Frames on a general Hilbert space was introduced by Duffin and Schaeffer [13] and it is now a fundamental building block in applied harmonic analysis. People regard frames as “overcomplete basis” and because of its redundancy, it makes the reconstruction more robust to errors in data and it is now widely used in signal

¹ The research of Yang Wang was partially supported by HK GRF grant 16317416.

transmission and reconstruction. Reader may refer to [2] for the background of general frame theory and [3] for some recent active topics.

One of the major hard problems in frame theory perhaps is constructing Fourier frames or exponential orthonormal bases in different measure space $L^2(\mu)$, particularly when μ is a singular measure without any atoms or people termed it as a “fractal measure” as the support is a fractal set. These constructions allow Fourier analysis to work on fractal space. This problem dates back to the time of Fuglede [15] who initiated the study and proposed the well-known spectral set conjecture. Although the conjecture was proved to be false by Tao [34], the conjecture has been extended into different facet and related questions are still being studied [35, 21, 22, 24, 25]. Another major advance in which fractals were involved was due to Jorgensen and Pedersen [23], who discovered that the standard one-third Cantor measure is not a spectral measure, while the standard one-fourth Cantor measure is. Following the discovery, more fractal measures were found to be spectral by many others [32, 27, 10]. Many unexpected properties of the Fourier bases were discovered [33, 6, 4]. While Fourier analysis appears to work perfectly on fractal spectral measures, for the measures which are non-spectral, it is natural to ask the following question.

(Q). Can a non-spectral fractal measure still admit some Fourier frames?

This question was possibly first proposed by Strichartz [32, p.212]. In particular, there has been discussions asking whether specifically the one-third Cantor measure can be frame spectral. Although we are unable to settle the case of the one-third Cantor measure, the main purpose of this paper is to answer positively **(Q)** with explicit examples. (see Theorem 1.4).

(Q) in its absolutely continuous counterpart is trivial since every bounded Borel set Ω with positive finite Lebesgue measure can be covered by a square. The orthonormal basis on the square naturally induces a tight frame on Ω . If $\mu = g(x)dx$ is a general absolutely continuous measure, a complete characterization on the density for μ to be frame spectral was also given by the first named author [26]. Such question becomes much more difficult if Ω is unbounded but still of finite measure as there cannot be any ad hoc “square-covering” argument to construct the Fourier frames. Despite the difficulty, it was recently solved to be positive by Nitzan et al [31] who used the recent solution of the celebrated Kadison-Singer conjecture [30].

Fractal measures are mostly supported on Lebesgue measure zero set, the situation is similar to unbounded sets of finite measures. However, it is even

more complicated because if any such frame spectrum exists, there cannot be any Beurling density [7, 4]. This prevents any weak convergence argument of discrete sets from happening. Furthermore, some fractal measures are known not to admit any Fourier frames if the measures are non-uniform on the support [11]. Intensive researches on this question [7, 8, 9, 11, 12, 20] has been going on and one major advance was obtained recently in [12]. Dutkay and Lai introduced the *almost-Parseval-frame condition* for the self-similar measure and proved that if such condition is satisfied, the self-similar measure admits a Fourier frame. We slightly modify the definition as below to suit the need in the paper.

Definition 1.1. Let ϵ_j be such that $0 \leq \epsilon_j < 1$ and $\sum_{j=1}^{\infty} \epsilon_j < \infty$. We say that $\{(N_j, B_j)\}$ is an *almost-Parseval-frame tower* associated to $\{\epsilon_j\}$ if the following conditions are satisfied.

- (1) N_j are integers and $N_j \geq 2$ for all j .
- (2) $B_j \subset \{0, 1, \dots, N_j - 1\}$ and $0 \in B_j$ for all j .
- (3) Let $M_j := \#B_j$. There exists $L_j \subset \mathbb{Z}$ (with $0 \in L_j$) such that for all j ,

$$\begin{aligned}
 (1 - \epsilon_j)^2 \sum_{b \in B_j} |w_b|^2 &\leq \sum_{\lambda \in L_j} \left| \frac{1}{\sqrt{M_j}} \sum_{b \in B_j} w_b e^{-2\pi i b \lambda / N_j} \right|^2 \\
 &\leq (1 + \epsilon_j)^2 \sum_{b \in B_j} |w_b|^2
 \end{aligned}
 \tag{1}$$

for all $\mathbf{w} = (w_b)_{b \in B_j} \in \mathbb{C}^{M_j}$. Letting the matrix

$$\mathcal{F}_j = \frac{1}{\sqrt{M_j}} [e^{2\pi i b \lambda / N_j}]_{\lambda \in L_j, b \in B_j}$$

and $\|\cdot\|$ the standard Euclidean norm, (1) is equivalent to

$$(1 - \epsilon_j) \|\mathbf{w}\| \leq \|\mathcal{F}_j \mathbf{w}\| \leq (1 + \epsilon_j) \|\mathbf{w}\|
 \tag{2}$$

for all $\mathbf{w} \in \mathbb{C}^{M_j}$.

Whenever $\{L_j\}_{j \in \mathbb{Z}}$ exists, we call $\{L_j\}_{j \in \mathbb{Z}}$ a *pre-spectrum* for the almost-Parseval-frame tower. We define the following measures associated to an almost-Parseval-frame tower.

$$\nu_j = \frac{1}{M_j} \sum_{b \in B_j} \delta_{b/N_1 N_2 \dots N_j}$$

(we denote by δ_a the Dirac measure supported on a) and

$$\mu = \nu_1 * \nu_2 * \dots
 \tag{3}$$

Roughly speaking, almost-Parseval-frame towers ensure every finite level approximated measure of the fractal is a frame spectral measure. Moreover, the frame bounds remain finite under iterations. Once all finite level has a frame with uniform frame bound, we take the weak limit under a mild condition so that fractal singular measure is also frame-spectral.

When all $\epsilon_j = 0$, the condition is equivalent to the *compatible tower condition* introduced by Strichartz [32]. This is known to be the key condition to construct fractal spectral measures. The measures in (3) are also known as *Moran-type measures*. These measures have been widely used in multifractal analysis [16, 17], harmonic analysis, particularly the construction of Salem sets [28, 29]. Some Moran-type measures were found to be spectral [1] and it was found later spectral Moran measures have a far reaching consequence in understanding the spectral set conjecture [19] and Hausdorff dimension of the support of the spectral measures [5]. We note that Moran-type measures covers self-similar measures because if there exists an integer $N \geq 2$ and a set $B \subset \{0, 1, \dots, N - 1\}$ such that

$$N_j = N^{n_j}, B_j = B + NB + \dots + N^{n_j-1} B,$$

then the associated measure is the *self-similar measure*. In particular if $N = 3$ and $B = \{0, 2\}$, μ is the standard one-third Cantor measure. In such situation, the almost-Parseval-frame tower is called *self-similar*.

