J. Fractal Geom. 5 (2018), 279–337
DOI 10.4171/JFG/62

Journal of Fractal Geometry

C^m Eigenfunctions of Perron–Frobenius operators and a new approach to numerical computation of Hausdorff dimension: applications in $\mathbb{R}^1$

Richard S. Falk and Roger D. Nussbaum[1]

Abstract. We develop a new approach to the computation of the Hausdorff dimension of the invariant set of an iterated function system or IFS. In the one dimensional case that we consider here, our methods require only C^3 regularity of the maps in the IFS. The key idea, which has been known in varying degrees of generality for many years, is to associate to the IFS a parametrized family of positive, linear, Perron–Frobenius operators L_s. The operators L_s can typically be studied in many different Banach spaces. Here, unlike most of the literature, we study L_s in a Banach space of real-valued, C^k functions, $k \geq 2$. We note that L_s is not compact, but has essential spectral radius ρ_s strictly less than the spectral radius λ_s and possesses a strictly positive C^k eigenfunction v_s with eigenvalue λ_s. Under appropriate assumptions on the IFS, the Hausdorff dimension of the invariant set of the IFS is the value $s = s_*$ for which $\lambda_s = 1$. This eigenvalue problem is then approximated by a collocation method using continuous piecewise linear functions. Using the theory of positive linear operators and explicit a priori bounds on the derivatives of the strictly positive eigenfunction v_s, we give rigorous upper and lower bounds for the Hausdorff dimension s_*, and these bounds converge to s_* as the mesh size approaches zero.

Mathematics Subject Classification (2010). 11K55, 37C30; 65J10.

Keywords. Hausdorff dimension, positive transfer operators, continued fractions.

[1] The work of R. D. Nussbaum was supported by NSF grant DMS-1201328.

Contents

1. Introduction

Our interest in this paper is in finding rigorous estimates for the Hausdorff dimension of invariant sets for iterated function systems or IFS's. The case of graph directed IFS's (see [42] and [41]) is also of great interest and can be studied by our methods, but for simplicity we shall restrict attention here to the IFS case.

Let $D \subset \mathbb{R}$ be a nonempty compact set and $\theta_j : D \to D$, $1 \le j \le m$, a contraction mapping, i.e., a Lipschitz mapping with Lipschitz constant $\mathrm{Lip}(\theta_j)$, satisfying $\mathrm{Lip}(\theta_j) := c_j < 1$. If $m < \infty$ and the above assumption holds, it is known that there exists a unique, compact, nonempty set $C \subset D$ such that $C = \bigcup_{j=1}^m \theta_j(C)$. The set C is called the invariant set for the IFS $\{\theta_j : 1 \le j \le m\}$.

Although we shall eventually specialize, it may be helpful to describe initially some functional analysis results in the generality of the previous paragraph. Let H be a bounded, open subset of $\mathbb{R}$, which is a finite union of open intervals, and let $C^k(\bar{H})$ denote the real Banach space of C^k real-valued maps, all of whose derivatives of order $\nu \le k$ extend continuously to $\bar{H}$. For a given positive integer N, assume that $g_j : \bar{H} \to (0, \infty)$ are strictly positive C^N functions for $1 \le j \le m < \infty$ and $\theta_j : \bar{H} \to \bar{H}$, $1 \le j \le m$, are C^N maps and contractions. For $s > 0$ and integers k, $0 \le k \le N$, one can define a bounded linear map $L_{s,k} : C^k(\bar{H}) \to C^k(\bar{H})$ by the formula

$$(L_{s,k} w)(x) = \sum_{j=1}^{m} [g_j(x)]^s w(\theta_j(x)). \tag{1.1}$$

Linear maps like $L_{s,k}$ are sometimes called positive transfer operators or Perron–Frobenius operators and arise in many contexts other than computation of Hausdorff dimension: see, for example, [2]. If $r(L_{s,k})$ denotes the spectral radius of

$L_{s,k}$, then $\lambda_s = r(L_{s,k})$ is positive and independent of k for $0 \le k \le N$; and λ_s is an algebraically simple eigenvalue of $L_{s,k}$ with a corresponding unique, normalized strictly positive eigenfunction $v_s \in C^N(\bar{H})$. Furthermore, the map $s \mapsto \lambda_s$ is continuous. If $\sigma(L_{s,k}) \subset \mathbb{C}$ denotes the spectrum of the complexification of $L_{s,k}$, $\sigma(L_{s,k})$ depends on k, but for $1 \le k \le N$,

$$\sup\{|z| : z \in \sigma(L_{s,k}) \setminus \{\lambda_s\}\} < \lambda_s. \tag{1.2}$$

If $k = 0$, the strict inequality in (1.2) may fail. A more precise version of the above result is stated in Theorem 4.1 of this paper and Theorem 4.1 is a special case of results in [49]. The method of proof involves ideas from the theory of positive linear operators, particularly generalizations of the Kreĭn–Rutman theorem to noncompact linear operators; see [35], [4], [56], [46], and [39]. We do not use the thermodynamic formalism (see [52]) and often our operators cannot be studied in Banach spaces of analytic functions.

The linear operators which are relevant for the computation of Hausdorff dimension comprise a small subset of the transfer operators described in (1.1), but the analysis problem which we shall consider here can be described in the generality of (1.1) and is of interest in this more general context. We want to find rigorous methods to estimate $r(L_{s,k})$ accurately and then use these methods to estimate s_*, where, in our applications, s_* will be the unique number $s \ge 0$ such that $r(L_{s,k}) = 1$. Under further assumptions, we shall see that s_* equals $\dim_H(C)$, the Hausdorff dimension of the invariant set associated to the IFS. This observation about Hausdorff dimension has been made, in varying degrees of generality by many authors. See, for example, [7], [8], [6], [10], [11], [14], [21], [23], [25], [24], [27], [28], [29], [30], [41], [40], [50], [52], [53], [54], and [57].

In the applications in this paper, we shall assume, for simplicity, that H is a bounded open interval, that $\theta_j : \bar{H} \to \bar{H}$ is a C^N contraction mapping, where $N \ge 3$, (or more generally satisfies (H5.1)) and $\theta_j'(x) \ne 0$ for all $x \in \bar{H}$. In the notation of (1.1), we define $g_j(x) = |\theta_j'(x)|$. It is often natural to assume that H is a finite union of open intervals, and our methods apply with no essential change to this case.

Given the existence of a strictly positive C^N eigenfunction v_s for (1.1), we show in Section 5 for $1 \le p \le 3$, that one can obtain explicit upper and lower bounds for the quantity $\frac{D^p v_s(x)}{v_s(x)}$ for $x \in \bar{H}$, where D^p denotes the p-th derivative of v_s. Such bounds can also be obtained for $p > 3$, but calculations become more onerous. In the important special case that $\theta_j(x)$ is of the form $(x + b_j)^{-1}$, where $b_j > 0$ and $g_j(x) = |\theta_j'(x)|$, we obtain in Section 6 sharp estimates on the quantity $\frac{D^p v_s(x)}{v_s(x)}$ for all $p \ge 1$ and all $x \in \bar{H}$. These estimates play a crucial role in allowing us to obtain rigorous upper and lower bounds for the Hausdorff dimension.

The basic idea of our numerical scheme is to cover $\bar{H}$ by nonoverlapping intervals of length h. We remark that our collection of intervals need not be a *Markov partition* for our IFS; compare the use of *Markov partitions* in [43]. We then approximate the strictly positive, C^2 eigenfunction v_s by a continuous piecewise linear function. Using explicit bounds on the first and second derivatives of v_s, we are able to associate to the operator $L_{s,k}$, square matrices A_s and B_s, which have nonnegative entries and also have the property that $r(A_s) \le \lambda_s \le r(B_s)$. We note that using a piecewise linear approximation to v_s, as opposed to a piecewise constant approximation, leads to a considerable increase in accuracy and speed of convergence. A key role here is played by an elementary fact which is not as well known as it should be. If M is a nonnegative matrix and w is a strictly positive vector and $M\mathrm{w} \le \lambda \mathrm{w}$, (coordinate-wise), then $r(M) \le \lambda$. An analogous statement is true if $M\mathrm{w} \ge \lambda \mathrm{w}$. We emphasize that our approach is robust and allows us to study the case $H \subset \mathbb{R}$ when $\theta_j(\cdot)$, $1 \le j \le m$, is only C^3.

If s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, so that s_* is the Hausdorff dimension of the invariant set for the IFS under study, we proceed as follows. If we can find a number s_1 such that $r(B_{s_1}) \le 1$, then, since the map $s \mapsto \lambda_s$ is decreasing, $\lambda_{s_1} \le r(B_{s_1}) \le 1$, and we can conclude that $s_* \le s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \ge 1$, then $\lambda_{s_2} \ge r(A_{s_2}) \ge 1$, and we can conclude that $s_* \ge s_2$. By choosing the mesh size for our approximating piecewise polynomials to be sufficiently small, we can make $s_1 - s_2$ small, providing a good estimate for s_*. For a given s, $r(A_s)$ and $r(B_s)$ are easily found by variants of the power method for eigenvalues, since (see Section 7) the largest eigenvalue has multiplicity one and is the only eigenvalue of its modulus.

If the coefficients $g_j(\cdot)$ and the maps $\theta_j(\cdot)$ in (1.1) are C^N with $N > 2$, it is natural to approximate $v_s(\cdot)$ by piecewise polynomials of degree $N-1$. The corresponding matrices A_s and B_s may no longer have all nonnegative entries and the arguments of this paper are no longer directly applicable. However, we hope to prove in a future paper that inequalities like $r(A_s) \le \lambda_s \le r(B_s)$ remain true and lead to much improved upper and lower bounds for $r(L_s)$. Heuristic evidence for this assertion is given in Table 3.2 of Section 3.2.

We illustrate our new approach by first considering in Section 3 the computation of the Hausdorff dimension of invariant sets in $[0,1]$ arising from classical continued fraction expansions. In this much studied case, one defines $\theta_m = \frac{1}{x+m}$, for m a positive integer and $x \in [0,1]$; and for a subset $\mathcal{B} \subset \mathbb{N}$, one considers the IFS $\{\theta_m : m \in \mathcal{B}\}$ and seeks estimates on the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for this IFS. This problem has previously been considered by many authors. See [5], [7], [8], [21], [23], [25], [24], [28], [29], and [22]. In this case, (1.1) becomes

$$(L_{s,k}w)(x) = \sum_{m \in \mathcal{B}} \Big(\frac{1}{x+m}\Big)^{2s} w\Big(\frac{1}{x+m}\Big), \quad 0 \le x \le 1,$$

and one seeks a value $s \ge 0$ for which $\lambda_s := r(L_{s,k}) = 1$. Table 3.1 in Section 3.2 gives upper and lower bounds for the value s such that $\lambda_s = 1$ for various sets $\mathcal{B}$. Jenkinson and Pollicott [29] use a completely different method and obtain, when $|\mathcal{B}|$ is small, high accuracy estimates for $\dim_H(C(\mathcal{B}))$, in which successive approximations converge at a super-exponential rate. It is less clear (see [28]) how well the approximation scheme in [29] or [28] works when $|\mathcal{B}|$ is moderately large or when different real analytic functions $\hat{\theta}_j : [0,1] \to [0,1]$ are used. Here, in the one dimensional case, we present an alternative approach with much wider applicability that only requires the maps in the IFS to be C^3. As an illustration, we consider in Section 3.3 perturbations of the IFS for the middle thirds Cantor set for which the corresponding contraction maps are C^3, but not C^4.

It is also worth comparing the approach used in our paper with that used by McMullen [43]. Superficially the methods seem different, but there are underlying connections. We exploit the existence of a C^k, strictly positive eigenfunction v_s of (1.1) with eigenvalue λ_s equal to the spectral radius of $L_{s,k}$; and we observe that explicit bounds on derivatives of v_s can be exploited to prove convergence rates on numerical approximation schemes which approximate λ_s. McMullen does not explicitly mention the operator $L_{s,k}$ or the analogue of $L_{s,k}$ for graph directed iterated function systems, and he does not use C^k, strictly positive eigenfunctions of equations like (1.1). Instead, he exploits finite positive measures μ which are called *$\mathcal{F}-$invariant densities of dimension* δ. If s_* is a value of s for which the above eigenvalue $\lambda_s = 1$, then in our context the measure μ is an eigenfunction of the Banach space adjoint $(L_{s_*,0})^*$ with eigenvalue 1, and our s_* corresponds to δ above. Standard arguments using weak* compactness, the Schauder-Tychonoff fixed point theorem, and the Riesz representation theorem imply the existence of a regular, finite, positive, complete measure μ, defined on a σ-algebra containing all Borel subsets of the underlying space $\bar{H}$ and such that $(L_{s_*,0})^*\mu = \mu$ and $\int v_s \, d\mu = 1$.

McMullen also uses refinements of *Markov partitions*, while our partitions, both here and in a sequel [16] in which we consider two dimensional problems, need not be Markov. However, in the end, both approaches generate (different) $n \times n$ nonnegative matrices M_s, parametrized by a parameter s and both methods use the spectral radius of M_s to approximate the desired Hausdorff dimension s_*. McMullen's matrices are obtained by approximating certain nonconstant functions defined on a refinement of the original Markov partition by piecewise constant functions defined with respect to this refinement. We approximate by linear functions on each subset in our partition in dimension one and (see [16]) by bilinear functions defined on each subset of our partition in dimension two. As we show below, by exploiting estimates on higher derivatives of $v_s(\cdot)$, our methods give explicit upper and lower bounds for s_* and more rapid convergence to s_* than one obtains using piecewise constant approximations.

The square matrices A_s and B_s mentioned above and described in more detail in Section 3 have nonnegative entries and satisfy $r(A_s) \leq \lambda_s \leq r(B_s)$. To apply standard numerical methods, it is useful to know that all eigenvalues $\mu \neq r(A_s)$ of A_s satisfy $|\mu| < r(A_s)$ and that $r(A_s)$ has algebraic multiplicity one and that corresponding results hold for $r(B_s)$. Such results are proved in Section 7 when the mesh size, h, is sufficiently small. Note that this result does not follow from the standard theory of nonnegative matrices, since A_s and B_s typically have zero columns and are not primitive. We also prove that $r(A_s) \leq r(B_s) \leq (1 + C_1 h^2) r(A_s)$, where the constant C_1 can be explicitly estimated. In Section 8, we prove that the map $s \mapsto \lambda_s$ is log convex and strictly decreasing; and the same result is proved for $s \mapsto r(M_s)$, where M_s is a naturally defined matrix such that $A_s \leq M_s \leq B_s$.

In a subsequent paper [16], we consider the computation of the Hausdorff dimension of some invariant sets arising for complex continued fractions. Suppose that $\mathcal{B}$ is a subset of $I_1 = \{m + ni : m \in \mathbb{N}, n \in \mathbb{Z}\}$, and for each $b \in \mathcal{B}$, define $\theta_b(z) = (z+b)^{-1}$. Note that θ_b maps $\overline{G} = \{z \in \mathbb{C}: |z - \frac{1}{2}| \leq \frac{1}{2}\}$ into itself. We are interested in the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for the IFS $\{\theta_b : b \in \mathcal{B}\}$. This is a two dimensional problem and we allow the possibility that $\mathcal{B}$ is infinite. In general (contrast work in [29] and [28]), it does not seem possible in this case to replace $L_{s,k}$, $k \geq 2$, by an operator Λ_s acting on a Banach space of analytic functions of one complex variable and satisfying $r(\Lambda_s) = r(L_{s,k})$. Instead, we work in $C^2(\overline{G})$ and apply our methods to obtain rigorous upper and lower bounds for the Hausdorff dimension $\dim_H(C(\mathcal{B}))$ for several examples. The case $\mathcal{B} = I_1$ has been of particular interest and is one motivation for the paper [16]. In [19], Gardner and Mauldin proved that $d := \dim_H(C(I_1)) < 2$. In Theorem 6.6 of [40], Mauldin and Urbanski proved that $1.2484 < d \leq 1.885$, and in [51], Priyadarshi proved that $d \geq 1.78$. We prove that $1.85550 \leq d \leq 1.85589$. A combination of the results in this paper plus the subsequent paper [16] can be found in a preliminary version published on the arXiv [15].

Although many of the key results in the paper are described above, an outline summarizing the sections may be helpful. In Section 2, we recall the definition of

Hausdorff dimension and present some mathematical preliminaries. In Section 3, we present the details of our approximation scheme for Hausdorff dimension, explain the crucial role played by estimates on derivatives of order ≤ 2 of v_s, and give the aforementioned estimates for Hausdorff dimension. We emphasize that this is a feasibility study. We have limited the accuracy of our approximations to what is easily found using the standard precision of *Matlab* and have run only a limited number of examples, using mesh sizes that allow the programs to run fairly quickly. In addition, we have not attempted to exploit the special features of our problems, such as the fact that our matrices are sparse. Thus, it is clear that one could write a more efficient code that would also speed up the computations. However, the *Matlab* programs we have developed are available on the web at

http://www.math.rutgers.edu/~falk/hausdorff/codes.html

and we hope other researchers will run other examples of interest to them.

The theory underlying the work in Section 3 is deferred to Sections 4–8. In Section 4 we describe some results concerning existence of C^m positive eigenfunctions for a class of positive (in the sense of order-preserving) linear operators. In Section 5, we derive explicit bounds on the derivatives of the eigenfunction v_s of L_s and in Section 6, we show how much sharper bounds on the derivatives of the eigenfunction can be obtained when the maps θ_b are Möbius transformations. In Section 7, we verify some spectral properties of the approximating matrices which justify standard numerical algorithms for computing their spectral radii. Finally, in Section 8, we show the log convexity of the spectral radius $r(L_s)$, which we exploit in our numerical approximation scheme.

2. Preliminaries

We recall the definition of the Hausdorff dimension, $\dim_H(K)$, of a subset $K \subset \mathbb{R}^N$. To do so, we first define for a given $s \geq 0$ and each set $K \subset \mathbb{R}^N$,

$$H^s_\delta(K) = \inf\Big\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta \text{ cover of } K\Big\},$$

where $|U|$ denotes the diameter of U and a countable collection $\{U_i\}$ of subsets of $\mathbb{R}^N$ is a δ-cover of $K \subset \mathbb{R}^N$ if $K \subset \bigcup_i U_i$ and $0 < |U_i| < \delta$ for all i. We then define the s-dimensional Hausdorff measure

$$H^s(K) = \lim_{\delta \to 0+} H^s_\delta(K).$$

Finally, we define the Hausdorff dimension of K, $\dim_H(K)$, as

$$\dim_H(K) = \inf\{s: H^s(K) = 0\}.$$

We now state the main result connecting Hausdorff dimension to the spectral radius of the map defined by (1.1). To do so, we first define the concept of an *infinitesimal similitude* (sometimes called a conformal map). Let (S, d) be a perfect metric space. If $\theta: S \to S$, then θ is an infinitesimal similitude at $t \in S$ if for any sequences $(s_k)_k$ and $(t_k)_k$ with $s_k \neq t_k$ for $k \geq 1$ and $s_k \to t$, $t_k \to t$, the limit

$$\lim_{k\to\infty} \frac{d(\theta(s_k), \theta(t_k))}{d(s_k, t_k)} =: (D\theta)(t)$$

exists and is independent of the particular sequences $(s_k)_k$ and $(t_k)_k$. Furthermore, θ is an infinitesimal similitude on S if θ is an infinitesimal similitude at t for all $t \in S$.

This concept generalizes the concept of affine linear similitudes, which are affine linear contraction maps θ satisfying for all $x, y \in \mathbb{R}^n$

$$d(\theta(x), \theta(y)) = c d(x, y), \quad c \neq 0.$$

In particular, the examples discussed in this paper, such as maps of the form $\theta(x) = \frac{1}{x+m}$, with m a positive integer, are infinitesimal similitudes. More generally, if S is a compact subset of $\mathbb{R}^1$ and $\theta: S \to S$ extends to a C^1 map defined on an open neighborhood of S in $\mathbb{R}^1$, then θ is an infinitesimal similitude.

Theorem 2.1 (Theorem 1.2 of [50]). *Let $\theta_i: S \to S$ for $1 \leq i \leq N$ be infinitesimal similitudes and assume that the map $t \mapsto (D\theta_i)(t)$ is a strictly positive Hölder continuous function on S. Assume that θ_i is a Lipschitz map with Lipschitz constant $c_i \leq c < 1$ and let C denote the unique, compact, nonempty invariant set such that*

$$C = \bigcup_{i=1}^{N} \theta_i(C).$$

Further, assume that θ_i satisfy

$$\theta_i(C) \cap \theta_j(C) = \emptyset, \quad \text{for } 1 \leq i, j \leq N,\ i \neq j,$$

and are one-to-one on C. Then the Hausdorff dimension of C is given by the unique σ_0 such that $r(L_{\sigma_0}) = 1$.

For related results on the computation of Hausdorff dimension, we refer the reader to the list of references near the bottom of p. 2.

Finally, we state a result that is key to obtaining explicit upper and lower bounds on the Hausdorff dimension. Although we give a proof to keep our presentation self-contained, the following lemma is actually a special case of much more general results concerning order-preserving, homogeneous cone mappings: see Lemmas 9.1–9.4 on pp. 89–91 in [34] and also Lemma 2.2 in [36] and Theorem 2.2 in [38]. If, for w as in Lemma 2.2 below, we let D denote the positive diagonal $N \times N$ matrix with diagonal entries w_j, $1 \leq j \leq N$, $r(M) = r(D^{-1}MD)$; and Lemma 2.2 can also be obtained by applying Theorem 1.1 on p. 24 of [44] to $D^{-1}MD$.

