J. Fractal Geom. 5 (2018), 165–226
DOI 10.4171/JFG/60

Journal of Fractal Geometry
© European Mathematical Society

Numerical integration for fractal measures

Jens Malmquist[1] and Robert S. Strichartz[2]

Abstract. We find estimates for the error in replacing an integral $\int f d\mu$ with respect to a fractal measure μ with a discrete sum $\sum_{x \in E} w(x) f(x)$ over a given sample set E with weights w. Our model is the classical Koksma–Hlawka theorem for integrals over rectangles, where the error is estimated by a product of a *discrepancy* that depends only on the geometry of the sample set and weights, and *variance* that depends only on the smoothness of f. We deal with p.c.f. self-similar fractals, on which Kigami has constructed notions of *energy* and *Laplacian*. We develop generic results where we take the variance to be either the energy of f or the L^1 norm of $\triangle f$, and we show how to find the corresponding discrepancies for each variance. We work out the details for a number of interesting examples of sample sets for the Sierpiński gasket, both for the standard self-similar measure and energy measures, and for other fractals.

Mathematics Subject Classification (2010). 28A80, 65D30.

Keywords. Numerical integration, fractal measures, p.c.f. self-similar fractals, energy measures, Laplacians, Koksma–Hlawka theorem, Sierpiński gasket.

Contents

[1] Research of the first author was supported by the National Science Foundation through the Research Experiences for Undergraduates Program at Cornell, grant DMS-1156350.

[2] Research of the second author was supported in part by the National Science Foundation, grant DMS-1162045.

1. Introduction

Numerical integration on domains in Euclidean space is a highly developed subject that is of interest from both a theoretical and practical point of view, with many open problems still being actively pursued ([5], [8]). The goal of this paper is to develop a similar theory on fractals, following up on earlier work in [1].

The gist of the matter, in any context, may be succinctly stated as follows. Given a measure μ on some space and a finite set of points E, the *sample set*, we would like to approximate the integral $\int f d\mu$ by the sum $\sum_{x\in E} w(x) f(x)$ for a set of weights $\{w(x)\}$. The main problem is to estimate the error of the approximation. A desirable form of the error estimate is in terms of a product of two factors, a *discrepancy* of the weights (or just of the set E if the weights are chosen uniformly $w(x) = \frac{1}{\#E}$) depending on the "geometry" of E, and a *variance* of f that measures the "smoothness" of f in a suitable norm. A well known version of such an estimate in the case of rectangles is the Koksma–Hlawka theorem ([8]), and some of our results are modeled on this theorem. Other interesting questions concern how to choose the sample set E to minimize the discrepancy, and how to choose "natural" weights on E.

We will restrict attention to Kigami's class of *p.c.f. self-similar fractals* with a *regular harmonic structure* [3]. A basic example is the Sierpiński gasket SG (see [10] for a detailed description of this example) and we will give the most detailed results for this example. We hope that our results will serve as a foundation for future work on products of fractals, motivated by the observation that rectangles are products of intervals, and intervals are in fact the most elementary examples of p.c.f. self-similar sets.

There are two types of measures that are natural to consider in this context. The first are the *self-similar measures* that are naturally associated with the self-similar structure of the fractal, and include the normalized Hausdorff measure in the appropriate Hausdorff dimension. The second are the energy measures associated with the harmonic structure. Very briefly, the harmonic structure provides an energy $\mathcal{E}(f, g)$, a bilinear Dirichlet form analogous to the energy $\int_\Omega (\nabla f \cdot \nabla g) dx$ on a domain Ω in Euclidean space. *Harmonic functions* are energy $\mathcal{E}(h, h)$ minimizers, analogous to linear functions on an interval.

The energy measure $\nu_{h,H}$ for harmonic functions h and H assigns to a set C the "restriction" of $\mathcal{E}_{h,H}$ to C. An interesting and surprising result of Kusuoka [6] shows that energy measures and self-similar measures are mutually singular, in start contrast to what happens in classical analysis. Associated to each measure μ is a *Laplacian* Δ_μ. The study of Laplacians for self-similar measures was originally the focus of the theory of analysis on these fractals, but recently energy measure Laplacians have come to the fore ([4], [2], [11], [1]). For this reason, it is worth investigating numerical integration for both types of measures.

We will consider two types of smoothness conditions. The first is a very minimal smoothness that $\mathcal{E}(f, f)$ is finite. This implies that f is continuous in our context (but not in Euclidean space of dimension above one). The second is the finiteness of $\| \Delta_\mu f \|_1$. There will be a different discrepancy associated to each of these variances of f, with the second one typically a lot smaller because we are assuming more smoothness for the function. We will have two "generic results" corresponding to these choices. We note that our results are not exactly analogs of Koksma–Hlawka; they are only similar in spirit.

For each sample set E we will typically investigate a "natural" set of weights $\{p(x)\}$. These weights will allow the exact evaluation

$$\int f(x) d\mu(x) = \sum_{x \in E} p(x) f(x)$$

for a finite dimensional space of functions called *piecewise harmonic splines*. These are basically the continuous functions that are harmonic on the complement of E, and are the exact analog of piecewise linear functions on an interval. So then it is natural to estimate the error for a general set of weights $\{w(x)\}$ in terms of the differences between the two sets of weights, using the approximation properties of the piecewise harmonic splines in terms of the smoothness norms of f.

We develop our generic results in Section 2. Then in Section 3 we study the example of SG and the standard self-similar measure μ, and work out in detail the natural weights and discrepancies for a variety of sample sets. In Section 4 we briefly examine some other p.c.f. fractals. In Section 5 we return to SG but consider energy measures. See [9] for related work concerning values of smooth functions on discrete sets of points. The programs used to generate the data in sections 4 and 5 may be found at the website [7]

2. Generic results

Let K be a p.c.f. self-similar fractal generated by a finite iterated system $\{F_j\}$ of contractive similarity on some ambient Euclidean space. So

$$K = \bigcup_i F_i K$$

and there exists a finite set V_0 of *boundary points* such that

$$F_i K \cap F_j K \subseteq F_i V_0 \cap F_j V_0.$$

We assume there is a self-similar energy form $\mathcal{E}(u)$ on K such that

$$\mathcal{E}(u) = \sum_i \frac{1}{r_i} \mathcal{E}(u \circ F_i)$$

for some energy renormalization constants $0 < r_i < 1$. See [3] for detailed definitions.

Let μ be a probability measure on K that is non-atomic and assigns positive values to nonempty open sets. Let E be a finite subset of K, and suppose we are given a set of positive weights $w(x)$ on E with

$$\sum_{x \in E} w(x) = 1.$$

Our goal is to understand how well the discrete sum $\sum_{x \in E} w(x) f(x)$ approximates the integral $\int_K f d\mu$ under various "smoothness" assumptions on f. We want estimates of the form

$$\left| \int f d\mu - \sum_{x \in E} w(x) f(x) \right| \leq \operatorname{disc}(E, w) \operatorname{Var}(f)$$

where the discrepancy $\operatorname{disc}(E, w)$ is some "geometric" measurement of the distance between the original measure μ and the approximate measure $\sum_{x \in E} w(x) \delta_x$, and $\operatorname{Var}(f)$ is some norm measuring the smoothness of f. The classical Koksma–Hlawka theorem is a model example of such an estimate.

Our approach to obtaining such estimates is to consider two separate subproblems. The first is to obtain estimates of $\int f d\mu$ under the assumption that $f|_E = 0$. The second is to consider a family of splines defined in terms of E and to find a family of weights $\{p(x)\}$ such that $\int g d\mu = \sum_{x \in E} p(x) g(x)$ for every spline g. Given a suitably smooth f, we write $f = (f - g) + g$ where g is a spline satisfying $g|_E = f|_E$. We use the first subproblem to handle $f - g$ and the second subproblem to handle g, and then add.

Associated to the energy $\mathcal{E}$ and the measure μ we have a Laplacian Δ_μ defined by the weak formulation

$$\int_K v \, \Delta_\mu \, u d\mu = -\mathcal{E}(u, v) \tag{2.1}$$

for all test functions $v \in \operatorname{dom} \mathcal{E}$ ($\operatorname{dom} \mathcal{E}$ is the set of functions with $\mathcal{E}(v) < \infty$, and $\mathcal{E}(u, v)$ is the associated bilinear form).

[Note that this definition actually gives the Neumann Laplacian with vanishing normal derivatives at boundary points. In the case that $V_0 \subseteq E$ we could just as well restrict (2.1) to hold for just test functions v vanishing on V_0.]

We define dom $\triangle_\mu$ to be the space of functions u where $\triangle_\mu u$ is continuous, and $\mathrm{dom}_{L^1} \triangle_\mu$ to be the larger space where $\triangle_\mu u \in L^1(d\mu)$ with seminorm

$$\|u\|_{\mathrm{dom}_{L^1} \triangle_\mu} = \int |\triangle_\mu u| d\mu.$$

Associated with the set E we have the Green's function $G_E(x, y)$ that gives the inverse of $-\triangle_\mu$ subject to Dirichlet boundary conditions on E. That means

$$F(x) = \int_K G_E(x, y) f(y) d\mu(y) \tag{2.2}$$

gives the unique solution to

$$-\triangle_\mu F = f, \quad F|_E = 0.$$

Note that, in particular, the function

$$g_E(x) = \int_K G_E(x, y) d\mu(y)$$

is the solution of

$$-\triangle_\mu g_E = 1, \quad g_E|_E = 0.$$

Also, G_E is symmetric under interchange of x and y. Another useful expression for G_E is

$$G_E(x, y) = \sum_j \frac{1}{\lambda_j} \varphi_j(x) \varphi_j(y) \tag{2.3}$$

where $\{\varphi_j\}$ is an orthonormal basis of Dirichlet eigenfunctions

$$-\triangle_\mu \varphi_j = \lambda_j \varphi_j, \quad \varphi_j|_E = 0.$$

Note that while the individual terms in (2.3) depend on μ, in fact G_E depends only on E and not μ.

Definition 2.1. Let

$$\delta_0(E) = \left(\int_K \int_K G_E(x, y) d\mu(y) d\mu(x) \right)^{1/2} = \left(\int_K g_E(x) d\mu(x) \right)^{1/2}$$

and

$$\delta_1(E) = \sup_x g_E(x).$$

Theorem 2.2. *Suppose* $u \in \mathrm{dom}\,\mathcal{E}$ *and* $u|_E = 0$. *Then*

$$\left| \int u d\mu \right| \leq \delta_0(E) \mathcal{E}(u)^{1/2} \tag{2.4}$$

Proof. Write $u = \sum c_j \varphi_j$ for $c_j = \int u \varphi_j \, d\mu$. Since

$$\mathcal{E}(\varphi_j, \varphi_k) = \int (-\Delta_\mu \varphi_j) \varphi_k \, d\mu = \lambda_j \int \varphi_j \varphi_k \, d\mu,$$

we have $\mathcal{E}(\varphi_j, \varphi_k) = 0$ if $j \neq k$ and $\mathcal{E}(\varphi_j, \varphi_j) = \lambda_j$. So

$$\mathcal{E}(u) = \sum_j \lambda_j |c_j|^2, \tag{2.5}$$

and by Cauchy-Schwarz

$$\begin{aligned} \left| \int u \, d\mu \right| &= \left| \sum_j c_j \int \varphi_j \, d\mu \right| \\ &\leq \Big(\sum_j \lambda_j |c_j|^2 \Big)^{1/2} \Big(\sum_j \frac{1}{\lambda_j} \left| \int \varphi_j \, d\mu \right|^2 \Big)^{1/2}. \end{aligned}$$

But by (2.3),

$$\sum_j \frac{1}{\lambda_j} \left| \int \varphi_j \, d\mu \right|^2 = \int \int G_E(x, y) \, d\mu(y) \, d\mu(x),$$

and combined with (2.5) this yields (2.4). □

Theorem 2.3. *Suppose* $u \in \mathrm{dom}_{L^1} \Delta_\mu$ *and* $u|_E = 0$. *Then*

$$\left| \int u \, d\mu \right| \leq \delta_1(E) \int |\Delta_\mu u| \, d\mu. \tag{2.6}$$

Proof. Let $f = \Delta_\mu u$, so $f \in L^1(d\mu)$ and (2.2) holds (with $F = u$). Then

$$\begin{aligned} \left| \int u \, d\mu \right| &= \left| \int \int G_E(x, y) f(y) \, d\mu(x) \, d\mu(y) \right| \\ &= \left| \int g_E(y) f(y) \, d\mu(y) \right| \\ &\leq \delta_1(E) \int |f| \, d\mu \end{aligned}$$

which is (2.6) □

Generally speaking, we expect $\delta_1(E)$ to be a lot smaller than $\delta_0(E)$, because of the square root in the definition of $\delta_0(E)$. We gain this better estimate because we are requiring more smoothness in u in Theorem 2.3.

Definition 2.4. Let $\mathcal{H}_E$ denote the space of piecewise harmonic splines with nodes in E. In other words, the continuous functions v such that $\Delta_\mu v = 0$ in the complement of E (the condition $\Delta_\mu v = 0$ is independent of μ). $\mathcal{H}_E$ is a space of dimension $\#E$ and each $v \in \mathcal{H}_E$ is uniquely determined by its values on E. (Note that if E does not contain V_0, then the harmonic condition at points in $V_0 \setminus E$ is just the vanishing of the normal derivative.)

Theorem 2.5. *There exists a set of weights $\{p(x)\}$ on E such that*

$$\int v\,d\mu = \sum_{x\in E} p(x)v(x) \quad \textit{for all } v \in \mathcal{H}_E. \tag{2.7}$$

Proof. Let $v_j \in \mathcal{H}_E$ be determined by the condition $v_j(x_k) = \delta_{jk}$ for all $x_k \in E$. Set

$$p(x_j) = \int v_j\,d\mu.$$

Then (2.7) follows from $v = \sum_j v(x_j)v_j$. □

Theorem 2.6. (a) *Suppose that $u \in \operatorname{dom}\mathcal{E}$. Then*

$$\left|\int u\,d\mu - \sum_{x\in E} p(x)u(x)\right| \le \delta_0(E)\mathcal{E}(u)^{1/2}. \tag{2.8}$$

(b) *Suppose $u \in \operatorname{dom}_{L^1}\Delta_\mu$. Then*

$$\left|\int u\,d\mu - \sum_{x\in E} p(x)u(x)\right| \le \delta_1(E)\int |\Delta_\mu u|\,d\mu. \tag{2.9}$$

Proof. Write $u = (u-v)+v$ where $v \in \mathcal{H}_E$ and $v|_E = u|_E$. Then

$$\int u\,d\mu - \sum_{x\in E} p(x)u(x) = \int (u-v)\,d\mu + \int v\,d\mu - \sum_{x\in E} p(x)v(x) = \int (u-v)\,d\mu$$

by Theorem (2.5). For part (a), we apply Theorem (2.2) to $u-v$ to obtain

$$\left|\int (u-v)\,d\mu\right| \le \delta_0(E)\mathcal{E}(u-v)^{1/2}.$$

Since $u - v$ vanishes on E, we have

$$\mathcal{E}(u-v, v) = -\int (u-v) \triangle_\mu v d\mu = 0$$

since $\triangle_\mu v = 0$ away from E. Thus, $\mathcal{E}(u, v) = \mathcal{E}(v, v)$ and hence

$$\mathcal{E}(u-v, u-v) = \mathcal{E}(u, u) - \mathcal{E}(v, v) \leq \mathcal{E}(u, u),$$

so we obtain (2.8). For part (b), we apply Theorem (2.3) to $u - v$ to obtain

$$\left| \int (u-v) d\mu \right| \leq \delta_1(E) \int | \triangle_\mu u - \triangle_\mu v| d\mu.$$

However, $\triangle_\mu v = 0$ away from E and μ is assumed to be non-atomic, so we obtain (2.9). □

It may not always be feasible to compute the weights $\{p(x)\}$ precisely, or we may have a preference for a different set of weights, for example the uniform weights $w(x) = \frac{1}{\#E}$ for all $x \in E$. So we want a more flexible theorem that gives error estimates for general weights.

Definition 2.7. Let R denote the radius in the effective resistance metric, namely the minimum value for which there exists $x_0 \in K$ (the "center") such that the estimate

$$|u(x) - u(x_0)|^2 \leq R\mathcal{E}(u) \tag{2.10}$$

holds for all $x \in K$ and all $u \in \operatorname{dom} \mathcal{E}$. For any set of finite weights $\{w(x)\}$, define

$$\delta(E, w) = R^{1/2} \sum_{x \in E} |p(x) - w(x)| . \tag{2.11}$$

Theorem 2.8. (a) *If $u \in \operatorname{dom} \mathcal{E}$ then*

$$\left| \int u d\mu - \sum_{x \in E} w(x) u(x) \right| \leq (\delta_0(E) + \delta(E, w)) \, \mathcal{E}(u)^{1/2}.$$

(b) *If $u \in \operatorname{dom}_{L^1} \triangle_\mu$, then*

$$\left| \int u d\mu - \sum_{x \in E} w(x) u(x) \right| \leq \delta_1(E) \int | \triangle_\mu u| d\mu + \delta(E, w) \mathcal{E}(u)^{1/2}.$$

Proof. In view of Theorem 2.6 it suffices to show

$$\left|\sum_{x\in E}(p(x)-w(x))\,u(x)\right| \leq \delta(E,w)\mathcal{E}(u)^{1/2}.$$

for $u \in \operatorname{dom}\mathcal{E}$. Note that $\sum_{x\in E}(p(x)-w(x)) = 0$ since both $\{p(x)\}$ and $\{w(x)\}$ sum to 1. Let

$$\bar{u}(x) = u(x) - c, \quad \text{for } c = u(x_0).$$

Then $\mathcal{E}(\bar{u}) = \mathcal{E}(u)$ and

$$\sum_{x\in E}(p(x)-w(x))u(x) = \sum_{x\in E}(p(x)-w(x))\bar{u}(x).$$

So

$$\begin{aligned}\left|\sum_{x\in E}(p(x)-w(x))u(x)\right| &\leq \|\bar{u}\|_\infty \sum_{x\in E}|p(x)-w(x)| \\ &\leq \delta(E,w)\mathcal{E}(u)^{1/2}\end{aligned}$$

by (2.10) and (2.11). □

Note that we can not control $\left|\sum_{x\in E}(p(x)-w(x))u(x)\right|$ in terms of $\int|\Delta u|d\mu$ alone because u could be harmonic and we can not make it zero by subtracting a constant.