In [12], it was proved if the almost-Parseval-frame tower is *self-similar*, then the self-similar measure induced will admit an Fourier frame. However, there was no example of such towers for which $\epsilon_j > 0$. In this paper, we relax the self-similar restriction and produce the first example almost-Parseval-frame tower whose $\epsilon_j > 0$.

Theorem 1.2. *Let N_j and M_j be positive integers satisfying*

$$N_j = M_j K_j + \alpha_j \tag{4}$$

for some integer K_j and $0 \leq \alpha_j < M_j$ with

$$\sum_{j=1}^{\infty} \frac{\alpha_j \sqrt{M_j}}{K_j} < \infty. \tag{5}$$

Define

$$B_j = \{0, K_j, \dots, (M_j - 1)K_j\} \quad \text{and} \quad L_j = \{0, 1, \dots, M_j - 1\}. \tag{6}$$

Then (N_j, B_j) forms an almost-Parseval-frame tower associated with

$$\epsilon_j = \frac{2\pi \alpha_j \sqrt{M_j}}{K_j}$$

and its pre-spectrum is $\{L_j\}$.

We then extend the result of [12] to general almost-Parseval-frame tower. For the measure μ defined in (3), we let

$$\mu_n = \nu_1 * \dots * \nu_n, \mu_{>n} = \nu_{n+1} * \nu_{n+2} * \dots .$$

so that $\mu = \mu_n * \mu_{>n}$. Define also the Fourier transform of a measure μ in an usual way,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x).$$

Theorem 1.3. (a) *Suppose that $\{(N_j, B_j)\}$ is an almost-Parseval-frame tower associated with $\{\epsilon_j\}$ and $\{L_j\}_{j=1}^\infty$. Let*

$$\mathbf{L}_n = L_1 + N_1 L_2 + \dots + (N_1 \dots N_{n-1}) L_n \quad \text{and} \quad \Lambda = \bigcup_{n=1}^\infty \mathbf{L}_n.$$

If

$$\delta(\Lambda) := \inf_n \inf_{\lambda \in \mathbf{L}_n} |\widehat{\mu_{>n}}(\lambda)|^2 > 0,$$

then the measure μ in (3) admits a Fourier frame with frame spectrum Λ .

(b) *For the almost-Parseval-frame tower constructed in Theorem 1.2, the associated Λ satisfies $\delta(\Lambda) > 0$ and hence the measure μ is a frame spectral measure.*

In the end, using the theorems above, we construct the first kind of the following examples:

Theorem 1.4. *There exists non-spectral fractal measure with only finitely many orthogonal exponentials, but it still admits Fourier frames.*

We organize our paper as follows: In Section 2, we prove the existence of the almost-Parseval-frame tower and prove Theorem 1.2. In Section 3, we construct the Fourier frame given the tower and prove Theorem 1.3. In Section 4, we construct the non-spectral measures with Fourier frames. In Appendix A, we study the Hausdorff dimension of the support.

2. Existence of Almost-Parseval-frame tower

Let A be an $n \times n$ matrix. We define the operator norm of A to be

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

and the *Frobenius norm* of A to be

$$\|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{i,j}|^2}$$

It follows easily from Cauchy-Schwarz inequality that $\|A\|_2 \leq \|A\|_F$.

For N_j and M_j satisfying (4) and (5) and for B_j and L_j defined in (6), We let

$$\mathcal{F}_j = \frac{1}{\sqrt{M_j}} [e^{2\pi i b \lambda / N_j}]_{\lambda \in L_j, b \in B_j}, \quad \mathcal{H}_j = \frac{1}{\sqrt{M_j}} [e^{2\pi i b \lambda / M_j K_j}]_{\lambda \in L_j, b \in B_j}.$$

Lemma 2.1. \mathcal{H}_n is a unitary matrix. i.e. $\|\mathcal{H}_n \mathbf{x}\| = \|\mathbf{x}\|$.

Proof. Let $b = m \in L_j$ and $\lambda = n K_j \in B_j$, for $m, n = 0, 1, \dots, M_j - 1$. It follows directly that $e^{2\pi i b \lambda / M_j K_j} = e^{2\pi i m n / M_j}$. Hence,

$$\mathcal{H}_j = \frac{1}{\sqrt{M_j}} [e^{2\pi i m n / M_j}]_{m, n=0, \dots, M_j-1},$$

which is the standard Fourier matrix of order M_j . Thus, \mathcal{H}_j is unitary. \square

Proof of Theorem 1.2. We first show that for any $j > 0$,

$$\|\mathcal{F}_j - \mathcal{H}_j\| \leq \frac{2\pi \alpha_j \sqrt{M_j}}{K_j}. \quad (7)$$

To see this, We note that

$$\|\mathcal{F}_j - \mathcal{H}_j\|^2 \leq \|\mathcal{F}_j - \mathcal{H}_j\|_F^2 = \frac{1}{M_j} \sum_{b \in B_j} \sum_{\lambda \in L_j} |e^{2\pi i b \lambda / N_j} - e^{2\pi i b \lambda / M_j K_j}|^2. \quad (8)$$

We now estimate the difference of the exponentials inside the summation. Recall that for any θ_1, θ_2 ,

$$|e^{i\theta_1} - e^{i\theta_2}| = |e^{i(\theta_1 - \theta_2)} - 1| \leq |\theta_1 - \theta_2|.$$

This implies that

$$\begin{aligned} |e^{2\pi i b \lambda / N_j} - e^{2\pi i b \lambda / M_j K_j}|^2 &\leq \left| \frac{2\pi b \lambda}{N_j} - \frac{2\pi b \lambda}{M_j K_j} \right|^2 \\ &= 4\pi^2 \frac{b^2 \lambda^2 \alpha_j^2}{M_j^2 K_j^2 N_j^2} \quad (\text{by } N_j = M_j K_j + \alpha_j) \\ &\leq 4\pi^2 \frac{M_j^2 \alpha_j^2}{N_j^2} \quad (\text{by } b \leq M_j K_j \text{ and } \lambda \leq M_j). \end{aligned}$$

Hence, from (8),

$$\begin{aligned} \|\mathcal{F}_j - \mathcal{H}_j\|^2 &\leq \frac{1}{M_j} \sum_{b \in B_j} \sum_{\lambda \in L_j} 4\pi^2 \frac{M_j^2 \alpha_j^2}{N_j^2} \\ &= 4\pi^2 \frac{M_j^3 \alpha_j^2}{N_j^2} \\ &= 4\pi^2 \frac{M_j \alpha_j^2}{(K_j + \alpha_j/M_j)^2} \end{aligned} \tag{9}$$

As $\alpha_j \geq 0$, $\|\mathcal{F}_j - \mathcal{H}_j\|^2 \leq 4\pi^2 \alpha_j^2 M_j / K_j^2$ and thus (7) follows by taking square root.

