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Discrete Schrödinger operators with potentials defined by measurable sampling functions over Liouville torus rotations

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Abstract. We consider a one-dimensional discrete Schrödinger operator $H_\omega = \Delta + v_\omega$ whose potential v_ω has the form $v_\omega(j) = V(\omega + j\bar{\alpha})$, $j \in \mathbb{Z}$. Here $\omega \in \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $\alpha \in \mathbb{R}^k$, $\bar{\alpha}$ is the projection of α on $\mathbb{T}^k$, and the function $V\colon \mathbb{T}^k \to \mathbb{C}$ is Borel measurable. We show that if the frequency vector α is Liouville (the sequence $\{\nu\bar{\alpha}\}_{\nu\in\mathbb{N}}$ has a subsequence converging to 0 fast enough), then for Lebesgue almost every $\omega \in \mathbb{T}^k$ the point spectrum of the operator H_ω is empty.

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1. Introduction

We consider a one-dimensional discrete Schrödinger operator H_ω acting in $l^2(\mathbb{Z})$ as follows:

$$(H_\omega y)(j) = y(j+1) + y(j-1) + V(\omega + j\bar{\alpha})y(j), \quad j \in \mathbb{Z}. \tag{1}$$

Here $\omega \in \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$; $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$; $\bar{\alpha}$ is the projection of α on $\mathbb{T}^k$; and $V(\cdot)$ is a complex-valued function on $\mathbb{T}^k$. It is well known that if the function $V(\cdot)$ is continuous and the frequency vector α is Liouville (that is, the sequence $\{\nu\bar{\alpha}\}_{\nu=1}^{\infty}$ has a subsequence approaching 0 sufficiently fast), then the operator H_ω with any $\omega \in \mathbb{T}^k$ has no eigenvalues (in fact, no eigenfunctions decaying at $\pm\infty$).

The usual proof of this fact has three ingredients:

(i) the statement often called "the Gordon lemma", whose more appropriate name is "the Gordon–Gorin lemma" (see [7, 8]), which allows one to estimate a solution of a periodic equation at any point in terms of its values at certain other points going to infinity as the period increases[1];

(ii) the approximability of longer and longer segments of the trajectory $\{\omega + \nu\bar{\alpha}\}_{\nu\in\mathbb{Z}}$ of $\omega \in \mathbb{T}^k$ by periodic trajectories with fast increasing accuracy, which follows from the Liouville property of α;

(iii) the uniform continuity of V, due to which the closeness of long segments of trajectories implies the closeness of the values of $V(\cdot)$ being read from them, and consequently, the closeness of the corresponding solutions satisfying identical initial conditions.

The uniform continuity means that $V(\cdot + \tau)$ converges to $V(\cdot)$ uniformly as $\mathbb{T}^k \ni \tau \to 0$. It turns out that in this context the uniform convergence can be replaced by the much weaker *convergence in measure*, which holds for any *measurable* V. The price is that now the absence of eigenvalues of H_ω can only be guaranteed for Lebesgue a.e. (rather than every) $\omega \in \mathbb{T}^k$.

Remark 1. Models with discontinuous V (although continuous outside a codimension-1 submanifold of $\mathbb{T}^k$) have been studied for a long time. In the case of Sturmian potentials ($k = 1;\ V(x) = \mathbf{1}_{[1-\theta,\theta)}(x),\ x \in \mathbb{T}$), after a series of partial results it was proven that the point specrum of the operator (1) is empty for any irrational α and any ω [4]. In another much studied case, the Maryland model (we are only referring to the one-dimensional case: $k = 1;\ V(x) = \tan(\pi x),\ x \in \mathbb{T}$), for a Diophantine α and any ω, Anderson localization takes place [6, 13], while for a Liouville α and a.e. ω, the point spectrum is empty [13]. Moreover, in the latter case the exceptional ω's do exist no matter how Liouville α is: for any irrational α there is an uncountable set of ω's for which the operator H_ω exhibits Anderson localization [9, Corollary 4.2].

Remark 2. According to [3], the operator (1) with a real-valued potential, having an "essential" singularity, has purely singular spectrum for a.e. $\omega \in \mathbb{T}^k$. Combining this with the result of the present paper, one concludes that for such potentials and Liouville α's, the spectrum of H_ω is singular continuous for a.e. ω.

[1] This lemma (or its idea) has been used in various contexts, see, e.g., papers [1, 4, 5, 9, 10, 11, 12, 13] and survey [2].