Lemma 2.2. *Let M be an $N \times N$ matrix with non-negative entries and* w *an N vector with strictly positive components.*

- *If $(M\mathrm{w})_k \geq \lambda \mathrm{w}_k$, $k = 1, \dots N$, then $r(M) \geq \lambda$.*
- *If $(M\mathrm{w})_k \leq \lambda \mathrm{w}_k$, $k = 1, \dots N$, then $r(M) \leq \lambda$.*

Proof. If $(M\mathrm{w})_k \geq \lambda \mathrm{w}_k$, $k = 1, \dots N$, it easily follows that $(M^n\mathrm{w})_k \geq \lambda^n \mathrm{w}_k$ and so $\|M^n\mathrm{w}\|_\infty \geq \lambda^n \|\mathrm{w}\|_\infty$. Let e be a vector with all $\mathrm{e}_i = 1$. Then

$$\|M^n\|_\infty = \|M^n\mathrm{e}\|_\infty \geq \frac{\|M^n\mathrm{w}\|_\infty}{\|\mathrm{w}\|_\infty} \geq \lambda^n.$$

Hence,

$$r(M) = \lim_{n\to\infty} \|M^n\|_\infty^{1/n} \geq \lambda.$$

If $(M\mathrm{w})_k \leq \lambda \mathrm{w}_k$, $k = 1, \dots N$, it easily follows that $(M^n\mathrm{w})_k \leq \lambda^n \mathrm{w}_k$. Let k be chosen so that $\|M^n\|_\infty = \sum_j (M^n)_{k,j}$. Since $[r(M)]^n = r(M^n) \leq \|M^n\|_\infty$,

$$\begin{aligned} \min_j \mathrm{w}_j\, [r(M)]^n &\leq \min_j \mathrm{w}_j \sum_j (M^n)_{k,j} \\ &\leq \sum_j (M^n)_{k,j} \mathrm{w}_j \\ &= (M^n\mathrm{w})_k \\ &\leq \lambda^n \mathrm{w}_k. \end{aligned}$$

So,

$$\min_j \mathrm{w}_j \leq \left[\frac{\lambda}{r(M)}\right]^n \mathrm{w}_k.$$

If $r(M) > \lambda$, then letting $n \to \infty$, we get that $\min_j \mathrm{w}_j \leq 0$, which contradicts the fact that all $\mathrm{w}_j > 0$. Hence, $r(M) \leq \lambda$. □

3. Examples

3.1. Continued fraction Cantor sets. We first consider the problem of computing the Hausdorff dimension of some Cantor sets arising from continued fraction expansions. More precisely, given any number $0 < x < 1$, we can consider its continued fraction expansion

$$x = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_1, a_2, a_3, \dots \in \mathbb{N}$. We then consider the Cantor set $E_{[m_1,\dots,m_p]}$, of all points in $[0, 1]$ where we restrict the coefficients a_i to the values $m_1, \dots, m_p$. A number of papers (e.g., [7], [8], [21], [23], [25], and [29]) have considered this problem in the case of the set $E_{1,2}$, consisting of all points in $[0, 1]$ for which each a_i has the value 1 or 2. In [29], a method is presented that computes this dimension to 25 decimal places. Computations are also presented in that paper and in [28] for other choices of the values $m_1, \dots, m_p$. In [5], the Hausdorff dimension of the Cantor set $E_{2,4,6,8,10}$ is computed to three decimal places (0.517).

Corresponding to the choices of m_i, we associate contraction maps $\theta_m(x) = \frac{1}{x+m}$. A key fact is that the Cantor sets we consider can be generated as limit points of sequences of these contraction maps. For example, the set $E_{1.2}$ can be generated using the maps $\theta_1(x) = \frac{1}{x+1}$ and $\theta_2(x) = \frac{1}{x+2}$ as the set of limit points of sequences $\theta_{m_1} \dots \theta_{m_n}(0)$, for $m_1, m_2, \dots \in \{1, 2\}$.

For $w \in C[0, 1]$, we define

$$(L_s w)(x) = \sum_{j=1}^{p} |\theta'_{m_j}(x)|^s w(\theta_{m_j}(x)).$$

In fact, we can just as easily think of L_s as an operator on $C[0, \gamma^{-1}]$ or on $C[(1 + \Gamma)^{-1}, \gamma^{-1}]$, where $\gamma = \min m_j$ and $\Gamma = \max m_j$. In the discussion below, we will usually work on the interval $[0, \gamma^{-1}]$.

Our computations are based on the following result, which we shall prove in subsequent sections.

Theorem 3.1. *For all $s > 0$, L_s has a unique strictly positive eigenfunction v_s with $L_s v_s = \lambda_s v_s$, where $\lambda_s > 0$ and $\lambda_s = r(L_s)$, the spectral radius of L_s. Furthermore, the map $s \mapsto \lambda_s$ is strictly decreasing and continuous, and for all $p > 0$ and for all $x \in [0, \gamma^{-1}]$,*

$$\begin{aligned}(2s)(2s+1)\ldots(2s+p-1)(2\gamma^{-1}+\Gamma)^{-p} \\ \leq (-1)^p \frac{D^p[v_s(x)]}{v_s(x)} \\ \leq (2s)(2s+1)\ldots(2s+p-1)\gamma^{-p},\end{aligned} \tag{3.1}$$

where $\gamma = \min_j m_j$ and $\Gamma = \max_j m_j$. Finally, the Hausdorff dimension of the Cantor set generated from the maps

$\theta_{m_1}, \ldots, \theta_{m_p}$ is the unique value of s with $\lambda_s = 1$.

Note that it follows easily from (3.1) when $p = 1$ and $x_1, x_2 \in [0, 1]$, that

$$v_s(x_2) \leq v_s(x_1) \exp\left(2s\frac{|x_2 - x_1|}{\gamma}\right). \tag{3.2}$$

To see this, write

$$\begin{aligned}\log \frac{v_s(x_2)}{v_s(x_1)} &= \log v_s(x_2) - \log v_s(x_1) \\ &= \int_{x_1}^{x_2} \frac{d}{dx} \log v_s(x)\, dx \\ &= \int_{x_1}^{x_2} \frac{v_s'(x)}{v_s(x)}\, dx,\end{aligned}$$

apply the bound in (3.1), and exponentiate the result.

To obtain approximations of the dimension of the Cantor sets described in this section, we first approximate a function $f \in C^2[0, \gamma^{-1}]$ by a continuous, piecewise linear function defined on a mesh of interval size h on $[0, \gamma^{-1}]$. More specifically, we approximate $f(x)$, $x_k \leq x \leq x_{k+1}$ by its piecewise linear interpolant $f^I(x)$ given by

$$f^I(x) = \frac{x_{k+1} - x}{h} f(x_k) + \frac{x - x_k}{h} f(x_{k+1}), \quad x_k \leq x \leq x_{k+1},$$

where the mesh points x_k satisfy $0 = x_0 < x_1 \cdots < x_n = \gamma^{-1}$, with $x_{k+1} - x_k = h = \frac{1}{\gamma n}$.

Notice that if $\mathrm{w} = (\mathrm{w}_0, \ldots, \mathrm{w}_n)$ is a vector in $\mathbb{R}^{n+1}$, we can associate a continuous piecewise linear function $w^I : [0, \gamma^{-1}] \to \mathbb{R}$ defined with respect to the partition $0 = x_0 < x_1 < \cdots < x_n = \gamma^{-1}$ of $[0, \gamma^{-1}]$ by

$$w^I(y) = \frac{[x_{r+1} - y]}{h}(\mathrm{w})_r + \frac{[y - x_r]}{h}(\mathrm{w})_{r+1}, \quad y \in [x_r, x_{r+1}],\ 0 \le r < n.$$

This notation will be used below and will play an important role in our argument.

Our goal is to construct $(n+1) \times (n+1)$ matrices A_s and B_s which have nonnegative entries and satisfy

$$r(A_s) \le r(L_s) \le r(B_s),$$

where $r(A_s)$ (respectively, $r(B_s)$) denotes the spectral radius of A_s (respectively, B_s). Furthermore, the entries $(A_s)_{ij}$ and $(B_s)_{ij}$ of A_s and B_s satisfy (for n large)

$$0 \le (B_s)_{ij} - (A_s)_{ij} \le C h^2,$$

where C is a constant which can be estimated explicitly and is independent of n.

Standard results for the error in linear interpolation on an interval $[a, b]$ (e.g., see Theorem 3.2 of [1]) assert that for $x \in [a, b]$, there exists $\xi = \xi(x) \in (a, b)$ such that

$$\begin{aligned} f^I(x) - f(x) &:= \frac{b - x}{b - a} f(b) + \frac{x - a}{b - a} f(a) - f(x) \\ &= \frac{1}{2}(b - x)(x - a) f''(\xi). \end{aligned}$$

In the notation above, if $x \in [0, \gamma^{-1}]$ and $x_r \le x \le x_{r+1}$ for some r, $0 \le r < n$, we shall apply this error estimate with $a = x_r$ and $b = x_{r+1}$, so $\xi \in (x_r, x_{r+1})$.

We can also use results from Theorem 3.1 to bound the interpolation error. Letting $f(x) = v_s(x)$, we obtain from Theorem 3.1 that

$$2s(2s+1)(2\gamma^{-1} + \Gamma)^{-2} v_s(\xi) \le v_s''(\xi) \le 2s(2s+1)\gamma^{-2} v_s(\xi).$$

Using (3.2), and the fact that $|\xi - x_r| \le h$ for $\xi \in [x_r, x_{r+1}]$, we have

$$\begin{aligned} v_s(x_r) \exp\Big(-2s\frac{h}{\gamma}\Big) &\le v_s(x_r) \exp\Big(-2s\frac{|\xi - x_r|}{\gamma}\Big) \\ &\le v_s(\xi) \\ &\le v_s(x_r) \exp\Big(2s\frac{|\xi - x_r|}{\gamma}\Big) \\ &\le v_s(x_r) \exp\Big(2s\frac{h}{\gamma}\Big). \end{aligned}$$

Similarly,

$$v_s(x_{r+1})\exp\Big(-2s\frac{h}{\gamma}\Big) \le v_s(\xi) \le v_s(x_{r+1})\exp\Big(2s\frac{h}{\gamma}\Big).$$

Taking a suitable convex combination of these results, we get for $y \in [x_r, x_{r+1}]$,

$$v_s^I(y)\exp\Big(-2s\frac{h}{\gamma}\Big) \le v_s(\xi) \le v_s^I(y)\exp\Big(2s\frac{h}{\gamma}\Big).$$

Using the interpolation error estimate, we then get for $x_r \le y \le x_{r+1}$,

$$\begin{aligned}
&[x_{r+1}-y][y-x_r]s(2s+1)(2\gamma^{-1}+\Gamma)^{-2}\exp\Big(-2s\frac{h}{\gamma}\Big)\,v_s^I(y)\\
&\le v_s^I(y)-v_s(y)\\
&\le [x_{r+1}-y][y-x_r]s(2s+1)\gamma^{-2}\exp\Big(2s\frac{h}{\gamma}\Big)\,v_s^I(y).
\end{aligned}$$

Using this estimate, we have precise upper and lower bounds on the error in the interval $[x_r, x_{r+1}]$ that only depend on the function values of v_s at x_r and x_{r+1}. For $y \in [x_r, x_{r+1}]$, define error functionals

$$\mathrm{err}^1(y) = [x_{r+1}-y][y-x_r]s(2s+1)\gamma^{-2}\exp\Big(2s\frac{h}{\gamma}\Big),$$

$$\mathrm{err}^2(y) = [x_{r+1}-y][y-x_r]s(2s+1)(2\gamma^{-1}+\Gamma)^{-2}\exp\Big(-2s\frac{h}{\gamma}\Big).$$

Note that $\mathrm{err}^1(y)$ and $\mathrm{err}^2(y)$ depend on the subinterval in which y lies, although this is not reflected directly in the notation.

It then follows that for all $y \in [x_r, x_{r+1}]$,

$$[1-\mathrm{err}^1(y)]v_s^I(y) \le v_s(y) \le [1-\mathrm{err}^2(y)]v_s^I(y).$$

For a fixed k, $0 \le k \le n$, if we replace y in the above inequality by $\theta_{m_j}(x_k)$ and sum over j, we obtain

$$\begin{aligned}
&\sum_{j=1}^{p} |\theta'_{m_j}(x_k)|^s[1-\mathrm{err}^1(\theta_{m_j}(x_k))]v_s^I(\theta_{m_j}(x_k))\\
&\le \sum_{j=1}^{p} |\theta'_{m_j}(x_k)|^s v_s(\theta_{m_j}(x_k))\\
&= (L_s v_s)(x_k)\\
&= r(L_s)v_s(x_k)\\
&\le \sum_{j=1}^{p} |\theta'_{m_j}(x_k)|^s[1-\mathrm{err}^2(\theta_{m_j}(x_k))]v_s^I(\theta_{m_j}(x_k)).
\end{aligned}$$

Motivated by the above inequality, we now define $(n+1) \times (n+1)$ matrices A_s and B_s which have nonnegative entries and satisfy the property that $r(A_s) \le r(L_s) \le r(B_s)$. Letting w be a vector in $\mathbb{R}^{n+1}$, we define $(B_s \mathrm{w})_k$ and $(A_s \mathrm{w})_k$, the kth component of $B_s \mathrm{w}$ and $A_s \mathrm{w}$ respectively, by

$$(B_s \mathrm{w})_k = \sum_{j=1}^{p} |\theta'_{m_j}(x_k)|^s \, [1 - \mathrm{err}^2(\theta_{m_j}(x_k))] \, w^I(\theta_{m_j}(x_k)),$$

$$(A_s \mathrm{w})_k = \sum_{j=1}^{p} |\theta'_{m_j}(x_k)|^s \, [1 - \mathrm{err}^1(\theta_{m_j}(x_k))] \, w^I(\theta_{m_j}(x_k)).$$

Because of the fact that in all of our previous definitions, we take $0 \le k \le n$, we shall also do so in our definitions of A_s and B_s, so that these matrices have row and columns, numbered 0 through n. In the above definitions, if $\theta_{m_j}(x_k) \in [x_{r_j}, x_{r_j+1}]$, (the subinterval also depends on k, but we have omitted this dependence in the notation, thinking of k as fixed), then applying the previous definition of $w^I(y)$,

$$w^I(\theta_{m_j}(x_k)) = \frac{x_{r_j+1} - \theta_{m_j}(x_k)}{h} \mathrm{w}_{r_j} + \frac{\theta_{m_j}(x_k) - x_{r_j}}{h} \mathrm{w}_{r_j+1}.$$

To understand these formulas, note that $w^I(\theta_{m_j}(x_k))$ is just a linear combination of two components of the vector w, namely w_{r_j} and w_{r_j+1}, where x_{r_j} and x_{r_j+1} are the endpoints of the subinterval to which $\theta_{m_j}(x_k)$ belongs. Determining this subinterval for $1 \le j \le p$ and $0 \le k \le n$ are the first calculations we need to make. In the case $p = 1$, there is only one term in the sum (when $j = 1$), and since $(B_s \mathrm{w})_k = \sum_{i=0}^{n} (B_s)_{k,i} \mathrm{w}_i$, we then have

$$(B_s)_{k,r_j} = |\theta'_{m_j}(x_k)|^s \, [1 - \mathrm{err}^2(\theta_{m_j}(x_k)] \, \frac{[x_{r_j+1} - \theta_{m_j}(x_k)]}{h},$$

$$(B_s)_{k,r_j+1} = |\theta'_{m_j}(x_k)|^s \, [1 - \mathrm{err}^2(\theta_{m_j}(x_k)] \, \frac{[\theta_{m_j}(x_k) - x_{r_j}]}{h},$$

$$(B_s)_{k,i} = 0, \quad i \ne r_j, r_j + 1.$$

If $p > 1$, then for each $j = 2, \dots, p$, we modify the entries in the kth row of the matrix B_s, according to which subinterval the points $\theta_{m_j}(x_k)$ lie. If the subinterval is disjoint from the previous subintervals, then we need to modify the corresponding two columns of the kth row of the matrix B_s, which introduces two new nonzero entries. If it coincides with a previous subinterval, then we simply add to the coefficients in the two corresponding columns. We perform this procedure for each $x_k, k = 0, \dots, n$, thus generating the $n + 1$ rows of the matrix B_s. The entries of the matrix A_s are generated in a similar fashion.

An example, where we simplify the presentation by working on the interval $[0,1]$ instead of $[0,\gamma^{-1}]$, is when $h = \frac{1}{4}$, so that we have $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, and $x_4 = 1$. We only show the computations for B_s, which is a 5×5 matrix, since the computations for A_s are similar. If we consider $p = 2$, $\theta_{m_1}(x) = \frac{1}{x+3}$ and $\theta_{m_2}(x) = \frac{1}{x+5}$, then

$$\theta_{m_1}(x_0) = \frac{1}{3}, \quad \theta_{m_1}(x_1) = \frac{4}{13}, \quad \theta_{m_1}(x_2) = \frac{2}{7},$$
$$\theta_{m_1}(x_3) = \frac{4}{15}, \quad \theta_{m_1}(x_4) = \frac{1}{4},$$

$$\theta_{m_2}(x_0) = \frac{1}{5}, \quad \theta_{m_2}(x_1) = \frac{4}{21}, \quad \theta_{m_2}(x_2) = \frac{2}{11},$$
$$\theta_{m_2}(x_3) = \frac{4}{23}, \quad \theta_{m_2}(x_4) = \frac{1}{6}.$$

Note that in this case, $\theta_{m_1}(x_k) \in \left[\frac{1}{4}, \frac{1}{2}\right]$ and $\theta_{m_2}(x_k) \in \left[0, \frac{1}{4}\right]$, for $k = 0, \ldots, 4$. Although $\theta_{m_1}(x_4)$ is also in $\left[\frac{1}{4}, \frac{1}{2}\right]$, there is no ambiguity, since the only nonzero coefficient multiplies w_1 and the coefficient is the same with either choice of subinterval.

We next compute $w^I(\theta_{m_j}(x_k))$ and $\mathrm{err}^2(\theta_{m_j}(x_k))$.

$$w^I(\theta_{m_1}(x_k)) = \frac{x_2 - \theta_{m_1}(x_k)}{h}\mathrm{w}_1 + \frac{\theta_{m_1}(x_k) - x_1}{h}\mathrm{w}_2,$$
$$w^I(\theta_{m_2}(x_k)) = \frac{x_1 - \theta_{m_2}(x_k)}{h}\mathrm{w}_0 + \frac{\theta_{m_2}(x_k) - x_0}{h}\mathrm{w}_1,$$

$$\mathrm{err}^2(\theta_{m_1}(x_k))$$
$$= [x_2 - \theta_{m_1}(x_k)][\theta_{m_1}(x_k) - x_1]s(2s+1)(2\gamma^{-1} + \Gamma)^{-2}\exp\Big(-2s\frac{h}{\gamma}\Big),$$

$$\mathrm{err}^2(\theta_{m_2}(x_k))$$
$$= [x_1 - \theta_{m_2}(x_k)][\theta_{m_2}(x_k) - x_0]s(2s+1)(2\gamma^{-1} + \Gamma)^{-2}\exp\Big(-2s\frac{h}{\gamma}\Big).$$

Combining these results, we find that for $k = 0, \ldots, 4$,

$$(B_s)_{k,0} = \frac{1}{h}(|\theta'_{m_2}(x_k)|^s[1 - \mathrm{err}^2(\theta_{m_2}(x_k))][x_1 - \theta_{m_2}(x_k)]),$$

$$(B_s)_{k,1} = \frac{1}{h}(|\theta'_{m_1}(x_k)|^s[1 - \mathrm{err}^2(\theta_{m_1}(x_k))][x_2 - \theta_{m_1}(x_k)]$$
$$+ |\theta'_{m_2}(x_k)|^s[1 - \mathrm{err}^2(\theta_{m_2}(x_k))][\theta_{m_2}(x_k) - x_0]),$$

$$(B_s)_{k,2} = \frac{1}{h}|\theta'_{m_1}(x_k)|^s[1 - \mathrm{err}^2(\theta_{m_1}(x_k))][\theta_{m_1}(x_k) - x_1],$$

$$(B_s)_{k,3} = (B_s)_{k,4} = 0.$$

Returning to the general case, note that since $\mathrm{err}^i(y) = O(h^2)$ for $i = 1, 2$, all of the entries of A_s and B_s will be nonnegative, provided h is sufficiently small. However, the example given above is typical and shows that, in general, the entries of A_s and B_s will not all be strictly positive. If we define a vector w by $\mathrm{w}_k = v_s(x_k)$, then $w^I(y) = v_s^I(y)$ for all $y \in [0,1]$, and our previous inequalities show that for $0 \le k \le n$,

$$(A_s\mathrm{w})_k \le r(L_s)v_s(x_k) = r(L_s)\mathrm{w}_k, \quad (B_s\mathrm{w})_k \ge r(L_s)v_s(x_k) = r(L_s)\mathrm{w}_k.$$

Since $\mathrm{w}_k = v_s(x_k) > 0$ for $k = 0, \dots, n$, we can apply Lemma 2.2 in Section 2 about nonnegative matrices to see that

$$r(A_s) \le r(L_s) \le r(B_s).$$

As described in Section 1, if s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, then s_* is the Hausdorff dimension of the set $E_{[m_1,\dots,m_p]}$. If we can find a number s_1 such that $r(B_{s_1}) \le 1$, then $r(L_{s_1}) \le r(B_{s_1}) \le 1$, and we can conclude that $s_* \le s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \ge 1$, then $r(L_{s_2}) \ge r(A_{s_2}) \ge 1$, and we can conclude that $s_* \ge s_2$. By choosing the mesh sufficiently fine, we can make $s_1 - s_2$ small, providing a good estimate for s_*.

We can also reduce the number of computations by first iterating the maps θ_{m_i} to produce a smaller initial domain that we need to approximate. For example, if we seek the Hausdorff dimension of the set $E_{1,2}$, since $\theta_1([0,1]) = \left[\frac{1}{2}, 1\right]$ and $\theta_2([0,1]) = \left[\frac{1}{3}, \frac{1}{2}\right]$, the maps θ_1 and θ_2 map $\left[\frac{1}{3}, 1\right] \mapsto \left[\frac{1}{3}, 1\right]$, so we can restrict the problem to this subinterval. Further iterating, we see that $\theta_1\left(\left[\frac{1}{3}, 1\right]\right) = \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\theta_2\left(\left[\frac{1}{3}, 1\right]\right) = \left[\frac{1}{3}, \frac{3}{7}\right]$. Hence the maps θ_1 and θ_2 map $\left[\frac{1}{3}, \frac{3}{7}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right]$ to itself and we can further restrict the problem to this domain.