We now show that $\{(N_j, B_j)\}$ forms an almost-Parseval-frame tower with pre-spectrum L_j . The first two conditions for the almost-Parseval-frame tower are clearly satisfied. To see the last condition, we recall that $\epsilon_j = 2\pi \sqrt{M_j} \alpha_j / K_j$. From the triangle inequality and (7), we have

$$\begin{aligned} \|\mathcal{F}_j \mathbf{w}\| &\leq \|\mathcal{H}_j \mathbf{w}\| + \|\mathcal{F}_j - \mathcal{H}_j\| \|\mathbf{w}\| \\ &\leq \left(1 + \frac{2\pi \alpha_j \sqrt{M_j}}{K_j}\right) \|\mathbf{w}\| \\ &= (1 + \epsilon_j) \|\mathbf{w}\|. \end{aligned}$$

Similarly, for the lower bound,

$$\begin{aligned} \|\mathcal{F}_j \mathbf{w}\| &\geq \|\mathcal{H}_j \mathbf{w}\| - \|\mathcal{F}_j - \mathcal{H}_j\| \|\mathbf{w}\| \\ &\geq \left(1 - \frac{2\pi \alpha_j \sqrt{M_j}}{K_j}\right) \|\mathbf{w}\| \\ &= (1 - \epsilon_j) \|\mathbf{w}\|. \end{aligned}$$

Thus, from (2), the last condition follows and (N_j, B_j) satisfies the almost-Parseval-frame condition associated with $\{\epsilon_j\}$ and $\sum_{j=1}^{\infty} \epsilon_j < \infty$ is guaranteed by (5) in the assumption. \square

Remark 2.2. In view of (9), condition (5) can be replaced by a weaker condition

$$\sum_{j=1}^{\infty} \frac{\alpha_j \sqrt{M_j}}{K_j + \alpha_j/M_j} < \infty.$$

(5) would be enough for the convenience of our discussion. It is also worth to note that if all $\alpha_j = 0$, then the the matrices $\mathcal{F}_j = \mathcal{H}_j$ are reduced to the Hadamard matrices. The associated measures are all spectral measures. see e.g. [1].

We end this section by illustrating some explicit examples of Theorem 1.2.

Example 2.3. Let p be an odd prime and suppose that $N_j = p^j$. Let $M_j = 2$ for all j . Then it is clear that $N_j = 2K_j + 1$ for some K_j . In this case,

$$\sum_{j=1}^{\infty} \frac{\alpha_j \sqrt{M_j}}{K_j} = \sum_{j=1}^{\infty} \frac{\sqrt{2}}{K_j} = \sum_{j=1}^{\infty} \frac{2\sqrt{2}}{p^j - 1} < \infty.$$

Thus $N_j = p^j$ and $B_j = \{0, K_j\}$ forms an almost-Parseval-frame tower with pre-spectrum $L_j = \{0, 1\}$ for all j .

Example 2.4. For $0 \leq \beta < 2, \gamma \geq 0$ and $N \geq 1$ such that

$$N(1 - \beta/2 - \gamma) > 1,$$

let K_j, M_j, α_j be integers $K_j \geq j^N, M_j \leq K_j^\beta$ and $\alpha_j \leq K_j^\gamma$. Then

$$\sum_{j=1}^{\infty} \frac{\alpha_j \sqrt{M_j}}{K_j} \leq \sum_{j=1}^{\infty} \frac{K_j^\gamma K_j^{\beta/2}}{K_j} = \sum_{j=1}^{\infty} \frac{1}{K_j^{1-\gamma-\beta/2}} \leq \sum_{j=1}^{\infty} \frac{1}{j^{N(1-\gamma-\beta/2)}} < \infty.$$

Hence, $N_j = K_j M_j + \alpha_j$ and $B_j = \{0, K_j, \dots, (M_j - 1)K_j\}$ satisfies the almost-Parseval-frame condition.

3. Construction of Fourier frames

In this section, we consider the almost-Parseval-frame tower defined in Section 1 and show that the measure μ defined in (3) is a frame spectral measure. We first recall some notations.

$$\nu_j = \frac{1}{M_j} \sum_{b \in B_j} \delta_{b/N_1 \dots N_j} \quad \text{and} \quad \mu = \nu_1 * \nu_2 * \dots.$$

We define

$$\mu_n = \nu_1 * \dots * \nu_n, \quad \mu_{>n} = \nu_{n+1} * \nu_{n+2} * \dots$$

so that $\mu = \mu_n * \mu_{>n}$. It is also direct to see that the support of μ is the compact set

$$K_\mu = \left\{ \sum_{j=1}^{\infty} \frac{b_j}{N_1 \dots N_j} \mid b_j \in B_j \text{ for all } j \right\}.$$

We also consider the first n^{th} -partial sum in K_μ and denote it by

$$\mathbf{B}_n = \frac{1}{N_1} B_1 + \frac{1}{N_1 N_2} B_2 + \cdots + \frac{1}{N_1 N_2 \dots N_n} B_n$$

which is the support of μ_n . For the $\{L_j\}_{j \in \mathbb{Z}}$ in the tower, we consider

$$\mathbf{L}_n = L_1 + N_1 L_2 + \cdots + (N_1 \dots N_{n-1}) L_n.$$

Proposition 3.1. *For any $n \geq 1$, let $\mathbf{M}_n = \prod_{j=1}^n M_j$ we have*

$$\left(\prod_{j=1}^n (1 - \epsilon_j) \right)^2 \|\mathbf{w}\|^2 \leq \sum_{\lambda \in \mathbf{L}_n} \left| \frac{1}{\sqrt{\mathbf{M}_n}} \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} e^{-2\pi i \mathbf{b} \lambda} \right|^2 \leq \left(\prod_{j=1}^n (1 + \epsilon_j) \right)^2 \|\mathbf{w}\|^2$$

for any $\mathbf{w} = (w_{\mathbf{b}})_{\mathbf{b} \in \mathbf{B}_n} \in \mathbb{C}^{M_1 \dots M_n}$

Proof. We prove it by mathematical induction. When $n = 1$, it is the almost-Parseval condition for (N_1, B_1) so the statement is true trivially. Assume now the inequality is true for $n - 1$. Then we decompose $\mathbf{b} \in \mathbf{B}_n$ and $\lambda \in \mathbf{L}_n$ by

$$\mathbf{b} = \frac{1}{N_1 \dots N_n} b_n + \mathbf{b}_{n-1}, \quad \lambda = \lambda_{n-1} + N_1 \dots N_{n-1} l_n,$$

where $b_n \in B_n$, $\mathbf{b}_{n-1} \in \mathbf{B}_{n-1}$, $\lambda_{n-1} \in \mathbf{L}_{n-1}$ and $l_n \in L_n$. Now, we have

$$\begin{aligned} & \sum_{\lambda \in \mathbf{L}_n} \left| \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} e^{-2\pi i \mathbf{b} \lambda} \right|^2 \\ &= \sum_{\lambda_{n-1} \in \mathbf{L}_{n-1}} \sum_{l_n \in L_n} \left| \sum_{\mathbf{b}_{n-1} \in \mathbf{B}_{n-1}} \sum_{b_n \in B_n} \frac{1}{\sqrt{\mathbf{M}_n}} w_{\mathbf{b}_{n-1} b_n} e^{\mathfrak{E} \cdot (\lambda_{n-1} + N_1 \dots N_{n-1} l_n)} \right|^2, \end{aligned}$$

where

$$\mathfrak{E} = -2\pi i \left(\frac{1}{N_1 \dots N_n} b_n + \mathbf{b}_{n-1} \right).$$