Remark 3. The novelty of this paper is in the observation that for a measurable V the potential $v_\omega(j) = V(\omega + j\bar{\alpha})$, under a suitable Liouville condition on α, has a periodic approximability property similar to the one it would have if V were continuous, unless ω belongs to a certain zero measure set. Once this is established, the rest of the argument is more or less standard, but now allows one to treat any measurable function. We choose to present the details in order to make the paper fully self-contained.

2. Statement

From now on we assume that the function $V\colon \mathbb{T}^k \to \mathbb{C}$ is Borel measurable. Before formulating the result, we introduce the necessary notation.

For $q \in \mathbb{R}^k$, we denote by $\bar{q}$ its projection on $\mathbb{T}^k$ and by $[\![q]\!]$ its l^∞ distance from $\mathbb{Z}^k$:

$$[\![q]\!] = \min_{m \in \mathbb{Z}^k} \|q - m\|_\infty.$$

For $\omega \in \mathbb{T}^k$ we set

$$\|\omega\| = \min_{q \in \omega} \|q\|_\infty$$

so that $d(\omega, \omega') = \|\omega - \omega'\|$ is the l^∞ distance in $\mathbb{T}^k$. Note that for any $q \in \mathbb{R}^k$

$$\|\bar{q}\| = [\![q]\!] \le \|q\|_\infty. \tag{2}$$

Since V is Borel measurable, $V(\cdot + \tau)$ converges to $V(\cdot)$ in measure as $\mathbb{T}^k \ni \tau \to 0$ (this can be proven by approximation of V by bounded and then continuous functions). Consequently, if we set

$$F(\tau, \varepsilon) = \{\omega \in \mathbb{T}^k\colon |V(\omega + \tau) - V(\omega)| > \varepsilon\} \quad (\tau \in \mathbb{T}^k,\ \varepsilon > 0)$$

and denote the Lebesgue measure of a Borel set A by $|A|$, then for any $\varepsilon > 0$ and any $\delta > 0$, there is $\beta(\varepsilon, \delta) > 0$ such that

$$\|\tau\| < \beta(\varepsilon, \delta) \implies |F(\tau, \varepsilon)| < \delta.$$

We will also need the sets

$$E(M) = \{\omega \in \mathbb{T}^d\colon |V(\omega)| > M\} \quad (M \ge 0)$$

and their measures $\rho(M) = |E(M)|$. Note that $\rho(M) \to 0$ as $M \to \infty$.

Theorem 1. *Suppose there is an infinite set $\mathcal{P} \subset \mathbb{N}$ and, for each $\nu \in \mathcal{P}$, numbers $M_\nu > 0$, $\delta_\nu > 0$ and $\varepsilon_\nu > 0$ such that, as $\mathcal{P} \ni \nu \to \infty$,*

$$\nu\rho(M_\nu) \to 0; \tag{3}$$

$$\nu\delta_\nu \to 0; \tag{4}$$

$$\nu(M_\nu + C)^{2\nu-1}\varepsilon_\nu \to 0 \quad \textit{for any constant } C > 0. \tag{5}$$

If

$$[\![\nu\bar{\alpha}]\!] < 2^{-1}\beta(\varepsilon_\nu, \delta_\nu) \quad \textit{for all } \nu \in \mathcal{P}, \tag{6}$$

then for Lebesgue almost every $\omega \in \mathbb{T}^k$ the operator (1) has no nontrivial eigenfunctions decaying at $\pm\infty$.

Remark. The rational approximation condition (6) on α depends on the choice of the sampling function V and will become "harder" to satisfy as V becomes "more ill-behaved".

3. Proof

Suppose the equation $H_\omega y = \lambda y$, that is,

$$y(j+1) + y(j-1) + V(\omega + j\bar{\alpha})y(j) = \lambda y(j), \quad j \in \mathbb{Z}, \tag{7}$$

has a decaying solution y:

$$y(j) \longrightarrow 0 \quad \text{as } |j| \to \infty. \tag{8}$$

We are going to show that $y = 0$ unless ω belongs to a certain measure zero subset of $\mathbb{T}^k$.

1. For any $\nu \in \mathcal{P}$, select $m_\nu \in \mathbb{Z}^k$ so that

$$\|\nu\alpha - m_\nu\|_\infty = [\![\![\nu\alpha]\!]\!] \equiv [\![\overline{\nu\alpha}]\!] \equiv [\![\nu\bar{\alpha}]\!], \tag{9}$$

and set $\alpha^\nu = \nu^{-1}m_\nu$.