3.2. Continued fraction Cantor sets – numerical results. In this section, we report in Table 3.1 the results of the application of the algorithm described above to the computation of the Hausdorff dimension of a sample of continued fraction Cantor sets. Where the true value was known to sufficient accuracy, it is not hard to check that the rate of convergence as h is refined is $O(h^2)$, which corresponds to the theoretical result described in Remark 7.3. The upper and lower errors are computed based on the results reported in [29]. For the last five entries, we do not have independent results for the true solution correct to a sufficient number of decimal places to compute the upper and lower errors, but our results give an interval which must contain the true solution.

Although the theory developed above does not apply to higher order piecewise polynomial approximation, since one cannot guarantee that the approximate matrices have nonnegative entries, we also report in Table 3.2 and Table 3.3 the results of higher order piecewise polynomial approximation to demonstrate the promise of this approach. In this case, we only provide the results for B_s, which does not contain any corrections for the interpolation error. In a future paper we hope to prove that rigorous upper and lower bounds for the Hausdorff dimension can also be obtained when higher order piecewise polynomial approximations are used.

Table 3.1. Computation of Hausdorff dimension s of some continued fraction Cantor sets.

Set	h	lower s	upper s	low err	up err
E[1,2]	0.00010	0.53128050509989	0.53128050644980	1.18e−09	1.73e−10
	0.00005	0.53128050598142	0.53128050632077	2.96e−10	4.36e−11
E[1,3]	0.00010	0.45448907685942	0.45448907780427	8.02e−10	1.42e−10
	0.00005	0.45448907745903	0.45448907769761	2.03e−10	3.58e−11
E[1,4]	0.00010	0.41118272409575	0.41118272491153	6.79e−10	1.37e−10
	0.00005	0.41118272460331	0.41118272480924	1.71e−10	3.44e−11
E[2,3]	0.00010	0.33743678074485	0.33743678082457	6.12e−11	1.85e−11
	0.00005	0.33743678079023	0.33743678081090	1.58e−11	4.84e−12
E[2,4]	0.00010	0.30631276799370	0.30631276807670	5.91e−11	2.39e−11
	0.00005	0.30631276803924	0.30631276805816	1.35e−11	5.37e−12
E[10,11]	0.00020	0.14692123539045	0.14692123539103	3.38e−13	2.43e−13
	0.00005	0.14692123539076	0.14692123539080	1.92e−14	1.40e−14
E[100,10000]	0.00040	0.05224659263866	0.05224659263866	2.21e−15	3.50e−15
	0.00010	0.05224659263866	0.05224659263866	1.73e−16	2.71e−16
E[2,4,6,8,10]	0.00010	0.51735703083073	0.51735703098246		
	0.00005	0.51735703091123	0.51735703094801		
E[1,...,10]	0.00010	0.92573758921886	0.92573759153175		
	0.00005	0.92573759066470	0.92573759124295		
E[1,3, 5, ..., 33]	0.00010	0.77051600758209	0.77051600898599		
	0.00005	0.77051600843322	0.77051600878460		
E[2, 4, 6, ..., 34]	0.00010	0.63347197012177	0.63347197028753		
	0.00005	0.63347197021161	0.63347197025258		
E[1, ...,34]	0.00010	0.98041962337899	0.98041962562238		
	0.00005	0.98041962476506	0.98041962532582		

Table 3.2. Computation of Hausdorff dimension s of E[1,2] using higher order piecewise polynomials.

degree	h	s	error
1	0.01	0.531282991861209	2.49 e−06
2	0.02	0.531280509905738	3.63 e−09
4	0.04	0.531280506277707	5.07 e−13
5	0.05	0.531280506277198	2.44 e−15

Table 3.3. Computation of Hausdorff dimension s of E[2,4,6,8,10] using piecewise cubic polynomials.

h	s
0.100	0.517357031893604
0.050	0.517357031040157
0.020	0.517357030941730
0.010	0.517357030937109
0.005	0.517357030937029
0.002	0.517357030937019
0.001	0.517357030937018

In the computations shown using higher order piecewise polynomials, since the number of unknowns for a continuous, piecewise polynomial of degree k on n uniformly spaced subintervals of width h is given by $kn + 1$, to get a fair comparison, we have adjusted the mesh sizes so that each computation involves the same number of unknowns. For this problem, the eigenfunction v_s is smooth and the computations show a dramatic increase in the accuracy of the approximation as the degree of the approximating piecewise polynomial is increased.

3.3. An example with less regularity. For $0 \leq a \leq 1$, we consider the maps

$$\theta_1(x) = \frac{1}{3+2a}(x + ax^{7/2}), \quad \theta_2(x) = \frac{1}{3+2a}(x + ax^{7/2}) + \frac{2+a}{3+2a}, \tag{3.3}$$

which map the unit interval to itself. Both these maps $\in C^3([0,1])$, but $\notin C^4([0,1])$. We note that because of the lack of regularity, the methods of [29] and [28] cannot be applied. When $a = 0$, these maps become

$$\theta_1(x) = \frac{x}{3}, \quad \theta_2(x) = \frac{x}{3} + \frac{2}{3},$$

and the corresponding Cantor set has Hausdorff dimension

$$\frac{\ln 2}{\ln 3} \approx 0.630929753571458.$$

Our computations, shown in Table 3.4, are based on the following result, which we shall prove in subsequent sections.

Table 3.4. Computation of Hausdorff dimension s of less regular examples.

a	h	lower s	upper s	upper s - lower s
0.00	0.0001	0.630929753571456	0.630929753571458	2.00e−15
0.25	0.0001	0.691029100877742	0.691029110502742	9.63e−09
0.50	0.0001	0.733474573000780	0.733474622222678	4.92e−08
0.75	0.0001	0.767207065889322	0.767207292955631	2.27e−07
1.00	0.0001	0.796726361744928	0.796727861914648	1.50e−06

Theorem 3.2. *Let*

$$(L_s w)(x) = \sum_{j=1}^{2} |\theta_j'(x)|^s w(\theta_j(x)),$$

where θ_1 and θ_2 are given by (3.3), and we have not indicated the dependence on a in our notation. For all $s > 0$, L_s has a unique (up to normalization) strictly positive C^2 eigenfunction v_s with $L_s v_s = r_s v_s$, where $r_s > 0$ and $r_s = r(L_s)$, the spectral radius of L_s. Furthermore, the map $s \mapsto r_s$ is strictly decreasing and continuous, and for all $x_1, x_2 \in [0, 1]$, we have the estimate

$$0 < \frac{v_s''(x)}{v_s(x)} \le \left[sG_2(a) + \frac{2s^2 C_1(a)^2 \kappa(a)}{1-\kappa(a)} + \frac{sC_1(a)E_2(a)}{1-\kappa(a)} \right] [1-\kappa(a)^2]^{-1},$$

where $\kappa(a)$, $C_1(a)$, $E_2(a)$, $C_2(a)$, and $G_2(a)$ are given by (5.28), (5.29), (5.30), (5.31), and (5.32), respectively, and a is as in (3.3). Finally, the Hausdorff dimension of the Cantor set generated from the maps θ_1 and θ_2 is the unique value of s with $r_s = r(L_s) = 1$.

4. Existence of C^m positive eigenfunctions

In this section we shall describe some results concerning existence of C^m positive eigenfunctions for a class of positive (in the sense of order-preserving) linear operators. We shall later indicate how one can often obtain explicit bounds on derivatives of the positive eigenfunctions. As noted above, such estimates play a crucial role in our numerical method and therefore in obtaining rigorous estimates of Hausdorff dimension for invariant sets associated with iterated function systems. The methods we shall describe can also be applied to the important case of graph directed iterated function systems, but for simplicity we shall restrict our attention in this paper to a class of linear operators arising in the iterated function system case.

The starting point of our analysis is Theorem 5.5 in [49], which we now describe for a simple case. If H is a bounded open subset of $\mathbb{R}$ and m is a positive integer, $C^m(\bar{H})$ will denote the set of real-valued C^m maps $w \colon H \to \mathbb{R}$ such that all derivatives $D^k w$ with $0 \le k \le m$ extend continuously to $\bar{H}$. Here $D^k w = \frac{d^k w}{dx^k}$ and $C^m(\bar{H})$ is a real Banach space with

$$\|w\| = \sup\{|D^k w(x)| \colon x \in H, 0 \le k \le m\}.$$

Let $\mathcal{B}$ denote a finite index set with $|\mathcal{B}| = p$. For $b \in \mathcal{B}$, we assume

(H4.1) $g_b \in C^m(\bar{H})$ for all $b \in \mathcal{B}$ and $g_b(x) > 0$ for all $x \in \bar{H}$ and all $b \in \mathcal{B}$;

(H4.2) $\theta_b\colon H \to H$ is a C^m map for all $b \in \mathcal{B}$.

In (H4.1) and (H4.2), we always assume that $m \geq 1$.

We define

$$\Lambda\colon C^m(\bar{H}) \longrightarrow C^m(\bar{H})$$

by

$$(\Lambda(w))(x) = \sum_{b \in \mathcal{B}} g_b(x) w(\theta_b(x)). \tag{4.1}$$

For integers $\mu \geq 1$, we define

$$\mathcal{B}_\mu := \{\omega = (j_1, \dots j_\mu)\colon j_k \in \mathcal{B} \text{ for } 1 \leq k \leq \mu\}.$$

For $\omega = (j_1, \dots j_\mu) \in \mathcal{B}_\mu$, we define

$$\begin{aligned}
\omega_\mu &= \omega, \\
\omega_{\mu-1} &= (j_1, \dots j_{\mu-1}), \\
\omega_{\mu-2} &= (j_1, \dots j_{\mu-2}), \\
&\vdots \\
\omega_1 &= j_1.
\end{aligned}$$

We define

$$\theta_{\omega_{\mu-k}}(x) = (\theta_{j_{\mu-k}} \circ \theta_{j_{\mu-k-1}} \circ \dots \circ \theta_{j_1})(x),$$

so

$$\theta_\omega(x) := \theta_{\omega_\mu}(x) = (\theta_{j_\mu} \circ \theta_{j_{\mu-1}} \circ \dots \circ \theta_{j_1})(x).$$

For $\omega \in \mathcal{B}_\mu$, we define $g_\omega(x)$ inductively by

$$\begin{aligned}
g_\omega(x) &= g_{j_1}(x) && \text{if } \omega = (j_1) \in \mathcal{B} := \mathcal{B}_1, \\
g_\omega(x) &= g_{j_2}(\theta_{j_1}(x)) g_{j_1}(x) && \text{if } \omega = (j_1, j_2) \in \mathcal{B}_2, \\
g_\omega(x) &= g_{j_\mu}(\theta_{\omega_{j_{\mu-1}}}(x)) g_{\omega_{\mu-1}}(x) && \text{if } \omega = (j_1, j_2, \dots j_\mu) \in \mathcal{B}_\mu.
\end{aligned}$$

If is not hard to show (see [48], [5], and [49]) that

$$(\Lambda^{\mu}(w))(x) = \sum_{\omega \in \mathcal{B}_{\mu}} g_{\omega}(x) w(\theta_{\omega}(x)). \tag{4.2}$$

It is easy to prove (see [49]) that Λ defines a bounded linear map of $C^m(\bar{H}) \to C^m(\bar{H})$. We shall let $\hat{\Lambda}$ denote the complexification of Λ and let $\sigma(\hat{\Lambda})$ denote the spectrum of $\hat{\Lambda}$. We shall define $\sigma(\Lambda) = \sigma(\hat{\Lambda})$. If all the functions g_b and θ_b are C^N, then we can consider Λ as a bounded linear operator $\Lambda_m \colon C^m(\bar{H}) \to C^m(\bar{H})$ for $1 \leq m \leq N$, but one should note that in general $\sigma(\Lambda_m)$ will depend on m.

To obtain a useful theory for Λ, we need a further crucial assumption.

(H4.3) There exists a positive integer μ and a constant $\kappa < 1$ such that for all $\omega \in \mathcal{B}_{\mu}$ and all $x, y \in H$, $|\theta_{\omega}(x) - \theta_{\omega}(y)| \leq \kappa |x - y|$.

If we define $c = \kappa^{1/\mu} < 1$, it follows from (H4.3) that there exists a constant M such that for all $\omega \in B_{\nu}$ and all $\nu \geq 1$,

$$|\theta_{\omega}(x) - \theta_{\omega}(y)| \leq M c^{\nu} |x - y| \quad \text{for all } x, y \in H. \tag{4.3}$$

The following theorem is a special case of Theorem 5.5 in [49].

Theorem 4.1. *Let H be a bounded open subset of $\mathbb{R}$, which is a finite union of open intervals. Let $X = C^m(\bar{H})$ and assume that (H4.1), (H4.2), and (H4.3) are satisfied (where $m \geq 1$ in (H4.1) and (H4.2)) and that $\Lambda \colon X \to X$ is given by (4.1). If $Y = C(\bar{H})$, the Banach space of real-valued continuous functions $w \colon \bar{H} \to \mathbb{R}$ and $L \colon Y \to Y$ is defined by (4.1), then $r(L) = r(\Lambda) > 0$, where $r(L)$ denotes the spectral radius of L and $r(\Lambda)$ denotes the spectral radius of Λ. If $\rho(\Lambda)$ denotes the essential spectral radius of Λ (see [38],[48],[50], and [46]), then $\rho(\Lambda) \leq c^m r(\Lambda)$ where $c = \kappa^{1/\mu}$ is as in (4.3). There exists $v \in X$ such that $v(x) > 0$ for all $x \in \bar{H}$ and*

$$\Lambda(v) = rv, \quad r = r(\Lambda).$$

There exists $r_1 < r$ such that if $\xi \in \sigma(\Lambda) \setminus \{r\}$, then $|\xi| \leq r_1$; and $r = r(\Lambda)$ is an isolated point of $\sigma(\Lambda)$ and an eigenvalue of algebraic multiplicity 1. If $u \in X$ and $u(x) > 0$ for all $x \in \bar{H}$, there exists a real number $s_u > 0$ such that

$$\lim_{k \to \infty} \left(\frac{1}{r}\Lambda\right)^k (u) = s_u v, \tag{4.4}$$

where the convergence in (4.4) is in the C^m topology on X.

Remark 4.1. If l is an integer satisfying $0 \leq l \leq m$, where $m \geq 1$ is as in (H4.1) and (H4.2), it follows from (4.4) that

$$\lim_{k\to\infty} \left(\frac{1}{r}\right)^k D^l \Lambda^k(u) = s_u D^l v, \tag{4.5}$$

and

$$\lim_{k\to\infty} \left(\frac{1}{r}\right)^k \Lambda^k(u) = s_u v, \tag{4.6}$$

where the convergence in (4.5) and (4.6) is in the topology of $C(\bar{H})$, the Banach space of continuous functions $w\colon \bar{H} \to \mathbb{R}$.

It follows from (4.5) and (4.6) that for any integer l with $0 \leq l \leq m$,

$$\lim_{k\to\infty} \frac{(D^l \Lambda^k(u))(x)}{\Lambda^k(u)(x)} = \frac{(D^l(v))(x)}{v(x)}, \tag{4.7}$$

where the convergence in (4.7) is uniform in $x \in \bar{H}$. If we choose $u(x) = 1$ for all $x \in \bar{H}$, it follows from (4.2) that for all integers l with $0 \leq l \leq m$, we have

$$\lim_{k\to\infty} \frac{D^l(\sum_{\omega\in\mathcal{B}_k} g_\omega(x))}{\sum_{\omega\in\mathcal{B}_k} g_\omega(x)} = \frac{D^l v(x)}{v(x)}, \tag{4.8}$$

where the convergence in (4.8) is uniform in $x \in \bar{H}$. We shall use (4.8) in our further work to obtain explicit bounds on $\sup\left\{\frac{|D^l v(x)|}{v(x)} : x \in \bar{H}\right\}$.

5. Estimates for derivatives of v_s: mappings of form (1.1)

Throughout this section, we shall assume for simplicity that $H = (a_1, a_2)$ is a bounded, open interval, although it is frequently natural to take H to be the finite union of disjoint intervals. $\mathcal{B}$ will denote a finite index set. For $b \in \mathcal{B}$ and some integer $m \geq 1$, we assume

(H5.1) For each $b \in \mathcal{B}$, $g_b \in C^m(\bar{H})$, $\theta_b \in C^m(\bar{H})$, $g_b(x) > 0$ for all $x \in \bar{H}$ and $\theta_b(H) \subset H$. There exist an integer $\mu \geq 1$ and a real number $\kappa < 1$ such that for all $\omega \in \mathcal{B}_\mu := \{(b_1, b_2, \ldots, b_\mu) \colon b_j \in \mathcal{B}$ for $1 \leq j \leq \mu\}$ and for all $x, y \in \bar{H}$, $|\theta_\omega(x) - \theta_\omega(y)| \leq \kappa|x - y|$, where $\theta_\omega := \theta_{b_\mu} \circ \theta_{b_{\mu-1}} \circ \cdots \circ \theta_{b_1}$ for $\omega = (b_1, b_2, \ldots, b_\mu) \in \mathcal{B}_\mu$.

As in Section 4, we define $Y = C(\bar{H})$ and $X_m = C^m(\bar{H})$. Assuming (H5.1), we define for $s \geq 0$, a bounded linear operator $L_s: Y \to Y$ by

$$(L_s w)(x) = \sum_{b \in \mathcal{B}} [g_b(x)]^s w(\theta_b(x)). \tag{5.1}$$

As in Section 4, $L_s(X_m) \subset X_m$ and $L_s|_{X_m}$ defines a bounded linear map of X_m to X_m which we denote by Λ_s. Theorem 4.1 is now directly applicable (replace $g_b(x)$ in Theorem 4.1 by $[g_b(x)]^s$) and yields information about $\sigma(\Lambda_s)$. In particular, $r(L_s) = r(\Lambda_s) > 0$ and there exists a unique (to within normalization) strictly positive, C^m eigenfunction v_s of Λ_s with eigenvalue $\lambda_s = r(\Lambda_s)$.

If $\omega = (b_1, b_2, \ldots, b_p) \in \mathcal{B}_p$, recall that we define $g_\omega(x)$ by

$$g_\omega(x) = g_{b_p}(\theta_{b_{p-1}} \circ \theta_{b_{p-2}} \circ \cdots \circ \theta_{b_1}(x)) \ldots g_{b_3}((\theta_{b_2} \circ \theta_{b_1})(x)) g_{b_2}((\theta_{b_1}(x)) g_{b_1}(x),$$

and

$$(L_s^p w)(x) = \sum_{\omega \in \mathcal{B}_p} [g_\omega(x)]^s w(\theta_\omega(x)). \tag{5.2}$$

Notice that L_s^p is of the same form as L_s and Theorem 4.1 is also directly applicable to L_s^p. Since v_s is also an eigenfunction of L_s^p, we can also work with (5.2) instead of (5.1): $\mathcal{B}_p$ is an index set corresponding to $\mathcal{B}$, g_ω, $\omega \in \mathcal{B}_p$, corresponds to g_b, $b \in \mathcal{B}$, and θ_ω, $\omega \in \mathcal{B}_p$, corresponds to θ_b, $b \in \mathcal{B}$.

If m is as in (H5.1) and k is a positive integer with $k \leq m$, we define $D = \frac{d}{dx}$, so $(Df)(x) = f'(x)$ and $(D^k f)(x) = f^{(k)}(x)$. We are interested in obtaining estimates for

$$\sup\left\{\frac{|D^k v_s(x)|}{v_s(x)} : x \in \bar{H}\right\}. \tag{5.3}$$

We note that the estimates we shall give below can be refined as in Section 6 of [15], but for simplicity we shall omit these refinements.

First observe that Hypothesis (H5.1) implies that there exist constants $M > 0$ and $c = \kappa^{1/\mu}$ (so $c < 1$) such that for all integers $\nu \geq 1$ and all $\omega \in \mathcal{B}_\nu$, (4.3) is satisfied.

If μ is as in (H5.1), we define a constant C_1 by

$$C_1 = \sup\left\{\frac{|g'_\omega(x)|}{g_\omega(x)} : \omega \in \mathcal{B}_\mu, x \in \bar{H}\right\}. \tag{5.4}$$

A calculation shows that for all $\omega \in \mathcal{B}_\nu$, $\nu \geq 1$,

$$\frac{D[g_\omega(x)^s]}{[g_\omega(x)]^s} = s\frac{g'_\omega(x)}{g_\omega(x)}, \tag{5.5}$$

so

$$\sup\left\{\frac{|D[g_\omega(x)^s]|}{[g_\omega(x)]^s} : \omega \in \mathcal{B}_\mu, x \in \bar{H}\right\} = sC_1.$$

We begin by considering (5.3) for the case $k = 1$. In our applications, we shall only need the case $s > 0$, so we shall restrict our attention to this case.

Theorem 5.1. *Assume that* (H5.1) *is satisfied, let μ, m, and κ be as in* (H5.1) *and let C_1 be as in* (5.4)*, For $s > 0$, let v_s denote the unique* (*to within normalization*) *strictly positive eigenfunction of $\Lambda_s := L_s|_{X_m}$. Then we have*

$$\sup\left\{\frac{|v'_s(x)|}{v_s(x)} : x \in \bar{H}\right\} \leq \frac{C_1 s}{1-\kappa} := M_1. \tag{5.6}$$

If $\delta \in \{0, 1\}$ and $(-1)^\delta \frac{g'_\omega(x)}{g_\omega(x)} \leq 0$ for all $\omega \in \mathcal{B}_\nu$, all $\nu \geq 1$ and all $x \in \bar{H}$, then $(-1)^\delta v'_s(x) \leq 0$ for all $x \in \bar{H}$ and all $s > 0$.