Note that $\mathbf{b}_{n-1} \cdot (N_1 \dots N_{n-1}) l_n$ is always an integer, the right hand side above can be written as

$$\sum_{\lambda_{n-1} \in \mathbf{L}_{n-1}} \sum_{l_n \in L_n} \left| \sum_{b_n \in B_n} \frac{1}{\sqrt{M_n}} \left(\sum_{\mathbf{b}_{n-1} \in \mathbf{B}_{n-1}} \frac{1}{\sqrt{M_{n-1}}} w_{\mathbf{b}_{n-1} b_n} e^{\mathfrak{E} \lambda_{n-1}} \right) e^{-2\pi i b_n l_n / N_n} \right|^2$$

Using the almost-Parseval-frame condition for (N_n, B_n) and also the induction hypothesis, this term

$$\begin{aligned}
 &\leq (1 + \epsilon_n)^2 \sum_{\lambda_{n-1} \in \mathbf{L}_{n-1}} \sum_{b_n \in B_n} \left| \sum_{\mathbf{b}_{n-1} \in \mathbf{B}_{n-1}} \frac{1}{\sqrt{M_{n-1}}} w_{\mathbf{b}_{n-1} b_n} e^{\epsilon \cdot \lambda_{n-1}} \right|^2 \\
 &= (1 + \epsilon_n)^2 \sum_{b_n \in B_n} \sum_{\lambda_{n-1} \in \mathbf{L}_{n-1}} \left| \sum_{\mathbf{b}_{n-1} \in \mathbf{B}_{n-1}} \frac{1}{\sqrt{M_{n-1}}} w_{\mathbf{b}_{n-1} b_n} e^{-2\pi i \mathbf{b}_{n-1} \cdot \lambda_{n-1}} \right|^2 \\
 &\leq \left(\prod_{j=1}^n (1 + \epsilon_j) \right)^2 \sum_{b_n \in B_n} \sum_{\mathbf{b}_{n-1} \in \mathbf{B}_{n-1}} |w_{\mathbf{b}_{n-1} b_n}|^2 \\
 &= \left(\prod_{j=1}^n (1 + \epsilon_j) \right)^2 \|\mathbf{w}\|^2.
 \end{aligned}$$

This completes the proof of the upper bound and the proof of the lower bound is analogous. \square

We now decompose K_μ as

$$K_\mu = \bigcup_{\mathbf{b} \in \mathbf{B}_n} (\mathbf{b} + K_{\mu, n}). \quad (10)$$

where

$$K_{\mu, n} = \left\{ \sum_{j=n+1}^{\infty} \frac{b_j}{N_1 \dots N_j} \mid b_j \in B_j \text{ for all } j \right\}.$$

Denote by $K_{\mathbf{b}} = \mathbf{b} + K_{\mu, n}$ and $\mathbf{1}_{K_{\mathbf{b}}}$ the characteristic function of $K_{\mathbf{b}}$. Let

$$\mathcal{S}_n = \left\{ \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \mathbf{1}_{K_{\mathbf{b}}} \mid w_{\mathbf{b}} \in \mathbb{C} \right\}.$$

\mathcal{S}_n denotes the collection of all n^{th} level step functions on K_μ . As

$$K_{\mu, n} = \bigcup_{b \in B_{n+1}} \left(\frac{b}{N_1 \dots N_{n+1}} + K_{\mu, n+1} \right),$$

we have $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots$. Let also

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

It is clear that \mathcal{S} forms a dense set of functions in $L^2(\mu)$.

Lemma 3.2. *Let $f = \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \mathbf{1}_{K_{\mathbf{b}}} \in \mathcal{S}_n$. Then*

$$\int |f|^2 d\mu = \frac{1}{\mathbf{M}_n} \sum_{\mathbf{b} \in \mathbf{B}_n} |w_{\mathbf{b}}|^2. \tag{11}$$

$$\int f(x) e^{-2\pi i \lambda x} d\mu(x) = \frac{1}{\mathbf{M}_n} \widehat{\mu}_{>n}(\lambda) \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} e^{-2\pi i \mathbf{b} \lambda}. \tag{12}$$

Here $\mathbf{M}_n = M_1 \dots M_n$.

Proof. As $K_{\mathbf{b}}$ and $K'_{\mathbf{b}}$ has either empty intersection or intersects at most one point, taking μ -measure on (10), we obtain $\mu(K_{\mathbf{b}}) = 1/\mathbf{M}_n$. (11) follows from a direct computation. For (12), we use $\mu = \mu_n * \mu_{>n}$ and we have

$$\begin{aligned} \int f(x) e^{-2\pi i \lambda x} d\mu(x) &= \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \int \mathbf{1}_{K_{\mathbf{b}}}(x) e^{-2\pi i \lambda x} d(\mu_n * \mu_{>n}(x)) \\ &= \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \int \mathbf{1}_{\mathbf{b} + K_{\mu,n}}(x + y) e^{-2\pi i \lambda(x+y)} d\mu_n(x) d\mu_{>n}(y). \end{aligned}$$

Note that $\mu_{>n}$ is supported on $K_{\mu,n}$ and $K_{\mathbf{b}}$ and $K'_{\mathbf{b}}$ has either empty intersection or intersects at most one point. The above is equal to

$$\begin{aligned} &= \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \frac{1}{\mathbf{M}_n} \int \mathbf{1}_{\mathbf{b} + K_{\mu,n}}(\mathbf{b} + y) e^{-2\pi i \lambda(\mathbf{b} + y)} d\mu_{>n}(y) \\ &= \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \frac{1}{\mathbf{M}_n} e^{-2\pi i \lambda \mathbf{b}} \int e^{-2\pi i \lambda y} d\mu_{>n}(y) \\ &= \frac{1}{\mathbf{M}_n} \widehat{\mu}_{>n}(\lambda) \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} e^{-2\pi i \mathbf{b} \lambda}. \end{aligned}$$

The lemma follows. □

Let

$$\Lambda = \bigcup_{n=1}^{\infty} \mathbf{L}_n. \tag{13}$$

As $0 \in L$, the sets in the union is an increasing union. We now define the following quantity

$$\delta_{n_0}(\Lambda) = \inf_{n \geq n_0} \inf_{\lambda \in \mathbf{L}_n} |\widehat{\mu}_{>n}(\lambda)|^2,$$

for some $n_0 \geq 1$. The following theorem gives a sufficient condition for Λ to be a Fourier frame for μ .