In some examples the constant δ_1 is larger than desirable because $g_E(x)$ has a large spike near the point where it assumes its maximum but is otherwise considerably smaller. In that case we may obtain a smaller constant by applying Hölder's inequality in the proof of Theorem (2.3), at the cost of assuming that $\Delta_\mu u$ is in some L^p space for $p > 1$.

Theorem 2.9. *Assume $u \in \operatorname{dom}_{L^p}\Delta_\mu$ for some $p \geq 1$, and let q be the dual index, $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\left|\int u d\mu - \sum_{x\in E} w(x)u(x)\right| \leq \|g_E\|_q \|\Delta_\mu u\|_p + \delta(E,w)\mathcal{E}(u)^{1/2}. \tag{2.12}$$

Proof. The same as the proof of Theorem (2.8)(b), except for the use of Hölder's inequality in the proof of Theorem (2.3). □

Note that if we take $p = \infty$, then

$$\|g_E\|_1 = \delta_0^2.$$

3. Basic examples on SG

In this section we consider some examples of the set E for the case of SG with μ the standard symmetric self-similar measure. We will use "$\triangle$" to refer to the Laplacian with respect to this measure. For each example we compute our estimate of $\delta_0(E)$ and $\delta_1(E)$, the weights $\{p(x)\}$, and $\delta(E, w)$ when w is the uniform weight $w(x) = \frac{1}{\#E}$. The results of Section 2 give us a recipe to make these computations. We find the function

$$g_E(x) = \int G_E(x, y)d\mu(y).$$

This function is non-negative and vanishes on E. Its integral over SG is $(\delta_0(E))^2$ and its maximum value is $\delta_1(E)$. For all $x \in E$, we compute the weight $p(x) = \int v_x d\mu$ (where v_x is specified by 2.5) by computing the harmonic spline v_x.

Example 3.1. $E = V_0$.

[12] provides an algorithm to compute the values of multiharmonic functions on V_* for an expansive family of fractals. For SG, Section 5.1 of [12] gives the specific values resulting from this algorithm. By Table 5.1 of [12], if f_{1k} is the biharmonic function such that $f_{1k}|_{V_0} = 0$ and $\triangle f_{1k} = h_k$, then

$$\left.\begin{aligned} 5 \cdot f_{1k}(F_i q_k) &= p_1 = -.12, \\ f_{1k}(F_i q_k) &= -9/375 \end{aligned}\right\} \quad (\text{for } i \neq k)$$

and

$$\left.\begin{aligned} 5 \cdot f_{1k}(F_i q_j) &= q_1 = -.09333\ldots, \\ f_{1k}(F_i q_j) &= -7/375 \end{aligned}\right\} \quad (\text{for } i, j, k \text{ all distinct}).$$

Thus, if $v(x) = \int G_{V_0} h_0(y)d\mu(y) = f_{1]}(x)$ (or equivalently, $\triangle v = h_0$ and $v|_{V_0} = 0$), then the values of v on V_1 are shown in Figure 3.1. In general, if $v(x) = \int G_{V_0}(x, y)h_i(y)d\mu(y)$, then the values of v on $V_1 \setminus V_0$ are $-9/375$, $-9/375$, and $-7/375$, with the $-7/375$ occurring at the midpoint of the side opposite from q_i.

Also

$$\begin{aligned} g_{V_0}(x) &= \int G_{V_0}(x, y)(-h_0(y) - h_1(y) - h_2(y))d\mu(y) \\ &= -f_{10}(x) - f_{11}(x) - f_{12}(x), \end{aligned}$$

so g_{V_0} takes the values shown in Figure 3.2.

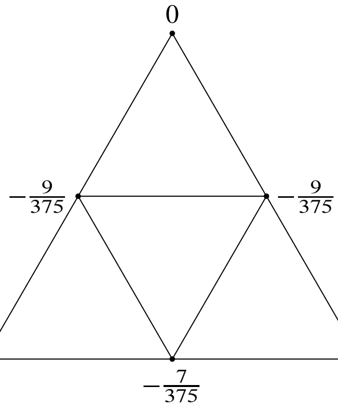


Figure 3.1. The values of $v(x) = \int G_{V_0}(x, y)h_0(y)d\mu(y)$ on V_1.

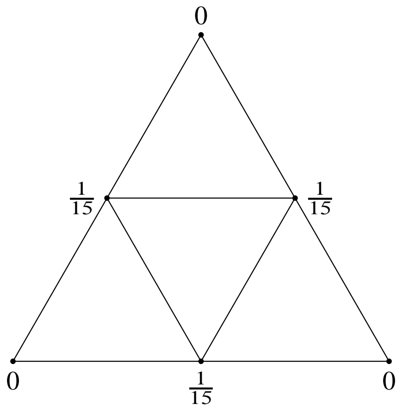


Figure 3.2. The values of g_{V_0} on V_1.

Moreover, $\triangle^2 g_E = \triangle(-1) = 0$, so g_E is the biharmonic function whose Laplacian is equal to -1 everywhere. From the values g_{V_0} takes on V_m, we deduce what values it takes for $x \in V_{m+1}$. Once we have $(g_{V_0} \circ F_w)|_{V_0}$ for a word w of length m, because we also know that $(\triangle g_{V_0}) \circ F_w = -1$, we can use the Green's function to calculate $(g_{V_0} \circ F_w)|_{V_1}$.

Lemma 3.1. *If $|w| = m$ and $(g_{V_0} \circ F_w)|_{V_0}$ takes values as shown in Figure 3.3, then $(g_{V_0} \circ F_w)|_{V_1}$ takes values as shown in Figure 3.4.*

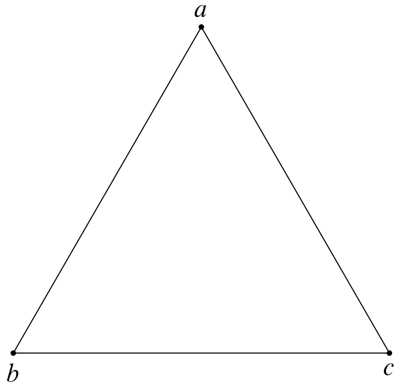


Figure 3.3. The values of $(g_{V_0} \circ F_w)$ on V_0, in the context of Lemma 3.1.

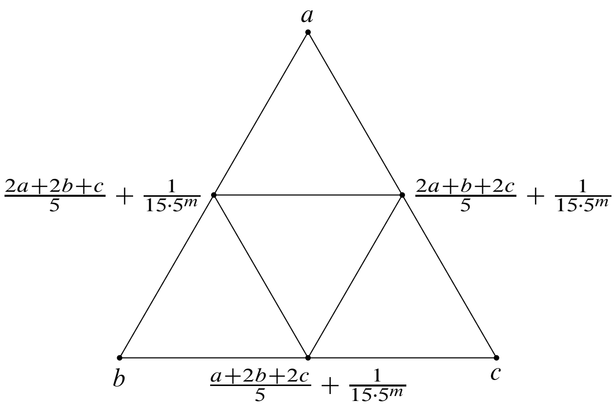


Figure 3.4. The values of $(g_{V_0} \circ F_w)$ on V_1, in the context of Lemma 3.1.

Proof. Let $u = g_{V_0} \circ F_w$. Let $\tilde{u}$ be the harmonic function that shares the values of $\tilde{u}$ on V_0. A simple consequence of the pointwise formulation of the Laplacian is

$$\triangle(f \circ F_w) = r_w \mu_w (\triangle f) \circ F_w$$

or

$$\triangle(f \circ F_w) = \left(\frac{1}{5}\right)^m (\triangle f) \circ F_w.$$

Therefore,

$$\triangle u = \triangle(g_{V_0} \circ F_w) = \left(\frac{1}{5}\right)^m (\triangle g_{V_0}) \circ F_w = \left(\frac{1}{5}\right)^m.$$

$u - \tilde{u}$ has the same Laplacian as u, so

$$\int G(x, y)\left(\frac{1}{5}\right)^m d\mu(y) = u(x) - \tilde{u}(x),$$

and this yields

$$\tilde{u}(x) + \left(\frac{1}{5}\right)^m g_{V_0}(x) = u(x). \tag{3.1}$$

By applying (3.1) to all $x \in V_1$ we get the values shown in Figure 3.4. $\square$

We compute $\delta_0(V_0)$ by considering g_{V_0} as a series of piecewise harmonic functions.

Theorem 3.2. *We have*

$$\delta_0(V_0) = \frac{1}{3\sqrt{2}}. \tag{3.2}$$

Proof. For all m, let f_m be the piecewise harmonic m-spline whose values on V_m are the same as those of g_{V_0}. For all $m > 0$, $x \in V_m$, Lemma 3.1 gives

$$f_m(x) - f_{m-1}(x) = \begin{cases} 0 & \text{for } x \in V_{m-1}, \\ \frac{1}{15 \cdot 5^m} & \text{for } x \notin V_{m-1}. \end{cases}$$

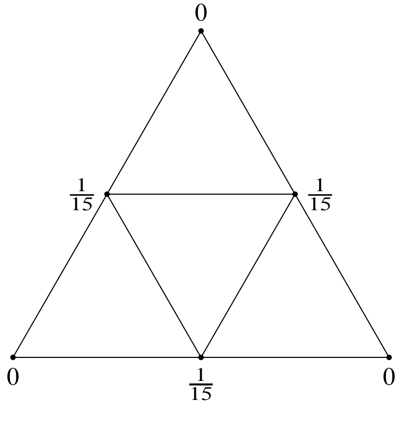


Figure 3.5. The values of $f_1 - f_0$ on V_1, in the context of Theorem 3.2.

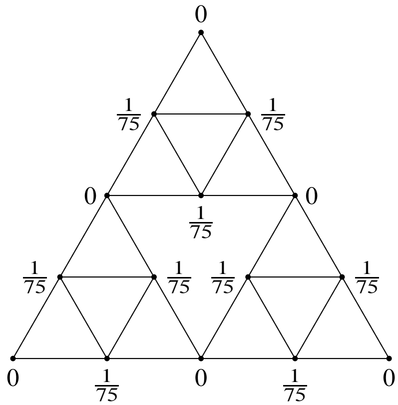


Figure 3.6. The values of $f_2 - f_1$ on V_2, in the context of Theorem 3.2.

Because $f_m - f_{m-1}$ is a harmonic m-spline, we can compute its integral from its values on V_m. The values of $f_m - f_{m-1}$ on the boundary of some m-cell are (in some order) 0, $\frac{1}{15 \cdot 5^{m-1}}$, and $\frac{1}{15 \cdot 5^{m-1}}$, because the boundary of each m-cell contains one point from V_{m-1} and two from $V_m \setminus V_{m-1}$. Thus,

$$\int f_m - f_{m-1} d\mu = \frac{2}{3} \cdot \frac{1}{15 \cdot 5^{m-1}}.$$

Since $\int g_{V_0} d\mu$ is clearly the limit of $\int f_m d\mu$, we have

$$\begin{aligned} \int g_{V_0} d\mu &= \lim_{m \to \infty} f_m d\mu \\ &= \int f_0 d\mu + \sum_{m=1}^{\infty} \int f_m - f_{m-1} d\mu \\ &= 0 + \sum_{m=1}^{\infty} \frac{2}{3} \frac{1}{15 \cdot 5^{m-1}} \\ &= \frac{1}{18}. \end{aligned}$$

$\delta_0(V_0)$ is equal to $\left(\int g_{V_0} d\mu\right)^{1/2}$, which gives us (3.2). $\square$

To compute $\delta_1(V_0)$, we must determine the maximum value of $g_{V_0}(x)$. To facilitate this computation, let us take advantage of the symmetry of g_{V_0} and instead consider the function $u = 15 g_{V_o} \circ F_0$. It is clear that

$$\sup_{x\in \mathrm{SG}} g_{V_0}(x) = \frac{1}{15} \sup_{x\in \mathrm{SG}} u(x).$$

Theorem 3.3. *If $F_w\mathrm{SG}$ is an m-cell along the bottom line of* SG (*that is, an m-cell whose bottom line is the bottom line of the gasket*), *then*

$$u(F_w q_0) = 1 - \left(\frac{1}{5}\right)^m, \quad u(F_w q_1) = 1, \quad u(F_w q_2) = 1. \tag{3.3}$$

Proof. We will prove this claim by using induction on m.

Base case. If $m = 0$, this is easy to verify.

Inductive step. Assume that $m \geq 0$ and that (3.3) holds for all m-cells along the bottom line. For any $(m+1)$-cell $F_w K$ along the bottom line: w is a word without any 0s. We can assume without loss of generality that the last character of w is a 1 (rather than a 2), so $w = w'1$ for some word w', and $F_{w'}\mathrm{SG}$ is an m-cell along the bottom line of SG. By the inductive hypothesis,

$$u(F_w q_0) = 1 - \left(\frac{1}{5}\right)^m, \quad u(F_w q_1) = 1, \quad u(F_w q_2) = 1.$$

Because $u = 15 g_{V_0} \circ F_0$, to describe the way $u|_{V_{m+1}\setminus V_m}$ depends on $u|_{V_m}$, we must use Lemma 3.1 but replace $\frac{1}{15\cdot 5^m}$ with $\frac{1}{5^{m+1}}$. We obtain

$$\begin{aligned}
u(F_w q_0) &= u(F_{w'} F_1 q_0) \\
&= \left(\frac{2\left(1 - \left(\frac{1}{5}\right)^{m+1}\right) + 2 + 1}{5}\right) + \frac{1}{5^{m+1}} \\
&= 1 - \left(\frac{1}{5}\right)^{m+1},
\end{aligned}$$

$$u(F_w q_1) = u(F_{w'} F_1 q_1) = u(F_{w'} q_1) = 1,$$

and

$$\begin{aligned}
u(F_w q_2) &= u(F_{w'} F_1 q_2) \\
&= \left(\frac{1 - \left(\frac{1}{5}\right)^m + 2 + 2}{5}\right) + \left(\frac{1}{5}\right)^{m+1} \\
&= 1.
\end{aligned}$$

Thus, the inductive hypothesis holds for $(m+1)$-cells. □

Theorem 3.4. *If x is not on the bottom line of* SG*, then $u(x) < 1$.*

Proof. Because x is not on the bottom line, there exists some m such that an m-cell containing x is along the bottom line, but an $(m+1)$-cell within this m-cell contains x and is not on the bottom line. Thus, for some word w consisting of m characters, all of them are 1 or 2, and $x \in F_{w0}\mathrm{SG}$. By Theorem 3.3,

$$u(F_{w0}q_0) = 1 - \left(\frac{1}{5}\right)^m, \quad u(F_{w0}q_1) = 1, \quad \text{and } u(F_{w0}q_2) = 1.$$

For all non-negative integers k, define

$$\varphi(k) = \sup_{\substack{x \in V_{m+k+1},\\ x \in F_{w0}\mathrm{SG}}} u(x).$$

We need only consider one $(m+1)$-cell, so

$$\varphi(0) = 1 - \left(\frac{1}{5}\right)^{m+1}.$$

For all k, $\varphi(k+1) \leq \varphi(k) + \left(\frac{1}{5}\right)^{m+k+2}$ (by applying our adjustment of Lemma 3.1 on an $(m+1)$-cell whose values on the boundary are all less than or equal to $\varphi(k)$). For all k,

$$\begin{aligned} \varphi(k) &\leq 1 - \left(\frac{1}{5}\right)^{m+1} + \sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^{m+i+2} \\ &= 1 - \left(\frac{3}{4}\right)\left(\frac{1}{5}\right)^{m+1}. \end{aligned}$$

Because V_* is dense in SG, $u(x) \leq 1 - \left(\frac{3}{4}\right)^{m+1}$, so $u(x) < 1$. □

Corollary 3.5. *$\delta_1(V_0)$, or the maximum value of g_{V_0}, is $\frac{1}{15}$.*

Proof. Let x be any point in the Sierpiński gasket. If $x \in F_0\mathrm{SG}$, then either x lies on the bottom line of $F_0\mathrm{SG}$ (in which case $g_{V_0}(x) = \frac{1}{15}$), or x is above this line (in which case $g_{V_0} < \frac{1}{15}$). In either case, $g_{V_0} \leq \frac{1}{15}$. If $x \notin F_0\mathrm{SG}$, then by symmetry of g_{V_0}, there is some point $y \in F_0\mathrm{SG}$ such that $g(x) = g(y) \leq \frac{1}{15}$. □

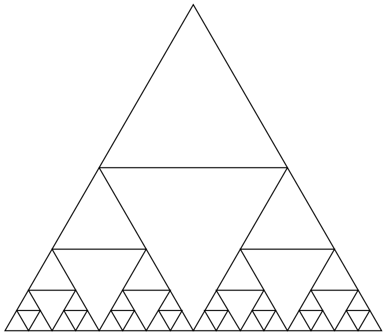


Figure 3.7. The cells along the bottom line of SG.

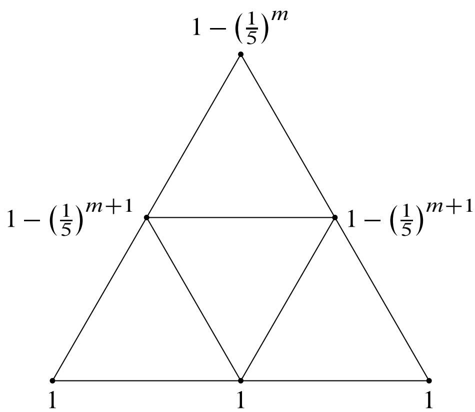


Figure 3.8. The values of u along an m-cell along the bottom line of SG.

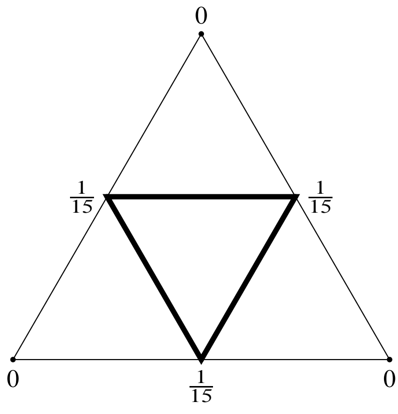


Figure 3.9. The values of g_{V_0} on V_1. g_{V_0} attains its maximum value, $\frac{1}{15}$ all along the thickened lines.

The weights $\{p(x)\}$ for all $x \in V_0$ are $\frac{1}{3}$ (the integral of any of h_0, h_1, and h_2). Because $\{p(x)\}$ for all $x \in V_0$ and the uniform weights $\{w(x)\}$ are one and the same, $\delta(E, w) = 0$.

The calculations for all of the examples E that follow may be greatly simplified by observing that the difference between g_{V_0} and g_E is a piecewise harmonic function, and combining calculations involving that piecewise harmonic function with the calculations that we already made in Example 3.1.