Proof. Recall from Section 4 that v_s is (after normalization) also the unique eigenfunction of Λ_s^μ with eigenvalue r^μ, where $r = r(\Lambda_s)$ and r^μ is the spectral radius of Λ_s^μ. Define $\hat{M}_1$ by

$$\hat{M}_1 = \sup\left\{\frac{|v'_s(x)|}{v_s(x)} : x \in \bar{H}\right\}.$$

We shall prove that $\hat{M}_1 \leq M_1$. For notational convenience we write for $\omega \in \mathcal{B}_\mu$

$$f_\omega(x) = [g_\omega(x)]^s v_s(\theta_\omega(x)).$$

Then we see that

$$\left|\frac{\lambda_s v'_s(x)}{\lambda_s v_s(x)}\right| = \frac{|\lambda_s v'_s(x)|}{\lambda_s v_s(x)} = \frac{\left|\sum_{\omega\in\mathcal{B}_\mu} f'_\omega(x)\right|}{\sum_{\omega\in\mathcal{B}_\mu} f_\omega(x)} \leq \frac{\sum_{\omega\in\mathcal{B}_\mu} |f'_\omega(x)|}{\sum_{\omega\in\mathcal{B}_\mu} f_\omega(x)}. \tag{5.7}$$

A calculation shows that

$$\frac{f'_\omega(x)}{f_\omega(x)} = s\frac{g'_\omega(x)}{g_\omega(x)} + \frac{v'_s(\theta_\omega(x))\theta'_\omega(x)}{v_s(\theta_\omega(x))},$$

so

$$\frac{|f'_\omega(x)|}{f_\omega(x)} \leq sC_1 + \hat{M}_1\kappa$$

and

$$\frac{\sum_{\omega\in\mathcal{B}_\mu}|f'_\omega(x)|}{\sum_{\omega\in\mathcal{B}_\mu} f_\omega(x)} \le sC_1 + \widehat{M}_1\kappa. \tag{5.8}$$

Taking the maximum of the left hand side of (5.7), we deduce from (5.7) and (5.8) that

$$\widehat{M}_1 \le sC_1 + \widehat{M}_1\kappa \tag{5.9}$$

and (5.9) implies that

$$\widehat{M}_1 \le \frac{sC_1}{1-\kappa} = M_1. \qquad \square$$

Throughout the remainder of this section, C_1 will be as in (5.4) and M_1 will be as in (5.6). Assuming that m and μ are as in (H5.1) and $m \ge 2$, it will also be convenient to define constants C_2, E_2, and K_2 by

$$C_2 = \sup\left\{\frac{|g''_\omega(x)|}{g_w(x)} : \omega \in \mathcal{B}_\mu, x \in \bar{H}\right\}, \tag{5.10}$$

$$E_2 = \sup\left\{|\theta''_\omega(x)| : \omega \in \mathcal{B}_\mu, x \in \bar{H}\right\},$$

$$K_2 = \sup\left\{\frac{|g''_\omega(x)g_w(x) - (1-s)[g'_\omega(x)]^2|}{[g_\omega(x)]^2} : \omega \in \mathcal{B}_\mu, x \in \bar{H}\right\}.$$

Notice that we always have the estimate $K_2 \le C_2 + |1-s|C_1^2$, but sometimes more precise estimates for K_2 can be obtained.

Theorem 5.2. *Assume that* (H5.1) *is satisfied with* $m \ge 2$ *and let* μ, m, *and* κ *be as in* (H5.1). *Assume that* $s > 0$ *and let* C_1, M_1, C_2, E_2, *and* K_2 *be as defined above. Let* v_s *denote the unique* (*to within normalization*) *strictly positive eigenfunction of* $\Lambda_s : X_m \to X_m$ *with eigenvalue* $r(\Lambda_s)$. *Then we have*

$$\sup\left\{\frac{|v''_s(x)|}{v_s(x)} : x \in \bar{H}\right\} \le M_2, \tag{5.11}$$

where

$$M_2 := \frac{sK_2 + 2sC_1M_1\kappa + M_1E_2}{1-\kappa^2}. \tag{5.12}$$

Proof. As in the proof of Theorem 5.1, for $\omega \in \mathcal{B}_\mu$, let $f_\omega(x) = [g_\omega(x)]^s v_s(\theta_\omega(x))$ and observe that

$$\left|\frac{\lambda_s v_s''(x)}{\lambda_s v_s(x)}\right| = \frac{|\lambda_s v_s''(x)|}{\lambda_s v_s(x)} = \frac{\left|\sum_{\omega\in\mathcal{B}_\mu} f_\omega''(x)\right|}{\sum_{\omega\in\mathcal{B}_\mu} f_\omega(x)} \le \frac{\sum_{\omega\in\mathcal{B}_\mu} |f_\omega''(x)|}{\sum_{\omega\in\mathcal{B}_\mu} f_\omega(x)}. \tag{5.13}$$

A calculation shows that

$$\begin{aligned}\frac{f_\omega''(x)}{f_\omega(x)} &= \Big[s(s-1)\Big(\frac{g_\omega'(x)}{g_\omega(x)}\Big)^2 + s\frac{g_\omega''(x)}{g_\omega(x)}\Big] + 2s\Big[\frac{g_\omega'(x)}{g_\omega(x)}\frac{v_s'(\theta_\omega(x))\theta_\omega'(x)}{v_s(\theta_\omega(x))}\Big]\\ &\quad + \Big[\frac{v_s''(\theta_\omega(x))}{v_s(\theta_\omega(x))}(\theta_\omega'(x))^2 + \frac{v_s'(\theta_\omega(x))}{v_s(\theta_\omega(x))}(\theta_\omega''(x))\Big].\end{aligned} \tag{5.14}$$

If we define $\hat{M}_2 = \sup\{\frac{|v_s''(x)|}{v_s(x)} : x \in \bar{H}\}$, we obtain from (5.14) that

$$\frac{|f_\omega''(x)|}{f_\omega(x)} \le sK_2 + 2s[C_1 M_1 \kappa] + \hat{M}_2\kappa^2 + M_1 E_2, \tag{5.15}$$

and using (5.15) and (5.13), we see that

$$\hat{M}_2 \le sK_2 + 2sC_1M_1\kappa + M_1E_2 + \hat{M}_2\kappa^2,$$

which implies that $\hat{M}_2 \le M_2$ (defined in (5.12)). □

Remark 5.1. If one has obtained bounds for $\sup\{\frac{|D^j v_s(x)|}{v_s(x)} : x \in \bar{H}\}$ for $1 \le j \le k$, it is not hard to show that the kind of argument in the proof of Theorem 5.2 can be used to estimate $\sup\{\frac{|D^{k+1} v_s(x)|}{v_s(x)} : x \in \bar{H}\}$.

Rather than give a formal proof of the general case, we shall restrict ourselves here to obtaining an estimate for $\sup\{\frac{|D^3 v_s(x)|}{v_s(x)} : x \in \bar{H}\}$. To state our theorem, it will be convenient to introduce further constants C_3, E_3, and K_3:

$$C_3 = \sup\Big\{\frac{|D^3 g_\omega(x)|}{g_w(x)} : \omega \in \mathcal{B}_\mu, x \in \bar{H}\Big\},$$

$$E_3 = \sup\Big\{|D^3\theta_\omega(x)| : \omega \in \mathcal{B}_\mu, x \in \bar{H}\Big\},$$

$$\begin{aligned}K_3 = \sup\Big\{\frac{1}{[g_\omega(x)]^3}(|(s-1)(s-2)[g_\omega'(x)]^3 &\\ + 3(s-1)g_\omega(x)g_\omega'(x)g_\omega''(x) &\\ + [g_\omega(x)]^2 g_\omega'''(x)|)\Big\},&\end{aligned}$$

where the supremum is taken over $\omega \in \mathcal{B}_\mu$ and $x \in \bar{H}$.

A crude estimate for K_3 in terms of C_1, C_2, and C_3 can be given:

$$K_3 \le |s-1||s-2|C_1^3 + 3|s-1|C_1C_2 + C_3.$$

However, better estimates are frequently available.

Theorem 5.3. *Assume that (H5.1) is satisfied with $m \ge 3$ and let μ, m, and κ be as in (H5.1). Assume that $s > 0$ and let C_1, M_1, C_2, E_2, K_2, M_2, C_3, E_3, and K_3 be as defined above. If v_s is the normalized strictly positive eigenfunction of $\Lambda_s \colon X_m \to X_m$ with eigenvalue $r(\Lambda_s)$, then we have*

$$\sup\left\{\frac{|D^3 v_s(x)|}{v_s(x)} \colon x \in \bar{H}\right\} \le M_3, \tag{5.16}$$

where

$$(1-\kappa^3)M_3$$
$$= (sK_3 + 3sK_2M_1\kappa + 3sC_1(M_2\kappa^2 + M_1E_2) + 3M_2\kappa E_2 + M_1E_3) := S.$$

Proof. Again set $f_\omega(x) = [g_\omega(x)]^s v_s(\theta_\omega(x))$ and define

$$\hat{M}_3 = \sup\left\{\frac{|D^3 v_s(x)|}{v_s(x)} \colon x \in \bar{H}\right\}.$$

As in the proof of Theorem 5.2, we find that

$$\frac{D^3 v_s(x)}{v_s(x)} = \frac{\sum_{\omega\in\mathcal{B}_\mu} D^3 f_\omega(x)}{\sum_{\omega\in\mathcal{B}_\mu} f_\omega(x)}. \tag{5.17}$$

A calculation shows that

$$\frac{D^3 f_\omega(x)}{f_\omega(x)} = \frac{D^3[g_\omega(x)^s]}{g_\omega(x)^s} + 3\frac{D^2[g_\omega(x)^s]}{g_\omega(x)^s}\frac{D[v_s(\theta_\omega(x))]}{v_s(\theta_\omega(x))}$$
$$+ 3\frac{D[g_\omega(x)^s]}{g_\omega(x)^s}\frac{D^2[v_s(\theta_\omega(x))]}{v_s(\theta_\omega(x))} + \frac{D^3[v_s(\theta_\omega(x))]}{v_s(\theta_\omega(x))}.$$

Further tedious calculations give

$$\frac{|D^3[g_\omega(x)^s]|}{g_\omega(x)^s} \le sK_3,$$

$$3\frac{|D^2[g_\omega(x))^s]|}{g_\omega(x)^s}\frac{|D[v_s(\theta_\omega(x))]|}{v_s(\theta_\omega(x))} \le 3sK_2M_1\kappa,$$

$$3\frac{|D[g_\omega(x)^s]|}{g_\omega(x)^s}\frac{|D^2[v_s(\theta_\omega(x))]|}{v_s(\theta_\omega(x))} \le 3sC_1(M_2\kappa^2 + M_1E_2),$$

$$\frac{|D^3[v_s(\theta_\omega(x))]|}{v_s(\theta_\omega(x))} \le 3M_2\kappa E_2 + M_1E_3 + \hat{M}_3\kappa^3.$$

It follows that $|D^3 f_\omega(x)| \leq (S + \widehat{M}_3\kappa^3) f_\omega(x)$, where S is as in the statement of Theorem 5.3. This proves that the absolute value of the right side of (5.17) is less than or equal to $S + \widehat{M}_3\kappa^3$. Taking the supremum of the left hand side of (5.17) for $x \in \bar{H}$ gives $\widehat{M}_3 \leq S + \widehat{M}_3\kappa^3$, which implies (5.16). $\square$

Theorems 5.1–5.3 are crude. If one has more information about the coefficients $g_b(\cdot)$ and the maps $\theta_b(\cdot)$, $b \in \mathcal{B}$, one can frequently obtain much sharper results. An example is provided by the following theorem.

Theorem 5.4. *Assume that* (H5.1) *is satisfied with* $m \geq 2$. *Assume also that* $\theta_b'(u) \geq 0$, $\theta_b''(u) \geq 0$, $g_b'(u) \geq 0$, $g_b''(u) \geq 0$, *and*

$$g_b''(u) g_b(u) - (1-s)[g_b'(u)]^2 \geq 0 \tag{5.18}$$

for all $b \in \mathcal{B}$, *for all* $u \in H$, *and for a given positive real number* s. *If* v_s *is the strictly positive* C^m *eigenfunction of* Λ_s, *it follows that for all* $u \in \bar{H}$

$$v_s'(u) \geq 0 \quad \textit{and} \quad v_s''(u) \geq 0.$$

If, in addition, there exists a set $F \subset \bar{H}$ *(possibly empty) such that for all* $u \in \bar{H} \setminus F$ *and all* $b \in \mathcal{B}$, $g_b'(u) > 0$ *and strict inequality holds in* (5.18), *then for all* $u \in \bar{H} \setminus F$,

$$v_s'(u) > 0 \quad \textit{and} \quad v_s''(u) > 0.$$

Proof. For $\nu \geq 1$, let $\omega = (b_1, b_2, \ldots, b_\nu)$ denote a fixed element of $\mathcal{B}_\nu$ and for $0 \leq k \leq \nu$, define $\xi_0(x) = x$, $\xi_1(x) = \theta_{b_1}(x)$ and generally $\xi_k(x) = (\theta_{b_k} \circ \theta_{b_{k-1}} \circ \cdots \circ \theta_{b_1}(x))$. We leave to the reader the simple proof that $\xi_k'(x) \geq 0$ and $\xi_k''(x) \geq 0$ for all $x \in \bar{H}$ and $0 \leq k \leq \nu$. Using (5.5), a straightforward calculation yields

$$\frac{D[g_\omega(x)^s]}{g_\omega(x)^s} = s\frac{g_\omega'(x)}{g_\omega(x)} = s\sum_{k=0}^{\nu-1} \frac{g_{b_{k+1}}'(\xi_k(x))\xi_k'(x)}{g_{b_{k+1}}(\xi_k(x))} \geq s\frac{g_{b_1}'(x)}{g_{b_1}(x)} \geq 0. \tag{5.19}$$

Using (4.8) and taking the limit as $\nu \to \infty$, we conclude that $\frac{v_s'(x)}{v_s(x)} \geq 0$ for all $x \in \bar{H}$. If, in addition, there exists a set F as in the statement of Theorem 5.4 and if $x \notin F$, it follows that

$$\inf\left\{s\frac{g_b'(x)}{g_b(x)} : b \in \mathcal{B}\right\} := s\delta_1(x) > 0,$$

so (5.19) then implies that

$$\frac{D[g_\omega(x)^s]}{g_\omega(x)^s} \geq s\delta_1(x).$$

Again using (4.8) and letting $\nu \to \infty$, we conclude that $v_s'(x) \geq s\delta_1(x) > 0$ for all $x \in \bar{H} \setminus F$.

For $\omega = (b_1, b_2, \dots, b_\nu) \in \mathcal{B}_\nu$, we obtain from (5.19) that

$$D[g_\omega(x)^s] = s g_\omega(x)^s \sum_{j=0}^{\nu-1} \frac{g'_{b_{j+1}}(\xi_j(x))\xi'_j(x)}{g_{b_{j+1}}(\xi_j(x))} := s g_\omega(x)^s \sum_{j=0}^{\nu-1} T_j(x)$$

and $D[g_\omega(x)] = g_\omega(x) \sum_{j=0}^{\nu-1} T_j(x)$. A calculation now gives

$$\begin{aligned} D^2[g_\omega(x)^s] &= sD\Big[g_\omega(x)^s \sum_{j=0}^{\nu-1} T_j(x)\Big] \\ &= s\Big[s g_\omega(x)^s \Big(\sum_{j=0}^{\nu-1} T_j(x)\Big)^2 + g_\omega(x)^s \sum_{j=0}^{\nu-1} D(T_j(x))\Big]. \end{aligned} \tag{5.20}$$

Because $T_j(x) \geq 0$ for all $x \in \bar{H}$ and $1 \leq j \leq \nu - 1$,

$$\Big(\sum_{j=0}^{\nu-1} T_j(x)\Big)^2 \geq \sum_{j=0}^{\nu-1} (T_j(x))^2 = \sum_{j=0}^{\nu-1} \frac{[g'_{b_{j+1}}(\xi_j(x))]^2(\xi'_j(x))^2}{[g_{b_{j+1}}(\xi_j(x))]^2}.$$

A calculation gives

$$\begin{aligned} \sum_{j=0}^{\nu-1} D(T_j(x)) = &\sum_{j=0}^{\nu-1} \frac{[g''_{b_{j+1}}(\xi_j(x))(\xi'_j(x))^2 + g'_{b_{j+1}}(\xi_j(x))\xi''_j(x)]g_{b_{j+1}}(\xi_j(x))}{[g_{b_{j+1}}(\xi_j(x))]^2} \\ &- \sum_{j=0}^{\nu-1} [T_j(x)]^2. \end{aligned} \tag{5.21}$$

Combining (5.20) and (5.21) and noticing that all terms in the summation are nonnegative, we find that

$$\begin{aligned} &D^2[g_\omega(x)^s] \\ &\geq s[g_\omega(x)]^s \sum_{j=0}^{\nu-1} (g''_{b_{j+1}}(\xi_j(x))g_{b_{j+1}}(\xi_j(x)) - (1-s)[g'_{b_{j+1}}(\xi_j(x))]^2) \\ &\qquad (\xi'_j(x))^2[g_{b_{j+1}}(\xi_j(x))]^{-2}. \end{aligned} \tag{5.22}$$

Since we assume that $g''_b(u)g_b(u) - (1-s)[g'_b(u)]^2 \geq 0$ for all $u \in \bar{H}$ and $b \in \mathcal{B}$, we find that for all $\omega \in \mathcal{B}_\nu$, $x \in \bar{H}$, $D^2[g_\omega(x)^s] \geq 0$. Letting $\nu \to \infty$ and using (4.8), we derive that $v''_s(x) \geq 0$ for all $x \in \bar{H}$.

If a set $F \subset H$ exists such that strict inequality holds in (5.18) for all $b \in \mathcal{B}$ and all $x \in \bar{H} \setminus F$, then by only taking the term $j = 0$ in the summation in (5.22), we find that there is a number $\delta_2(x;s) > 0$ for $x \in \bar{H} \setminus F$ and $s > 0$ such that

$$\frac{D^2[g_\omega(x)^s]}{g_\omega(x)^s} \geq \delta_2(x;s).$$

Again, using (4.8) and letting $\nu \to \infty$, this implies that for $x \in \bar{H} \setminus F$,

$$\frac{v_s''(x)}{v_s(x)} \geq \delta_2(x;s) > 0,$$

which completes the proof. □

Remark 5.2. An examination of the proof of Theorem 5.4 shows that we have proved that for all $x \in \bar{H}$, for all $\nu \geq 1$, and for all $\omega \in B_\nu$, $\theta_\omega'(x) \geq 0$, $\theta_\omega''(x) \geq 0$, $g_\omega'(x) \geq 0$, $g_\omega''(x) \geq 0$, and $D^2[g_\omega(x)^s] \geq 0$. Because

$$D^2[g_\omega(x)^s] = s g_\omega(x)^{s-2}(g_\omega''(x) g_\omega(x) - (1-s)[g_\omega'(x)]^2),$$

we also see that $g_\omega''(x) g_\omega(x) - (1-s)[g_\omega'(x)]^2 \geq 0$ for all $\omega \in B_\nu$, all $\nu \geq 1$, and all $x \in \bar{H}$. If the constants C_1, C_2, M_1, κ, and E_2 are defined as above in this section, one obtains immediately that for all $x \in \bar{H}$,

$$0 \leq \frac{v_s'(x)}{v_s(x)} \leq \frac{C_1 s}{1-\kappa}.$$

An examination of the proof of Theorem 5.2 yields the following refinement of (5.11) and (5.12).

$$0 \leq \frac{v_s''(x)}{v_s(x)} \leq \left[s G_2 + 2 s^2 C_1^2 \frac{\kappa}{1-\kappa} + s C_1 E_2 \frac{1}{1-\kappa} \right] \left[\frac{1}{1-\kappa^2} \right], \tag{5.23}$$

where

$$G_2 = \max \left\{ \frac{g_\omega''(x) g_\omega(x) - (1-s)[g_\omega'(x)]^2}{g_\omega(x)^2} : \omega \in \mathcal{B}_\mu, x \in \bar{H} \right\}. \tag{5.24}$$

Example. To illustrate the methods of this section, we consider a simple example which nevertheless has some interest because of a failure of smoothness which makes techniques in [29] inapplicable. We shall always assume that $0 \leq a \leq 1$ and define

$$\theta_1(x) = \frac{1}{3+2a}(x + a x^{7/2}), \quad \theta_2(x) = \theta_1(x) + \frac{2+a}{3+2a},$$

so $\theta_j\colon [0,1] \to [0,1]$, $\theta_1(0) = 0$, and $\theta_2(1) = 1$. For simplicity we suppress the dependence of $\theta_j(x)$ on a in our notation. If $\mathcal{B} = \{1,2\}$ and $a > 0$ and $\omega = (j_1, j_2, \dots, j_\nu) \in \mathcal{B}_\nu$, notice that $D^3\theta_\omega(x)$ is defined and Hölder continuous for all $x \in [0,1]$; but if $j_1 = 1$, $D^4\theta_\omega(x)$ is not defined at $x = 0$. Using that $0 \le a \le 1$, one can check that $0 < \theta_j'(x) < 1$ for $0 \le x \le 1$; and it follows that there exists a unique compact, nonempty set $J_a \subset [0,1]$ such that $J_a = \theta_1(J_a) \cup \theta_2(J_a)$. Note that J_0 is the *middle thirds* Cantor set.

For $a \in [0,1]$ fixed, and $0 < s$, let $X = C^2[0,1]$ and $Y = C[0,1]$, and define

$$g_1(x) := g_2(x) := g(x) := \theta_1'(x) = \frac{1}{3+2a}\Big(1 + \frac{7}{2}ax^{5/2}\Big).$$

As in Section 1, define $\Lambda_s\colon X \to X$ and $L_s\colon Y \to Y$ by the same formula:

$$(\Lambda_s(w))(x) = g(x)^s[w(\theta_1(x)) + w(\theta_2(x))]. \tag{5.25}$$

Theorem 4.1 implies that $r(L_s) = r(\Lambda_s)$; and it follows, for example, from theorems in [50] that the Hausdorff dimension of J_a is the unique value of s, $0 < s \le 1$, for which $r(\Lambda_s) = 1$.

If $w \in Y$ is a nonnegative function, we have that

$$\begin{aligned}(L_s(w))(x) &\ge \Big(\frac{1}{3+2a}\Big)^s[w(\theta_1(x)) + w(\theta_2(x))]\\ &\ge \Big(\frac{1}{5}\Big)^s[w(\theta_1(x)) + w(\theta_2(x))].\end{aligned}$$

If $u(x) = 1$ for $0 \le x \le 1$, it follows that

$$L_s(u) \ge \Big(\frac{1}{5}\Big)^s(2u),$$

which implies that $r(L_s) \ge 2\big(\frac{1}{5}\big)^s$. If log denotes the natural logarithm and $0 \le s < \frac{\log(2)}{\log(5)} \approx 0.4307$, it follows that $r(L_s) > 1$. Thus if one is only interested in s with $r(L_s) \le 1$, one may restrict attention to $s \ge \frac{\log(2)}{\log(5)}$.