Theorem 3.3. *Suppose that (N_j, B_j) is an almost-Parseval-frame tower and μ be the associated measure. Let L be the associated spectrum for the tower and Λ defined (13) satisfies $\delta_{n_0}(\Lambda) > 0$. Then μ admits a Fourier frame $E(\Lambda)$ with lower and upper frame bounds respectively equal*

$$\delta(\Lambda) \left(\prod_{j=1}^{\infty} (1 - \epsilon_j) \right)^2, \left(\prod_{j=1}^{\infty} (1 + \epsilon_j) \right)^2.$$

Proof. To check the Fourier frame inequality holds, it suffices to show that it is true for a dense set of functions in $L^2(\mu)$ [2, Lemma 5.1.7], in which we will check it for step functions in \mathcal{S} . Moreover, since \mathcal{S}_n is an increasing union of sets, we consider $f = \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} \mathbf{1}_{K_{\mathbf{b}}} \in \mathcal{S}_n$ with $n \geq n_0$. By Lemma 3.2, we have

$$\begin{aligned} \sum_{\lambda \in \mathbf{L}_n} \left| \int f(x) e^{-2\pi i \lambda x} d\mu(x) \right|^2 &= \sum_{\lambda \in \mathbf{L}_n} \left| \frac{1}{\mathbf{M}_n} \widehat{\mu}_{>n}(\lambda) \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} e^{-2\pi i \mathbf{b} \lambda} \right|^2 \\ &= \frac{1}{\mathbf{M}_n} \sum_{\lambda \in \mathbf{L}_n} |\widehat{\mu}_{>n}(\lambda)|^2 \left| \frac{1}{\sqrt{\mathbf{M}_n}} \sum_{\mathbf{b} \in \mathbf{B}_n} w_{\mathbf{b}} e^{-2\pi i \mathbf{b} \lambda} \right|^2. \end{aligned}$$

Note that $\delta(\Lambda) \leq |\widehat{\mu}_{>n}(\lambda)|^2 \leq 1$. By Proposition 3.1, we have this implies that

$$\begin{aligned} \frac{1}{\mathbf{M}_n} \delta(\Lambda) \left(\prod_{j=1}^n (1 - \epsilon_j) \right)^2 \|\mathbf{w}\|^2 &\leq \sum_{\lambda \in \mathbf{L}_n} \left| \int f(x) e^{-2\pi i \lambda x} d\mu(x) \right|^2 \\ &\leq \frac{1}{\mathbf{M}_n} \left(\prod_{j=1}^n (1 + \epsilon_j) \right)^2 \|\mathbf{w}\|^2. \end{aligned}$$

Using Lemma 3.2 again, we have

$$\begin{aligned} \delta(\Lambda) \left(\prod_{j=1}^n (1 - \epsilon_j) \right)^2 \int |f|^2 d\mu &\leq \sum_{\lambda \in \mathbf{L}_n} \left| \int f(x) e^{-2\pi i \lambda x} d\mu(x) \right|^2 \\ &\leq \left(\prod_{j=1}^n (1 + \epsilon_j) \right)^2 \int |f|^2 d\mu. \end{aligned}$$

To complete the proof, we note that for all $m > n$, $f \in \mathcal{S}_n \subset \mathcal{S}_m$, the inequality can also be written as

$$\begin{aligned} \delta(\Lambda) \left(\prod_{j=1}^m (1 - \epsilon_j) \right)^2 \int |f|^2 d\mu &\leq \sum_{\lambda \in \mathbf{L}_m} \left| \int f(x) e^{-2\pi i \lambda x} d\mu(x) \right|^2 \\ &\leq \left(\prod_{j=1}^m (1 + \epsilon_j) \right)^2 \int |f|^2 d\mu, \end{aligned}$$

for all $f \in \mathcal{S}_n$. Taking m to infinity, we have

$$\begin{aligned} \delta(\Lambda) \left(\prod_{j=1}^{\infty} (1 - \epsilon_j) \right)^2 \int |f|^2 d\mu &\leq \sum_{\lambda \in \Lambda} \left| \int f(x) e^{-2\pi i \lambda x} d\mu(x) \right|^2 \\ &\leq \left(\prod_{j=1}^{\infty} (1 + \epsilon_j) \right)^2 \int |f|^2 d\mu. \end{aligned}$$

Note that the frame bounds are finite since $\sum_{j=1}^{\infty} \epsilon_j < \infty$. This shows the frame inequality for any $f \in \mathcal{S}$. Hence, $E(\Lambda)$ is a Fourier frame for $L^2(\mu)$. \square

We now show that the almost-Parseval-frame tower constructed in Theorem 1.2 satisfies $\delta(\Lambda) > 0$. Recall that

$$N_j = M_j K_j + \alpha_j,$$

with $B_j = \{0, K_j, \dots, (M_j - 1)K_j\}$ and $L_j = \{0, 1, \dots, M_j - 1\}$. The associated measure is given by

$$\mu = \nu_1 * \nu_2 * \dots \quad \text{and} \quad \nu_j = \frac{1}{M_j} \sum_{b \in B_j} \delta_{j/N_1 \dots N_j}.$$

The Fourier transform is given by

$$\hat{\mu}(\xi) = \prod_{j=1}^{\infty} \hat{\nu}_j(\xi) = \prod_{j=1}^{\infty} \left[\frac{1}{M_j} \sum_{k=0}^{M_j-1} e^{-2\pi i k K_j \xi / N_1 \dots N_j} \right].$$

It follows directly from summation of geometric series that

$$\hat{\nu}_j(\xi) = \begin{cases} \frac{1}{M_j} e^{\pi i c_j (M_j - 1) \xi} \frac{\sin \pi c_j M_j \xi}{\sin \pi c_j \xi} & \text{if } \xi \notin \frac{1}{c_j} \mathbb{Z}, \\ 1 & \text{if } \xi \in \frac{1}{c_j} \mathbb{Z}. \end{cases}$$

where $c_j = K_j / N_1 \dots N_j$.

Proposition 3.4. *With all the notation above, there exists k_0 such that $\Lambda = \bigcup_{k=1}^{\infty} \mathbf{L}_k$ satisfies*

$$\delta_{k_0}(\Lambda) = \inf_{k \geq k_0} \inf_{\lambda \in \mathbf{L}_k} |\widehat{\mu}_{>k}(\lambda)|^2 > 0$$

where $\mathbf{L}_k = L_1 + N_1 L_2 + \dots + N_1 \dots N_{k-1} L_k$.