Lemma 3.6. *If E is a finite superset of V_0, then $g_{V_0} - g_E$ is harmonic away from E.*

Proof. For $x \notin E$,

$$\begin{aligned}\Delta(g_{V_0} - g_E)(x) &= \Delta g_{V_0}(x) - \Delta g_E(x)\\ &= -1 - (-1)\\ &= 0,\end{aligned}$$

so $g_{V_0} - g_E$ is harmonic away from E. □

Example 3.2. $E = V_0 \cup \{x_0\}$, where x_0 is a member of $V_1 \setminus V_0$.

The next example we consider is when E contains the three points from V_0 and one additional point from V_1. Without loss of generality, we take $x_0 = F_0 q_1$. The analysis would be exactly the same for either of the other choices of x_0 (if the characters 0, 1, and 2 were permuted accordingly). For this set E, Lemma 3.6 guarantees that $g_{V_0} - g_E$ is harmonic away from E. Because $E \subset V_1$, $g_{V_0} - g_E$ must be a harmonic 1-spline. Furthermore, $(g_{V_0} - g_E)|_{V_0} = 0$ (because both G_E and G_{V_0} vanish on V_0) and $g_{V_0} - g_E$ is harmonic at $F_0 q_2$ and $F_1 q_2$.

Because $F_0 q_1 \in E$, $g_E(F_0 q_1) = 0$, so $(g_{V_0} - g_E)(F_0 q_1) = g_{V_0}(F_0 q_1) = \frac{1}{15}$. Let $x = (g_{V_0} - g_E)(F_0 q_2)$. By symmetry, $(g_{V_0} - g_E)(F_1 q_2) = x$. By the harmonicity of $g_{V_0} - g_E$ at $F_0 q_1$, we have

$$x = \frac{0 + \frac{1}{15} + x + 0}{4},$$

hence

$$x = \frac{1}{45}.$$

Thus, $g_{V_0} - g_E$ is the harmonic 1-spline with values as shown in Figure 3.10.

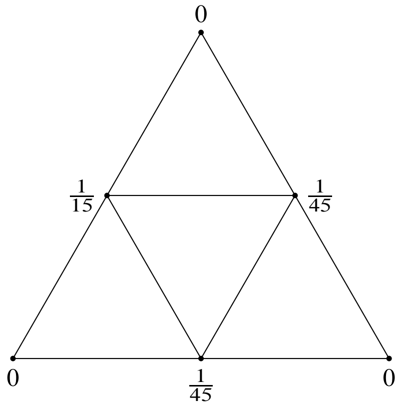


Figure 3.10. The values of $g_{V_0} - g_E$ on V_1, in the context of Example 3.2.

Theorem 3.7. *If $E = V_0 \cup \{x_0\}$ for some $x_0 \in V_1 \setminus V_0$, then*

$$\delta_0(E) = \left(\frac{5}{162}\right)^{1/2}.$$

Proof. Because $g_{V_0} - g_E$ is the harmonic spline with values shown in Figure 3.10,

$$\int g_{V_0} - g_E \, d\mu = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{15} + \frac{1}{45}\right) + \frac{1}{3}\left(\frac{1}{15} + \frac{1}{45}\right) + \frac{1}{3}\left(\frac{1}{45} + \frac{1}{45}\right)\right) = \frac{2}{81}$$

and so

$$\int g_E \, d\mu = \int g_{V_0} d\mu - \int g_{V_0} - g_E \, d\mu = \frac{1}{18} - \frac{2}{81} = \frac{5}{162}$$

and

$$\delta_0(E) = \left(\int g_E \, d\mu\right)^{1/2} = \left(\frac{5}{162}\right)^{1/2}. \qquad \square$$

We know how to compute g_{V_0} for all points in V_*, and $g_{V_0} - g_E$ is a harmonic spline. Therefore, we can equally well compute g_E for all points in V_*. After computing the values of g_E for the finite graph approximations up to V_{10}, the maximum value is $\frac{11}{225}$, a value that first occurs in V_2. We conjecture that this is the absolute maximum value the function takes, and that

$$\delta_1(E) = \frac{11}{225}.$$

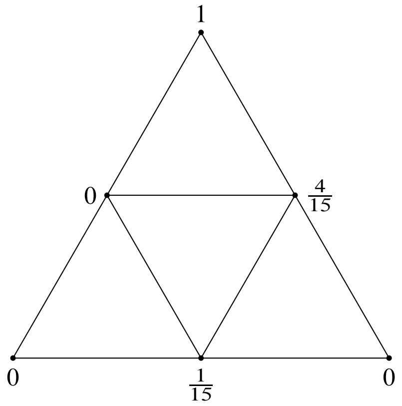


Figure 3.11. The values of v_{q_0} on V_1, in the context of Example 3.2.

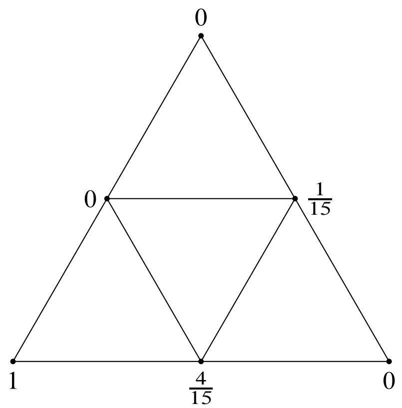


Figure 3.12. The values of v_{q_1} on V_1. in the context of Example 3.2.

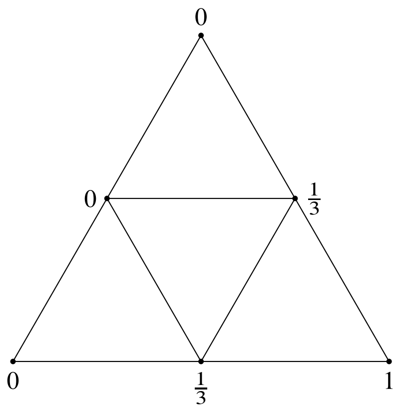


Figure 3.13. The values of v_{q_2} on V_1 in the context of Example 3.2.

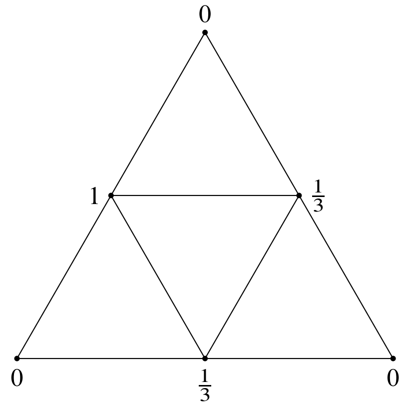


Figure 3.14. The values of $v_{F_0 q_1}$ on V_1 in the context of Example 3.2.

Each of the functions v_x for $x \in E$ will be a function that is harmonic away from E and is determined the same way we determined $g_{V_0} - g_E$: by assigning the appropriate values to the points in E (in this case, 1 for x, 0 for all other points in E) and choosing the values for $V_1 \setminus V_0$ such that v_x is harmonic at these points. Then, $p(x)$ will be $\int v_x d\mu$. The values of that the functions v_x $(x \in E)$ take on V_1 are shown in Figures 3.11–3.14. The weights, calculated from these functions, are

$$p(q_0) = {}^5/_{27},$$

$$p(q_1) = {}^5/_{27},$$

$$p(q_2) = {}^7/_{27},$$

$$p(F_0 q_1) = {}^{10}/_{27}.$$

If $\{w(x)\}$ are the uniform weights,

$$\delta(E, w) = R^{1/2} \cdot \frac{7}{27}.$$

Recall that we took x_0 to be $F_0 q_1$, but one could also take $F_0 q_2$ or $F_1 q_2$, and can figure out the resulting weights from the above analysis by symmetry.

Example 3.3. $E = V_0 \cup \{x_0, x_1\}$, where x_0 and x_1 are distinct elements of $V_1 \setminus V_0$.

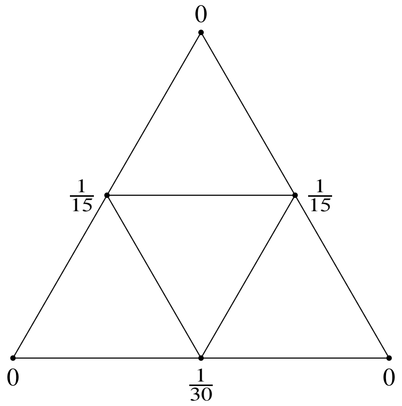


Figure 3.15. The values of $g_{V_0} - g_E$ in the context of Example 3.3.

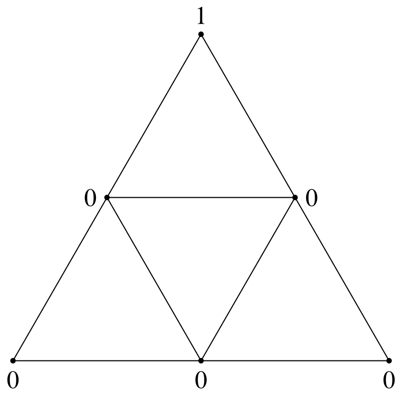


Figure 3.16. The values of v_{q_0} on V_1 in the context of Example 3.3.

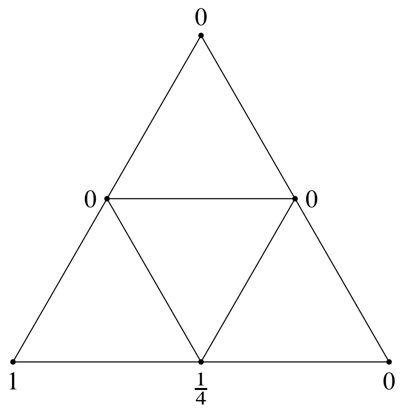


Figure 3.17. The values of v_{q_1} on V_1 in the context of Example 3.3.

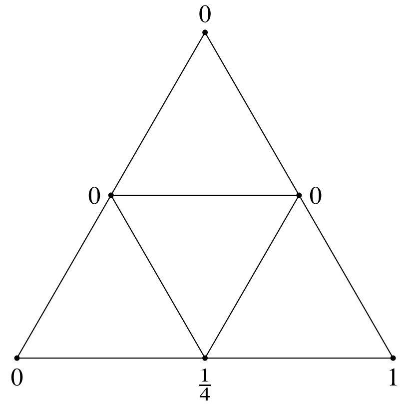


Figure 3.18. The values of v_{q_2} on V_1 in the context of Example 3.3.

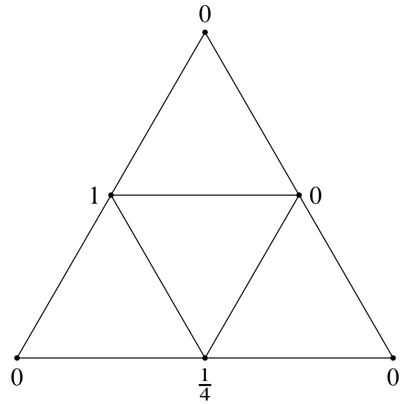


Figure 3.19. The values of $v_{F_0 q_1}$ on V_1 in the context of Example 3.3.

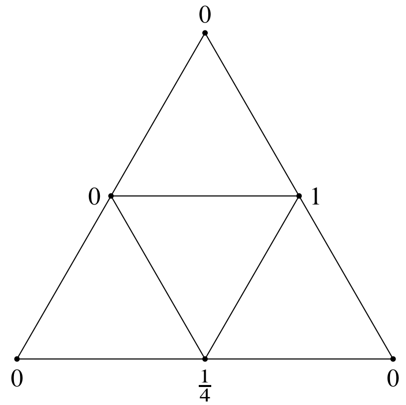


Figure 3.20. The values of $v_{F_0 q_2}$ on V_1 in the context of Example 3.3.

As in Example 3.2, we assume specific values of x_0 and x_1 (in this case: $x_0 = F_0 q_1$ and $x_1 = F_0 q_2$), and know that one could determine the weights for any other choice of x_0, x_1 from these calculations. The methods for our calculations in this example are exactly the same as those in Example 3.2.

$g_{V_0} - g_E$ is the harmonic 1-spline with values shown in Figure 3.15. Thus

$$\int g_{V_0} - g_E \, d\mu = \frac{1}{27}$$

and

$$\int g_E \, d\mu = \frac{1}{54}.$$

From computing the values g_E takes on all points in the finite graph approximations up to V_{10}, it appears that the maximum is $\frac{1}{30}$, which first occurs in V_1. We conjecture that this is the true maximum value of h, that

$$\delta_1(E) = \frac{1}{30}.$$

The values of the functions v_x are shown in Figures 3.16–3.20. The weights are

$$p(q_0) = {}^1\!/\!_9,$$

$$p(q_1) = {}^1\!/\!_6,$$

$$p(q_2) = {}^1\!/\!_6,$$

$$p(F_0 q_1) = {}^5\!/\!_{18},$$

$$p(F_0 q_2) = {}^5\!/\!_{18}.$$

For the uniform weights $\{w(x)\}$,

$$\delta(E, w) = R^{1/2} \cdot \frac{14}{45}.$$

The method used in Examples 3.2 and 3.3 of finding g_E from the harmonic spline difference between g_{V_0} and g_E will work for any finite $E \supset V_0$. However, for larger E, there are, in some cases, improvements to the method. The post-critically finite nature of the Sierpiński gasket allows us to easily analyze examples E that divide the gasket into m-cells, where the inverse image of E under each F_w ($|w| = m$) is a more wieldy set (such as one of the Examples 3.1, 3.2, or 3.3). The most important result that allows this analysis via decompositions into m-cells is the scaling of Green's functions.

Theorem 3.8. *If $V_m \subseteq E$, and for all $|w| = m$, we denote $\{F_w^{-1}x : x \in E \cap F_w \mathrm{SG}\}$ (the inverse image of E under F_w) by E_w, then*

$$G_E(x, y) = \begin{cases} \left(\frac{3}{5}\right)^m G_{E_w}(F_w^{-1}x, F_w^{-1}y) & \text{if } x, y \in F_w \mathrm{SG} \text{ and } |w| = m, \\ 0 & \text{if } x \text{ and } y \text{ belong to separate } m\text{-cells.} \end{cases} \tag{3.4}$$

Proof. Let $a(x, y)$ be the right-hand side of (3.4), the function that we claim is G_E. It suffices to show that for a function $u \in \operatorname{dom} \Delta$ such that $u|_E = 0$,

$$-\int a(x, y)(\Delta u(y))d\mu(y) = u(x). \tag{3.5}$$

If $x \in V_m$, both sides of (3.5) are 0. If $x \notin V_m$, let w be the unique word of length m such that $x \in F_w K$. Let $x' = F_w^{-1}x$. The left-hand side of (3.5) is

$$\begin{aligned} &-\int a(F_w x', y)(\Delta u(y)d\mu(y)) \\ &= -\left(\frac{3}{5}\right)^m \int G_{E_w}(x', F_w y)(\Delta u(y)d\mu(y)) \\ &= -\left(\frac{3}{5}\right)^m \left(\frac{1}{3}\right)^m \int_{F_w K} G_{E_w}(x', y')[\Delta u \circ F_w](y')d\mu(y') \\ &= -\left(\frac{1}{5}\right)^m \int_{F_w K} G_{E_w}(x', y')[5^m \Delta (u \circ F_w)](y')d\mu(y') \\ &= -\int G_{E_w}(x', y') \Delta (u \circ F_w)(y')d\mu(y') \\ &= u(F_w x') \\ &= u(x) \end{aligned}$$

which verifies (3.5). Thus, $a(x, y)$ is the Green's function for E. □

Corollary 3.9. *Suppose $V_m \subseteq E$.*

(a)

$$\delta_0(E) = \left(\frac{\sum_{|w|=m} (\delta_0(E_w))^2}{15^m}\right)^{1/2}. \tag{3.6}$$

(b)

$$\delta_1(E) = \frac{1}{5^m} \sup_{|w|=m} \delta_1(E_w). \tag{3.7}$$

(c) *To simplify the notation, for all $\widetilde{E}$, we let $p_{\widetilde{E}}(x)$ refer to the weight of x on $\widetilde{E}$. Then for all $x \in E$:*

$$p_E(x) = \begin{cases} \dfrac{1}{3^m} p_{E_w}(F_w^{-1}x) \\ \qquad \textit{if } x \textit{ belongs to a unique } m\textit{-cell, } F_w\mathrm{SG}, \\ \dfrac{1}{3^m} p_{E_w}(F_w^{-1}x) + \dfrac{1}{3^m} p_{E_{w'}}(F_{w'}^{-1}x) \\ \qquad \textit{if } x \textit{ belongs to two distinct } m\textit{-cells, } F_w\mathrm{SG} \textit{ and } F_{w'}\mathrm{SG}. \end{cases}$$

Proof. If x belongs to the m-cell $F_w\mathrm{SG}$ and $x' = F_w^{-1}x$, then

$$\begin{aligned} g_E(x) &= \int G_E(x,y)d\mu(y) \\ &= \int_{F_wK} \left(\frac{3}{5}\right)^m G_{E_w}(F_w^{-1}x, F_w^{-1}y)d\mu(y) \\ &= \left(\frac{3}{5}\right)^m \left(\frac{1}{3}\right)^m \int G_{E_w}(x', y')d\mu(y) \\ &= \left(\frac{1}{5}\right)^m \int g_{E_w}(F_w^{-1}x). \end{aligned}$$

For (a),

$$\begin{aligned} (\delta_0(E))^2 &= \int g_E \, d\mu \\ &= \sum_{|w|=m} \int_{F_wK} \left(\frac{1}{5}\right)^m g_{E_w} \circ F_w^{-1} d\mu \\ &= \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^m \sum_{|w|=m} \int g_{E_w} d\mu \\ &= \left(\frac{1}{15}\right)^m \sum_{|w|=m} (\delta_0(E_w))^2 \end{aligned}$$

which implies (3.6).

For (b),

$$\begin{aligned}
\delta_1(E) &= \sup_z g_E(x) \\
&= \sup_{\substack{|w|=m,\\ x\in F_w K}} \left(\frac{1}{5}\right)^m g_{E_w}(F_w^{-1}x) \\
&= \left(\frac{1}{5}\right)^m \sup_{|w|=m} \sup_{x'\in K} g_{E_w}(x') \\
&= \left(\frac{1}{5}\right)^m \sup_{|w|=m} \delta_1(E_w)
\end{aligned}$$

which is (3.7).

(c) is a trivial consequence of adding the harmonic indicators. □

The most obvious examples to apply Theorem 3.8 and Corollary 3.9 to are sets E such that that $V_m \subseteq E \subseteq V_{m+1}$. In such examples, for all $|w| = m$, E_w is either one of the sets described in Examples 3.1, 3.2, and 3.3, or V_1, which is also simple. We first consider the notable case $E = V_m$, and then consider $V_m \subseteq E \subseteq V_{m+1}$ in general.