In order to apply Theorem 5.4, we must determine a range of $s > 0$ such that

$$g''(x)g(x) - (1-s)[g'(x)]^2 > 0, \quad 0 < x \le 1. \tag{5.26}$$

The other hypotheses of Theorem 5.4 can be trivially verified. A calculation gives, for $0 < x \le 1$ that

$$\begin{aligned}&g''(x)g(x) - (1-s)[g'(x)]^2\\ &\quad= \frac{1}{(3+2a)^2}\Big(\frac{7a}{2}\Big)\Big(\frac{5}{2}\Big)x^{1/2}\Big[\Big(\frac{3}{2}\Big) + \Big(\frac{7a}{2}\Big)\Big(-1 + \frac{5}{2}s\Big)x^{5/2}\Big].\end{aligned}$$

Assuming that $a > 0$ and noting that $0 < u := x^{5/2} \leq 1$ if and only if $0 < x \leq 1$, we see that (5.26) is satisfied if and only if

$$\Big(\frac{3}{2}\Big) + \Big(\frac{7a}{2}\Big)\Big(-1 + \frac{5}{2}s\Big)u > 0 \quad \text{for } 0 \leq u \leq 1,$$

which is equivalent to

$$\Big(\frac{3}{2}\Big) + \Big(\frac{7a}{2}\Big)\Big(-1 + \frac{5}{2}s\Big) > 0.$$

The latter inequality is certainly satisfied for $0 < a \leq \frac{3}{7}$ and $s > 0$; and if $\frac{3}{7} \leq a \leq 1$, the inequality is satisfied for $s > \frac{2}{5}\big[1 - \frac{3}{7a}\big]$. Thus we conclude that for $0 \leq a \leq 1$, $0 < x \leq 1$, and $s > 0$, (5.26) is satisfied if and only if

$$s > \frac{2}{5}\Big[1 - \frac{3}{7a}\Big]. \tag{5.27}$$

It follows from Theorem 5.4 that if $0 < a \leq 1$, $s > 0$ and (5.27) is satisfied, then $v_s'(x) > 0$ and $v_s''(x) > 0$ for $0 < x \leq 1$.

It remains to apply Theorems 5.1 and 5.2 to our example. We assume, in the notation of (H5.1) that $\mu = 1$ and $m \geq 2$. The eigenfunction $v_s(\cdot)$ for (5.25) depends on the parameter a, although this is not indicated in our notation, and, of course our various constants depend on a. Since $\theta_j'(x) = g(x)$, the constant $\kappa = \kappa(a)$ in (H5.1) is given by

$$\kappa(a) = \max\{g(x) : 0 \leq x \leq 1\} = \frac{2 + 7a}{6 + 4a} < 1. \tag{5.28}$$

The constant $C_1 = C_1(a)$ in Theorem 5.1 is defined by

$$\begin{aligned} C_1 = C_1(a) &= \sup\Big\{\Big[\frac{7a}{2}\frac{5}{2}x^{3/2}\Big]\Big[1 + \frac{7a}{2}x^{5/2}\Big]^{-1} : 0 \leq x \leq 1\Big\} \\ &= \sup\Big\{\Big[\frac{7a}{2}\frac{5}{2}u^{3}\Big]\Big[1 + \frac{7a}{2}u^{5}\Big]^{-1} : 0 \leq u \leq 1\Big\}. \end{aligned}$$

An elementary but tedious calculus argument, which we leave to the reader, yields

$$C_1(a) = \begin{cases} \dfrac{7a}{2}\dfrac{5}{2}\Big[1 + \dfrac{7}{2}a\Big]^{-1} & \text{for } 0 < a \leq \dfrac{3}{7}, \\ \dfrac{7a}{2}\Big(\dfrac{3}{7a}\Big)^{3/5} & \text{for } \dfrac{3}{7} \leq a \leq 1. \end{cases} \tag{5.29}$$

It follows from Theorems 5.1 and (4.3) that for $0 < x \leq 1$,

$$\begin{aligned} 0 &< \frac{v_s'(x)}{v_s(x)} \\ &\leq sC_1(a)[1-\kappa(a)]^{-1} \\ &= sC_1(a)\frac{6+4a}{4-3a} := M_1(a). \end{aligned}$$

An easy calculation also yields that

$$\max\{\theta_1''(x) : 0 \leq x \leq 1\} = \frac{7a}{2}\frac{5}{6+4a} := E_2(a). \tag{5.30}$$

By definition (see (5.10) with $\mu = 1$) we have that

$$\begin{aligned} C_2 := C_2(a) &= \sup\left\{\frac{g''(x)}{g(x)} : 0 \leq x \leq 1\right\} \\ &= \frac{15}{4}\sup\left\{\frac{7a}{2}x^{1/2}\left[1+\frac{7a}{2}x^{5/2}\right]^{-1} : 0 \leq x \leq 1\right\} \\ &= \frac{15}{4}\sup\left\{\frac{7a}{2}u\left[1+\frac{7a}{2}u^5\right]^{-1} : 0 \leq u \leq 1\right\}, \end{aligned}$$

and a simple calculus exercise yields

$$C_2(a) = \begin{cases} \frac{15}{4}\frac{7a}{2}\left[1+\frac{7a}{2}\right]^{-1} & \text{for } 0 < a \leq \frac{1}{14}, \\ 3\left(\frac{1}{4}\right)^{1/5}\left(\frac{7a}{2}\right)^{4/5} & \text{for } \frac{1}{14} \leq a \leq 1. \end{cases} \tag{5.31}$$

Using (5.23) and (5.24), we now find that for $\frac{8}{35} < s \leq 1$, $0 < a \leq 1$, and $0 < x \leq 1$, we have

$$0 < \frac{v_s''(x)}{v_s(x)} \leq \left[sG_2(a) + \frac{2s^2C_1(a)^2\kappa(a)}{1-\kappa(a)} + \frac{sC_1(a)E_2(a)}{1-\kappa(a)}\right]\left[1-\kappa(a)^2\right]^{-1},$$

where $\kappa(a)$, $C_1(a)$, $E_2(a)$, and $C_2(a)$ are given by (5.28), (5.29), (5.30), and (5.31), respectively, and $G_2(a)$ is given by

$$G_2(a) = \max_{0\leq x\leq 1}\frac{g''(x)g(x)-(1-s)[g'(x)]^2}{g(x)^2} < C_2(a). \tag{5.32}$$

Equation (5.27) implies that $G_2(a) > 0$ for $0 < a \leq 1$ if we assume that $\frac{8}{35} < s \leq 1$.

6. Estimates for derivatives of v_s: the case of Möbius transformations

When the maps θ_b, $b \in \mathcal{B}$ are Möbius transformations, one can obtain much sharper estimates for $\max\left\{\frac{D^k v_s(x)}{v_s(x)} \colon x \in \bar{H}\right\}$ than were available in Section 5.

We shall be interested in the one dimensional case, and our maps will eventually be of the form $\theta_b(x) := \frac{1}{x+b}$, where $b > 0$. The special case where $\mathcal{B}$ is a subset of the positive integers has been of great interest because of connections with continued fractions. See, for example, [5], [7], [8], [10], [11], [20], [21], [23], [24], and [25]. However, for our immediate purposes, nothing is gained by restricting to $\mathcal{B} \subset \mathbb{N}$.

Lemma 6.1. *Let $\mathcal{B}$ denote a finite collection of complex numbers b such that $\mathrm{Re}(b) \geq \gamma > 0$ for all $b \in \mathcal{B}$. For $b \in \mathcal{B}$, define $M_b = \left(\begin{smallmatrix} 0 & 1 \\ 1 & b \end{smallmatrix}\right)$ and $\theta_b(z) = \frac{1}{z+b}$ for $\mathrm{Re}(z) \geq 0$. Let b_j, $j \geq 1$, denote a sequence of elements of $\mathcal{B}$. Then for $n \geq 1$, we have*

$$M_{b_1} M_{b_2} \dots M_{b_n} = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} \tag{6.1}$$

and

$$M_{b_n} M_{b_{n-1}} \dots M_{b_1} = \begin{pmatrix} A_{n-1} & B_{n-1} \\ A_n & B_n \end{pmatrix},$$

where $A_0 = 0$, $A_1 = 1$, $B_0 = 1$, $B_1 = b_1$ and for $n \geq 2$,

$$A_{n+1} = A_{n-1} + b_{n+1} A_n \quad \text{and} \quad B_{n+1} = B_{n-1} + b_{n+1} B_n. \tag{6.2}$$

If

$$G := \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geq 0\}$$

and

$$D_{\gamma^{-1}} := \{z \in \mathbb{C} \colon |z - \gamma^{-1}| \leq \gamma^{-1}\},$$

then for all $b \in \mathcal{B}$,

$$\theta_b(G) \subset D_{\gamma^{-1}}. \tag{6.3}$$

Also, we have for all $z \in G$, $(\theta_{b_1} \circ \theta_{b_2} \circ \dots \circ \theta_{b_n})(z) \in D_{\gamma^{-1}}$ and

$$(\theta_{b_1} \circ \theta_{b_2} \circ \dots \circ \theta_{b_n})(z) = \frac{A_{n-1} z + A_n}{B_{n-1} z + B_n}, \tag{6.4}$$

and

$$(\theta_{b_n} \circ \theta_{b_{n-1}} \circ \dots \circ \theta_{b_1})(z) = \frac{A_{n-1} z + B_{n-1}}{A_n z + B_n}. \tag{6.5}$$

For all $n \geq 0$, $B_n \neq 0$ *and* $\mathrm{Re}\left(\frac{B_{n+1}}{B_n}\right) \geq \gamma$, *while for all* $n \geq 1$, $A_n \neq 0$ *and* $\mathrm{Re}\left(\frac{A_{n+1}}{A_n}\right) \geq \gamma$. *For all* $b, c \in \mathcal{B}$, $\theta_b \circ \theta_c|_G$ *is a Lipschitz map* (*with respect to the Euclidean norm on* $\mathbb{C}$) *and*

$$|\theta_b(\theta_c(z)) - \theta_b(\theta_c(w))| \leq \frac{1}{4\gamma^2}|z - w|, \quad \textit{for all } z, w \in G. \tag{6.6}$$

Proof.

$$M_{b_1} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} = \begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix}.$$

We argue by induction and assume that (6.1) is satisfied for some $n \geq 1$. Then we obtain

$$\begin{aligned} M_{b_1} M_{b_2} \dots M_{b_n} M_{b_{n+1}} &= \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} A_n & A_{n-1} + b_{n+1} A_n \\ B_n & B_{n-1} + b_{n+1} B_n \end{pmatrix} \\ &= \begin{pmatrix} A_n & A_{n+1} \\ B_n & B_{n+1} \end{pmatrix}, \end{aligned}$$

which completes the inductive proof. The formula for $M_{b_n} M_{b_{n-1}} \dots M_{b_1}$ follows by taking the transpose of the formula for $M_{b_1} M_{b_2} \dots M_{b_n}$.

Equations (6.4) and (6.5) are now standard results for Möbius transformations. If $z \in G$, $z + b \in \{w : \mathrm{Re}(w) \geq \gamma\}$. A standard exercise shows that the map $w \mapsto \frac{1}{w}$ takes the set $\{w : \mathrm{Re}(w) \geq \gamma\}$ into $D_{\gamma^{-1}}$, and this establishes (6.3).

Notice that $B_0 = 1$ and $B_1 = b_1$ so B_0 and B_1 are nonzero and $\mathrm{Re}\left(\frac{B_1}{B_0}\right) \geq \gamma$. We argue by induction and assume that we have proved $B_j \neq 0$ for $0 \leq j \leq n$ and $\mathrm{Re}\left(\frac{B_{j+1}}{B_j}\right) \geq \gamma$ for $0 \leq j \leq n-1$. We then obtain that

$$\mathrm{Re}\left(\frac{B_{n+1}}{B_n}\right) = \mathrm{Re}\left(\frac{B_{n-1}}{B_n}\right) + \mathrm{Re}(b_{n+1}) \geq \mathrm{Re}\left(\frac{B_{n-1}}{B_n}\right) + \gamma.$$

Writing $\beta = \frac{B_n}{B_{n-1}}$, so $\mathrm{Re}(\beta) \geq \gamma$, we see that

$$\mathrm{Re}\left(\frac{B_{n-1}}{B_n}\right) = \mathrm{Re}\left(\frac{1}{\beta}\right) = \mathrm{Re}\left(\frac{\bar{\beta}}{|\beta|^2}\right) \geq \frac{\gamma}{|\beta|^2},$$

so

$$\mathrm{Re}\left(\frac{B_{n+1}}{B_n}\right) = \gamma\left(1 + \left|\frac{B_{n-1}}{B_n}\right|^2\right) > \gamma$$

and $B_{n+1} \neq 0$.

The proof that $A_n \neq 0$ for all $n \geq 1$ and $\operatorname{Re}\big(\frac{A_{n+1}}{A_n}\big) \geq \gamma$ for all $n \geq 1$ follows by a similar induction argument and is left to the reader.

Notice that $\det(M_{b_1} M_{b_2} \dots M_{b_n}) = (-1)^n$, so

$$\frac{d}{dz}(\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_n})(z) = \frac{(-1)^n}{(B_{n-1}z + B_n)^2} = \frac{(-1)^n}{B_{n-1}^2\big(z + \frac{B_n}{B_{n-1}}\big)^2}. \tag{6.7}$$

If we can prove that $\big|B_{n-1}^2\big(z + \frac{B_n}{B_{n-1}}\big)^2\big| \geq L$ for all $z \in G$, it will follow that for all $z, w \in G$,

$$|(\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_n})(z) - (\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_n})(w)| \leq \frac{1}{L}|z - w|. \tag{6.8}$$

However, for $n \geq 2$ and $z \in G$,

$$\begin{aligned}
\left|z + \frac{B_n}{B_{n-1}}\right| &\geq \operatorname{Re}\Big(z + \frac{B_n}{B_{n-1}}\Big) \\
&\geq \operatorname{Re}\Big(\frac{B_n}{B_{n-1}}\Big) \\
&= \operatorname{Re}\Big(\frac{B_{n-2}}{B_{n-1}} + b_n\Big) \\
&= \operatorname{Re}\Big(\frac{B_{n-1}}{B_{n-2}}\Big)\frac{|B_{n-2}|^2}{|B_{n-1}|^2} + \operatorname{Re}(b_n) \\
&\geq \gamma\frac{|B_{n-2}|^2}{|B_{n-1}|^2} + \gamma.
\end{aligned}$$

This implies that

$$\begin{aligned}
\left|B_{n-1}^2\Big(z + \frac{B_n}{B_{n-1}}\Big)^2\right| &\geq |B_{n-1}|^2\gamma^2\Big(1 + \frac{|B_{n-2}|^2}{|B_{n-1}|^2}\Big)^2 \\
&= \gamma^2|B_{n-2}|^2\Big(\frac{|B_{n-1}|^2}{|B_{n-2}|^2} + 2 + \frac{|B_{n-2}|^2}{|B_{n-1}|^2}\Big).
\end{aligned} \tag{6.9}$$

Using (6.7) and (6.9), we see that for $z \in G$ and $n \geq 2$,

$$\left|\frac{d}{dz}(\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_n})(z)\right| \leq (4\gamma^2|B_{n-2}|^2)^{-1},$$

with strict inequality unless $|B_{n-1}| = |B_{n-2}|$, and this implies (6.8), with

$$L := 4\gamma^2|B_{n-2}|^2.$$

If we take $n = 2$, so $B_{n-2} = 1$, we find that for any two elements b_1 and b_2 in $\mathbb{C}$ and for all $z, w \in G$, we have

$$|\theta_{b_1}(\theta_{b_2}(z)) - \theta_{b_1}(\theta_{b_2}(w))| \leq \frac{1}{4\gamma^2}|z - w|.$$

Taking $b_1 = b$ and $b_2 = c$, we obtain (6.6). □

For the remainder of this section we shall restrict ourselves to the case in Lemma 6.1 that $\mathcal{B}$ is a subset of the positive reals and $b \geq \gamma > 0$ for all $b \in \mathcal{B}$.

Lemma 6.2. *Let $\mathcal{B}$ denote a finite set of positive reals such that $b \geq \gamma > 0$ for all $b \in \mathcal{B}$ and let notation be as in Lemma 6.1. If b_j, $j \geq 1$ denotes a sequence of elements of $\mathcal{B}$, then for all $n \geq 0$, $B_n > 0$ and $\frac{B_{n+1}}{B_n} \geq \gamma$ and for all $n \geq 1$, $A_n > 0$ and $\frac{A_{n+1}}{A_n} \geq \gamma$ and $\frac{B_n}{A_n} \geq \gamma$. For all $k \geq 0$, we have*

$$B_{2k} \geq (1 + \gamma^2)^k \quad \textit{and} \quad B_{2k+1} \geq \gamma(1 + \gamma^2)^k. \tag{6.10}$$

For all $\omega = (b_1, b_2, \ldots, b_{2m}) \in \mathcal{B}_{2m}$, $m \geq 1$, and $z, w \in G$, we have

$$|(\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_{2m}})(z) - (\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_{2m}})(w)| \leq (1+\gamma^2)^{-2m}|z - w| \tag{6.11}$$

and

$$|\theta_\omega(z) - \theta_\omega(w)| \leq (1 + \gamma^2)^{-2m}|z - w|. \tag{6.12}$$

Proof. Using (6.2) it is an easy induction argument (left to the reader) to prove that $A_n > 0$ for all $n \geq 1$ and $B_n > 0$ for all $n \geq 0$. It then follows immediately from Lemma 6.1 that $\frac{A_{n+1}}{A_n} \geq \gamma$ for $n \geq 1$ and $\frac{B_{n+1}}{B_n} \geq \gamma$ for $n \geq 0$.

Since $B_1 = b_1 \geq \gamma$ and $A_1 = 1$, we see that $\frac{B_1}{A_1} \geq \gamma$. Arguing by induction, assume that we have proved that $\frac{B_j}{A_j} \geq \gamma$ for $1 \leq j \leq n$. Then we obtain

$$\frac{B_{n+1}}{A_{n+1}} = \frac{B_{n-1} + b_{n+1}B_n}{A_{n-1} + b_{n+1}A_n} \geq \frac{A_{n-1}\gamma + b_{n+1}A_n\gamma}{A_{n-1} + b_{n+1}A_n} = \gamma,$$

which completes the inductive argument.

We next claim that for all $k \geq 0$, the first inequality in (6.10) holds. For $k = 0$, this is immediate, since $B_0 = 1$. We argue by induction and assume that we have proved the first inequality in (6.10) for some $k \geq 0$. We have that

$$B_{2k+1} = B_{2k-1} + b_{2k+1}B_{2k} \geq B_{2k-1} + \gamma B_{2k},$$

and this implies that

$$\begin{aligned} B_{2k+2} &= B_{2k} + b_{2k+2}B_{2k+1} \\ &\geq B_{2k} + \gamma B_{2k+1} \\ &\geq B_{2k} + \gamma B_{2k-1} + \gamma^2 B_{2k} \\ &\geq (1+\gamma^2)B_{2k} \\ &\geq (1+\gamma^2)^{k+1}. \end{aligned}$$

This completes the induction argument.

Since $B_1 = b_1 \geq \gamma$, and $B_{2k+1} = B_{2k-1} + b_{2k}B_{2k} \geq \gamma(1+\gamma^2)^k$ for $k \geq 1$, we obtain the second part of (6.10).

For $z \in H$, we obtain from Lemma 6.1 that

$$\left|\frac{d}{dz}(\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_{2m}})(z)\right| \leq |B_{2m-1}z + B_{2m}|^{-2} \tag{6.13}$$

and

$$\left|\frac{d}{dz}\theta_\omega(z)\right| := \left|\frac{d}{dz}(\theta_{b_{2m}} \circ \theta_{b_{2m-1}} \circ \cdots \circ \theta_{b_1})(z)\right| \leq |A_{2m}z + B_{2m}|^{-2}. \tag{6.14}$$

Because B_{2m-1}, A_{2m}, and B_{2m} are positive, $\mathrm{Re}(B_{2m-1}z + B_{2m}) \geq \mathrm{Re}(B_{2m}) \geq (1+\gamma^2)^m$ and $\mathrm{Re}(A_{2m}z + B_{2m}) \geq \mathrm{Re}(B_{2m}) \geq (1+\gamma^2)^m$. This implies that for all $z \in H$,

$$|B_{2m-1}z + B_{2m}|^{-2} \leq (1+\gamma^2)^{-2m} \quad \text{and} \quad |A_{2m}z + B_{2m}|^{-2} \leq (1+\gamma^2)^{-2m}. \tag{6.15}$$

Using (6.13), (6.14), and (6.15), we obtain (6.11) and (6.12). □

Remark 6.1. Given $\omega = (b_1, b_2, \dots, b_n) \in \mathcal{B}_n$, we have defined $\theta_\omega = \theta_{b_n} \circ \theta_{b_{n-1}} \circ \cdots \circ \theta_{b_1}$ (to conform to notation used in [50]). However, we could also have defined $\tilde{\theta}_\omega = \theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_n}$, which is perhaps more natural. Similarly, we have defined $g_\omega(z)$ by

$$g_{b_n}(\theta_{b_{n-1}} \circ \theta_{b_{n-2}} \circ \cdots \circ \theta_{b_1}(z)) g_{b_{n-1}}(\theta_{b_{n-2}} \circ \theta_{b_{n-3}} \circ \cdots \circ \theta_{b_1}(z)) \dots g_{b_2}(\theta_{b_1}(z)) g_{b_1}(z).$$

However, we could have defined

$$\tilde{g}_\omega(z) = g_{b_1}(\theta_{b_2} \circ \theta_{b_3} \circ \cdots \circ \theta_{b_n}(z)) g_{b_2}(\theta_{b_3} \circ \theta_{b_4} \circ \cdots \circ \theta_{b_n}(z)) \dots g_{b_{n-1}}(\theta_{b_n}(z)) g_{b_n}(z).$$

We leave to the reader the verification that

$$(\Lambda_s^n f)(z) = \sum_{\omega \in \mathcal{B}_n} [g_\omega(z)]^s f(\theta_\omega(z)) = \sum_{\omega \in \mathcal{B}_n} [\tilde{g}_\omega(z)]^s f(\tilde{\theta}_\omega(z)).$$

Theorem 6.3. *Let $\mathcal{B}$ be a finite set of positive reals such that $b \geq \gamma > 0$ for all $b \in \mathcal{B}$. For such b and all $x \geq 0$, define $\theta_b(x) = (x+b)^{-1}$. If $A \geq \gamma^{-1}$, define $H = \{x \in \mathbb{R}: 0 < x < A\}$, so $\theta_b(\bar{H}) \subset [0, \gamma^{-1}]$. Assume that m is a positive integer and $g_b: [0, A] \to \mathbb{R}$ is a C^m function such that $g_b(x) > 0$ for all $x \in [0, A]$. Let $X = X_m$ denote the Banach space $C^m(\bar{H})$ and for $s > 0$ define*

$$(\Lambda_s f)(x) = \sum_{b \in \mathcal{B}} [g_b(x)]^s f(\theta_b(x)).$$

Then all the hypotheses of Theorem 4.1 are satisfied, so Λ_s has a unique (to within normalization) strictly positive eigenfunction $v_s \in X$ with eigenvalue $r(\Lambda_s) > 0$. Furthermore, in our usual notation, for $1 \leq j \leq m$ and $x \in [0, A]$,

$$\frac{D^j v_s(x)}{v_s(x)} = \lim_{n \to \infty} \frac{\sum_{\omega \in \mathcal{B}_n} D^j g_\omega(x)}{\sum_{\omega \in \mathcal{B}_n} g_\omega(x)}. \tag{6.16}$$

Proof. Theorem 6.3 follows from Theorem 4.1 and Remark 4.1 once we verify that conditions (H4.1), (H4.2), and (H4.3) in Section 4 are satisfied. Conditions (H4.1) and (H4.2) are obviously satisfied. Also, it follows from (6.11) or (6.12) in Lemma 6.2 that for all $x, y \in [0, A]$ and all $b_1, b_2 \in \mathcal{B}$,

$$|\theta_{b_1}(\theta_{b_2}(x)) - \theta_{b_1}(\theta_{b_2}(y))| \leq (1+\gamma^2)^{-2}|x-y|,$$

which verifies (H4.3) with $\mu = 2$ and $\kappa = (1+\gamma^2)^{-2}$. □

Notice that if $g_b(\cdot)$ is C^∞ on $[0, A]$, Theorem 6.3 implies that $v_s(\cdot)$ is C^∞ on $[0, A]$ and (6.16) holds for all $j \geq 1$.