Proof. We note that

$$\begin{aligned}
 |\widehat{\mu}_{>k}(\lambda)|^2 &= \prod_{j=1}^{\infty} |\widehat{v}_{k+j}(\lambda)|^2 \\
 &= \prod_{j=1}^{\infty} \left| \frac{1}{M_{k+j}} \sum_{\ell=0}^{M_{k+j}-1} e^{-2\pi i \ell K_{k+j} \lambda / (N_1 \dots N_k N_{k+1} \dots N_{k+j})} \right|^2.
 \end{aligned} \tag{14}$$

For any $\lambda \in \mathbf{L}_k$ for which the terms $|\widehat{v}_{k+j}(\lambda)|^2 < 1$, we have

$$\left| \frac{1}{M_{k+j}} \sum_{k=0}^{M_j-1} e^{-2\pi i k K_j \lambda / (N_1 \dots N_k N_{k+1} \dots N_{k+j})} \right|^2 = \left| \frac{1}{M_{k+j}} \frac{\sin \pi c_{k+j} M_{k+j} \lambda}{\sin \pi c_{k+j} \lambda} \right|^2.$$

Using the elementary estimate $\sin x \leq x$ and $\sin x \geq x - x^3/3!$, we have

$$\left| \frac{1}{M_{k+j}} \frac{\sin \pi c_{k+j} M_{k+j} \lambda}{\sin \pi c_{k+j} \lambda} \right|^2 \geq \left| \frac{\sin(\pi c_{k+j} M_{k+j} \lambda)}{\pi c_{k+j} M_{k+j} \lambda} \right|^2 = \left(1 - \frac{(\pi c_{k+j} M_{k+j} \lambda)^2}{3!} \right)^2.$$

Recall that $c_{k+j} = K_{k+j}/N_1 \dots N_{k+j}$, we have

$$\begin{aligned}
 &\left(1 - \frac{(\pi c_{k+j} M_{k+j} \lambda)^2}{3!} \right)^2 \\
 &= \left(1 - \frac{\pi^2}{6N_{k+1}^2 \dots N_{k+j-1}^2} \cdot \left(\frac{K_{k+j} M_{k+j}}{N_{k+j}} \right)^2 \cdot \left(\frac{\lambda}{N_1 \dots N_k} \right)^2 \right)^2.
 \end{aligned} \tag{15}$$

We need to ensure all the terms inside the outermost square are positive and their product is strictly positive. For $\lambda \in \mathbf{L}_k$, we write

$$\lambda = \ell_1 + N_1 \ell_2 + \dots + (N_1 \dots N_{k-1}) \ell_k, \quad \text{for some } \ell_i \in L_i.$$

From $N_i = M_i K_i + \alpha_i$, we have $\ell_i \leq M_i - 1 < N_i$ and thus

$$\begin{aligned}
 \frac{\lambda}{N_1 \dots N_k} &= \frac{\ell_k}{N_k} + \frac{\ell_{k-1}}{N_k N_{k-1}} + \dots + \frac{\ell_1}{N_k \dots N_1} \\
 &< \frac{M_k}{N_k} + \frac{M_{k-1}}{N_k N_{k-1}} + \dots + \frac{M_1}{N_k \dots N_1} \\
 &\leq \frac{1}{K_k} + \frac{1}{N_k K_{k-1}} + \dots + \frac{1}{N_k \dots N_2 K_1} \\
 &\leq \frac{1}{K_k} \left(1 + \frac{1}{M_k K_{k-1}} + \frac{1}{M_k N_{k-1} K_{k-2}} + \dots \right. \\
 &\quad \left. + \frac{1}{M_k N_{k-1} N_{k-2} \dots N_2 K_1} \right) \\
 &\leq \frac{2}{K_k} \quad (\text{since all } M_j, N_j \geq 2).
 \end{aligned}$$

For the term $j > 1$ in (15), we use

$$\frac{\lambda}{N_1 \dots N_k} < 1 \quad \text{and} \quad \frac{K_{k+j} M_{k+j}}{N_{k+j}} \leq 1.$$

With $N_j \geq 2$ for all j , we have

$$(15) \geq \left(1 - \frac{\pi^2}{6 \cdot 2^{2(j-1)}}\right)^2, \quad \text{for } j > 1.$$

If $j = 1$,

$$\begin{aligned} \left(1 - \frac{\pi^2}{6} \cdot \left(\frac{K_{k+j} M_{k+j}}{N_{k+j}}\right)^2 \cdot \left(\frac{\lambda}{N_1 \dots N_k}\right)^2\right)^2 &\geq \left(1 - \frac{\pi^2}{6} \cdot \left(\frac{2}{K_k}\right)^2\right)^2 \\ &\geq \left(1 - \frac{2\pi^2}{3K_k^2}\right)^2. \end{aligned}$$

Note that our assumption that $\sum_{k=1}^{\infty} \frac{\alpha_k \sqrt{M_k}}{K_k} < \infty$ implies that K_k tends to infinity. Hence, there exists k_0 such that for all $k \geq k_0$, $K_k \geq 3$. This ensure the term insider the square is greater than or equal to $\delta := 1 - 2\pi^2/27 > 0$. Putting all the inequality back to (14), we obtain

$$|\widehat{\mu}_{>k}(\lambda)|^2 \geq \delta^2 \cdot \prod_{j=2}^{\infty} \left(1 - \frac{\pi^2}{6 \cdot 2^{2(j-1)}}\right)^2 := c_0.$$

Hence, $\delta(\Lambda) \geq c_0$. As $\sum_{j=2}^{\infty} \pi^2/(6 \cdot 2^{2(j-1)}) < \infty$ and $\pi^2/(6 \cdot 2^{2(j-1)}) < 1$ for all $j > 1$, $c_0 > 0$ and this completes the proof. \square

Proof of Theorem 1.3. (a) follows directly from Theorem 3.3. For (b), Proposition 3.4 implies that $\delta(\Lambda) > 0$ and hence the measure μ is frame spectral by Theorem 3.3. \square

4. Non-spectral measures

In this section, we will see the measures defined by the almost-Parseval-frame tower in Theorem 1.2 is in general not spectral. For a given probability measure μ , we let

$$Z(\hat{\mu}) = \{\xi \in \mathbb{R} : \hat{\mu}(\xi) = 0\}$$

be its zero set of $\hat{\mu}$. We recall that the collection of the exponentials $\{e^{2\pi i \lambda x} \mid \lambda \in \Lambda\}$ is a *mutually orthogonal set* if the exponential functions are mutually orthogonal in $L^2(\mu)$. In order to show μ cannot be a spectral measure, we need the following simple observation.