Let $I_\nu = \{j \in \mathbb{Z}: |j| \le 2\nu\}$ and

$$Z^\nu = \Big(\bigcup_{j\in I_\nu} X_j^\nu\Big) \cup \Big(\bigcup_{j\in I_\nu} Y_j^\nu\Big),$$

where

$$X_j^\nu = \{\omega \in \mathbb{T}^k: |V(\omega + j\bar{\alpha})| > M_\nu\} \equiv E(M_\nu) - j\bar{\alpha}$$

and

$$Y_j^\nu = \{\omega \in \mathbb{T}^k : |V(\omega + j\overline{\alpha^\nu}) - V(\omega + j\bar{\alpha})| > \varepsilon_\nu\} \equiv F(j(\overline{\alpha^\nu} - \bar{\alpha}), \varepsilon_\nu) - j\bar{\alpha}.$$

Note that, in view of (2) and (9), for $j \in I_\nu$

$$\begin{aligned} [\![j(\overline{\alpha^\nu} - \bar{\alpha})]\!] &\le \|j(\alpha^\nu - \alpha)\|_\infty \\ &= \|(j/\nu)(m_\nu - \nu\alpha)\|_\infty \\ &= (|j|/\nu)\|m_\nu - \nu\alpha\|_\infty \\ &\le 2[\![\nu\bar{\alpha}]\!]. \end{aligned}$$

By (6), for $\nu \in \mathcal{P}$ and $j \in I_\nu$

$$[\![j(\overline{\alpha^\nu} - \bar{\alpha})]\!] < \beta(\varepsilon_\nu, \delta_\nu)$$

and hence $|Y_j^\nu| < \delta_\nu$. We also have $|X_j^\nu| = \rho(M_\nu)$; consequently, for $\nu \in \mathcal{P}$

$$|Z^\nu| < (4\nu + 1)(\delta_\nu + \rho(M_\nu)),$$

and, by (3) and (4), $|Z^\nu| \to 0$ as $\mathcal{P} \ni \nu \to \infty$.

2. By thinning out the set $\mathcal{P}$, we may assume that $\sum_{\nu\in\mathcal{P}} |Z^\nu| < \infty$. Now, by the Borel–Cantelli lemma, the set Z_∞ of all $\omega \in \mathbb{T}^k$ that belong to infinitely many sets Z^ν ($\nu \in \mathcal{P}$) has measure zero.

In what follows, we assume that $\omega \in \mathbb{T}^k \setminus Z_\infty$ so that for any large enough $\nu \in \mathcal{P}$ (say, $\nu \ge \nu_\omega$), ω is outside the sets X_j^ν and Y_j^ν for all $j \in I_\nu$. This means that

$$|V(\omega + j\bar{\alpha})| \le M_\nu; \quad |V(\omega + j\overline{\alpha^\nu}) - V(\omega + j\bar{\alpha})| \le \varepsilon_\nu \tag{10}$$

$(\mathcal{P} \ni \nu \ge \nu_\omega;\ j \in I_\nu)$.

3. Along with the solution y of equation (7), we consider the solution z^ν of the ν-periodic equation

$$z(j+1) + z(j-1) + V(\omega + j\overline{\alpha^\nu})z(j) = \lambda z(j), \quad j \in \mathbb{Z}, \tag{11}$$

satisfying the same initial condition as $y(\cdot)$:

$$z^\nu(0) = y(0); \quad z^\nu(1) = y(1).$$

We want to estimate the difference between y and z^ν. Setting

$$\xi(j) = (y(j+1), y(j))^\top; \quad \eta^\nu(j) = (z^\nu(j+1), z^\nu(j))^\top,$$

we transform equations (7) and (11) into vector equations

$$\xi(j) = A_\omega(j)\xi(j-1) \quad \text{and} \quad \eta^\nu(j) = B^\nu_\omega(j)\eta^\nu(j-1),\ j \in \mathbb{Z}, \tag{12}$$

where

$$A_\omega(j) = \begin{pmatrix} \lambda - V(\omega + j\bar{\alpha}) & -1 \\ 1 & 0 \end{pmatrix}; \quad B^\nu_\omega(j) = \begin{pmatrix} \lambda - V(\omega + j\overline{\alpha^\nu}) & -1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

It follows from (12) that for $j \geq 1$

$$\xi(j) = A_\omega(j)A_\omega(j-1)\dots A_\omega(1)\xi(0); \quad \eta^\nu(j) = B^\nu_\omega(j)B^\nu_\omega(j-1)\dots B^\nu_\omega(1)\xi(0)$$

so that, by the telescopic identity,

$$\eta^\nu(j) - \xi(j) = \sum_{l=1}^{j} B^\nu_\omega(j)B^\nu_\omega(j-1)\dots B^\nu_\omega(l+1) \\ (B^\nu_\omega(l) - A_\omega(l))A_\omega(l-1)A_\omega(l-2)\dots A_\omega(1)\xi(0). \tag{14}$$