We are interested in Theorem 6.3 in the special case that $g_b(x) = |\theta_b'(x)|^s = (x+b)^{-2s}$. In this case, it is easy to verify that for $\mu \geq 1$,

$$(\Lambda_s^\mu f)(x) = \sum_{\omega \in \mathcal{B}_\mu} |\theta_\omega'(x)|^s f(\theta_\omega(x)).$$

If $\omega = (b_1, b_2, \ldots, b_\mu) \in \mathcal{B}_\mu$ and A_j and B_j are as defined in Lemma 6.1, recall that

$$[g_\omega(x)]^s = |\theta_\omega'(x)|^s = (A_\mu x + B_\mu)^{-2s} = A_\mu^{-2s}\Big(x + \frac{B_\mu}{A_\mu}\Big)^{-2s}.$$

If $1 \leq j \leq m$, it follows that

$$\begin{aligned} \frac{D^j [g_\omega(x)]^s}{[g_\omega(x)]^s} &= \frac{D^j\big[\big(x + \frac{B_\mu}{A_\mu}\big)^{-2s}\big]}{\big(x + \frac{B_\mu}{A_\mu}\big)^{-2s}} \\ &= (-1)^j (2s)(2s+1)\ldots(2s+j-1)\Big(x + \frac{B_\mu}{A_\mu}\Big)^{-j}. \end{aligned} \tag{6.17}$$

Lemma 6.2 implies that $\frac{B_\mu}{A_\mu} \geq \gamma$ for all $\mu \geq 1$. On the other hand, if $\Gamma = \max\{b : b \in \mathcal{B}\}$, a calculation gives

$$\frac{B_1}{A_1} = b_1 \leq \Gamma \quad \text{and} \quad \frac{B_2}{A_2} = b_1 + b_2^{-1} \leq \Gamma + \gamma^{-1}.$$

Let $K = \Gamma + \gamma^{-1}$ and, arguing inductively, assume that we have proved, for some $n \geq 2$, that

$$\frac{B_j}{A_j} \leq K, \quad 1 \leq j \leq n. \tag{6.18}$$

Then we obtain

$$\frac{B_{n+1}}{A_{n+1}} = \frac{B_{n-1} + b_{n+1}B_n}{A_{n-1} + b_{n+1}A_n} \leq \frac{KA_{n-1} + Kb_{n+1}A_n}{A_{n-1} + b_{n+1}A_n} = K,$$

which proves that (6.18) holds for all n. It follows that for $0 \leq x \leq A$ and $\mu \geq 1$, we have

$$(K + A)^{-j} \leq \left(x + \frac{B_\mu}{A_\mu}\right)^{-j} \leq \gamma^{-j}. \tag{6.19}$$

Using (6.19) in (6.17), we obtain for $0 \leq x \leq A$ and $\mu \geq 1$,

$$\begin{aligned}(2s)(2s+1)\dots(2s+j-1)(K+A)^{-j} \\ \leq (-1)^j \frac{D^j[g_\omega(x)]}{g_\omega(x)} \\ \leq (2s)(2s+1)\dots(2s+j-1)\gamma^{-j}.\end{aligned} \tag{6.20}$$

Thus we have proved the following corollary of Theorem 6.3.

Corollary 6.4. *Let $\mathcal{B}$ be a finite set of positive real numbers and define $\gamma = \min\{b : b \in \mathcal{B}\}$, $\Gamma = \max\{b : b \in \mathcal{B}\}$, and $K = \gamma^{-1} + \Gamma$. Let A be any real number with $A \geq \gamma^{-1}$ and for any positive integer m, define $X = X_m = C^m([0, A])$. For $s > 0$ define a bounded linear operator $\Lambda_s : X_m \to X_m$ by*

$$(\Lambda_s f)(x) = \sum_{b \in \mathcal{B}} (x + b)^{-2s} f(\theta_b(x)),$$

where $\theta_b(x) = (x + b)^{-1}$. Then Λ_s has a unique (to within normalization) strictly positive eigenfunction $v_s \in X_m$ and v_s is actually infinitely differentiable. Furthermore, for integers $j \geq 1$, we have the estimates

$$\begin{aligned}(2s)(2s+1)\dots(2s+j-1)(K+A)^{-j} \\ \leq (-1)^j \frac{D^j[v_s(x)]}{v_s(x)} \\ \leq (2s)(2s+1)\dots(2s+j-1)\gamma^{-j}, \quad x \in [0, A].\end{aligned} \tag{6.21}$$

Proof. Equation (6.21) follows from (6.16) and (6.20) by letting $n \to \infty$, where $\omega \in \mathcal{B}_n$. □

Remark 6.2. Suppose that assumptions and notation are as in Corollary 6.4, so $v_s: [0, A] \mapsto \mathbb{R}$ is strictly positive and $v_s \in C^m([0, A])$. Then $v_s(\cdot)$ has an analytic, complex-valued extension to $H = \{z \in \mathbb{C}: \mathrm{Re}(z) > 0\}$. The idea of the proof is to consider the linear operator

$$(R_s f)(z) = \sum_{b \in \mathcal{B}} (z + b)^{-2s} f([z + b]^{-1}),$$

where f is an element of an appropriate Banach space of complex analytic functions $f(\cdot)$ defined on $\{z \in \mathbb{C}: \left|z - \frac{A}{2}\right| < \frac{A}{2}\} := D$ and continuous on $\bar{D}$.

Since we shall not use this analyticity result, we omit the proof, but its interest for us is precisely that in more general situations, it does not seem possible to study our problem in a Banach space of analytic functions. Suppose that $\mathcal{B}$ is a finite set of complex numbers as in Lemma 6.1 and $\theta_b(z) = (z + b)^{-1}$ for $b \in \mathcal{B}$ and $\mathrm{Re}(z) \geq 0$. If $A > \gamma^{-1}$ and D is as above, one can prove that $\{\theta_b(z): z \in \bar{D}, b \in \mathcal{B}\}$ is contained in a compact subset of D. For $m \geq 2$ and $s > 0$, one defines

$$\Lambda_s: C^m(\bar{D}) \longrightarrow C^m(\bar{D})$$

by

$$(\Lambda_s f)(z) = \sum_{b \in \mathcal{B}} |z + b|^{-2s} f(\theta_b(z)),$$

(note $(z+b)^{-2s}$ has been replaced by $|z+b|^{-2s}$), and Λ_s has a unique, normalized eigenfunction $v_s(\cdot)$ such that $v_s(z) > 0$ for all $z \in \bar{D}$. The eigenvalue of v_s is $r(\Lambda_s)$, the spectral radius of Λ_s. In the context of *complex continued fractions* (see [19], [40], [50], and [51]), one wants to estimate $r(\Lambda_s)$. However $z \mapsto |z + b|^{-2s}$ and $z \mapsto v_s(z)$ are C^∞, but not complex analytic on D. If $\mathcal{B}$ is not contained in $\mathbb{R}$, in general there does not seem to be a natural bounded linear operator in a Banach space of analytic functions with spectral radius $r(\Lambda_s)$. In this generality, the linear operator R_s can still be defined in a Banach space of analytic functions, but will almost always have spectral radius less than $r(\Lambda_s)$.

7. Computing the spectral radius of A_s and B_s

In previous sections, we have constructed matrices A_s and B_s such that $r(A_s) \leq r(L_s) \leq r(B_s)$. The $(n + 1) \times (n + 1)$ matrices A_s and B_s have nonnegative entries, so the Perron–Frobenius theory for such matrices implies that $r(B_s)$ is an eigenvalue of B_s with corresponding nonnegative eigenvector, with a similar statement for A_s. One might also hope that standard theory (see [44]) would imply that $r(B_s)$, respectively $r(A_s)$, is an eigenvalue of B_s with algebraic multiplicity one and that all other eigenvalues z of B_s (respectively, of A_s) satisfy $|z| < r(B_s)$ (respectively, $|z| < r(A_s)$). Indeed, this would be true if B_s were *primitive*, i.e., if B_s^k had all positive entries for some integer k. However, typically B_s has many zero columns and B_s is neither primitive nor *irreducible* (see [44]); and the same problem occurs for A_s. Nevertheless, the desirable spectral properties mentioned above are satisfied for both A_s and B_s. Furthermore B_s has an eigenvector w_s with all positive entries and with eigenvalue $r(B_s)$; and if x is any $(n+1) \times 1$ vector with all positive entries,

$$\lim_{k\to\infty} \frac{B_s^k(x)}{\|B_s^k(x)\|} = \frac{\mathrm{w}_s}{\|\mathrm{w}_s\|},$$

where the convergence rate is geometric. Of course, corresponding theorems hold for A_s. Such results justify standard numerical algorithms for approximating $r(B_s)$ and $r(A_s)$.

In this section, we shall prove these assertions. The basic point is simple. Although A_s and B_s both map the cone K of nonnegative vectors in $\mathbb{R}^{n+1}$ into itself, K is *not* the natural cone in which such matrices should be studied.

To outline our method of proof, it is convenient to describe, at least in the finite dimensional case, some classical theorems concerning linear maps $L\colon \mathbb{R}^N \to \mathbb{R}^N$ which leave a cone $\mathcal{C} \subset \mathbb{R}^N$ invariant. Recall that a closed subset $\mathcal{C}$ of $\mathbb{R}^N$ is called a closed cone if (i) $ax+by \in \mathcal{C}$ whenever $a \geq 0$, $b \geq 0$, $x \in \mathcal{C}$ and $y \in \mathcal{C}$ and (ii) if $x \in \mathcal{C}\setminus\{0\}$, then $-x \notin \mathcal{C}$. If $\mathcal{C}$ is a closed cone, $\mathcal{C}$ induces a partial ordering on $\mathbb{R}^N$ denoted by $\leq_{\mathcal{C}}$ (or simply $\leq$, if $\mathcal{C}$ is obvious) by $u \leq_{\mathcal{C}} v$ if and only if $v - u \in \mathcal{C}$. If $u, v \in \mathcal{C}$, we shall say that u and v are *comparable* (with respect to $\mathcal{C}$) and we shall write $u \sim_{\mathcal{C}} v$ if there exist positive scalars a and b such that $v \leq_{\mathcal{C}} au$ and $u \leq_{\mathcal{C}} bv$. *Comparable with respect to* $\mathcal{C}$ partitions $\mathcal{C}$ into equivalence classes of comparable elements. We shall henceforth assume that int($\mathcal{C}$), the interior of $\mathcal{C}$, is nonempty. Then an easy argument shows that all elements of int($\mathcal{C}$) are comparable. Generally, if $x_0 \in \mathcal{C}$ and $\mathcal{C}_{x_0} := \{x \in \mathcal{C}\colon x \sim_{\mathcal{C}} x_0\}$, all elements of $\mathcal{C}_{x_0}$ are comparable.

Following standard notation, if $u, v \in \mathcal{C}$ are comparable elements, we define

$$M\Big(\frac{u}{v}; \mathcal{C}\Big) = \inf\{\beta > 0\colon u \leq \beta v\},$$

$$m\Big(\frac{u}{v}; \mathcal{C}\Big) = M\Big(\frac{v}{u}; \mathcal{C}\Big)^{-1} = \sup\{\alpha > 0\colon \alpha v \leq u\}.$$

If u and v are comparable elements of $\mathcal{C}\setminus\{0\}$, we define Hilbert's projective metric $d(u, v; \mathcal{C})$ by

$$d(u, v; \mathcal{C}) = \log\Big(M\Big(\frac{u}{v}; \mathcal{C}\Big)\Big) + \log\Big(M\Big(\frac{v}{u}; \mathcal{C}\Big)\Big).$$

We make the convention that $d(0, 0; \mathcal{C}) = 0$. If $x_0 \in \mathcal{C} \setminus \{0\}$, then for all $u, v, w \in \mathcal{C}_{x_0}$, one can prove that (i) $d(u, v; \mathcal{C}) \geq 0$, (ii) $d(u, v; \mathcal{C}) = d(v, u; \mathcal{C})$, and (iii) $d(u, v; \mathcal{C}) + d(v, w; \mathcal{C}) \geq d(u, w; \mathcal{C})$. Thus d restricted to $\mathcal{C}_{x_0}$ is almost a metric, but $d(u, v; \mathcal{C}) = 0$ if and only if $v = tu$ for some $t > 0$ and generally, $d(su, tv; \mathcal{C}) = d(u, v; \mathcal{C})$ for all $u, v \in \mathcal{C}_{x_0}$ and all $s > 0$ and $t > 0$. If $\|\cdot\|$ is any norm on $\mathbb{R}^N$ and $S := \{u \in \mathrm{int}(\mathcal{C}) : \|u\| = 1\}$ (or, more generally, if $x_0 \in \mathcal{C} \setminus \{0\}$ and $S = \{x \in \mathcal{C}_{x_0} : \|x\| = 1\}$, then $d(\cdot, \cdot; \mathcal{C})$, restricted to $S \times S$, gives a metric on S; and it is known that S is a complete metric space with this metric.

With these preliminaries we can describe a special case of the Birkhoff-Hopf theorem. We refer to [3], [26], and [55] for the original papers and to [12] and [13] for an exposition of a general version of this theorem and further references to the literature. We remark that P. P. Zabreiko, M. A Krasnosel′skij, Y. V. Pokornyi, and A. V. Sobolev independently obtained closely related theorems; and we refer to [34] for details. If $\mathcal{C}$ is a closed cone as above, $S = \{x \in \mathrm{int}(\mathcal{C}) : \|x\| = 1\}$, and $L : \mathbb{R}^N \to \mathbb{R}^N$ is a linear map such that $L(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$, we define $\Delta(L; \mathcal{C})$, *the projective diameter* of L by

$$\begin{aligned}\Delta(L; \mathcal{C}) &= \sup\{d(Lx, Ly; \mathcal{C}) : x, y \in \mathcal{C} \text{ and } Lx \sim_{\mathcal{C}} Ly\}\\ &= \sup\{d(Lx, Ly; \mathcal{C}) : x, y \in \mathrm{int}(\mathcal{C})\}.\end{aligned}$$

The Birkhoff-Hopf theorem implies that if $\Delta := \Delta(L; \mathcal{C}) < \infty$, then L is a contraction mapping with respect to Hilbert's projective metric. More precisely, if we define $\lambda = \tanh(\frac{1}{4}\Delta) < 1$, then for all $x, y \in \mathcal{C} \setminus \{0\}$ such that $x \sim_{\mathcal{C}} y$, we have

$$d(Lx, Ly; \mathcal{C}) \leq \lambda d(x, y; \mathcal{C}),$$

and the constant λ is optimal.

If we define $\Phi : S \to S$ by $\Phi(x) = \frac{L(x)}{\|L(x)\|}$, it follows that Φ is a contraction mapping with a unique fixed point $v \in S$, and v is necessarily an eigenvector of L with eigenvector $r(L) := r =$ the spectral radius of L. Furthermore, given any $x \in \mathrm{int}(\mathcal{C})$, there are explicitly computable constants M and $c < 1$ (see Theorem 2.1 in [12]) such that for all $k \geq 1$,

$$\left\| \frac{L^k(x)}{\|L^k(x)\|} - v \right\| \leq M c^k;$$

and the latter inequality is exactly the sort of result we need. Furthermore, it is proved in Theorem 2.3 of [12] that $r = r(L)$ is an algebraically simple eigenvalue of L and that if $\sigma(L)$ denotes the spectrum of L and $q(L)$ denotes the *spectral clearance of* L,

$$q(L) := \sup\left\{\frac{|z|}{r(L)} : z \in \sigma(L), z \neq r(L)\right\},$$

then $q(L) < 1$ and $q(L)$ can be explicitly estimated.

If A_s, B_s, and L_s are as in Section 3, it remains to find a suitable cone as above. For the remainder of this section, $[a, b]$ will denote a fixed, closed bounded interval and s a fixed nonnegative real. For a given positive integer $n \geq 2$ and for integers j, $0 \leq j \leq n$, we shall write $h = \frac{b-a}{n}$ and $x_j = a + jh$. C will denote a fixed constant and we shall always assume at least that

$$C\frac{h}{4} \leq 1. \tag{7.1}$$

In our applications, C will depend on s, but we shall not indicate this dependence in our notation. If $\mathrm{w}: \{x_j : 0 \leq j \leq n\} \to \mathbb{R}$, one can extend w to a piecewise linear map $w^I : [a, b] \to \mathbb{R}$ by defining

$$w^I(x) = \frac{x - x_j}{h}\mathrm{w}_{j+1} + \frac{x_{j+1} - x}{h}\mathrm{w}_j, \quad \text{for } x_j \leq x \leq x_{j+1},\ 0 \leq j < n, \tag{7.2}$$

where we have written $\mathrm{w}_j = \mathrm{w}(x_j)$.

We shall denote by X_n (or X, if n is obvious), the real vector space of maps $\mathrm{w}: \{x_j : 0 \leq j \leq n\} \to \mathbb{R}$; obviously X_n is linearly isomorphic to $\mathbb{R}^{n+1}$, and we shall consider A_s, B_s, and L_s as maps of X_n to X_n. Note that in applying the results described above, we set $N = n + 1$. For a given real $M > 0$, we shall denote by $K_M \subset X_n$ the closed cone with nonempty interior given by

$$K_M = \{\mathrm{w} \in X_n : \mathrm{w}_{j+1} \leq \mathrm{w}_j \exp(Mh) \text{ and } \mathrm{w}_j \leq \mathrm{w}_{j+1} \exp(Mh),\ 0 \leq j < n\}. \tag{7.3}$$

The reader can verify that if $\mathrm{w} = (\mathrm{w}_0, \mathrm{w}_1, \dots, \mathrm{w}_n) \in K_M \setminus \{0\}$, then $\mathrm{w}_j > 0$ for $0 \leq j \leq n$.

If $K_M \subset X_n$ are as above, suppose that $L: X_n \to X_n$ is a linear map and that there exists M', $0 < M' < M$, such that $L(K_M \setminus \{0\}) \subset K_{M'} \setminus \{0\}$. After correcting the typo in the formula for $d_2(f, g)$ on p. 286 of [37], it follows from Lemma 2.12 on p. 284 of [37] that

$$\sup\{d(f, g; K_M) : f, g \in K_{M'} \setminus \{0\}\} \leq 2\log\left(\frac{M + M'}{M - M'}\right) + 2M'(b - a) < \infty.$$

This implies that $\Delta(L; K_M) < \infty$, which in turn implies that L has a normalized eigenvector $v \in K_{M'}$ with positive eigenvalue $r = r(L) =$ the spectral radius of L. Furthermore, r has algebraic multiplicity 1, $q(L) < 1$, and

$$\lim_{k\to\infty} \left\| \frac{L^k(x)}{\|L^k(x)\|} - v \right\| = 0 \quad \text{for all } x \in K_M \setminus \{0\}.$$

Thus it suffices to prove for appropriate maps L that $L(K_M \setminus \{0\}) \subset K_{M'} \setminus \{0\}$ for some $M' < M$.

If x_j, $0 \le j \le n$ are as above, define a map

$$Q: [a, b] \longrightarrow \left[0, \frac{h^2}{4}\right]$$

by

$$Q(u) = (x_{j+1} - u)(u - x_j), \quad \text{for } x_j \le u \le x_{j+1},\ 0 \le j < n.$$

Lemma 7.1. *Assume that* $\beta \in K_{M_0} \setminus \{0\}$ *for some* $M_0 > 0$*, that* $0 < h \le 1$ *and that* h *and* C *satisfy* (7.1). *Let* $\theta: [a, b] \to [a, b]$ *and define* $\hat{\beta}_s \in X_n$ *by*

$$\hat{\beta}_s(x_k) = \left[1 + \frac{1}{2} C Q(\theta(x_k))\right] [\beta(x_k)]^s.$$

Then $\hat{\beta}_s \in K_{M_1}$*, where* $M_1 = s M_0 + \frac{1+h}{2} \le M_0 + 1$.