Lemma 4.1. *If μ is a spectral measure whose support is an infinite set, then any mutually orthogonal set Λ must be of infinite cardinality and satisfies $\Lambda - \Lambda \subset Z(\hat{\mu}) \cup \{0\}$.*

Proof. If μ is a spectral measure whose support is an infinite set, then $L^2(\mu)$ is of infinite dimension as a vector space, so any mutually orthogonal sets must be infinite in cardinality. For mutually orthogonality to hold, we need for any $\lambda \neq \lambda' \in \Lambda$,

$$0 = \int e^{2\pi i(\lambda - \lambda')x} d\mu(x) = \hat{\mu}(\lambda - \lambda')$$

Hence, $\Lambda - \Lambda \subset Z(\hat{\mu}) \cup \{0\}$ follows. □

Focusing on the tower we constructed in Theorem 1.2,

$$N_j = K_j M_j + \alpha_j.$$

and $B_j = \{0, K_j, \dots, (M_j - 1)K_j\}$, $L = \{0, 1, \dots, M_j - 1\}$, we have

Lemma 4.2. *We have*

$$Z(\hat{\mu}) = \bigcup_{j=1}^{\infty} Z(\hat{v}_j) = \bigcup_{j=1}^{\infty} \left[\frac{N_1 \dots N_j}{K_j M_j} (\mathbb{Z} \setminus M_j \mathbb{Z}) \right].$$

Proof. we can compute directly the zero set of the Fourier transform of v_j as

$$\begin{aligned} \hat{v}_j(\xi) &= \frac{1}{M_j} \sum_{k=0}^{M_j-1} e^{-2\pi i k K_j \xi / N_1 \dots N_j} \\ &= \begin{cases} \frac{1}{M_j} e^{\pi i c_j (M_j-1)\xi} \frac{\sin \pi c_j M_j \xi}{\sin \pi c_j \xi} & \text{if } \xi \notin \frac{1}{c_j} \mathbb{Z}, \\ 1 & \text{if } \xi \in \frac{1}{c_j} \mathbb{Z}, \end{cases} \end{aligned}$$

where $c_j = K_j / N_1 \dots N_j$. It follows directly that

$$Z(\hat{v}_j) = \frac{1}{c_j M_j} (\mathbb{Z} \setminus M_j \mathbb{Z}) = \frac{N_1 \dots N_j}{K_j M_j} (\mathbb{Z} \setminus M_j \mathbb{Z})$$

so that

$$Z(\hat{\mu}) = \bigcup_{j=1}^{\infty} Z(\hat{v}_j) = \bigcup_{j=1}^{\infty} \left[\frac{N_1 \dots N_j}{K_j M_j} (\mathbb{Z} \setminus M_j \mathbb{Z}) \right]. \quad \square$$

It is natural to conjecture that:

Conjecture 4.3. *Suppose that (N_j, B_j) are the almost-Parseval-frame tower defined in Theorem 1.2 and the associated measure μ is spectral. Then all $\alpha_j = 0$.*

However, this will let us into rather involved number theoretic and combinatoric questions. To serve the purpose of this paper, the following proposition shows that under simple conditions on M_j, K_j and α_j , the measure μ cannot be spectral.

Proposition 4.4. *Suppose that Λ is a mutually orthogonal set for μ defined in (3) with $N_j = K_j M_j + 1$ ($\alpha_j = 1$) and $B_j = \{0, K_j, \dots, (M_j - 1)K_j\}$, $L_j = \{0, 1, \dots, M_j - 1\}$ satisfying*

- (1) all $M = 2$ and
- (2) all K_j are odd.

Then

$$\#\Lambda \leq 2.$$

Proof. Suppose that there exists mutually orthogonal set Λ with cardinality greater than 2. We can find distinct $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 - \lambda_2, \lambda_3 - \lambda_2, \lambda_1 - \lambda_3 \in Z(\hat{\mu})$. Hence, we can write

$$\lambda_1 - \lambda_2 = \frac{(N_1 \dots N_j)}{K_j M_j} (r_j + M_j q_j), \quad \lambda_3 - \lambda_2 = \frac{(N_1 \dots N_k)}{K_k M_k} (r_k + M_k q_k),$$

$$\lambda_1 - \lambda_3 = \frac{(N_1 \dots N_\ell)}{K_\ell M_\ell} (r_\ell + M_\ell q_\ell)$$

where $0 < r_n < M_n$ for $n = j, k, \ell$ and q_j, q_k, q_ℓ are integers. Denote by

$$\mathbf{N}_n = N_1 \dots N_n.$$

As $(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_2) = \lambda_1 - \lambda_3$, we have the following algebraic relation,

$$\frac{\mathbf{N}_j}{K_j M_j} (r_j + M_j q_j) - \frac{\mathbf{N}_k}{K_k M_k} (r_k + M_k q_k) = \frac{\mathbf{N}_\ell}{K_\ell M_\ell} (r_\ell + M_\ell q_\ell).$$

It follows that

$$\begin{aligned} & \mathbf{N}_j K_k K_\ell M_k M_\ell (r_j + M_j q_j) - \mathbf{N}_k K_j K_\ell M_j M_\ell (r_k + M_k q_k) \\ & = \mathbf{N}_\ell K_j K_k M_j M_k (r_\ell + M_\ell q_\ell). \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbf{N}_j K_k K_\ell M_k M_\ell r_j - \mathbf{N}_k K_j K_\ell M_j M_\ell r_k - \mathbf{N}_\ell K_j K_k M_j M_k r_\ell \\ & = M_j M_k M_\ell \cdot (\mathbf{N}_\ell K_j K_k q_\ell - \mathbf{N}_k K_j K_\ell q_k - \mathbf{N}_j K_k K_\ell q_j). \end{aligned} \tag{16}$$

In the first case, if all $M_j = 2$, then all $0 < r_i < 2$ which means all $r_i = 1$. (16) is reduced to

$$\begin{aligned} & \mathbf{N}_j K_k K_\ell - \mathbf{N}_k K_j K_\ell - \mathbf{N}_\ell K_j K_k \\ & = 2 \cdot (\mathbf{N}_\ell K_j K_k q_\ell - \mathbf{N}_k K_j K_\ell q_k - \mathbf{N}_j K_k K_\ell q_j). \end{aligned}$$

The right hand side is an even number. However, as all K_j are odd numbers, and all $N_j = 2K_j + 1$ are odd,, each term in the left hand side of (16) must be odd and hence the left hand side is an odd number overall. This is a contradiction. Hence, we cannot have a mutually orthogonal set of cardinality large than 2. \square

We are now ready to construct the example in Theorem 1.4.

Proof of Theorem 1.4. Let p be a prime number of the form $4k + 3$. It is well known that there are infinitely many primes of such form by the Dirichlet theorem. Writing

$$p^j = 2K_j + 1,$$

j we claim that K_j is an odd number whenever j is odd. Expanding in binomial theorem, we have for some integer L_j ,

$$K_j = \frac{p^j - 1}{2} = \frac{(4k + 3)^j - 1}{2} = \frac{4L_j + 3^j - 1}{2}.$$

It suffices to show that $(3^j - 1)/2$ is an odd number if j is odd. But from Binomial expansion, $\frac{3^j - 1}{2} = \frac{(2+1)^j - 1}{2} = 2^{j-1} + j2^{j-2} + \dots + \binom{j}{2}2 + j$. This shows that $\frac{3^j - 1}{2}$ is an odd number.