Example 3.4. $E = V_m$. We have

$$\begin{aligned}
\delta_0(V_m) &= \frac{1}{3\sqrt{2\cdot 5^m}}; \\
\delta_1(V_m) &= \frac{1}{15\cdot 5^m}; \\
p(x) &= \begin{cases} \dfrac{1}{3^{m+1}} & \text{if } x \in V_0, \\[2ex] \dfrac{2}{3^{m+1}} & \text{if } x \in V_m \setminus V_0; \end{cases} \\
\delta(E, w) &= \frac{2(3^m-1)}{3^m(3^m+1)} R^{1/2}
\end{aligned}$$

for the uniform weights $\{w(x)\}$.

Example 3.5. $V_m \subseteq E \subseteq V_{m+1}$. For all $|w| = m$, E_w contains either 3, 4, 5, or 6 points. (In other words, E_w is one of the sets described in Examples 3.1, 3.2, and 3.3, or $E_w = V_1$, which is a special case of Example 3.4.)

Let

$$A = \#\{|w| = m \colon E_w = V_0\},$$

$$B = \#\{|w| = m \colon \#E_w = 4\},$$

$$C = \#\{|w| = m \colon \#E_w = 5\},$$

$$D = \#\{|w| = m \colon E_w = V_1\}.$$

We can express $\delta_0(E)$ and $\delta_1(E)$ in terms of A, B, C, and D:

$$\delta_0 = \Big(\frac{A \cdot \frac{1}{18} + B \cdot \frac{5}{162} + C \cdot \frac{1}{54} + D \cdot \frac{1}{90}}{15^m}\Big)^{1/2};$$

$$\delta_1(E) = \begin{cases} \frac{1}{15 \cdot 5^m} & \text{if } A \neq 0, \\ \frac{11}{225 \cdot 5^m} & \text{if } A = 0 \text{ and } B \neq 0, \\ \frac{1}{30 \cdot 5^m} & \text{if } A = 0,\ B = 0, \text{ and } C \neq 0, \\ \frac{1}{75 \cdot 5^m} & \text{if } A = B = C = 0. \end{cases}$$

The weights $\{p(x)\}$ can be calculated using part (c) of Corollary 3.9. From this, $\delta(E, w)$ for the uniform weights $\{w(x)\}$ can be calculated if R is known.

Example 3.6. $E = V_0 \cup \{F_0F_1q_2, F_1F_2q_0, F_2F_0q_1\}$ (E consists of the three elements of V_0 and the three most interior points of V_2, as shown in Figure 3.21).

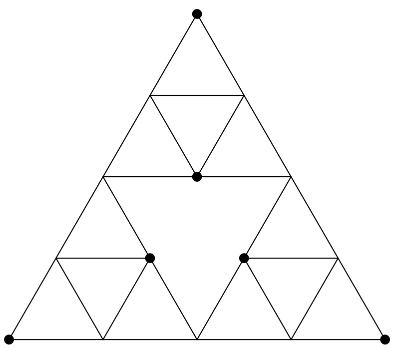


Figure 3.21. The set E in Example 3.6.

The sample set E in Example 3.5 can be thought of as very "wide" (in that at a given level k, many k-cells are represented) but not very "deep" (as the points of E all come from V_k for particularly small values of k). Given a finite number of points that we are allowed to pick for our sample set, some trade-off must necessarily be made between width and depth. In Example 3.6, we choose a basic set that can be described as "deeper" than the other sets of similar size we have considered so far, since it includes elements of $V_2 \setminus V_1$.

As usual, in order to calculate $\delta_0(E)$, we consider the harmonic spline $g_{V_0} - g_E$. For all $x \in V_0$, $g_{V_0}(x) - g_E(x) = 0 - 0 = 0$. For the interior values of $x \in E$, $g_E(x) = 0$ since $x \in E$, and applying Lemma 3.1 (or Theorem 3.3) gives $g_{V_0}(x) = \frac{1}{15}$. Therefore, the values of $g_{V_0} - g_E$ are as shown in Figure 3.22, where a and b are some constants. The function $g_{V_0} - g_E$ is harmonic away from E, so a and b must satisfy the average rule:

$$a = \frac{0 + a + \frac{1}{15} + b}{4}, \qquad b = \frac{a + \frac{1}{15} + \frac{1}{15} + a}{4}.$$

Therefore, $a = \frac{1}{25}$ and $b = \frac{4}{75}$. Now we can calculate the integral $g_{V_0} - g_E$:

$$\int g_{V_0} - g_E \, d\mu = \frac{2}{45}.$$

So

$$(\delta_0(E))^2 = \int g_E \, d\mu = \int g_{V_0} d\mu - \int g_{V_0} - g_E \, d\mu = \frac{1}{18} - \frac{2}{45} = \frac{1}{90},$$

$$\delta_0(E) = \frac{1}{3\sqrt{10}}.$$

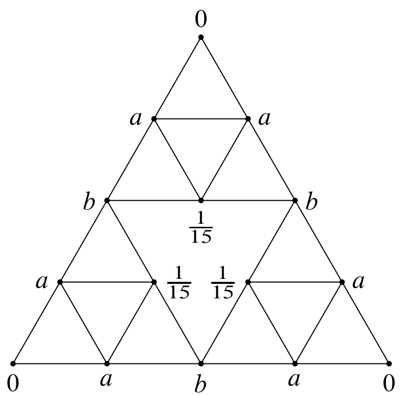


Figure 3.22. The values of $g_{V_0} - g_E$ on V_2, in the context of Example 3.6. Constants a and b are as of yet unknown.

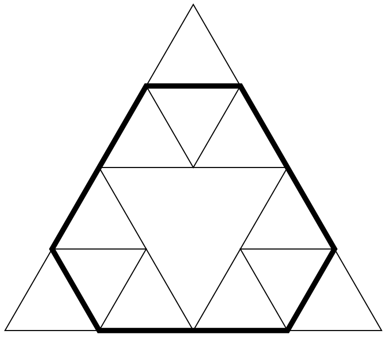


Figure 3.23. In example 3.6, the biharmonic function g_E obtains its maximum value, $\frac{1}{75}$, along these shaded lines.

By the same inductive arguments that were used to prove Theorems 3.3 and 3.4 and Corollary 3.5, g_E obtains its maximum value, $\frac{1}{75}$ along the shaded lines in Figure 3.23, so

$$\delta_1(E) = \frac{1}{75}.$$

Now we calculate the weights

$$p(x) = \int v_x d\mu$$

for $x \in E$. By symmetry, there are only two weights to calculate: $p(q_0)$ and $p(F_0 F_1 q_2)$. The indicators are shown in Figures 3.24 and 3.25.

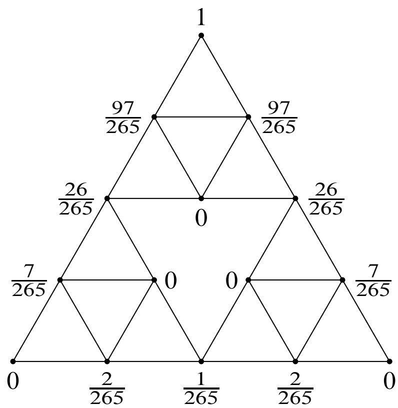


Figure 3.24. v_{q_0} in Example 3.6.

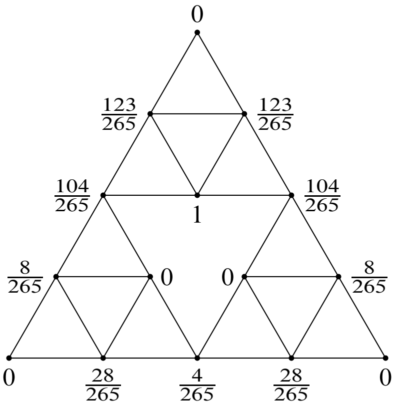


Figure 3.25. $v_{F_0 F_1 q_2}$ in Example 3.6.

The weights are

$$p(q_0) = p(q_1) = p(q_2) = {}^1\!/_9,$$

$$p(F_0 F_1 q_2) = p(F_1 F_2 q_0) = p(F_2 F_0 q_1) = {}^2\!/_9.$$

If $\{w(x)\}_{x \in E}$ are the uniform weights,

$$\delta(E, w) = \left(3\left|\frac{1}{9} - \frac{1}{6}\right| + 3\left|\frac{2}{9} - \frac{1}{6}\right|\right) R^{1/2} = \frac{1}{3} R^{1/2}.$$

Interestingly, V_1 is another highly symmetric 6-element sample set and $\delta_0(E) = \delta_0(V_1)$, $\delta_1(E) = \delta_1(V_1)$, and when $\{w(x)\}$ are the uniform weights for each set E, $\delta(E, w) = \delta(V_1, w)$. Therefore, by taking E rather V_1 as our sample set (choosing depth over width), it is not clear whether we would be making a better or a worse choice.

We briefly mention one more family of sample sets $\widetilde{E}_m$. For a fixed m, $\widetilde{E}_m$ consists of $F_w x$ for all $|w| = m$, $x \in E$ (where E is still the sample set in Example 3.6). Because the values of δ_0, δ_1, and $\delta(E, w)$ (for the uniform weights $\{w(x)\}$) are the same for E and V_1, Corollary 3.9 tells us that they will continue to be the same for V_{m+1} and $\widetilde{E}_m$, for all m. We have

$$\delta_0(\widetilde{E}_m) = \delta_1(V_{m+1}) = \frac{1}{3\sqrt{2 \cdot 5^{m+1}}},$$

$$\delta_1(\widetilde{E}_m) = \delta_1(V_{m+1}) = \frac{1}{15 \cdot 5^{m+1}},$$

$$\delta(\widetilde{E}_m, w) = \delta(V_{m+1}, w) = \frac{2(3^m - 1)}{3^m + 1} R^{1/2}.$$

To calculate the weights $\{p(x)\}$ for $\widetilde{E}_m$, notice that the harmonic spline indicators for $\widetilde{E}_m$ are the indicators for E but with F_w^{-1} for some $|w| = m$. (For those elements $x \in \widetilde{E}_m$ that are shared between two distinct m-cells F_wSG and $F_{w'}$SG, the indicator for x in $\widetilde{E}_m$ is the sum of two indicators of E, one composed with F_w^{-1} and the other composed with $F_{w'}^{-1}$.) Thus

$$p_{\widetilde{E}_m}(x) = \begin{cases} \frac{1}{9\cdot 3^m} & \text{if } x \in V_0, \\ \frac{2}{9\cdot 3^m} & \text{if } x \in (V_m \setminus V_0), \\ \frac{2}{9\cdot 3^m} & \text{if } x \text{ is one of the interior points of an } m\text{-cell.} \end{cases}$$

4. Other self-similar measures

In this section, we apply the results of Section 2 to more fractals: the Sierpiński tetrahedron (ST) and the 3-level gasket (SG_3), both of which will be covered in less depth than the Sierpiński gasket was in Section 3. Like in Section 3, our starting point is using [12] to determine the values of g_{V_0} on V_1 for these fractals. However, whereas [12] gives us these values directly for SG, it does not for ST or SG_3. Therefore, we will have to apply the general algorithm of Section 2 of [12] in its entirety to ST and SG_3. We begin this section with a summary of that algorithm. We slightly modify the notation and indexing of [12] to be consistent with our own and to be the most useful for our purposes.

Let K be a p.c.f. self-similar fractal with boundary $V_0 = \{q_k\}_{0 \le k < N_0}$ generated by a set of contractions $\{F_i\}_{0 \le i < N}$, for some N_0 and N. For the fractals we consider in this paper, it will help to add the simplifying assumption that $N_0 \le N$ and each q_k is the fixed point of F_k. For m, let $V_m = \{F_w x \colon |w| = m, x \in V_0\}$, and let $V_* = \bigcup_m V_m$. Let K have a regular harmonic structure with Dirichlet form $\mathcal{E}$ on V_1 satisfying

$$\mathcal{E}(u, v) = \sum_{i=0}^{N-1} r_i^{-1} \mathcal{E}(u \circ F_i, v \circ F_i)$$

and a self-similar probability measure μ satisfying

$$\mu = \sum_{i=0}^{N-1} \mu_i (\mu \circ F_i).$$

Let $\triangle$ be the associated Laplacian. For all j, let $\mathcal{H}_j = \{f : \triangle^{j+1} f = 0\}$. An easy basis for $\mathcal{H}_j$ is $\{f_{lk}\}_{0 \le l \le j, 0 \le k < N_0}$, where f_{lk} is the solution to

$$\triangle^m f_{lk}(q_n) = \delta_{ml}\delta_{kn} \quad \text{for all } m, n \text{ such that } 0 \le m \le l \text{ and } 0 \le n < N_0.$$

Define the harmonic functions h_i as usual such that $h_i(q_k) = \delta_{ik}$. (This means that $h_i = f_{0i}$ for all $i \in \{0, 1, 2, \dots, N_0 - 1\}$.) For all $k, k', n, n' \in \{0, 1, 2, N_0 - 1\}$, let

$$A(kk', nn') = \sum_{i=0}^{N-1} \mu_i h_k(F_i q_n) h_{k'}(F_i q_{n'}). \tag{4.1}$$

It is a result [12] that if

$$I(kk') = \sum_{i=0}^{N-1} \mu_i \int (h_k \circ F_i)(h_{k'} \circ F_i) d\mu,$$

then the vector $I(kk')$ is an eigenvector of the matrix $A(kk', nn')$ corresponding to eigenvalue 1, and

$$\sum_{k=0}^{N_0-1} \sum_{k'=0}^{N_0-1} I(kk') = 1.$$

It is easy to compute $A(kk', nn')$ for any example K (such as ST and SG_3), so $I(kk')$ can be determined.

Let X be the matrix whose rows and columns are indexed by the elements of $V_1 \setminus V_0$, such that

$$X_{pq} = \mathcal{E}(v_p, v_q) \tag{4.2}$$

where v_p and v_q are harmonic 1-splines such that $v_p(r) = \delta_{pr}$ and $v_q(r) = \delta_{qr}$. Let $G = X^{-1}$. For all $i, i' \in \{0, 1, 2, \dots, N-1\}$ and $n, n' \in \{0, 1, 2, \dots, N_0 - 1\}$, let

$$\gamma(i, i', n, n') = \begin{cases} G_{F_i q_n, F_{i'} q_{n'}} & \text{if } F_i q_n, F_{i'} q_{n'} \in (V_1 \setminus V_0), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, it is another result in [12] that

$$f_{1k}(F_i q_n) = \sum_{i'=0}^{N-1} \sum_{n'=0}^{N_0-1} \sum_{k'=0}^{N_0-1} -\mu_{i'} \gamma(i, i', n, n') I(k'n') h_k(F_{i'} q_{k'}). \tag{4.3}$$

After using this recipe to calculate the values of f_{1k} on V_1 for our fractal K, the values of g_{V_0} on V_1 can be determined. $\triangle g_{V_0} = -1$, so

$$g_{V_0} = -\sum_{k=0}^{N_0-1} f_{1k}.$$

We will then require three more results, Lemma 4.1, Theorem 4.2, and Corollary 4.3. These are the generalizations of Lemma 3.1, Theorem 3.8, and Corollary 3.9 respectively.

Lemma 4.1. *If u is a function on K with $\triangle u = -1$, $|w| = m$, $u \circ F_w = v$, $\tilde{v}$ is the harmonic function with the same values on V_0 as v, and $x \in (V_1 \setminus V_0)$, then*

$$u(F_w x) = v(x) = \tilde{v}(x) + \mu_w r_w g_{V_0}(x)$$

(*Recall that if $w = w_1 w_2 \dots w_m$, $\mu_w = \mu_1 \mu_2 \cdots \mu_m$ and $r_w = r_1 r_2 \cdots r_m$.*)

Proof. The proof is essentially the same as that of Lemma 3.1.

$$\triangle v = \triangle(u \circ F_w) = \mu_w r_w (\triangle u) \circ F_w = \mu_w r_w \cdot (-1)$$

so

$$\triangle(\mu_w^{-1} r_w^{-1} v) = -1.$$

$\mu_w^{-1} r_w^{-1} \tilde{v}$ is harmonic and has the same values on the boundary as $\mu_w^{-1} r_w^{-1} v$, while g_{V_0} has the same Laplacian as $\mu_w^{-1} r_w^{-1} v$ but vanishes on the boundary. Thus,

$$\mu_w^{-1} r_w^{-1} v = \mu_w^{-1} r_w^{-1} \tilde{v} + g_{V_0},$$

$$v = \tilde{v} + \mu_w r_w g_{V_0},$$

$$v(x) = \tilde{v}(x) + \mu_w r_w g_{V_0}(x),$$

$$u(F_w x) = \tilde{v}(x) + \mu_w r_w g_{V_0}(x). \qquad \square$$

Theorem 4.2. *If $V_m \subseteq E$, then*

$$G_E(x, y) = \begin{cases} r_m G_{E_w}(F_w^{-1} x, F_w^{-1} y) & \text{if } x, y \in F_w K, \\ 0 & \text{if } x \text{ and } y \text{ belong to separate } m\text{-cells.} \end{cases} \tag{4.4}$$

Proof. Let

$$a(x, y) = \begin{cases} G_{E_w}(F_w^{-1} x, F_w^{-1} y) & \text{if } x, y \in F_w K, |w| = m, \\ 0 & \text{if } x \text{ and } y \text{ belong to separate } m\text{-cells.} \end{cases}$$

Fix $u \in \operatorname{dom} \triangle$ such that $u|_{V_m} = 0$. For all $x \in K$, if $F_w K$ is the m-cell that x belongs to

$$\begin{aligned}
-\int a(x, y) \triangle u(y) d\mu(y) &= -\sum_{|w'|=m} \int_{F_w K} a(x, y) \triangle u(y) d\mu(y) \\
&= -\sum_{\substack{|w'|=m, \\ w' \neq w}} \int_{F_w K} 0 \triangle u(y) d\mu(y) \\
&\quad - \int_{F_w K} G_{E_w}(F_w^{-1} x, F_w^{-1} y) \triangle u(y) d\mu(y) \\
&= -\mu_w \int G_{E_w}(F_w^{-1} x, y') \triangle u(F_w y') d\mu(y').
\end{aligned}$$

Note that $\triangle(u \circ F_w) = r_w \mu_w \triangle u \circ F_w$, so $r_w^{-1} \mu_w^{-1} \triangle (u \circ F_w) = \triangle u \circ F_w$. Therefore, this becomes

$$\begin{aligned}
&-\int a(x, y) \triangle u(y) d\mu(y) \\
&= -\mu_w \int_K G_{E_w}(F_w^{-1} x, y')[\triangle u \circ F_w](y') d\mu(y') \\
&= -\mu_w \int G_{E_w}(F_w^{-1} x, y')(r_w^{-1} \mu_w^{-1})(\triangle(u \circ F_w))(y') d\mu(y') \\
&= -r_w^{-1} \int G_{E_w}(F_w^{-1} x, y')(\triangle(u \circ F_w))(y') d\mu(y').
\end{aligned}$$

$u \circ F_w$ is in $\operatorname{dom} \triangle$ and vanishes on E_w (because u vanishes on the boundary of E) so this becomes

$$-\int a(x, y) d\mu(y) = r_w^{-1} (u \circ F_w)(F_w^{-1} x),$$

$$-\int r_w a(x, y) d\mu(y) = u(x).$$

This holds for all $u \in \operatorname{dom} \triangle$, so

$$G_E(x, y) = r_w a(x, y). \tag{4.5}$$

(4.5) is equivalent to (4.4). $\square$

For Corollary 4.3, we bring back the notations E_w and $p_{\widetilde{E}}(x)$ from Section 3.