Proof. Define $\psi \in X_n$ by

$$\psi(x_k) = 1 + \frac{1}{2} C Q(\theta(x_k))$$

and suppose we can prove that $\psi \in K_{\frac{1+h}{2}}$. For notational convenience define $b(x_k) = [\beta(x_k)]^s$. Then for $0 \le k < n$, we obtain

$$\begin{aligned} \psi(x_k) b(x_k) &\le \psi(x_{k+1}) \exp\left(\frac{[1+h]h}{2}\right) b(x_{k+1}) \exp(s M_0 h) \\ &= \psi(x_{k+1}) b(x_{k+1}) \exp(M_1 h), \end{aligned}$$

and the same calculation gives

$$\psi(x_{k+1}) b(x_{k+1}) \le \exp(M_1 h) \psi(x_k) b(x_k),$$

which implies that $x_k \mapsto \psi(x_k) b(x_k)$ is an element of K_{M_1}.

Define $\delta = \frac{1+h}{2}$. Since $\psi(x_k) > 0$ for $0 \le k \le n$, one can check that $\psi(\cdot) \in K_\delta$ if and only if, for $0 \le k < n$,

$$|\log(\psi(x_{k+1})) - \log(\psi(x_k))| = \left|\log\left(\frac{\psi(x_{k+1})}{\psi(x_k)}\right)\right| \le \delta h.$$

Given x_k and x_{k+1} with $0 \le k < n$, write $\xi = \theta(x_k)$ and $\eta = \theta(x_{k+1})$. Define $u := \frac{1}{2}CQ(\theta(x_k))$ and $v = \frac{1}{2}CQ(\theta(x_{k+1}))$, so $\psi(x_k) = 1 + u$ and $\psi(x_{k+1}) = 1 + v$. Because u and v both lie in the interval $\left[0, \frac{Ch^2}{8}\right]$, (7.1) implies that $|u - v| \le \frac{h}{2}$, $|u| \le \frac{h}{2}$ and $|v| \le \frac{h}{2}$. It follows that

$$|\log(\psi(x_k)) - \log(\psi(x_{k+1}))| = |\log(1+u) - \log(1+v)| = \left|\int_{1+v}^{1+u} \left(\frac{1}{t}\right) dt\right|.$$

Because $0 \le \frac{1}{t} \le \frac{1}{1-\frac{h}{2}} \le 1 + h$ for all $t \in [1+v, 1+u]$, we obtain

$$|\log(\psi(x_k)) - \log(\psi(x_{k+1}))| \le (1+h)|u - v| \le (1+h)\frac{h}{2},$$

which proves the lemma. □

Lemma 7.2. *Let assumptions and notation be as in Lemma 7.1. Let δ denote a fixed positive real and s a fixed nonnegative real. Assume, in addition that $\theta\colon [a,b] \to [a,b]$ is a Lipschitz map with* $\mathrm{Lip}(\theta) \le c < 1$ *and that, for $h = \frac{b-a}{n}$ and M_1 as in Lemma 7.1,* $\exp(-[M_1 + \delta]h) \ge \frac{1+c}{2}$ *and $M > 0$ is such that* $\exp(Mh) \ge 2$. *Define a linear map $L_s\colon X_n \to X_n$ by*

$$L_s(\mathrm{w})(x_k) := w^I(\theta(x_k))\hat{\beta}_s(x_k), \quad 0 \le k \le n.$$

Then, if $K_M \subset X_n$ is defined by (7.3), $L_s(K_M) \subset K_{M-\delta}$.

Proof. For a fixed k, $0 \le k < n$, recall we have defined $\xi = \theta(x_k)$ and $\eta = \theta(x_{k+1})$. We must prove that if h and M satisfy the above constraints and $\mathrm{w} \in K_M$, then

$$w^I(\xi)\hat{\beta}_s(x_k) \le \exp([M - \delta]h) w^I(\eta)\hat{\beta}_s(x_{k+1}),$$

$$w^I(\eta)\hat{\beta}_s(x_{k+1}) \le \exp([M - \delta]h) w^I(\xi)\hat{\beta}_s(x_k).$$

Using Lemma 7.1, we see that $x_k \mapsto \hat{\beta}_s(x_k)$ is an element of K_{M_1}, so the above inequalities will be satisfied if

$$w^I(\xi) \le \exp([M - M_1 - \delta]h) w^I(\eta), \tag{7.4}$$

$$w^I(\eta) \le \exp([M - M_1 - \delta]h) w^I(\xi). \tag{7.5}$$

For notational convenience, we write $M_2 = M_1 + \delta$. By interchanging the roles of ξ and η, we can assume that $\eta \le \xi$, and it suffices to prove that (7.4) and (7.5) are satisfied for M and h as in the statement of the Lemma. Define $j = n-1$ if $\xi \ge x_{n-1}$ and otherwise define j to be the unique integer, $0 \le j < n-1$, such that $x_j \le \xi < x_{j+1}$. Because $0 \le \xi - \eta \le ch < h$, there are only two cases to consider: either (i) $x_j \le \eta \le \xi$ or (ii) $x_{j-1} < \eta < x_j$ and $x_j \le \xi < x_{j+1}$.

We first assume that we are in case (i), so $\xi, \eta \in [x_j, x_{j+1}]$ and $0 \le \xi - \eta \le ch$, Using (7.2), we see that (7.4) is equivalent to proving

$$\begin{aligned}(x_{j+1} - \xi)\mathrm{w}_j &+ (\xi - x_j)\mathrm{w}_{j+1} \\ &\le \exp([M - M_2]h)[(x_{j+1} - \eta)\mathrm{w}_j + (\eta - x_j)\mathrm{w}_{j+1}].\end{aligned} \tag{7.6}$$

Subtracting $(x_{j+1} - \eta)\mathrm{w}_j + (\eta - x_j)\mathrm{w}_{j+1}$ from both sides of (7.6) shows that (7.6) will be satisfied if

$$\begin{aligned}(\xi - \eta)&[\mathrm{w}_{j+1} - \mathrm{w}_j] \\ &\le [\exp([M - M_2]h) - 1][(x_{j+1} - \eta)\mathrm{w}_j + (\eta - x_j)\mathrm{w}_{j+1}].\end{aligned} \tag{7.7}$$

Equation (7.7) will certainly be satisfied if $\mathrm{w}_{j+1} \le \mathrm{w}_j$, so we can assume that $\mathrm{w}_{j+1} - \mathrm{w}_j > 0$ and $1 < \frac{\mathrm{w}_{j+1}}{\mathrm{w}_j} \le \exp(Mh)$. If we divide both sides of (7.7) by w_j and recall that $\xi - \eta \le ch$, we see that the left hand side of (7.7) is dominated by $ch[\exp(Mh) - 1]$, while the right hand side of (7.7) is $\ge [\exp([M - M_2]h) - 1]h$, Thus, (7.7) will be satisfied if

$$c \le \frac{\exp([M - M_2]h) - 1}{\exp(Mh) - 1} = \exp(-M_2 h) + \frac{\exp(-M_2 h) - 1}{\exp(Mh) - 1}. \tag{7.8}$$

If $h > 0$ is chosen so that $\exp(-M_2 h) \ge \frac{1+c}{2}$, a calculation shows that (7.8) will be satisfied if $M \ge \frac{\log(2)}{h}$, where log denotes the natural logarithm. Thus, if $h > 0$ satisfies (7.1), $M \ge \frac{\log(2)}{h}$, and $\exp(-M_2 h) \ge \frac{1+c}{2}$, (7.4) is satisfied in case (i). Under the same conditions on h and M, an exactly analogous argument shows that (in case (i)), (7.5) is also satisfied.

We next consider case (ii), so $\xi \in [x_j, x_{j+1}]$, $\eta \in [x_{j-1}, x_j]$ and $0 \le \xi - \eta \le ch$. It follows that $\xi - x_j = c_1 h$ and $x_j - \eta = c_2 h$, where $c_1 \ge 0$, $c_2 \ge 0$, and $c_1 + c_2 \le c < 1$. As before, we need to show that inequalities (7.4) and (7.5) are satisfied. Inequality (7.5) takes the form

$$\begin{aligned}w^I(\eta) &= \frac{\eta - x_{j-1}}{h}\mathrm{w}_j + \frac{x_j - \eta}{h}\mathrm{w}(x_{j-1}) \\ &\le \exp([M - M_2]h)\Big[\frac{\xi - x_j}{h}\mathrm{w}_{j+1} + \frac{x_{j+1} - \xi}{h}\mathrm{w}(x_j)\Big],\end{aligned} \tag{7.9}$$

which is equivalent to

$$(\eta - x_{j-1}) + (x_j - \eta)\frac{\mathrm{w}(x_{j-1})}{\mathrm{w}_j} \le \exp([M - M_2]h)\Big[(\xi - x_j)\frac{\mathrm{w}_{j+1}}{\mathrm{w}_j} + (x_{j+1} - \xi)\Big]. \tag{7.10}$$

Since $\frac{\mathrm{w}(x_{j-1})}{\mathrm{w}_j} \le \exp(Mh)$, $\frac{\mathrm{w}_{j+1}}{\mathrm{w}_j} \ge \exp(-Mh)$, $x_j - \eta = c_2 h$ and $\xi - x_j = c_1 h$, (7.10) will be satisfied if

$$(1 - c_2) + c_2 \exp(Mh) \le \exp([M - M_2]h)[c_1 \exp(-Mh) + (1 - c_1)]. \tag{7.11}$$

Because $c_2 \le c - c_1$, we have

$$(1 - c_2) + c_2 \exp(Mh) \le (1 - c + c_1) + (c - c_1)\exp(Mh),$$

and inequality (7.11) will be satisfied if

$$(1 + c_1 - c) + (c - c_1)\exp(Mh) \le \exp(-M_2 h)[c_1 + (1 - c_1)\exp(Mh)]. \tag{7.12}$$

A necessary condition that (7.12) be satisfied is that $\exp(-M_2 h) \ge \frac{c-c_1}{1-c_1}$. Since $\frac{c-c_1}{1-c_1} \le c$ and $c < \frac{1+c}{2}$, we choose $h = \frac{b-a}{n} > 0$ sufficiently small that

$$\exp(-M_2 h) \ge \frac{1+c}{2}. \tag{7.13}$$

For this choice of h, (7.12) will be satisfied if

$$(1 + c_1 - c) + (c - c_1)\exp(Mh) \le \frac{1+c}{2}[c_1 + (1 - c_1)\exp(Mh)],$$

which is equivalent to

$$\Big(1 + \frac{c_1}{2}\Big)(1 - c) \le \Big[(1 + c_1)\frac{1-c}{2}\Big]\exp(Mh). \tag{7.14}$$

Since $\frac{2+c_1}{1+c_1} \le 2$, (7.14) will be satisfied if

$$2 \le \exp(Mh). \tag{7.15}$$

Thus (7.9) will be satisfied if h satisfies (7.13) and, for this h, M satisfies (7.15).

Inequality (7.4) will be satisfied in case (ii) if

$$(\xi - x_j)\frac{\mathrm{w}_{j+1}}{\mathrm{w}_j} + (x_{j+1} - \xi) \le \exp([M - M_2]h)\Big[(\eta - x_{j-1}) + (x_j - \eta)\frac{\mathrm{w}(x_{j-1})}{\mathrm{w}_j}\Big]. \tag{7.16}$$

The same reasoning as above shows that if $h > 0$ satisfies (7.13) and M then satisfies (7.15), (7.16) will be satisfied. Details are left to the reader. □

Theorem 7.3. *Let N denote a positive integer. For $1 \leq j \leq N$, assume that $\theta_j\colon [a,b] \to [a,b]$ is a Lipschitz map with $Lip(\theta_j) \leq c < 1$, c independent of j. For $1 \leq j \leq N$, assume that $\beta_j \in K_{M_0} \setminus \{0\} \subset X_n$, where M_0 is independent of j. For $j \geq 1$, let C_j be a real number with $|C_j| \leq C$, where C is independent of j; and for a fixed $s \geq 0$, define $\hat{\beta}_{j,s} \in X_n$ by*

$$\hat{\beta}_{j,s}(x_k) = \Big[1 + \frac{1}{2} C_j Q(\theta_j(x_k))\Big][\beta_j(x_k)]^s, \quad 0 \leq k \leq n.$$

Let $\delta > 0$ be a given real number and for $j \geq 1$ define a linear map $L_{j,s}\colon X_n \to X_n$ by

$$(L_{j,s}\mathrm{w})(x_k) = \hat{\beta}_{j,s}(x_k) w^I(\theta_j(x_k)), \quad 0 \leq k \leq n,$$

and a linear map $L_s\colon X_n \to X_n$ by $L_s = \sum_{j=1}^N L_{j,s}$. Assume that $h = \frac{b-a}{n} \leq 1$ and $\frac{Ch}{4} \leq 1$ and define $M_2 = M_1 + \delta$. Assume also that $\exp(-M_2 h) \geq \frac{1+c}{2}$ and that $M \in \mathbb{R}$ is such that $\exp(Mh) \geq 2$. Then we have that $L_s(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$.

Proof. Lemma 7.1 implies that $x_k \mapsto \hat{\beta}_{j,s}(x_k)$ is an element of K_{M_1}, where $M_1 = sM_0 + \frac{1+h}{2}$. Under our hypotheses, Lemma 7.2 implies that $L_{j,s}(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$, so $L_s(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$. □

Our next theorem follows immediately from Theorem 7.3 and the remarks at the beginning of this section.

Theorem 7.4. *Let notation and assumptions be as in Theorem 7.3. Then L_s has an eigenfunction $v \in K_{M-\delta} \setminus \{0\}$, $\|v\| = 1$, with eigenvalue $r > 0$. If $\hat{L}_s$ denotes the complexification of L_s, r is an eigenvalue of $\hat{L}_s$ of algebraic multiplicity one; and if $L_s w = \lambda w$ for some $w \in K_M \setminus \{0\}$, $\lambda = r$, and w is a positive multiple of v. If z is an eigenvalue of $\hat{L}_s$ and $z \neq r$, then $|z| < r$. If $x \in K_M \setminus \{0\}$,*

$$\lim_{k \to \infty} \left\| \frac{L^k(x)}{\|L^k(x)\|} - v \right\| = 0$$

and the convergence rate is geometric.

Remark 7.1. With the aid of Theorem 7.3, we could also have used the theory of u_0-positive linear operators (see [33] and [34]) to derive Theorem 7.4.

Remark 7.2. Since the linear maps A_s and B_s are both of the form of the map L_s in Theorem 7.3, Theorem 7.4 implies the desired spectral properties of A_s and B_s. With greater care it is possible to use results in [12] to estimate the spectral clearance $q(L_s)$ of L_s.

Remark 7.3. We claim that there is a constant E, which can be easily estimated, such that, for $h = \frac{b-a}{n}$ sufficiently small,

$$r(B_s) \le r(A_s)(1 + Eh^2).$$

(Of course we already know that $r(A_s) \le r(B_s)$.) For a fixed $s \ge 0$, let $\beta_j(\cdot)$ and $\theta_j(\cdot)$ be as in Theorem 7.3. We know that A_s and B_s are of the form of L_s in Theorem 7.3, so we can write, for $0 \le k \le n$,

$$(A_s \mathrm{w})(x_k) = \sum_{j=1}^{N} \Big[1 + \frac{C_j}{2} Q(\theta_j(x_k))\Big][\beta_j(x_k)]^s w^I(\theta_j(x_k)),$$

$$(B_s \mathrm{w})(x_k) = \sum_{j=1}^{N} \Big[1 + \frac{D_j}{2} Q(\theta_j(x_k))\Big][\beta_j(x_k)]^s w^I(\theta_j(x_k)).$$

We assume that $h \le 1$ and $C\frac{h}{4} \le 1$, where C is a positive constant such that $\max(|C_j|, |D_j|) \le C$ for $1 \le j \le N$. We assume also that for $1 \le j \le N$, $C_j \le D_j$. Let $K = \{\mathrm{w} \in X_n : \mathrm{w}(x_k) \ge 0 \text{ for } 0 \le k \le n\}$, so $A_s(K) \subset K$ and $B_s(K) \subset K$. Define $\mu \ge 1$ by

$$\mu = \sup\Big\{\Big[1 + \frac{D_j}{2} Q(\theta_j(x_k))\Big]\Big[1 + \frac{C_j}{2} Q(\theta_j(x_k))\Big]^{-1} : 1 \le j \le N, 0 \le k \le N\Big\} \ge 1.$$

Then for all $\mathrm{w} \in K$ and $0 \le k \le n$, $(B_s(\mathrm{w}))(x_k) \le \mu(A_s(\mathrm{w}))(x_k)$, which implies that $r(B_s) \le \mu r(A_s)$. Since $Q(u) \le \frac{h^2}{4}$, a little thought shows that $\mu \le \big(1 + \frac{Ch^2}{8}\big)\big(1 - \frac{Ch^2}{8}\big)^{-1} \le 1 + Eh^2$, which gives the desired estimate.

8. Log convexity of the spectral radius of Λ_s

Throughout this section we shall assume that hypotheses (H4.1), (H4.2), and (H4.3) in Section 4 are satisfied and we shall also assume that H is a bounded, open, subset of $\mathbb{R}$. As in Section 4, we shall write $X = C^m(\bar{H})$ and $Y = C(\bar{H})$. For $s \in \mathbb{R}$, we define $\Lambda_s : X \to X$ and $L_s : Y \to Y$ by

$$(\Lambda_s(w))(x) = \sum_{b \in \mathcal{B}} [g_b(x)]^s w(\theta_b(x)), \tag{8.1}$$

$$(L_s(w))(x) = \sum_{b \in \mathcal{B}} [g_b(x)]^s w(\theta_b(x)). \tag{8.2}$$

Theorem 4.1 implies that $r(\Lambda_s)$ is an algebraically simple eigenvalue of Λ_s for $s \in \mathbb{R}$ and that $\sup\{|z| : z \in \sigma(\Lambda_s), z \ne r(\Lambda_s)\} < r(\Lambda_s)$, where $\sigma(\Lambda_s)$ denotes the spectrum of Λ_s.

Let $\widehat{X}$ denote the complexification of X, so $\widehat{X}$ is the Banach space of C^m maps $f: H \to \mathbb{C}$ such that $x \mapsto (D^k f)(x)$ extends continuously to $\bar{H}$ for all $0 \le k \le m$. For $s \in \mathbb{C}$ one can define

$$\widehat{\Lambda}_s : \widehat{X} \longrightarrow \widehat{X}$$

by

$$(\widehat{\Lambda}_s(w))(x) = \sum_{b \in \mathcal{B}} (g_b(x))^s w(\theta_b(x)) := \sum_{b \in \mathcal{B}} \exp(s \log g_b(x)) w(\theta_b(x)).$$

The reader can verify that $s \mapsto \widehat{\Lambda}_s \in \mathcal{L}(\widehat{X}, \widehat{X})$ is an analytic map. Because $r(\widehat{\Lambda}_s)$ is an algebraically simple eigenvalue of $\widehat{\Lambda}_s$ for $s \in \mathbb{R}$ and $\sup\{|z| : z \in \sigma(\Lambda_s), z \neq r(\Lambda_s)\} < r(\Lambda_s)$, it follows from the kind of argument used on pp. 227f of [45] that there is an open neighborhood U of $\mathbb{R}$ in $\mathbb{C}$ and the map $s \in U \mapsto r(\widehat{\Lambda}_s)$ is analytic on U.

Theorem 8.1. *Assume that hypotheses* (H4.1), (H4.2), *and* (H4.3) *are satisfied with $m \ge 1$ and that $H \subset \mathbb{R}$ is a bounded, open set. For $s \in \mathbb{R}$, let Λ_s and L_s be defined by* (8.1) *and* (8.2). *Then we have that $s \mapsto r(\Lambda_s)$ is log convex, i.e., $s \mapsto log(r(\Lambda_s))$ is convex on* $[0, \infty)$.

Proof. Because Theorem 4.1 implies that $r(L_s) = r(\Lambda_s)$ for all real s, it suffices to take $s_0 < s_1$, and $0 < t < 1$ and prove that

$$r(L_{(1-t)s_0 + ts_1}) \le r(L_{s_0})^{1-t} r(L_{s_1})^t.$$

We shall use an old trick (see [47] and the references therein). Let $v_{s_j}(x)$, $j = 0, 1$ denote the strictly positive eigenfunction of L_{s_j} which is ensured by Theorem 4.1. Then $L_{s_j} v_{s_j} = r(L_{s_j}) v_{s_j}$. For a fixed t, $0 < t < 1$, define $s_t = (1-t)s_0 + ts_1$ and

$$w_t(x) = [v_{s_0}(x)]^{1-t} [v_{s_1}(x)]^t.$$

Then, using Hölder's inequality, we find that

$$\begin{aligned}(L_{s_t}(w_t))(x) &= \sum_{b \in \mathcal{B}} [g_b(x)^{s_0} v_{s_0}(x)]^{1-t} [g_b(x)^{s_1} v_{s_1}(x)]^t \\ &\le \Big(\sum_{b \in \mathcal{B}} g_b(x)^{s_0} v_{s_0}(x) \Big)^{1-t} \Big(\sum_{b \in \mathcal{B}} g_b(x)^{s_1} v_{s_1}(x) \Big)^t \qquad (8.3) \\ &= [r(L_{s_0})^{1-t} r(L_{s_1})^t] w_t(x).\end{aligned}$$

Because $w_t(x) > 0$ for all $x \in \bar{H}$, a standard argument (see Lemma 5.9 in [50]) shows that

$$r(L_{s_t}) = \lim_{k\to\infty} \|L_{s_t}^k\|^{1/k} = \lim_{k\to\infty} \|L_{s_t}^k(w_t)\|^{1/k}. \tag{8.4}$$

Using inequalities (8.3) and (8.4), we see that $r(L_{s_t}) \le r(L_{s_0})^{1-t} r(L_{s_1})^t$. □

In general, if V is a convex subset of a vector space X, we shall call a map $f: V \to [0,\infty)$ log convex if (i) $f(x) = 0$ for all $x \in V$ or (ii) $f(x) > 0$ for all $x \in V$ and $x \mapsto \log(f(x))$ is convex. Products of log convex functions are log convex, and Hölders inequality implies that sums of log convex functions are log convex.