Letting $N_j = p^{2j-1} = 2K_{2j-1} + 1$ where p is a prime number of the form $4k + 3$ and $B_j = \{0, K_{2j-1}\}$. From Example 2.3, we have an almost-Parseval-frame tower. By Theorem 1.3(b), the associated measure μ is frame spectral. On the other hand, it is non-spectral by Proposition 4.4. \square

Remark 4.5. In [20], it was proved that if ν is a spectral measure on $[0, 1]$ with spectrum inside \mathbb{Z} , then any $\mu = \nu * \delta_A$, with $A \subset \mathbb{Z}$, is a frame spectral measure and some of them are not spectral. In view of Theorem 1.2, the measure μ we constructed cannot be of the form $\nu * \delta_A$, where ν is spectral and δ_A is a discrete measure supported on some set A . If it was the case, then

$$\hat{\mu}(\xi) = \hat{\nu}(\xi)\widehat{\delta}_A(\xi)$$

and this would have implied that any mutually orthogonal set of ν must be mutually orthogonal set of μ and hence cardinality of such sets for μ for be infinite, which is a contradiction by Proposition 4.4 we just proved.

Appendix A. Hausdorff dimension

In this section, we study the Hausdorff dimension, denoted by \dim_H , of the support of μ , which is an important question in fractal geometry. We refer the reader to [14] the definition of Hausdorff dimension. Given a sequence of positive integers M_j and a sequence of numbers r_j . Suppose that $0 < r_j < 1$, $M_j \geq 2$ and $r_j M_j \leq 1$ for all j . For $k \geq 1$, we let $D_0 = \emptyset$, $D_k = \{(i_1, \dots, i_k) : i_j \in \{0, 1, \dots, M_j - 1\}\}$. For $\mathbf{i} \in D_k$ and $\mathbf{j} \in D_\ell$, $\mathbf{ij} \in D_{k+\ell}$ is the standard concatenation of two words. For each $\sigma \in \bigcup_{k=1}^\infty D_k$, we define an interval J_σ . We say that

$$E = \bigcap_{k=1}^\infty \bigcup_{\sigma \in D_k} J_\sigma$$

is a *homogeneous Moran set* if the following conditions are satisfied.

- (1) $J_\emptyset = [0, 1]$. For any $\sigma \in D_k$, $J_{\sigma 0}, \dots, J_{\sigma(M_{k+1}-1)}$ are subinterval of J_σ enumerated from left to right and $J_{\sigma i} \cap J_{\sigma j}$ has intersects at most one point.
- (2) For any $k \geq 1$, for any $\sigma \in D_{k-1}$ and $j \in \{0, 1, \dots, M_k - 1\}$,

$$r_k = \frac{|J_{\sigma j}|}{|J_\sigma|}.$$

$|\cdot|$ denotes the Lebesgue measure of the interval.

- (3) For any $\sigma \in D_k$, the gaps between $J_{\sigma i}$ and $J_{\sigma(i+1)}$ are equal in length, the left endpoint of $J_{\sigma 0}$ is equal to the left endpoint of J_σ and the right endpoint of $J_{\sigma(M_{k+1}-1)}$ is equal to the right endpoint of J_σ .

It was shown that [18] (see also [17, Proposition 3.1]) that the Hausdorff dimension of E is equal to

$$\dim_H(E) = \liminf_{j \rightarrow \infty} \frac{\log(M_1 \dots M_j)}{-\log(r_1 \dots r_j)}.$$

Turning to our case where $N_j = M_j K_j + \alpha_j$ and $B_j = \{0, K_j, \dots, K_j(M_j - 1)\}$, the support of the measure μ is

$$K_\mu = \left\{ \sum_{j=1}^\infty \frac{i_j K_j}{N_1 \dots N_j} : i_j K_j \in B_j \right\} = \bigcap_{k=1}^\infty \bigcup_{\sigma \in D_k} J_\sigma$$

where $J_\sigma = [\sum_{j=1}^k i_j K_j (N_1 \dots N_j)^{-1}, \sum_{j=1}^k i_j K_j (N_1 \dots N_j)^{-1} + (N_1 \dots N_k)^{-1}]$ and $\sigma = (i_1, \dots, i_k)$. Note that The support K_μ is contained in the interval $[0, \rho]$, where $\rho = \sum_{j=1}^\infty (K_j (M_j - 1))(N_1 \dots N_j)^{-1}$. By a simple rescaling, $C = \rho^{-1} K_\mu$. It is easy to see that C is actually a homogeneous Moran set with $r_1 = 1/N_1$ and $r_k = (N_1 \dots N_k)^{-1}/(N_1 \dots N_{k-1})^{-1} = 1/N_k$. Hence, we have thus proved

Proposition A.1. *We have*

$$\dim_H(K_\mu) = \liminf_{j \rightarrow \infty} \frac{\log(M_1 \dots M_j)}{\log(N_1 \dots N_j)}.$$

We now compute the Hausdorff dimension of some frame spectral measures μ .

Example A.2. In Example 2.3, $N_j = p^j = 2K_j + 1$ and $M_j = 2$ for all j . In this case, the Hausdorff dimension

$$\dim_H(K_\mu) = \liminf_{j \rightarrow \infty} \frac{\log 2^j}{\log(p^{1+2+\dots+j})} = \liminf_{j \rightarrow \infty} \frac{2j \log 2}{j(j+1) \log p} = 0.$$

The non-spectral measure with Fourier frame given in Theorem 1.4 is a special case of this type, and thus the support has Hausdorff dimension 0.

This example shows that frame spectral measure can be very “thin”, similar situation happens for spectral measures [5]. The following example shows that our construction does give frame spectral measures with positive Hausdorff dimension.

Example A.3. In Example 2.4, $M_j = K_j^\beta$ and $\alpha_j = 1$ (i.e. $\gamma = 0$) for all j . Then $N_j = K_j^{1+\beta} + 1$. In this case, the Hausdorff dimension

$$\dim_H(K_\mu) = \liminf_{j \rightarrow \infty} \frac{\log((K_1 \dots K_j)^\beta)}{\log((K_1^{1+\beta} + 1) \dots (K_j^{1+\beta} + 1))}.$$

As $\lim_{x \rightarrow \infty} \log(1+x)/\log x = 1$, for x large, $C^{-1} \log x \leq \log(1+x) \leq C \log x$ for some constant $C > 0$. Hence, if K_j is large enough,

$$\begin{aligned} C^{-1} \log((K_1 \dots K_j)^{1+\beta}) &\leq \log((K_1^{1+\beta} + 1) \dots (K_j^{1+\beta} + 1)) \\ &\leq C \log((K_1 \dots K_j)^{1+\beta}). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{\beta}{C(1+\beta)} &= \liminf_{j \rightarrow \infty} \frac{\beta \log((K_1 \dots K_j))}{C(1+\beta) \log((K_1 \dots K_j))} \\ &\leq \dim_H(K_\mu) \\ &\leq \liminf_{j \rightarrow \infty} \frac{\beta \log((K_1 \dots K_j))}{C^{-1}(1+\beta) \log((K_1 \dots K_j))} \\ &= \frac{\beta}{(1+\beta)C^{-1}}. \end{aligned}$$

Hence, the support of the frame spectral measure has Hausdorff dimension at least $\frac{\beta}{C(1+\beta)} > 0$.

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Received November 19, 2015

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