In the rest of the proof, $\|\cdot\|$ denotes the l^∞ norm of vectors in $\mathbb{C}^2$ and the corresponding operator norm. Note that

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \max(|a| + |b|, |c| + |d|).$$

By (5), $\varepsilon_\nu \to 0$ as $\mathcal{P} \ni \nu \to \infty$. Let $C = |\lambda| + 1 + \sup_\nu \varepsilon_\nu$. In view of (13) and (10), for large $\nu \in \mathcal{P}$ and any $j \in I_\nu$

$$\|A_\omega(j)\| \leq M_\nu + C; \quad \|B^\nu_\omega(j)\| \leq M_\nu + C; \quad \|B^\nu_\omega(j) - A_\omega(j)\| \leq \varepsilon_\nu.$$

Equation (14) and these inequalities imply that if $1 \leq j \leq 2\nu$, then

$$\|\eta^\nu(j) - \xi(j)\| \leq 2\nu(M_\nu + C)^{2\nu-1}\varepsilon_\nu\, \|\xi(0)\|. \tag{15}$$

In a similar manner, using instead of (12) and (13) the equations

$$\xi(j-1) = (A_\omega(j))^{-1}\xi(j); \quad \eta^\nu(j-1) = (B^\eta_\omega(j))^{-1}\eta^\nu(j), \quad j \in \mathbb{Z},$$

and

$$(A_\omega(j))^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda - V(\omega + j\bar{\alpha}) \end{pmatrix};$$

$$(B^\nu_\omega(j))^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda - V(\omega + j\overline{\alpha^\nu}) \end{pmatrix}$$

(the latter formulas can be derived directly from (7) and (11)), one can show that the inequality (15) also holds if $-2\nu \le j \le -1$; therefore, it is true for all $j \in I_\nu$ (recall that $\eta^\nu(0) = \xi(0)$). It follows then from (5) that

$$\max_{j\in I_\nu} \|\eta^\nu(j) - \xi(j)\| \longrightarrow 0 \quad \text{as } \mathcal{P} \ni \nu \to \infty.$$

4. By (8), $\|\xi(j)\| \to 0$ as $|j| \to \infty$. Consequently,

$$\max_{j=\pm\nu,\pm 2\nu} \|\eta^\nu(j)\| \longrightarrow 0 \quad \text{as } \mathcal{P} \ni \nu \to \infty. \tag{16}$$

On the other hand, for all $r \in \mathbb{Z}$ we have

$$\eta^\nu(r\nu) = (T^\nu_\omega)^r \eta^\nu(0),$$

where $T^\nu_\omega\colon \eta(0) \mapsto \eta(\nu)$ is the monodromy operator of the ν-periodic vector equation $\eta(j) = B^\nu_\omega(j)\eta(j-1)$, $j \in \mathbb{Z}$:

$$T^\nu_\omega = B^\nu_\omega(\nu) B^\nu_\omega(\nu - 1) \dots B^\nu_\omega(1).$$

The aforementioned Gordon–Gorin lemma [7, 8], in the 2-dimensional case, states: *Let X be a complex vector space* (dim $X = 2$) *with a seminorm $|||\cdot|||$, and T an invertible linear operator in X; then for any vector $\zeta \in X$, one has*

$$|||\zeta||| \le 2 \max_{r=\pm 1,\pm 2} |||T^r \zeta|||.$$

It follows that for all $\nu \in \mathcal{P}$

$$\|\eta^\nu(0)\| \le 2 \max_{j=\pm\nu,\pm 2\nu} \|\eta^\nu(j)\|.$$

By (16), the right-hand side converges to 0 as $\mathcal{P} \ni \nu \to \infty$, so that also $\|\eta^\nu(0)\| \to 0$. But $\eta^\nu(0) = \xi(0)$ for all ν; consequently, $\xi(0) = 0$ and hence $y(j) = 0$ for all $j \in \mathbb{Z}$.

We have shown that if $\omega \in \mathbb{T}^k \setminus Z_\infty$ and $H_\omega y = \lambda y$, where $y\colon \mathbb{Z} \to \mathbb{C}$ is a function decaying at infinity, then $y = 0$. Since $|Z_\infty| = 0$, this completes the proof of the theorem.

Corollary 1. *Given a measurable function $V\colon \mathbb{T}^k \to \mathbb{C}$, there is an explicit dense G_δ set $W \subset \mathbb{T}^k$ such that if $\bar\alpha \in W$, then for Lebesgue a.e. $\omega \in \mathbb{T}^k$ the operator* (1) *does not have eigenfunctions decaying at infinity.*

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