Corollary 4.3. *If $V_m \subseteq E$,*

(a)

$$\delta_0(E) = \Big(\sum_{|w|=m} \mu_w^2 r_w (\delta_0(E_w))^2 \Big)^{1/2}; \tag{4.6}$$

(b)

$$\delta_1(E) = \sup_{|w|=m} \mu_w r_w \delta_1(E_w);$$

(c)

$$p_E(x) = \sum_{\substack{|w|=m, \\ x \in F_w K}} \mu_w p_{E_w}(F_w^{-1} x). \tag{4.7}$$

Proof. By Theorem 4.2,

$$G_E(x,y) = \begin{cases} r_w G_{E_w}(F_w^{-1}x, F_w^{-1}y) & \text{if } x, y \in F_w K, \\ 0 & \text{if } x \text{ and } y \text{ belong to separate } m\text{-cells.} \end{cases}$$

To find δ_0, we take the square-root of $\int g_E \, d\mu$. For all $x \in K$, if $F_w K$ is the m-cell to which x belongs,

$$g_E(x) = \int G_E(x,y) d\mu(y)$$

so by Theorem 4.2,

$$\begin{aligned} g_E(x) &= \int_{F_w K} r_w G_{E_w}(F_w^{-1}x, F_w^{-1}y) d\mu(y) \\ &= \mu_w \int_K r_w G_{E_w}(F_w^{-1}x, F_w^{-1}y) d\mu(y') \end{aligned}$$

or

$$g_E(x) = \mu_w r_w g_{E_w}(F_w^{-1}x). \tag{4.8}$$

Therefore,

$$\begin{aligned}\int g_E(x)d\mu(x) &= \sum_{|w|=m} \int_{F_w K} g_E(x)d\mu(x)\\ &= \sum_{|w|=m} \mu_w r_w \int_{F_w K} g_{E_w}(F_w^{-1}x)\\ &= \sum_{|w|=m} \mu_w r_w \cdot \mu_w \int_K g_{E_w}(x')d\mu(x')\\ &= \sum_{|w|=m} \mu_w^2 r_w \int g_E\, d\mu.\end{aligned}$$

Taking the square-root of both sides yields (4.6).

By (4.8),

$$\begin{aligned}\delta_1(E) = \sup_{x\in K} g_E(x) &= \sup_{|w|=m} \sup_{x\in F_w K} \mu_w r_w g_{E_w}(F_w^{-1}x)\\ &= \sup_{|w|=m} \sup_{x\in K} \mu_w r_w g_{E_w}(x)\\ &= \sup_{|w|=m} \mu_w r_w \sup_{x\in K} g_{E_w}(x)\\ &= \sup_{|w|=m} \mu_w r_w \delta_1(E_w).\end{aligned}$$

The weights are as in (4.7) because for each cell $F_w K$ containing x, if v_x is the indicator for $F_w^{-1}x$ with respect to E_w, the contribution to this cell to the weight of x with respect to E is $\mu_w \int v_x d\mu = \mu_w p_{E_w}(F_w^{-1}x)$. □

We now apply these results to the Sierpiński tetrahedron (ST). Recall that ST is generated by the four similarities in $\mathbb{R}^3$ with contraction ratio $\frac{1}{2}$ and fixed points the vertices of a regular tetrahedron. For ST, $N = 4$, $N_0 = 4$, $\mu_i = \frac{1}{4}$, and $r_i = \frac{2}{3}$. The values of the harmonic functions on V_1 are

$$h_j(F_i q_k) = \begin{cases} 1 & \text{if } i = j = k,\\ 0 & \text{if } j \neq i = k,\\ 1/3 & \text{if } j \neq k \text{ and } (i = j \text{ or } i = k),\\ 1/6 & i, j, k \text{ all distinct.}\end{cases}$$

Let us index $A(kk', nn')$ and $I(kk')$ by the ordering

$$kk' < nn' \iff (k < n \text{ or } (k = n \text{ and } k' < n')).$$

By (4.1), $A(kk', nn')$ is

$$\frac{1}{144}\begin{pmatrix}
48 & 16 & 16 & 16 & 16 & 6 & 5 & 5 & 16 & 5 & 6 & 5 & 16 & 5 & 5 & 6\\
8 & 32 & 12 & 12 & 2 & 8 & 3 & 3 & 3 & 12 & 5 & 4 & 3 & 12 & 4 & 5\\
8 & 12 & 32 & 12 & 3 & 5 & 12 & 4 & 2 & 3 & 8 & 3 & 3 & 4 & 12 & 5\\
8 & 12 & 12 & 32 & 3 & 5 & 4 & 12 & 3 & 4 & 5 & 12 & 2 & 3 & 3 & 8\\
8 & 2 & 3 & 3 & 32 & 8 & 12 & 12 & 12 & 3 & 5 & 4 & 12 & 3 & 4 & 5\\
6 & 16 & 5 & 5 & 16 & 48 & 16 & 16 & 5 & 16 & 6 & 5 & 5 & 16 & 5 & 6\\
5 & 3 & 12 & 4 & 12 & 8 & 32 & 12 & 3 & 2 & 8 & 3 & 4 & 3 & 12 & 5\\
5 & 3 & 4 & 12 & 12 & 8 & 12 & 32 & 4 & 3 & 5 & 12 & 3 & 2 & 3 & 8\\
8 & 3 & 2 & 3 & 12 & 5 & 3 & 4 & 32 & 12 & 8 & 12 & 12 & 4 & 3 & 5\\
5 & 12 & 3 & 4 & 3 & 8 & 2 & 3 & 12 & 32 & 8 & 12 & 4 & 12 & 3 & 5\\
6 & 5 & 16 & 5 & 5 & 6 & 16 & 5 & 16 & 16 & 48 & 16 & 5 & 5 & 16 & 6\\
5 & 4 & 3 & 12 & 4 & 5 & 3 & 12 & 12 & 12 & 8 & 32 & 3 & 3 & 2 & 8\\
8 & 3 & 3 & 2 & 12 & 5 & 4 & 3 & 12 & 4 & 5 & 3 & 32 & 12 & 12 & 8\\
5 & 12 & 4 & 3 & 3 & 8 & 3 & 2 & 4 & 12 & 5 & 3 & 12 & 32 & 12 & 8\\
5 & 4 & 12 & 3 & 4 & 5 & 12 & 3 & 3 & 3 & 8 & 2 & 12 & 12 & 32 & 8\\
6 & 5 & 5 & 16 & 5 & 6 & 5 & 16 & 5 & 5 & 6 & 16 & 16 & 16 & 16 & 48
\end{pmatrix}.$$

By taking $I(kk')$ the eigenvector of magnitude 1 corresponding to eigenvalue 1,

$$I(kk') = \begin{cases} 7/80 & \text{if } k = k', \\ 13/240 & \text{if } k \neq k'. \end{cases}$$

By computing the energies $\mathcal{E}_1(v_p, v_q)$, and indexing the rows and columns by the ordering $F_0q_1 < F_0q_2 < F_0q_3 < F_1q_2 < F_1q_3 < F_2q_3$,

$$X = \frac{1}{2}\begin{pmatrix}
18 & -3 & -3 & -3 & -3 & 0\\
-3 & 18 & -3 & -3 & 0 & -3\\
-3 & -3 & 18 & 0 & -3 & -3\\
-3 & -3 & 0 & 18 & -3 & -3\\
-3 & 0 & -3 & -3 & 18 & -3\\
0 & -3 & -3 & -3 & -3 & 18
\end{pmatrix}$$

so

$$G = X^{-1} = \frac{1}{72}\begin{pmatrix}
10 & 3 & 3 & 3 & 3 & 2\\
3 & 10 & 3 & 3 & 2 & 3\\
3 & 3 & 10 & 2 & 3 & 3\\
3 & 3 & 2 & 10 & 3 & 3\\
3 & 2 & 3 & 3 & 10 & 3\\
2 & 3 & 3 & 3 & 3 & 10
\end{pmatrix}$$

or

$$\gamma(i,i',n,n') = \begin{cases} 0 & \text{if } F_i q_n \in V_0 \text{ or } F_{i'} q_{n'} \in V_0, \\ {}^{10}/_{72} & \text{if } F_i q_n = F_{i'} q_{n'} \in (V_1 \setminus V_0), \\ {}^{2}/_{72} & \text{if } \{i,i',n,n'\} = \{0,1,2,3\}, \\ {}^{3}/_{72} & \text{otherwise.} \end{cases}$$

All that remains is to plug into (4.3). This yields

$$f_{1j}(F_i q_k) = \begin{cases} 0 & \text{if } i = k, \\ -{}^{5}/_{432} & \text{if } i \neq k, \text{ and } (j = i \text{ or } j = k), \\ -{}^{4}/_{432} & \text{if } i, j, k \text{ all distinct.} \end{cases}$$

We now proceed to calculate the weights and discrepancies for some sample sets E (where $K = \mathrm{ST}$).

Example 4.1. $K = \mathrm{ST}$, $E = V_0$. We have $g_{V_0} = -f_{10} - f_{11} - f_{12} - f_{13}$, so for $x \in V_1$,

$$g_{V_0}(x) = \begin{cases} -{}^{1}/_{16} & \text{if } x \in (V_1 \setminus V_0), \\ 0 & \text{if } x \in V_0. \end{cases}$$

By applying Lemma 4.1 to ST, if u is a function on ST with $\triangle u = -1$, w is a word of length m, $\{i,j,k,l\} = \{0,1,2,3\}$, $u(F_w q_i) = a$, $u(F_w q_j) = b$, $u(F_w q_k) = c$, and $u(F_w q_l) = d$, then

$$u(F_w F_i q_j) = \frac{2a + 2b + c + d}{6} + \frac{1}{16 \cdot 6^m}. \tag{4.9}$$

It follows from this that

$$g_{V_0} = \sum_{m=0}^{\infty} h_m, \tag{4.10}$$

where h_m is the $(m+1)$-spline such that for all $x \in V_{m+1}$,

$$h_m(x) = \begin{cases} \frac{1}{16 \cdot 6^m} & \text{if } x \in (V_{m+1} \setminus V_m), \\ 0 & \text{if } x \in V_m. \end{cases}$$

Theorem 4.4. *If $E = V_0$, then the weights $\{p(x)\}$ are the uniform weights, $\delta(E,w) = 0$,*

$$\delta_0(E) = \frac{3}{4\sqrt{10}}$$

and

$$\delta_1(E) = \frac{1}{16}.$$

Proof. By symmetry, the weights are equal, so $p(x) = 1/3$ and $\delta(E, w) = 0$. For each m,

$$\int h_m d\mu = \sum_{|w|=m} \Big(\frac{1}{4}\Big)^m \Big(\frac{3 \cdot \frac{1}{16 \cdot 6^m} + 0}{4}\Big) = \frac{3}{64 \cdot 6^m}.$$

By (4.10),

$$\int g_{V_0} d\mu = \sum_{m=0}^{\infty} \frac{3}{64 \cdot 6^m} = \frac{9}{160}.$$

By Definition 2.1,

$$\delta_0(V_0) = \Big(\int g_{V_0} d\mu\Big)^{1/2} = \frac{3}{4\sqrt{10}}.$$

For δ_1, we will first show by induction that for all $m \geq 1$, for all $|w| = m$ such that the character 0 does not occur in w, $F_{0w}(q_1) = F_{0w}(q_2) = F_{0w}(q_3) = \frac{1}{16}$, $F_{0w}(q_0) = \frac{1}{16} - \frac{1}{16 \cdot 6^m}$, and g_{V_0} attains its supremum in F_{0w} ST.

Base case. Let $m = 1$. Let j be the one character of w. We have

$$F_{0j}(q_j) = F_0(q_j) = 1/16.$$

Let $\{k, l\} = \{0, 1, 2, 3\} \setminus \{0, j\}$. We get

$$\begin{aligned}
F_{0j}(q_j) &= F_0(q_j) = \frac{1}{16}, \\
F_{0j}(q_k) &= \Big(\frac{2F_0(q_j) + 2F_0(q_k) + F_0(q_0) + F_0(q_l)}{6} + \frac{1}{16 \cdot 6}\Big) \\
&= \Big(\frac{5}{6} \cdot \frac{1}{16} + \frac{1}{6} \cdot 0 + \frac{1}{16 \cdot 6}\Big) \\
&= \frac{1}{16}, \\
F_{0j}(q_l) &= F_{0j}(q_k) \\
&= \frac{1}{16}, \\
F_{0j}(q_0) &= \Big(\frac{2F_0(q_0) + 2F_0(q_j) + F_0(q_k) + F_0(q_l)}{6} + \frac{1}{16 \cdot 6}\Big) \\
&= \Big(\frac{4}{6} \cdot \frac{1}{16} + \frac{2}{6} \cdot 0 + \frac{1}{16 \cdot 6}\Big) \\
&= \frac{5}{96} \\
&= \Big(\frac{1}{16} - \frac{1}{16 \cdot 6}\Big).
\end{aligned}$$

All 2-cells of the form F_{ii} ST are symmetric, as are all 2-cells of the form $F_{ii'}$ such that $i \neq i'$. Therefore, g_{V_0} must attain its supremum either on all cells F_{ii} ST, on all cells $F_{ii'}$ ST ($i \neq i'$) or both. The values on the boundary of $F_{ii'}$ ST ($5/96$, $1/16$, $1/16$, and $1/16$) are greater than those on the boundary of F_{ii} ST (0, $5/96$, $5/96$, and $5/96$) so the supremum must be attained in $F_{ii'}$ ST. If we let $i = 0, i' = j$, g_{V_0} attains its supremum on F_{0j} ST. Thus, the result holds for the base case $m = 1$.

Inductive step. Suppose the result holds for m. Consider w, a word of length m with no 0s, and j an element of $\{1, 2, 3\}$. Then if $\{k, l\} = \{1, 2, 3\} \setminus \{j\}$,

$$\begin{aligned}
F_{0wj}(q_j) &= F_{0w}(q_j) \\
&= \frac{1}{16}, \\
F_{0wj}(q_k) &= \Big(\frac{2F_{0w}(q_j) + 2F_{0w}(q_k) + F_{0w}(q_0) + F_{0w}(q_l)}{6} + \frac{1}{16 \cdot 6^{m+1}}\Big) \\
&= \Big(\frac{\frac{2}{16} + \frac{2}{16} + \big(\frac{1}{16} - \frac{1}{16 \cdot 6^m}\big) + \frac{1}{16}}{6} + \frac{1}{16 \cdot 6^{m+1}}\Big) \\
&= \Big(\frac{(6/16)}{6} - \frac{1}{16 \cdot 6^{m+1}} + \frac{1}{16 \cdot 6^{m+1}}\Big) \\
&= \frac{1}{16}, \\
F_{0wj}(q_l) &= F_{0wj}(q_k) \\
&= \frac{1}{16}, \\
F_{0wj}(q_0) &= \Big(\frac{2F_{0w}(q_0) + 2F_{0w}(q_j) + F_{0w}(q_k) + F_{0w}(q_l)}{6} + \frac{1}{16 \cdot 6^{m+1}}\Big) \\
&= \Big(\frac{2\big(\frac{1}{16} - \frac{1}{16 \cdot 6^m}\big) + \frac{2}{16} + \frac{1}{16} + \frac{1}{16}}{6} + \frac{1}{16 \cdot 6^{m+1}}\Big) \\
&= \Big(\frac{1}{16} - \frac{1}{16 \cdot 6^{m+1}}\Big).
\end{aligned}$$

By the inductive hypothesis, g_{V_0} attains its supremum on F_{0w} ST. The cells F_{0wj} ST, F_{0wk} ST, and F_{0wl} ST are symmetrical, and have boundary values greater than those of F_{0w0} ST, so g_{V_0} attains its supremum on each of them, including F_{0wj} ST. This completes the inductive step.

Thus, if we let $\{w_{(n)}\}_{n\in\mathbb{N}}$ be any sequence of words such that each $w_{(n)}$ has length n, and the leading k-character substring of $w_{(n)}$ is $w_{(k)}$ for all $k \le n$, then

$$\lim_{n\to\infty} g_{V_0}(F_{w_{(n)}}) = \sup_{x\in\mathrm{ST}} g_{V_0}(x),$$

$$\lim_{n\to\infty} \frac{1}{16} = \delta_1(V_0),$$

$$\frac{1}{16} = \delta_1(V_0). \qquad \square$$

Example 4.2. $K = \mathrm{ST}$, $E = V_m$.

Theorem 4.4 and Corollary 4.3 allow us to compute the weights and discrepancies for $E = V_m$.