Results related to Theorem 8.1 can be found in [47], [31], [32], [9], [18], and [17]. Note that the terminology *super convexity* is used to denote log convexity in [31] and [32], presumably because any log convex function is convex, but not conversely. Theorem 8.1, while adequate for our immediate purposes, can be greatly generalized by a different argument that does not require existence of strictly positive eigenfunctions. This generalization (which we omit) contains Kingman's matrix log convexity result in [32] as a special case.

In our applications, the map $s \mapsto r(L_s)$ will usually be strictly decreasing on an interval $[s_1, s_2]$ with $r(L_{s_1}) > 1$ and $r(L_{s_2}) < 1$, and we wish to find the unique $s_* \in (s_1, s_2)$ such that $r(L_{s_*}) = 1$. The following hypothesis ensures that $s \mapsto r(L_s)$ is strictly decreasing for all S.

(H8.1) Assume that $g_b(\cdot)$, $b \in \mathcal{B}$ satisfy the conditions of (H4.1). Assume also that there exists an integer $\mu \ge 1$ such that $g_\omega(x) < 1$ for all $\omega \in \mathcal{B}_\mu$ and all $x \in \bar{H}$.

Theorem 8.2. *Assume hypotheses* (H4.1), (H4.2), (H4.3), *and* (H8.1) *are satisfied. Then the map* $s \mapsto r(\Lambda_s)$, $s \in \mathbb{R}$, *is strictly decreasing and real analytic and* $\lim_{s\to\infty} r(\Lambda_s) = 0$.

Proof. If $L_s: C(\bar{H}) \to C(\bar{H})$ is given by (4.1), it is a standard result that $r(L_s^\nu) = (r(L_s))^\nu$ and $r(\Lambda_s^\nu) = (r(\Lambda_s))^\nu$ for all integers $\nu \ge 1$, and Theorem 4.1 implies that $r(L_s) = r(\Lambda_s)$. Thus it suffices to prove that for some positive integer ν, $s \mapsto r(L_s^\nu)$ is strictly decreasing and $\lim_{s\to\infty} r(L_s^\nu) = 0$.

Suppose that K denotes the set of nonnegative functions in $C(\bar{H})$ and $A: C(\bar{H}) \to C(\bar{H})$ is a bounded linear map such that $A(K) \subset K$. If there exists $w \in C(\bar{H})$ such that $w(x) > 0$ for all $x \in \bar{H}$ and if $(A(w))(x) \le a w(x)$ for all $x \in \bar{H}$, it is well-known (and easy to verify) that $r(A) \le a$, where $r(A)$ denotes the spectral radius of A. In our situation, we take $\nu = \mu$, where μ is as in (H8.1), and $A = (L_s)^\mu$. If $s < t$ and v_s is the strictly positive eigenfunction for $(L_s)^\mu$, (H8.1) implies that there is a constant $c < 1$, $c = c(s,t)$, such that $cg_\omega(x)^s \geq g_\omega(x)^t$ for all $\omega \in \mathcal{B}_\mu$ and $x \in H$. Thus we find that

$$\begin{aligned} cr(L_s)^\mu v_s(x) &= \sum_{\omega\in\mathcal{B}_\mu} cg_\omega(x)^s v_s(\theta_\omega(x)) \\ &\geq \sum_{\omega\in\mathcal{B}_\mu} g_\omega(x)^t v_s(\theta_\omega(x)) \\ &= (L_t^\mu(v_s))(x). \end{aligned}$$

It follows that $r(L_t)^\mu \leq c(s,t)r(L_s)^\mu$, so $r(L_t) < r(L_s)$, for $s < t$. Because $0 < g_\omega(x) < 1$ for all $x \in \bar{H}$ and $\omega \in \mathcal{B}_\mu$, it is also easy to see that $\lim_{t\to\infty} \|(L_t)^\mu\| = 0$; and since $\|(L_t)^\mu\| \geq r(L_t^\mu)$, we get $\lim_{t\to\infty} r(L_t^\mu) = 0$. □

Remark 8.1. It is easy to construct examples for which (H8.1) is satisfied for some $\mu > 1$, but not satisfied for $\mu = 1$. The functions $\theta_1(x) := \frac{9}{x+1}$ and $\theta_2(x) := \frac{1}{x+2}$ both map the closed interval $\bar{H} = \left[\frac{1}{11}, 9\right]$ into itself. There is a unique nonempty compact set $J \subset \bar{H}$ such that $J = \theta_1(J) \cup \theta_2(J)$. For $s \in \mathbb{R}$, define

$$L_s : C(\bar{H}) \longrightarrow C(\bar{H})$$

by

$$(L_s w)(x) := \sum_{j=1}^{2} |D\theta_j(x)|^s w(\theta_j(x)) := \sum_{j=1}^{2} g_j(x)^s w(\theta_j(x)),$$

where $D := \frac{d}{dx}$. The Hausdorff dimension of J is the unique $s = s_*$, $0 < s_* < 1$, such that $r(L_s) = 1$. Our previous remarks show that

$$(L_s^2 w)(x) = \sum_{j=1}^{2}\sum_{k=1}^{2} |D(\theta_j \circ \theta_k)(x)|^s w(\theta_j \circ \theta_k)(x)).$$

One can check that (H8.1) is not satisfied for $\mu = 1$, but is satisfied for $\mu = 2$.

Remark 8.2. Assume that the assumptions of Theorem 8.2 are satisfied and define $\psi(x) = \log(r(L_s)) = \log(r(\Lambda_s))$ (where log denotes the natural logarithm), so $s \mapsto \psi(s)$ is a convex, strictly decreasing function with $\psi(0) > 1$ (unless $|\mathcal{B}| = p = 1$) and $\lim_{s\to\infty} \psi(s) = -\infty$. We are interested in finding the unique value of s such that $\psi(s) = 0$. In general suppose that $\psi : [s_1, s_2] \to \mathbb{R}$ is a continuous, strictly decreasing, convex function such that $\psi(s_1) > 0$ and $\psi(s_2) < 0$, so there exists a unique $s = s_* \in (s_1, s_2)$ with $\psi(s_*) = 0$. If t_1 and

t_2 are chosen so that $s_1 \leq t_1 < t_2 \leq s_*$ and t_{k+1} is obtained from t_{k-1} and t_k by the secant method, an elementary argument show that $\lim_{k\to\infty} t_k = s_*$. If $s_* \leq t_2 < t_1 < s_2$ and $s_1 \leq t_3$, a similar argument shows that $\lim_{k\to\infty} t_k = s_*$. If $\psi \in C^3$, elementary numerical analysis implies that the rate of convergence is faster than linear ($= \frac{1+\sqrt{5}}{2}$). In our numerical work, we apply these observations, not directly to $\psi(s) = \log(r(\Lambda_s))$, but to convex decreasing functions which closely approximate $\log(r(\Lambda_s))$.

One can also ask whether the maps $s \mapsto r(B_s)$ and $s \mapsto r(A_s)$ are log convex, where A_s and B_s are the previously described approximating matrices for L_s. An easier question is whether the map $s \mapsto r(M_s)$ is log convex, where A_s and B_s are obtained from M_s by adding error correction terms. We shall prove that $s \mapsto r(M_s)$ is log convex.

First, we need to recall a useful theorem of Kingman [32]. Let $M(s) = (a_{ij}(s))$ be an $m \times m$ matrix whose entries $a_{ij}(s)$ are either strictly positive for all s in a fixed interval J or are identically zero for all $s \in J$. Assume that $s \mapsto a_{ij}(s)$ is log convex on J for $1 \leq i, j \leq m$. Under these assumptions, Kingman [32] has proved that $s \mapsto r(M_s)$ is log convex.

Let $n \geq 2$ be a positive integer, and for $a < b$ given real numbers, define $x_k = a + kh$, $-1 \leq k \leq n+1$, $h = \frac{b-a}{n}$. Let X_n denote the vector space of real valued maps $\mathrm{w}: \{x_k : 0 \leq k \leq n\} \to \mathbb{R}$, so X_n is a real vector space linearly isomorphic to $\mathbb{R}^{n+1}$. As usual, if $\mathrm{w} \in X_n$, extend w to a map $w^I : [a,b] \to \mathbb{R}$ by linear interpolation, so

$$w^I(u) = \frac{u - x_k}{h}\mathrm{w}(x_{k+1}) + \frac{x_{k+1} - u}{h}\mathrm{w}(x_k), \quad x_k \leq u \leq x_{k+1},\ 0 \leq k \leq n.$$

For $1 \leq j \leq N$, assume that $\theta_j : [a,b] \to [a,b]$ are given maps and assume that $g_j : [a,b] \to (0,\infty)$ are given positive functions. For $s \in \mathbb{R}$, define a linear map $M_s : X_n \to X_n$ by

$$M_s\mathrm{w}(x_k) = \sum_{j=1}^{N} [g_j(x_k)]^s f^I(\theta_j(x_k)), \quad 0 \leq k \leq n,$$

so if $\mathrm{w}(x_k) \geq 0$ for $0 \leq k \leq n$, $g(x_k) \geq 0$ for $0 \leq k \leq n$. We can write $M_s\mathrm{w}(x_k) = \sum_{m=0}^{n} a_{km}(s)\mathrm{w}(x_m)$, where for $0 \leq k, m \leq n$,

$$a_{km}(s) = \sum_{j, x_{m-1} \leq \theta_j(x_k) \leq x_m} [g_j(x_k)]^s \frac{[\theta_j(x_k) - x_{m-1}]}{h} + \sum_{j, x_m \leq \theta_j(x_k) \leq x_{m+1}} [g_j(x_k)]^s \frac{[x_{m+1} - \theta_j(x_k)]}{h}.$$

If, for a given k and m, there is no j, $1 \le j \le N$, with $x_{m-1} \le \theta_j(x_k) \le x_{m+1}$, we define $a_{km} = 0$. Since the sum of log convex functions is log convex, $s \mapsto a_{km}(s)$ is log convex on $\mathbb{R}$. It follows from Kingman's theorem that $s \mapsto r(M_s)$ is log convex, where $r(M_s)$ denotes the spectral radius of M_s.

References

[1] K. E. Atkinson, *An introduction to numerical analysis.* Second edition. John Wiley & Sons, Inc., New York, 1989. MR 1007135 Zbl 0718.65001

[2] V. Baladi, *Positive transfer operators and decay of correlations.* Advanced Series in Nonlinear Dynamics, 16. World Scientific, River Edge, N.J., 2000. MR 1793194 Zbl 1012.37015

[3] G. Birkhoff, Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.* **85** (1957), 219–227. MR 0087058 Zbl 0079.13502

[4] F. F. Bonsall, Linear operators in complete positive cones. *Proc. London Math. Soc.* (3) **8** (1958), 53–75. MR 0092938 Zbl 0097.09803

[5] J. Bourgain and A. Kontorovich, *On Zaremba's conjecture. Ann. of Math.* (2) **180** (2014), no. 1, 137–196. MR 3194813 Zbl 1370.11083

[6] R. Bowen, Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 11–25. MR 556580 Zbl 0606.30023

[7] R. T. Bumby, Hausdorff dimensions of Cantor sets. *J. Reine Angew. Math.* **331** (1982), 192–206. MR 647383 Zbl 0468.10032

[8] R. T. Bumby, Hausdorff dimension of sets arising in number theory. D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn, and M. B. Nathanson, *Number theory.* Proceedings of the seminar held at the City University of New York, New York, 1983–1984. Lecture Notes in Mathematics, 1135. Springer-Verlag, Berlin, 1985, 1–8. MR 803348 Zbl 0575.28004

[9] J. E. Cohen, Convexity of the dominant eigenvalue of an essentially nonnegative matrix. *Proc. Amer. Math. Soc.* **81** (1981), no. 4, 657–658. MR 601750 Zbl 0459.15011

[10] T. W. Cusick, Continuants with bounded digits. *Mathematika* **24** (1977), no. 2, 166–172. MR 0472721 Zbl 0371.10023

[11] T. W. Cusick, Continuants with bounded digits. II. *Mathematika* **25** (1978), no. 1, 107–109. MR 0498413 Zbl 0388.10021

[12] S. P. Eveson and R. D. Nussbaum, Applications of the Birkhoff–Hopf theorem to the spectral theory of positive linear operators. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 3, 491–512. MR 1317492 Zbl 0834.47029

[13] S. P. Eveson and R. D. Nussbaum, An elementary proof of the Birkhoff–Hopf theorem. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 1, 31–55. MR 1297895 Zbl 0834.47028

[14] K. Falconer, *Techniques in fractal geometry.* John Wiley & Sons, Chichester, 1997. MR 1449135 Zbl 0869.28003

[15] R. S. Falk and R. D. Nussbaum, C^m eigenfunctions of Perron–Frobenius operators and a new approach to numerical computation of Hausdorff dimension. Preprint, 2016. arXiv:1601.06737 [math.DS]

[16] R. S. Falk and R. D. Nussbaum, C^m eigenfunctions of Perron–Frobenius operators and a new approach to numerical computation of Hausdorff dimension: applications to complex continued fractions. Preprint, 2016. arXiv:1612.00869 [math.DS]

[17] S. Friedland and S. Karlin, Some inequalities for the spectral radius of non-negative matrices and applications. *Duke Math. J.* **42** (1975), no. 3, 459–490. MR 0376717 Zbl 0373.15008

[18] Sh. Friedland, Convex spectral functions. *Linear and Multilinear Algebra* **9** (1980/81), no. 4, 299–316. MR 611264 Zbl 0462.15017

[19] R. J. Gardner and R. D. Mauldin, On the Hausdorff dimension of a set of complex continued fractions. *Illinois J. Math.* **27** (1983), no. 2, 334–345. MR 694647 Zbl 0492.10042

[20] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings.* Monographs and Textbooks in Pure and Applied Mathematics, 83. Marcel Dekker, New York, 1984. MR 744194 Zbl 0537.46001

[21] I. J. Good, The fractional dimensional theory of continued fractions. *Proc. Cambridge Philos. Soc.* **37** (1941). 199–228. MR 0004878 JFM 67.0988.03 Zbl 0061.09408

[22] S. M. Heinemann and M. Urbański, Hausdorff dimension estimates for infinite conformal IFSs. *Nonlinearity* **15** (2002), no. 3, 727–734. MR 1901102 Zbl 1015.37019

[23] D. Hensley, The Hausdorff dimensions of some continued fraction Cantor sets. *J. Number Theory* **33** (1989), no. 2, 182–198. MR 1034198 Zbl 0689.10060

[24] D. Hensley, Continued fraction Cantor sets, Hausdorff dimension, and functional analysis. *J. Number Theory* **40** (1992), no. 3, 336–358. MR 1154044 Zbl 0745.28005

[25] D. Hensley, A polynomial time algorithm for the Hausdorff dimension of continued fraction Cantor sets. *J. Number Theory* **58** (1996), no. 1, 9–45. MR 1387719 Zbl 0858.11039

[26] E. Hopf, An inequality for positive linear integral operators. *J. Math. Mech.* **12** (1963), 683–692. MR 0165325 Zbl 0115.32501

[27] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747. MR 625600 Zbl 0598.28011

[28] O. Jenkinson, On the density of Hausdorff dimensions of bounded type continued fraction sets: the Texan conjecture. *Stoch. Dyn.* **4** (2004), no. 1, 63–76. MR 2069367 Zbl 1089.28006

[29] O. Jenkinson and M. Pollicott, Computing the dimension of dynamically defined sets: E_2 and bounded continued fractions. *Ergodic Theory Dynam. Systems* **21** (2001), no. 5, 1429–1445. MR 1855840 Zbl 0991.28009

[30] O. Jenkinson and M. Pollicott, Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets. *Amer. J. Math.* **124** (2002), no. 3, 495–545. MR 1902887 Zbl 1002.37023

[31] T. Kato, Superconvexity of the spectral radius, and convexity of the spectral bound and the type. *Math. Z.* **180** (1982), no. 2, 265–273. MR 661703 Zbl 0471.46012

[32] J. F. C. Kingman, A convexity property of positive matrices. *Quart. J. Math. Oxford Ser.* (2) **12** (1961), 283–284. MR 0138632 Zbl 0101.25302

[33] M. A. Krasnosel′skiĭ, *Positive solutions of operator equations.* Translated from the Russian by R. E. Flaherty. Edited by L. F. Boron. Noordhoff, Groningen, 1964. MR 0181881 Zbl 0121.10604

[34] M. A. Krasnosel′skij, Je. A. Lifshits, and A. V. Sobolev, *Positive linear systems.* The method of positive operators. Translated from the Russian by J. Appell. Sigma Series in Applied Mathematics, 5. Heldermann Verlag, Berlin, 1989. MR 1038527 Zbl 0674.47036

[35] M. G. Kreĭn and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space. *Uspehi Matem. Nauk* (*N. S.*) **3** (1948). no. 1(23), 3–95. In Russian. English translation, *Amer. Math. Soc. Translation* **1950**, no. 26, 128 pp. MR 0027128 MR 0038008 (transl.) Zbl 0030.12902

[36] B. Lemmens and R. Nussbaum, Continuity of the cone spectral radius. *Proc. Amer. Math. Soc.* **141** (2013), no. 8, 2741–2754. MR 3056564 Zbl 1282.47068

[37] B. Lemmens and R. Nussbaum, Birkhoff's version of Hilbert's metric and its applications in analysis. In A. Papadopoulos and M. Troyanov (eds.), *Handbook of Hilbert geometry.* IRMA Lectures in Mathematics and Theoretical Physics, 22. European Mathematical Society (EMS), Zürich, 2014, 275–303. MR 3329884

[38] J. Mallet-Paret and R. D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators. *Discrete Contin. Dyn. Syst.* **8** (2002), no. 3, 519–562. MR 1897866 Zbl 1007.47031

[39] J. Mallet-Paret and R. D. Nussbaum, Generalizing the Kreĭn–Rutman theorem, measures of noncompactness and the fixed point index. *J. Fixed Point Theory Appl.* **7** (2010), no. 1, 103–143. MR 2652513 Zbl 1206.47044

[40] R. D. Mauldin and M. Urbański, Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc.* (3) **73** (1996), no. 1, 105–154. MR 1387085 Zbl 0852.28005

[41] R. D. Mauldin and M. Urbański, *Graph directed Markov systems.* Geometry and dynamics of limit sets. Cambridge Tracts in Mathematics, 148. Cambridge University Press, Cambridge, 2003. MR 2003772 Zbl 1033.37025

[42] R. D. Mauldin and S. C. Williams, Hausdorff dimension in graph directed constructions. *Trans. Amer. Math. Soc.* **309** (1988), no. 2, 811–829. MR 961615 Zbl 0706.28007

[43] C. T. McMullen, Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.* **120** (1998), no. 4, 691–721. MR 1637951 Zbl 0953.30026

[44] H. Minc, *Nonnegative matrices.* Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication. John Wiley & Sons, New York, 1988. MR 932967 Zbl 0638.15008

[45] R. D. Nussbaum, Periodic solutions of some nonlinear integral equations. In A. R. Bednarek and L. Cesari (eds.), *Dynamical systems.* Proceedings of a University of Florida International Symposium on Dynamical Systems held at the University of Florida, Gainesville, Fla., March 24–26, 1976. Academic Press, New York and London, 1977, 221–249. MR 0463844 Zbl 0538.45005

[46] R. D. Nussbaum, Eigenvectors of nonlinear positive operators and the linear Kreĭn–Rutman theorem. In E. Fadell and G. Fournier (eds.), *Fixed point theory.* Proceedings of a Conference held at the Université de Sherbrooke, Sherbrooke, Que., June 2–21, 1980. Lecture Notes in Mathematics, 886. Springer etc., Berlin, 1981, 309–330. MR 0643014 Zbl 0489.47037

[47] R. D. Nussbaum, Convexity and log convexity for the spectral radius. *Linear Algebra Appl.* **73** (1986), 59–122. MR 818894 Zbl 0588.15016

[48] R. D. Nussbaum, Periodic points of positive linear operators and Perron–Frobenius operators. *Integral Equations Operator Theory* **39** (2001), no. 1, 41–97. MR 1806843 Zbl 0987.47026

[49] R. D. Nussbaum, C^m positive eigenvectors for linear operators arising in the computation of Hausdorff dimension. *Integral Equations Operator Theory* **84** (2016), no. 3, 357–393. MR 3463454 Zbl 1341.47013

[50] R. D. Nussbaum, A. Priyadarshi, and S. Verduyn Lunel, Positive operators and Hausdorff dimension of invariant sets. *Trans. Amer. Math. Soc.* **364** (2012), no. 2, 1029–1066. MR 2846362 Zbl 1280.37029

[51] A. Priyadarshi, *Hausdorff dimension of invariant sets and positive linear operators.* Ph.D. Thesis, Rutgers University, New Brunswick, N.J., 2011. MR 2996073

[52] D. Ruelle, *Thermodynamic formalism.* The mathematical structures of classical equilibrium statistical mechanics. With a foreword by G. Gallavotti and G.-C. Rota. Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley, Reading, MA, 1978. MR 511655 Zbl 0401.28016

[53] D. Ruelle, Bowen's formula for the Hausdorff dimension of self-similar sets. In J. Fröhlich (ed.), *Scaling and self-similarity in physics.* Renormalization in statistical mechanics and dynamics. Lectures presented at the seminar held at the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, 1981/1982. Birkhäuser Boston, Boston, MA, 1983, 351–358. MR 733478

[54] H. H. Rugh, On the dimensions of conformal repellers. Randomness and parameter dependency. *Ann. of Math.* (2) **168** (2008), no. 3, 695–748. MR 2456882 Zbl 1181.37066

[55] H. Samelson, On the Perron–Frobenius theorem. *Michigan Math. J.* **4** (1957), 57–59. MR 0086041 Zbl 0077.02303

[56] H. H. Schaefer and M. P. Wolff, *Topological vector spaces.* Second edition. Graduate Texts in Mathematics, 3. Springer-Verlag, New York, 1999. MR 1741419 Zbl 0983.46002

[57] A. Schief, Self-similar sets in complete metric spaces. *Proc. Amer. Math. Soc.* **124** (1996), no. 2, 481–490. MR 1301047 Zbl 0844.28004

Received August 23, 2016

Richard S. Falk, Department of Mathematics – Hill Center, Rutgers University, Piscataway, NJ 08854-8019, USA

e-mail: falk@math.rutgers.edu

Roger D. Nussbaum, Department of Mathematics – Hill Center, Rutgers University, Piscataway, NJ 08854-8019, USA

e-mail: nussbaum@math.rutgers.edu