Theorem 4.5. *If* $E = V_m$*,*

$$\delta_0(E) = \frac{3}{4\sqrt{10\cdot 6^m}},$$

$$\delta_1(E) = \frac{1}{16\cdot 6^m},$$

the weights $\{p(x)\}$ *are*

$$p(x) = \begin{cases} \left(\frac{1}{4}\right)^{m+1} & \text{if } x \in V_0,\\ 2\left(\frac{1}{4}\right)^{m+1} & \text{if } x \in (V_m \setminus V_0),\end{cases}$$

and for the uniform weights $\{w(x)\}$*,*

$$\delta(E,w) = \frac{3\,(4^m-1)}{4^m\,(4^m+1)} R^{1/2}.$$

Proof. The values of δ_0, δ_1, and the weights follow from Theorem 4.3 (for $K = \mathrm{ST}$). $\delta(E,w)$ is computed from the weights. $\square$

Now, we provide some examples for another p.c.f. self-similar fractal, SG_3. $N = 6$ and $N_0 = 3$. The points q_k and the cells $F_i\mathrm{SG}_3$ are shown in Figure 4.1. To make dealing with these contractions more intuitive, from this point on we will refer to F_3 as $F_{(01)}$, F_4 as $F_{(02)}$, and F_5 as $F_{(12)}$. The renamed cells are displayed in Figure 4.2. For all i, $\mu_i = \frac{1}{6}$ and $r_i = \frac{7}{15}$. The values of the harmonic function h_0 on V_1 are shown in Figure 4.3. (The values for the other harmonic functions can be determined from this by symmetry.) If $A(kk', nn')$ is indexed by the ordering $(kk' < nn' \iff k < k'$ or $(k = k'$ and $n < n'))$, then

$$A(kk', nn') = \frac{1}{1350}\begin{pmatrix} 410 & 219 & 219 & 219 & 123 & 113 & 219 & 113 & 123 \\ 125 & 280 & 161 & 55 & 125 & 71 & 71 & 161 & 97 \\ 125 & 161 & 280 & 71 & 97 & 161 & 55 & 71 & 125 \\ 125 & 55 & 71 & 280 & 125 & 161 & 161 & 71 & 97 \\ 123 & 219 & 113 & 219 & 410 & 219 & 113 & 219 & 123 \\ 97 & 71 & 161 & 161 & 125 & 280 & 71 & 55 & 125 \\ 125 & 71 & 55 & 161 & 97 & 71 & 280 & 161 & 125 \\ 97 & 161 & 71 & 71 & 125 & 55 & 161 & 280 & 125 \\ 123 & 113 & 219 & 113 & 123 & 219 & 219 & 219 & 410 \end{pmatrix},$$

so the eigenvector $I(kk')$ is

$$I(kk') = \begin{cases} 551/3735 & \text{if } k = k', \\ 347/3735 & \text{if } k \neq k'. \end{cases}$$

We move on to the matrix X. Let p refer to the point in the middle of SG_3: $p = F_{(01)}q_2 = F_{(02)}q_1 = F_{(12)}q_0$. The 7 elements of $V_1 \setminus V_0$ (seen in Figure 4.4) are F_0q_1, F_0q_2, F_1q_0, F_1q_2, F_2q_0, F_2q_1, and p. We will consider them in this order for the indexing of X. We have

$$X = \frac{1}{7}\begin{pmatrix} 60 & -15 & -15 & 0 & 0 & 0 & -15 \\ -15 & 60 & 0 & 0 & -15 & 0 & -15 \\ -15 & 0 & 60 & -15 & 0 & 0 & -15 \\ 0 & 0 & -15 & 60 & 0 & -15 & -15 \\ 0 & -15 & 0 & 0 & 60 & -15 & -15 \\ 0 & 0 & 0 & -15 & -15 & 60 & -15 \\ -15 & -15 & -15 & -15 & -15 & -15 & 90 \end{pmatrix},$$

$$G = X^{-1} = \frac{1}{2700}\begin{pmatrix} 469 & 203 & 203 & 133 & 133 & 119 & 210 \\ 203 & 469 & 133 & 119 & 203 & 133 & 210 \\ 203 & 133 & 469 & 203 & 119 & 133 & 210 \\ 133 & 119 & 203 & 469 & 133 & 203 & 210 \\ 133 & 203 & 119 & 133 & 469 & 203 & 210 \\ 119 & 133 & 133 & 203 & 203 & 469 & 210 \\ 210 & 210 & 210 & 210 & 210 & 210 & 420 \end{pmatrix}.$$

This allows us to evaluate $f_{1k}(F_iq_n)$. The values for f_{10} are shown in Figure 4.5. The values for other harmonic functions can be determined by symmetry.

The values for $g_{V_0} = -\sum_{k=0}^{2} f_{1k}$ are shown in Figure 4.6. For $x \in V_1$,

$$g_{V_0} = \begin{cases} 0 & \text{if } x \in V_0, \\ 1/15 & \text{if } x = p, \\ 1/18 & \text{if } x \in V_1 \setminus (V_0 \cup \{p\}). \end{cases}$$

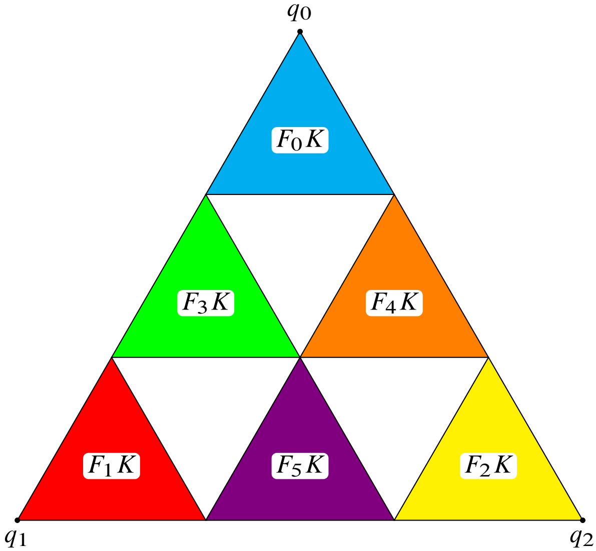


Figure 4.1. The boundary points and 1-cells of SG_3.

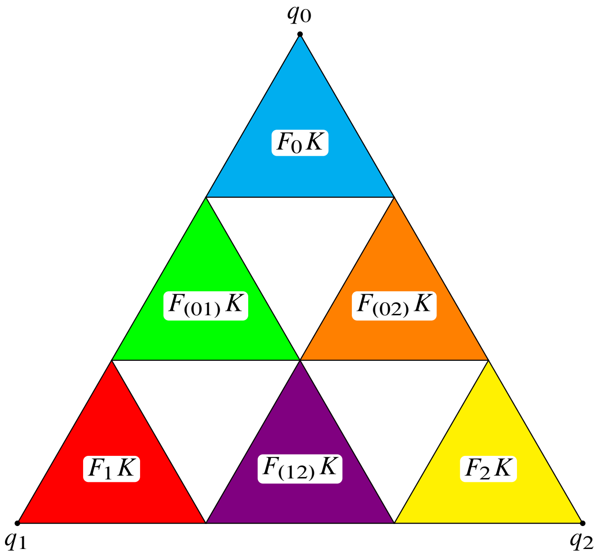


Figure 4.2. The boundary points and 1-cells of SG_3.

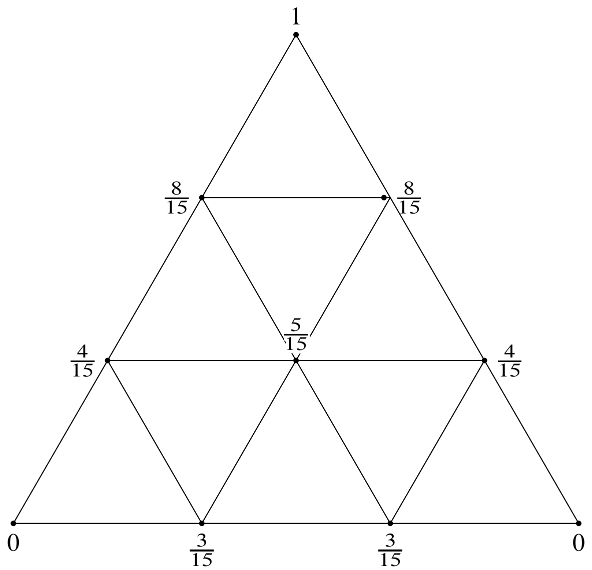


Figure 4.3. The values of h_0 on V_1.

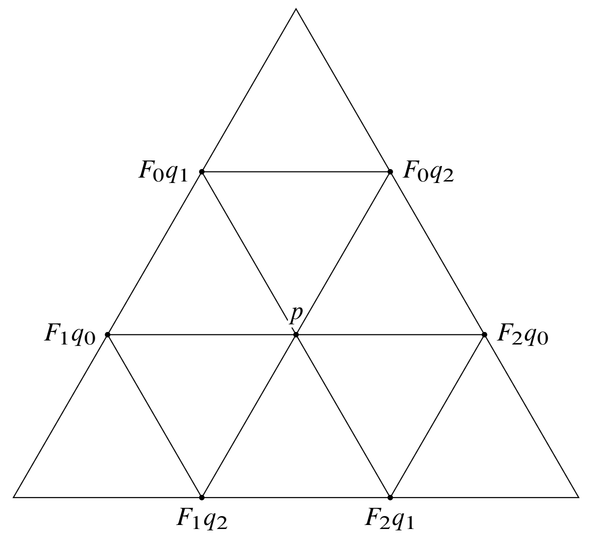


Figure 4.4. The elements of $V_1 \setminus V_0$.

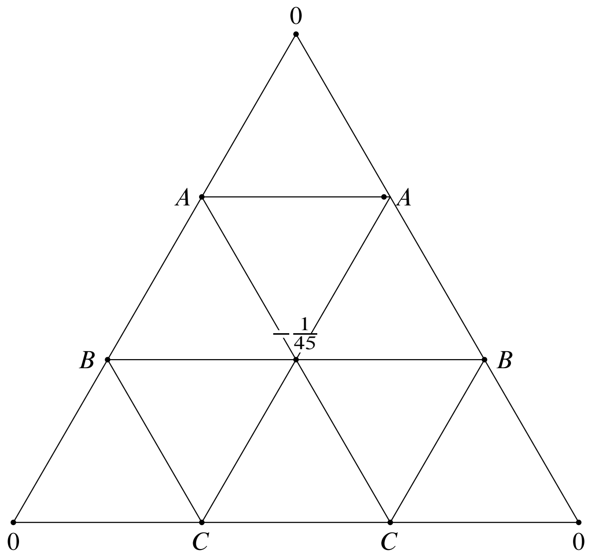


Figure 4.5. The values of f_{10} on V_1. $A = -{}^{431}/{}_{20250}$, $B = -{}^{121}/{}_{6750}$, and $C = -{}^{331}/{}_{20250}$.

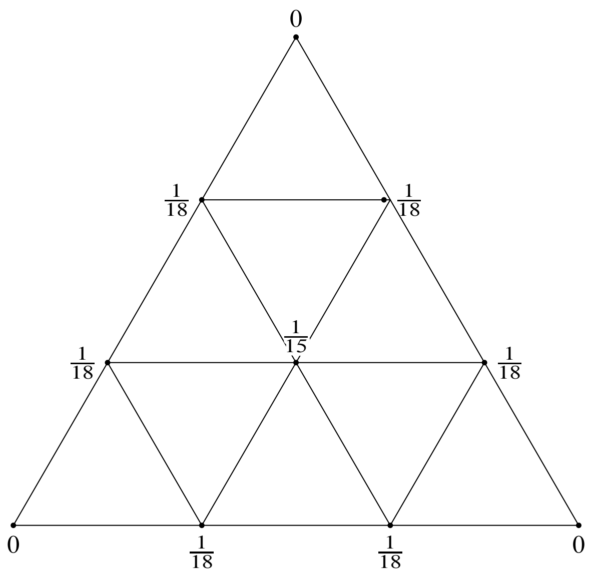


Figure 4.6. The values of g_{V_0} on V_1.

Example 4.3. $K = \mathrm{SG}_3$, $E = V_0$.

Our first sample set for SG_3 is V_0. The discrepancies can be calculated using the results we have shown for a general K and the values of g_{V_0} on V_1.

Theorem 4.6. *For* SG_3,

$$\delta_0(V_0) = \left(\frac{13}{249}\right)^{1/2}, \quad \delta_1(V_0) = \frac{540}{8051},$$

$p(x) = \frac{1}{3}$ *for* $x \in V_0$, *and* $\delta(V_0, w) = 0$.

Proof. Because $\mu_i = \frac{1}{6}$ and $r_i = \frac{7}{15}$ for SG_3, Lemma 4.1 applied to SG_3 says that if w is a word of length m, $g_{V_0}(F_w q_0) = a$, $g_{V_0}(F_w q_1) = b$, and $g_{V_0}(F_w q_2) = c$,

$$\begin{aligned}
u(F_w F_0 q_1) &= \frac{8a + 4b + 3c}{15} + \left(\frac{7}{90}\right)^m \frac{1}{18}, \\
u(F_w F_0 q_2) &= \frac{8a + 3b + 4c}{15} + \left(\frac{7}{90}\right)^m \frac{1}{18}, \\
u(F_w F_1 q_0) &= \frac{4a + 8b + 3c}{15} + \left(\frac{7}{90}\right)^m \frac{1}{18}, \\
u(F_w F_1 q_2) &= \frac{3a + 8b + 4c}{15} + \left(\frac{7}{90}\right)^m \frac{1}{18}, \\
u(F_w F_2 q_0) &= \frac{4a + 3b + 8c}{15} + \left(\frac{7}{90}\right)^m \frac{1}{18}, \\
u(F_w F_2 q_1) &= \frac{3a + 4b + 8c}{15} + \left(\frac{7}{90}\right)^m \frac{1}{18}, \\
\text{and } u(F_w p) &= \frac{a + b + c}{3} + \left(\frac{7}{90}\right)^m \frac{1}{15}.
\end{aligned} \tag{4.11}$$

Thus,

$$g_{V_0} = \sum_{m=0}^{\infty} h_m,$$

where h_m is the $(m+1)$-spline such that for all $x \in V_{m+1}$,

$$h_m(x) = \begin{cases} 0 & \text{if } x \in V_m, \\ \left(\frac{7}{90}\right)^m \frac{1}{18} & \text{if } x = F_w y \text{ for some } |w| = m,\ y \in V_1 \setminus (V_0 \cup \{p\}), \\ \left(\frac{7}{90}\right)^m \frac{1}{15} & \text{if } x = F_w p \text{ for some } |w| = m. \end{cases}$$

For each m,

$$\int h_m d\mu = \frac{3\cdot 0 + 6\cdot 2\cdot \left(\frac{7}{90}\right)^m\left(\frac{1}{18}\right) + 1\cdot 3\cdot \left(\frac{7}{90}\right)^m\left(\frac{1}{15}\right)}{3 + 6\cdot 2 + 1\cdot 3} = \frac{13}{270}\left(\frac{7}{90}\right)^m.$$

Thus,

$$\int g_{V_0} d\mu = \sum_{m=0}^{\infty} \frac{13}{270}\left(\frac{7}{90}\right)^m = \frac{13}{249}$$

so

$$\delta_0(V_0) = \left(\frac{13}{249}\right)^{1/2}.$$

Let $w = (01)2$. Note that w is a word of length 2, because (01) is a character that is really equal to 3. Let $w^2 = (01)2(01)2$, $w^3 = (01)2(01)2(01)2$, and so on. Let F_{w^∞} be the fixed point of F_w. (This definition is natural because for all $x \in \mathrm{SG}_3$, $\lim_{k\to\infty} F_{w^k} x = F_{w^\infty}$.) If (4.11) is used to compute the values of g_{V_0} on V_2, then

$$F_{(01)2}(q_0) = F_{(01)2}(q_1) = F_{(01)2}(q_2) = \frac{1}{15} \tag{4.12}$$

and for all $|w'| = 2$, $i \in \{0, 1, 2\}$,

$$F_{(01)2}(q_i) \geq F_{w'}(q_i). \tag{4.13}$$

By (4.13), g_{V_0} attains its supremum on $F_{(01)2}\mathrm{SG}_3 = F_w\mathrm{SG}$. Let

$$u = \left(\frac{7}{90}\right)^{-2}\left(g_{V_0} \circ F_w - \frac{1}{15}\right).$$

By (4.12), $u(x) = 0$ for all $x \in V_0$. The Laplacian of u is

$$\left(\frac{7}{90}\right)^{-2}\left(\triangle\, (g_{V_0} \circ F_w) - \triangle\left(\frac{1}{15}\right)\right) = \left(\frac{7}{90}\right)^{-2}\left(\left(\frac{7}{90}\right)^2 \triangle\, g_{V_0} - 0\right) = -1.$$

Thus,

$$u = g_{V_0}.$$

By repeating this argument indefinitely and using induction, g_{V_0} attains its supremum in $F_{w^k}\mathrm{SG}_3$ for all k. Thus,

$$\delta_1(V_0) = g_{V_0}(F_{w^\infty}).$$

This means that $\delta_1(V_0)$ satisfies

$$\delta_1(V_0) = \frac{1}{15} + \left(\frac{7}{90}\right)^2 \delta_1(V_0)$$

so

$$\delta_1(V_0) = \frac{540}{8051}.$$

The weights are uniform by symmetry. □

Example 4.4. $K = \mathrm{SG}_3$, $E = V_m$.

As with ST, Corollary 4.3 allows us to extend the results for V_0 to V_m. For the weights $\{p(x)\}$ of V_m, we introduce a function $\eta : V_m \to \mathbb{N}$. For all $x \in V_m$, $\eta(x)$ is the number of m-cells to which x belongs. If we consider the graph Γ_m with vertex set V_m and an edge between every $F_w x$ and $F_w y$ such that $|w| = m$, $x, y \in V_0$, and $x \neq y$, then $\eta(x)$ is equal to $\frac{1}{N_0-1}$ times the number of neighbors of x in Γ_m. In the case of SG_3, $\eta(x)$ is 1 if $x \in V_0$, 3 if $x = F_w p$ for some w of *any* length (not only m), and 2 otherwise.

Theorem 4.7. *For the* 3*-level gasket,*

$$\delta_0(V_m) = \left(\frac{7}{90}\right)^{m/2} \frac{13}{249}, \tag{4.14}$$

$$\delta_1(V_m) = \left(\frac{7}{90}\right)^{m} \frac{540}{8051}, \tag{4.15}$$

$$p_{V_m}(x) = \frac{\eta(x)}{3 \cdot 6^m}, \tag{4.16}$$

and

$$\delta(V_m, w) = \begin{cases} 0 & \text{if } m = 0, \\ \frac{4}{15} R^{1/2} & \text{if } m = 1, \\ \frac{4}{5} \cdot \frac{(6^m-1)(6^m+4)}{(6^m)(7\cdot 6^m+8)} R^{1/2} & \text{if } m \geq 2. \end{cases}$$

Proof. Apply Corollary 4.3 to $K = \mathrm{SG}_3$, $E = V_m$. (a) gives (4.7), (b) gives (4.15), and (c) gives (4.16). For $m = 0$ and $m = 1$, $\delta(V_m, w)$ can be directly computed. To calculate $\delta(V_m, w)$ for $m \geq 2$, first let $K_k = \{x \in V_m \mid \eta(x) = k\}$ for all k. $K_1 = V_0$, so $\#K_1 = 3$. For each j-cell, there are 6 elements $x \in (V_{j+1} \setminus V_j)$ with $\eta(x) = 2$ and one with $\eta(x) = 3$. Thus,

$$\#K_2 = \sum_{j=0}^{m-1} 6 \cdot 6^j = \frac{6}{5}(6^m - 1)$$

and

$$\#K_3 = \sum_{j=0}^{m-1} 6^j = \frac{1}{5}(6^m - 1).$$

The number of points of V_m is

$$\#K_1 + \#K_2 + \#K_3 = 3 + \frac{7}{5}(6^m - 1),$$

so the uniform weights are

$$w(x) = \frac{1}{3 + \frac{7}{5}(6^m - 1)} = \frac{5}{7 \cdot 6^m + 8}.$$

For $x \in K_1$,

$$w(x) - p(x) = \frac{5}{7 \cdot 6^m + 8} - \frac{1}{3 \cdot 6^m} = \frac{8(6^m - 1)}{3(6^m)(7 \cdot 6^m + 8)}.$$

For $x \in K_2$,

$$w(x) - p(x) = \frac{5}{7 \cdot 6^m + 8} - \frac{2}{3 \cdot 6^m} = \frac{6^m - 16}{3(6^m)(7 \cdot 6^m + 8)}.$$

For $x \in K_3$,

$$p(x) - w(x) = \frac{3}{3 \cdot 6^m} - \frac{5}{7 \cdot 6^m + 8} = \frac{6 \cdot 6^m + 24}{3(6^m)(7 \cdot 6^m + 8)}.$$

So

$$\begin{aligned} \delta(V_m, w) &= \sum_{x \in V_m} |p(x) - w(x)| \\ &= \frac{6^m - 1}{3(6^m)(7 \cdot 6^m + 8)} \Big(3 \cdot 8 + \frac{6}{5}(6^m - 16) + \frac{1}{5}(6 \cdot 6^m + 24) \Big) \\ &= \frac{4}{5} \cdot \frac{(6^m - 1)(6^m + 4)}{(6^m)(7 \cdot 6^m + 8)}. \end{aligned}$$ □

Interestingly, SG_3 is the first fractal we have encountered in which $\delta(V_m, w)$ does not decay exponentially to 0 as m increases. Rather,

$$\delta(V_m, w) \xrightarrow{m \to \infty} \frac{4}{35}.$$

This is because there is a set S_m of points (the elements x with $\eta(x) = 3$) whose weights differ substantially from the uniform weights and

$$\lim_{m \to \infty} \frac{\#S_m}{\#V_m} > 0.$$

In our previous examples SG and ST, the only points in V_m whose weights differed substantially from the uniform weights (for large m) were those in V_0, a set which does not grow at all with m. This means that the uniform weights $\{w(x)\}$ are a poor choice to use to numerically integrate functions on SG_3.

5. Energy measures

We now turn our attention to harmonic energy measures on the Sierpiński gasket: measures $\nu_{h,H}$ where h and H are harmonic functions and for any cell C, $\nu_{h,H} = \mathcal{E}_C(h, H)$. We develop a technique to calculate integrals of the form $\int u\,d\nu_{h,H}$, where u is a harmonic spline. Later in the section, we will generalize the results beyong SG to a more extensive class of fractals. Given a finite set $E \subset V_*$, we will derive a method to produce a set of weights $\{p(x)\}$ that can be used to numerically integrate any function that satisfies the conditions of Theorem 2.6.

First, we show that $\{\nu_0, \nu_1, \nu_2\}$ is a basis of the set of harmonic energy measures on SG, and provide a formula to express any harmonic energy measure as a linear combination of ν_0, ν_1, and ν_2.

Theorem 5.1. *If $h = a_0h_0 + a_1h_1 + a_2h_2$ and $H = b_0h_0 + b_1h_1 + b_2h_2$, then*

$$\begin{aligned}\nu_{h,H} = &\Big(a_0b_0 + \frac{a_1b_2 + a_2b_1 - a_0b_1 - a_1b_0 - a_0b_2 - a_2b_0}{2}\Big)\nu_0 \\ &+ \Big(a_1b_1 + \frac{a_0b_2 + a_2b_0 - a_1b_0 - a_0b_1 - a_1b_2 - a_2b_1}{2}\Big)\nu_1 \\ &+ \Big(a_2b_2 + \frac{a_0b_1 + a_1b_0 - a_2b_0 - a_0b_2 - a_2b_1 - a_1b_2}{2}\Big)\nu_2.\end{aligned} \tag{5.1}$$

Proof. We use the fact that energies are additive in the sense that $\mathcal{E}_C(u + v, w) = \mathcal{E}_C(u, w) + \mathcal{E}_C(v, w)$. This additivity clearly follows from the definition of energy, and holds for both the first and second variable.

By expanding for each variable,

$$\nu_{h,H} = \sum_i \sum_j \nu_{a_ih_i, b_jh_j}.$$

Clearly, for any cell C, $\mathcal{E}_C(au, bv) = ab\mathcal{E}_C(u, v)$, so

$$\nu_{h,H} = \sum_i \sum_j a_ib_j\nu_{h_i,h_j}. \tag{5.2}$$

It is a result in [1] that

$$\nu_{h_0,h_1} = \frac{1}{2}(-\nu_0 - \nu_1 + \nu_2),$$

$$\nu_{h_0,h_2} = \frac{1}{2}(-\nu_0 + \nu_1 - \nu_2),$$

$$\nu_{h_1,h_2} = \frac{1}{2}(\nu_0 - \nu_1 - \nu_2).$$

Thus, all 9 terms in (5.2) can be expressed as linear combinations of $\{\nu_0, \nu_1, \nu_2\}$, and when they are and their sum is taken, the result is (5.1). □

To calculate the weights $\{p(x)\}$, we must calculate integrals of the form $\int u\,d\nu$, where ν is a harmonic energy measure and u is a harmonic spline (more specifically, u is an indicator of some $x \in E$). This problem can be split into two problems: determining $\int u\,d\nu$ from the values of $\int u \circ F_w\,d\nu$ for $|w| = m$, and taking the integral with respect to ν of a harmonic function.

To solve the first problem, we will construct matrices M_w for each word w such that for every continuous function f, we can use M_w to evaluate integrals $\int_{F_w \mathrm{SG}} f\,d\nu_i$.

Theorem 5.2. *If*

$$M_0 = \frac{1}{15}\begin{pmatrix} 9 & 0 & 0 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \tag{5.3a}$$

$$M_1 = \frac{1}{15}\begin{pmatrix} 2 & 2 & -1 \\ 0 & 9 & 0 \\ -1 & 2 & 2 \end{pmatrix}, \tag{5.3b}$$

$$M_2 = \frac{1}{15}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 0 & 0 & 9 \end{pmatrix}, \tag{5.3c}$$

then for all continuous functions f, and for all $i \in \{0, 1, 2\}$,

$$\begin{pmatrix} \int_{F_i \mathrm{SG}} f\,d\nu_0 \\ \int_{F_i \mathrm{SG}} f\,d\nu_1 \\ \int_{F_i \mathrm{SG}} f\,d\nu_2 \end{pmatrix} = M_i \begin{pmatrix} \int f \circ F_i\,d\nu_0 \\ \int f \circ F_i\,d\nu_1 \\ \int f \circ F_i\,d\nu_2 \end{pmatrix}. \tag{5.4}$$

Proof. By the definition of an energy measure,

$$\int_{F_0 \mathrm{SG}} f\,d\nu_0 = \frac{5}{3}\int f \circ F_0\,d\nu_{h_0 + \frac{2}{5}h_1 + \frac{2}{5}h_2} \tag{5.5}$$

and

$$\int_{F_0 \mathrm{SG}} f\,d\nu_1 = \frac{5}{3}\int f \circ F_0\,d\nu_{h_0 + \frac{2}{5}h_1 + \frac{2}{5}h_2}. \tag{5.6}$$

By Theorem 5.1,

$$\nu_{h_0 + \frac{2}{5}h_1 + \frac{2}{5}h_2} = \frac{9}{25}\nu_0 \tag{5.7}$$

and

$$\nu_{\frac{2}{5}h_1 + \frac{1}{5}h_2} = \frac{2}{25}\nu_0 + \frac{2}{25}\nu_1 - \frac{1}{25}\nu_2. \tag{5.8}$$

By (5.5) and (5.7),

$$\int_{F_0 SG} f d\nu_0 = \frac{3}{5} \int f \circ F_0 d\nu_0$$

and by (5.6) and (5.8)

$$\int_{F_0 SG} f d\nu_1 = \frac{2}{15} \int f \circ F_0 d\nu_0 + \frac{2}{15} \int f \circ F_0 d\nu_1 - \frac{1}{15} \int f \circ F_0 d\nu_2.$$

By symmetry,

$$\int_{F_0 SG} f d\nu_2 = \frac{2}{15} \int f \circ F_0 d\nu_0 - \frac{1}{15} \int f \circ F_0 d\nu_1 + \frac{2}{15} \int f \circ F_0 d\nu_2.$$

Thus

$$M_0 = \frac{1}{15} \begin{pmatrix} 9 & 0 & 0 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

and by symmetry

$$M_1 = \frac{1}{15} \begin{pmatrix} 2 & 2 & -1 \\ 0 & 9 & 0 \\ -1 & 2 & 2 \end{pmatrix}, \quad M_2 = \frac{1}{15} \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 0 & 0 & 9 \end{pmatrix}. \qquad \square$$

.

Interestingly, [1] showed that for what turn out to be the same matrices M_0, M_1, and M_2:

$$\begin{pmatrix} \int_{F_0 SG} f d\nu_i \\ \int_{F_1 SG} f d\nu_i \\ \int_{F_2 SG} f d\nu_i \end{pmatrix} = M_i \begin{pmatrix} \int f \circ F_0 d\nu_i \\ \int f \circ F_1 d\nu_i \\ \int f \circ F_2 d\nu_i \end{pmatrix}. \tag{5.9}$$

That these matrices satisfy both (5.4) and (5.9) may be a simple coincidence, arising from the fact that $(M_i)_{i,j} = (M_i)_{j,i}$. whenever $i \neq j$.

Theorem 5.3. *We have*

$$\begin{pmatrix} \int_{F_w SG} f d\nu_0 \\ \int_{F_w SG} f d\nu_1 \\ \int_{F_w SG} f d\nu_2 \end{pmatrix} = M_w \begin{pmatrix} \int f \circ F_w d\nu_0 \\ \int f \circ F_w d\nu_1 \\ \int f \circ F_w d\nu_2 \end{pmatrix}.$$

Proof. This theorem is proven by induction.

If $m = 1$, the result follows from Theorem 5.2.

If the result holds for all words of length m and w is a word of length $m+1$, let $w = w_1 w'$ for some word w' of length m and some $w_1 \in \{0,1,2\}$, $w = w_1 w'$. By the same argument as the one used in the proof of Theorem 5.2 (that of appealing to the definition of an energy measure and then using Theorem 5.1),

$$\begin{pmatrix} \int_{F_w SG} f d\nu_0 \\ \int_{F_w SG} f d\nu_1 \\ \int_{F_w SG} f d\nu_2 \end{pmatrix} = M_{w_1} \begin{pmatrix} \int_{F_{w'} SG} f \circ F_{w_1} d\nu_0 \\ \int_{F_{w'} SG} f \circ F_{w_1} d\nu_1 \\ \int_{F_{w'} SG} f \circ F_{w_1} d\nu_2 \end{pmatrix}. \tag{5.10}$$

By the inductive hypothesis,

$$\begin{pmatrix} \int_{F_{w'} SG} f \circ F_{w_1} d\nu_0 \\ \int_{F_{w'} SG} f \circ F_{w_1} d\nu_1 \\ \int_{F_{w'} SG} f \circ F_{w_1} d\nu_2 \end{pmatrix} = M_{w'} \begin{pmatrix} \int f \circ F_{w_1} \circ F_{w'} d\nu_0 \\ \int f \circ F_{w_1} \circ F_{w'} d\nu_1 \\ \int f \circ F_{w_1} \circ F_{w'} d\nu_2 \end{pmatrix}. \tag{5.11}$$

$M_{w_1} M_{w'} = M_w$ and $f \circ F_{w_1} \circ F_{w'} = f \circ F_w$, so by (5.10) and (5.11):

$$\begin{pmatrix} \int_{F_w SG} f d\nu_0 \\ \int_{F_w SG} f d\nu_1 \\ \int_{F_w SG} f d\nu_2 \end{pmatrix} = M_w \begin{pmatrix} \int f \circ F_w d\nu_0 \\ \int f \circ F_w d\nu_1 \\ \int f \circ F_w d\nu_2 \end{pmatrix}. \qquad \square$$

Because every harmonic function is a linear combination of $\{h_0, h_1, h_2\}$, and Theorem 5.1 constitutes a formula to express every harmonic energy measure as a linear combination of $\{\nu_0, \nu_1, \nu_2\}$, the problem of taking the integral with respect to a harmonic energy measure ν is solved by calculating the integrals $\int h_i d\nu_j$ and taking linear combinations with the appropriate coefficients.

Theorem 5.4. *For $i, j \in \{0,1,2\}$:*

$$\int h_i d\nu_j = \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{2} & \text{if } i \neq j. \end{cases} \tag{5.12}$$

Proof. Let $\alpha = \int h_0 d\nu_0$ and $\beta = \int h_1 d\nu_0$. By symmetry, $\int h_i d\nu_i = \alpha$ for all i and $\int h_i d\nu_j = \beta$ whenever $i \neq j$. From the integral of a constant function, $\alpha + 2\beta = 2$. By (5.9),

$$\begin{pmatrix} \int_{F_0 SG} h_0 d\nu_0 \\ \int_{F_1 SG} h_0 d\nu_0 \\ \int_{F_2 SG} h_0 d\nu_0 \end{pmatrix} = \frac{1}{15} \begin{pmatrix} 9 & 0 & 0 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} \int f \circ F_0 d\nu_0 \\ \int f \circ F_1 d\nu_0 \\ \int f \circ F_2 d\nu_0 \end{pmatrix}.$$

Thus

$$\begin{aligned}\int h_0 d\nu_0 = \sum_i \int_{F_i \mathrm{SG}} h_0 d\nu_0 = \frac{3}{5}\int h_0 + \frac{2}{5}h_1 + \frac{2}{5}h_2 d\nu_0 \\ + \frac{2}{15}\int \frac{2}{5}h_0 + \frac{1}{5}h_2 d\nu_0 \\ + \frac{2}{15}\int \frac{2}{5}h_0 + \frac{1}{5}h_2 d\nu_1 \\ - \frac{1}{15}\int \frac{2}{5}h_0 + \frac{1}{5}h_2 d\nu_2 \\ + \frac{2}{15}\int \frac{2}{5}h_0 + \frac{1}{5}h_1 d\nu_0 \\ - \frac{1}{15}\int \frac{2}{5}h_0 + \frac{1}{5}h_1 d\nu_1 \\ + \frac{2}{15}\int \frac{2}{5}h_0 + \frac{1}{5}h_1 d\nu_2,\end{aligned}$$

so

$$75\alpha = 9(5\alpha+4\beta)+2(2\alpha+\beta)+2(3\beta)-(\alpha+2\beta)+2(2\alpha+\beta)-(\alpha+2\beta)+2(3\beta),$$

$$75\alpha = 51\alpha + 48\beta,$$

and

$$\alpha = 2\beta$$

Because $\alpha + 2\beta = 2$ and $\alpha = 2\beta$, $\alpha = 1$ and $\beta = \frac{1}{2}$. In other words,

$$\int h_i d\nu_{jk} = \begin{cases} 1 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j. \end{cases} \tag{5.13}$$

□

We can now, in principle, compute $\int u d\nu$ for any harmonic spline u and harmonic energy measure ν. In particular, if E is a finite subset of V_*, and for all $x \in E$ we let $p(x) = \int v_x d\nu$ (where v_x is the function that is harmonic away from E with $V_x|_E = \delta_x$), then any function that satisfies the conditions of Theorem 2.6 can be numerically integrated using the weights $\{p(x)\}$.

The results proven in this section so far relating to the Sierpiński gasket generalize to any self-similar p.c.f. fractal generated by a finite iterated system $\{F_j\}$ satisfying the conditions of Section 4, if the results are expressed using $\{\nu_{h_i,h_j}\}_{0\le i<j<N_0}$ (instead of $\{\nu_i\}_{0\le i<N_0}$) as the spanning set for the set of harmonic energy measures. This choice of spanning set may seem odd or unnatural,

because the measures ν that we are interested in are non-negative (for example, the Kusuoka measure $\nu = \sum_i \nu_i$), as are the measures ν_i, while the measures ν_{h_i,h_j} $(i \neq j)$ are signed; the Theorems of Section 2 require that ν be non-negative. However, it is possible to express measures ν_i as linear combinations of $\{\nu_{h_i,h_j}\}_{0\leq i<j<N_0}$, and the reverse is not true. Therefore, though our choice of spanning set may be less natural, it is necessary to generalize the results of this section.

Let K be defined as in Section 4, with $V_0 = \{q_i\}_{0\leq i<N_0}$ and the harmonic functions $\{h_i\}$ such that $\big(\sum_{i=0}^{N_0-1} a_i h_i\big)(q_j) = a_j$.

If $0 \leq i, j < N_0$, denote ν_{h_i,h_j} by ν_{ij}.

Theorem 5.5. *If ν is a harmonic energy measure on K (that is, $\nu = \nu_{h,H}$ for some $h = \sum_i a_i h_i$ and $H = \sum_j b_j h_j$), then ν is a linear combination of $\{\nu_{ij}\}_{i\neq j}$ given by*

$$\nu = \sum_{0\leq i<j<N_0} (a_i b_j + a_j b_i - a_i b_i - a_j b_j)\nu_{ij}.$$

Proof. By the additivity and scalar multiplication of energy measures,

$$\nu = \sum_{i=0}^{N_0-1} \sum_{j=0}^{N_0-1} a_i b_j \nu_{ij}.$$

For each i, $\sum_{j=0}^{N_0-1} \nu_{ij} = \nu_{h_i,(h_0+h_1+\ldots+h_{n-1})} = \nu_{h_i,1} = 0$, so $\nu_{ii} = -\sum_{j\neq i} \nu_{ij}$. Therefore,

$$\nu = \sum_{0\leq i<j<N_0} (a_i b_j + a_j b_i - a_i b_i - a_j b_j)\nu_{ij}. \qquad \square$$

For the next theorems, we will speak of matrices whose rows and columns are indexed by pairs (j,k) such that $0 \leq j < k < N_0$, ordered lexicographically, so (j,k) comes "before" (l,m) if $i < l$ or $i = l$ and $k < m$.

We will refer to the "(j,k)-th row" or "(l,m)-th column or "$((j,k),(l,m))$-th entry" of such a matrix.

We will also define constants $a_{i,j,k,l,m}$ and matrices M_i and M_w as follows:

Definition 5.6. If $i \in \{0,1,2,\ldots,N-1\}$, $0 \leq j < k < N_0$, and $0 \leq l < m < N_0$, let

$$\begin{aligned} a_{i,j,k,l,m} = {} & (h_j \circ F_i)(q_l) \cdot (h_k \circ F_i)(q_m) \\ & + (h_j \circ F_i)(q_m) \cdot (h_k \circ F_i)(q_l) \\ & - (h_j \circ F_i)(q_l) \cdot (h_k \circ F_i)(q_l) \\ & - (h_j \circ F_i)(q_m) \cdot (h_k \circ F_i)(q_m). \end{aligned}$$

For all i, let us M_i be the square matrix with rows and columns indexed by $\{(j,k)\}_{0\le j<k<N_0}$ whose $((j,k),(l,m))$-th entry is $r_i^{-1}a_{i,j,k,l,m}$.

For all words $w = w_1 w_2 \ldots w_m$, let $M_w = M_{w_1} M_{w_2} \ldots M_{w_m}$.

Theorem 5.7. *For all words w and continuous functions f,*

$$\begin{pmatrix} \int_{F_w K} f d\nu_{01} \\ \vdots \\ \int_{F_w K} f d\nu_{(N_0-2)(N_0-1)} \end{pmatrix} = M_w \begin{pmatrix} \int f \circ F_w d\nu_{01} \\ \vdots \\ \int f \circ F_w d\nu_{(N_0-2)(N_0-1)} \end{pmatrix}. \tag{5.14}$$

Proof. First, suppose w is a word of length 1, whose one character is i. For all $0 \le j < k < N_0$, by the definition of an energy measure,

$$\int_{F_i K} f d\nu_{jk} = r_i^{-1} \int f \circ F_i d\nu_{h_j \circ F_i, h_k \circ F_i}. \tag{5.15}$$

By applying Theorem 5.5 to $\nu_{h_j \circ F_i, h_k \circ F_i}$,

$$\nu_{h_j \circ F_i, h_k \circ F_i} = \sum_{0\le l<m<N_0} a_{i,j,k,l,m} \nu_{lm}. \tag{5.16}$$

By (5.15) and (5.16),

$$\int_{F_i K} f d\nu_{jk} = r_i^{-1} \sum_{0\le l<m<N_0} a_{i,j,k,l,m} \int f \circ F_i d\nu_{lm}.$$

This statement for all (j,k) is equivalent to (5.14) for $w = i$. This theorem extends to longer words by the same argument as used in the proof of Theorem 5.3. □

To compute weights for Theorem 2.6, all that is left to do is evaluate the integrals of the form $\int h_i d\nu_{jk}$ for $0 \le j < k < N_0$. For all i, j, k:

$$\int h_i d\nu_{jk} = \sum_{l=0}^{N-1} \int_{F_l K} h_i d\nu_{jk}.$$

By applying Theorem 5.7, each integral $\int_{F_l K} h_i d\nu_{jk}$ can be expressed as a linear combination of $\{\int h_i d\nu_{jk}\}_{0\le j<k<N_0}$. Doing this for all i, j, k yields a system of $\frac{1}{2}(N_0^3 - N_0^2)$ equations and $\frac{1}{2}(N_0^3 - N_0^2)$ unknowns for the integrals $\int h_i d\nu_{jk}$.

This system is linearly dependent because setting every integral equal to 0 would be one solution. However, combining this system with the equations

$$\sum_{i=0}^{N_0-1} \int h_i \, d\nu_{jk} = \nu_{jk}(K) = \mathcal{E}(h_j, h_k)$$

for all (j, k) will in most cases make it independent (it does in all of our examples). It is possible that this system will have an un unwieldy amount of terms. Likely, symmetry can be used to reduce it to a more manageable system.

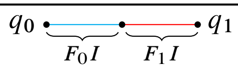
We now choose some specific self-similar p.c.f. fractals and list the results obtained when the above calculations are performed. For each fractal, these calculations determine the matrices M_i. The fractals chosen are the Unit Interval, the Sierpiński gasket, the Sierpiński tetrahedron, the Sierpiński n-hedron for a general n, and the 3-level gasket. (The Sierpiński n-hedron is generated by the similarities with contraction ratio $\frac{1}{2}$ whose fixed points are n pairwise equidistant vertices in $\mathbb{R}^{n-1}$.)

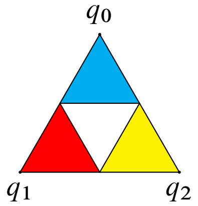
Note that the matrices M_0, M_1, M_2 for the Sierpiński gasket are not the same as the ones given by Theorem 5.2, because of our change in choice of spanning set: the matrices in Theorem 5.2 satisfy

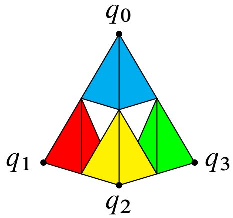
$$\begin{pmatrix} \int_{F_i \mathrm{SG}} f \, d\nu_0 \\ \int_{F_i \mathrm{SG}} f \, d\nu_1 \\ \int_{F_i \mathrm{SG}} d\nu_2 \end{pmatrix} = M_i \begin{pmatrix} \int f \circ F_i \, d\nu_0 \\ \int f \circ F_i \, d\nu_1 \\ \int f \circ F_i \, d\nu_2 \end{pmatrix},$$

while the matrices in this table satisfy

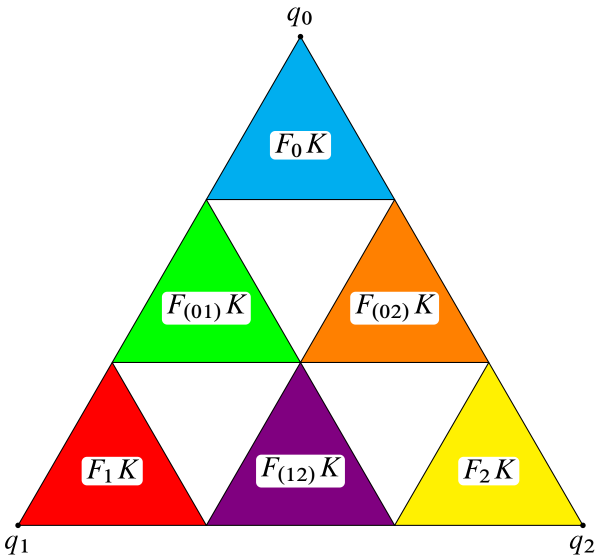
$$\begin{pmatrix} \int_{F_i \mathrm{SG}} f \, d\nu_{01} \\ \int_{F_i \mathrm{SG}} f \, d\nu_{02} \\ \int_{F_i \mathrm{SG}} d\nu_{12} \end{pmatrix} = M_i \begin{pmatrix} \int f \circ F_i \, d\nu_{01} \\ \int f \circ F_i \, d\nu_{02} \\ \int f \circ F_i \, d\nu_{12} \end{pmatrix}.$$

Fractal	Unit Interval (I)
Picture	
Matrices	$M_0 = M_1 = \frac{1}{2}$ For all w, for all continuous f, $\int_{F_w I} f \, d\nu_{01} = M_w \int f \circ F_w \, d\nu_{01}$

Fractal	Sierpiński gasket (SG)
Picture	
Matrices	$M_0 = \frac{1}{15}\begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ -2 & -2 & 1 \end{pmatrix}$ $M_1 = \frac{1}{15}\begin{pmatrix} 6 & 0 & 3 \\ -2 & 1 & -2 \\ 3 & 0 & 6 \end{pmatrix}$ $M_2 = \frac{1}{15}\begin{pmatrix} -2 & -2 & 1 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{pmatrix}$ For all w, for all continuous f, $\begin{pmatrix} \int_{F_w \mathrm{SG}} f d\nu_{01} \\ \int_{F_w \mathrm{SG}} f d\nu_{02} \\ \int_{F_w \mathrm{SG}} f d\nu_{12} \end{pmatrix} = M_w \begin{pmatrix} \int f \circ F_w d\nu_{01} \\ \int f \circ F_w d\nu_{02} \\ \int f \circ F_w d\nu_{12} \end{pmatrix}$

Fractal	Sierpiński tetrahedron (ST)
Picture	
Matrices	$M_0 = \frac{1}{24}\begin{pmatrix} 8 & 4 & 4 & 0 & 0 & 0 \\ 4 & 8 & 4 & 0 & 0 & 0 \\ 4 & 4 & 8 & 0 & 0 & 0 \\ -2 & -2 & -1 & 1 & 0 & 0 \\ -2 & -1 & -2 & 0 & 1 & 0 \\ -1 & -2 & -2 & 0 & 0 & 1 \end{pmatrix}$ $M_1 = \frac{1}{24}\begin{pmatrix} 8 & 0 & 0 & 4 & 4 & 0 \\ -2 & 1 & 0 & -2 & -1 & 0 \\ -2 & 0 & 1 & -1 & -2 & 0 \\ 4 & 0 & 0 & 8 & 4 & 0 \\ 4 & 0 & 0 & 4 & 8 & 0 \\ -1 & 0 & 0 & -2 & -2 & 1 \end{pmatrix}$ $M_2 = \frac{1}{24}\begin{pmatrix} 1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 8 & 0 & 4 & 0 & 4 \\ 0 & -2 & 1 & -1 & 0 & -2 \\ 0 & -2 & 1 & -1 & 0 & -2 \\ 0 & 4 & 0 & 8 & 0 & 4 \\ 0 & 4 & 0 & 4 & 0 & 8 \end{pmatrix}$ $M_3 = \frac{1}{24}\begin{pmatrix} 1 & 0 & -2 & 0 & -2 & -1 \\ 0 & 1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 8 & 0 & 4 & 4 \\ 0 & 0 & -1 & 1 & -2 & -2 \\ 0 & 0 & 4 & 0 & 8 & 4 \\ 0 & 0 & 4 & 0 & 4 & 8 \end{pmatrix}$ For all w, for all continuous f, $\begin{pmatrix} \int_{F_w \mathrm{ST}} f d\nu_{01} \\ \int_{F_w \mathrm{ST}} f d\nu_{02} \\ \int_{F_w \mathrm{ST}} f d\nu_{03} \\ \int_{F_w \mathrm{ST}} f d\nu_{12} \\ \int_{F_w \mathrm{ST}} f d\nu_{13} \\ \int_{F_w \mathrm{ST}} f d\nu_{23} \end{pmatrix} = M_w \begin{pmatrix} \int f \circ F_w d\nu_{01} \\ \int f \circ F_w d\nu_{02} \\ \int f \circ F_w d\nu_{03} \\ \int f \circ F_w d\nu_{12} \\ \int f \circ F_w d\nu_{13} \\ \int f \circ F_w d\nu_{23} \end{pmatrix}.$

Fractal	Sierpiński n-hedron
Matrices	$(M_i)_{(j,k),(l,m)} = \frac{1}{n(n+2)} \cdot \big[\xi(j,i,l)\xi(k,i,m) + \xi(j,i,m)\xi(k,i,l) - \xi(j,i,l)\xi(k,i,l) - \xi(j,i,m)\xi(k,i,m)\big],$ where $\xi(a,b,c) = \begin{cases} n+2 & a=b=c, \\ 2 & a=b\neq c, \\ 2 & a=c\neq b, \\ 0 & a\neq b=c, \\ 1 & a\neq b, b\neq c, a\neq c \end{cases}$

Fractal	3-level gasket (SG_3)
Picture	
Matrices	$M_0 = \frac{1}{105}\begin{pmatrix} 28 & 7 & 0 \\ 7 & 28 & 0 \\ -12 & -12 & 1 \end{pmatrix}$ $M_1 = \frac{1}{105}\begin{pmatrix} 28 & 0 & 7 \\ -12 & 1 & -12 \\ 7 & 0 & 28 \end{pmatrix}$ $M_2 = \frac{1}{105}\begin{pmatrix} 1 & -12 & -12 \\ 0 & 28 & 7 \\ 0 & 7 & 28 \end{pmatrix}$ $M_{(01)} = \frac{1}{105}\begin{pmatrix} 16 & 3 & 3 \\ 0 & 6 & -2 \\ 0 & -2 & 6 \end{pmatrix}$ $M_{(02)} = \frac{1}{105}\begin{pmatrix} 6 & 0 & -2 \\ 3 & 16 & 3 \\ -2 & 0 & 6 \end{pmatrix}$ $M_{(12)} = \frac{1}{105}\begin{pmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 3 & 3 & 16 \end{pmatrix}$ For all w, for all continuous f, $\begin{pmatrix} \int_{F_w \mathrm{SG}_3} f\,d\nu_{01} \\ \int_{F_w \mathrm{SG}_3} f\,d\nu_{02} \\ \int_{F_w \mathrm{SG}_3} f\,d\nu_{12} \end{pmatrix} = M_w \begin{pmatrix} \int f \circ F_w\,d\nu_{01} \\ \int f \circ F_w\,d\nu_{02} \\ \int f \circ F_w\,d\nu_{12} \end{pmatrix}$

By the method that we used to reach (5.13), we can compute the integrals of the form $\int h_i d\nu_{jk}$. In the table below, we list these integrals for the same fractals as in the above table. From these, since each measure of the form ν_i is equal to $-\sum_{j\neq i} \nu_{ij}$, we can calculate integrals of the form $\int h_i d\nu_j$:

$$\int h_i d\nu_j = -\sum_{k\neq j} \int h_i d\nu_{jk}.$$

These basic integrals for our example fractals (I, SG, ST, the n-hedron for $3 \leq n \leq 100$, and SG_3) are listed in the table below. For I, SG, and ST, the integrals were calculated by hand, while for the n-hedrons and the 3-level gasket, they were calculated with the assistance of a computer program. We hypothesize that the formulae for the n-hedron continue to hold for all positive integers $n \geq 3$.

Fractal	$\int h_i d\nu_{jk}$	$\int h_i d\nu_j$
Unit Interval	$-\frac{1}{2}$	$\frac{1}{2}$
Sierpiński gasket	$\begin{cases} -1/2 & \text{if } i = j \text{ or } i = k, \\ 0 & \text{if } i, j, k \text{ distinct} \end{cases}$	$\begin{cases} 1 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j \end{cases}$
Sierpiński tetrahedron	$\begin{cases} -1/2 & \text{if } i = j \text{ or } i = k, \\ 0 & \text{if } i, j, k \text{ distinct} \end{cases}$	$\begin{cases} 3/2 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j \end{cases}$
n-hedron ($3 \leq n \leq 100$)	$\begin{cases} -1/2 & \text{if } i = j \text{ or } i = k, \\ 0 & \text{if } i, j, k \text{ distinct} \end{cases}$	$\begin{cases} \frac{1}{2}(n-1) & \text{if } i = j, \\ 1/2 & \text{if } i \neq j \end{cases}$
3-level gasket	$\begin{cases} -1/2 & \text{if } i = j \text{ or } i = k, \\ 0 & \text{if } i, j, k \text{ distinct} \end{cases}$	$\begin{cases} 1 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j \end{cases}$

As with SG, we can numerically integrate any function that satisfies the conditions of Theorem 2.6 with respect to a non-negative harmonic energy measure ν: For some finite $E \subseteq V_*$, let $\{v_x\}_{x\in E}$ be the usual indicator splines, and let the weights be $p(x) = \int v_x d\nu$.

References

[1] R. Bell, C.-W. Ho, and R. S. Strichartz, Energy measures of harmonic functions on the Sierpiński gasket. *Indiana Univ. Math. J.* **63** (2014), no. 3, 831–868. MR 3254525 Zbl 06363211

[2] N. Kajino, Heat kernel asymptotics for the measurable Riemannian structure on the Sierpiński gasket. *Potential Anal.* **36** (2012), no. 1, 67–115. MR 2886454 Zbl 1248.28012

[3] J. Kigami, *Analysis on fractals.* Cambridge Tracts in Mathematics, 143. Cambridge University Press, Cambridge, 2001. MR 1840042 Zbl 0998.28004

[4] J. Kigami, Measurable Riemannian geometry on the Sierpiński gasket: the Kusuoka measure and the Gaussian heat kernel estimate. *Math. Ann.* **340** (2008), no. 4, 781–804. MR 2372738 Zbl 1143.28004

[5] A. Kontorovich, From Apollonius to Zaremba: local-global phenomena in thin orbits. *Bull. Amer. Math. Soc.* (*N.S.*) **50** (2013), no. 2, 187–228. MR 3020826 Zbl 1309.11040

[6] S. Kusuoka, Dirichlet forms on fractals and products of random matrices. *Publ. Res. Inst. Math. Sci.* **25** (1989), no. 4, 659–680. MR 1025071 Zbl 0694.60071

[7] J. Malmquist and R. S. Strichartz, Numerical integration for fractal measures - programs. http://www.math.cornell.edu/~jensmalmquist/codeguide.pdf

[8] H. Niederreiter, Quasi-Monte Carlo methods and pseudo-random numbers. *Bull. Amer. Math. Soc.* **84** (1978), no. 6, 957–1041. MR 0508447 Zbl 0404.65003

[9] P.-H. Li, N. Ryder, R. S. Strichartz, and B. E. Ugurcan, Extensions and their minimizations on the Sierpiński Gasket. *Potential Anal.* **41** (2014), no. 4, 1167–1201. MR 3269719 Zbl 1306.28010

[10] Robert S. Strichartz, *Differential equations on fractals.* A tutorial. Princeton University Press, Princeton, N.J., 2006. MR 2246975 Zbl 1190.35001

[11] R. S. Strichartz and T. T. Tse, Local behavior of smooth functions for the energy Laplacian on the Sierpiński gasket. *Analysis* (*Munich*) **30** (2010), no. 3, 285–299. MR 2676212 Zbl 1223.28014

[12] R. S. Strichartz and M. Usher, Splines on fractals. *Math. Proc. Cambridge Philos. Soc.* **129** (2000), no. 2, 331–360. MR 1765920 Zbl 0978.31005

Received June 9, 2016

Jens Malmquist, Mathematics Department, Evans Hall, University of California, Berkeley, CA 94720, USA

e-mail: jmalmquist@berkeley.edu

Robert S. Strichartz, Mathematics Department, Malott Hall, Cornell University, Ithaca, NY 14853, USA

e-mail: str@math.cornell.edu