Global fractal transformations and global addressing

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Abstract. The attractor is a central object of an iterated function system (IFS), and *fractal transformations* are the natural maps from the attractor of one IFS to the attractor of another. This paper presents a global point of view, showing how to extend the domain of a fractal transformation from an attractor with non-empty interior to the ambient space. Intimately related is the extension of addressing from such an attractor to the set of points of the ambient space. Properties of such global fractal transformations are obtained, and tilings are constructed based on global addresses.

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Contents

1. Introduction

An iterated function system (IFS) is a standard method for the construction of deterministic fractals, the attractor of the IFS (usually) being a fractal. A fundamental property of the attractor is that its points have addresses, an address being a word in the alphabet $[N] = \{1, 2, \ldots, N\}$, where N is the number of functions in the IFS. Addressing of points of the attractor is well known; see for example [1], [15], [16]. A method, called masking, can be used to assign to each point of the attractor a unique address. The natural maps from the attractor of one IFS to the attractor of another are fractal transformations. Given attractors $A_{\mathcal{F}}$ and $A_{\mathcal{G}}$ of two IFSs $\mathcal{F}$ and $\mathcal{G}$ with the same number of functions, a *fractal transformation* is basically a transformation that takes each point in $A_{\mathcal{F}}$ to a point in $A_{\mathcal{G}}$ with the same address. Fractal transformations between attractors were introduced by M. Barnsley [4], although mappings between self-similar sets defined by pairing points with the same address are implicit in early examples like the Cantor function (devil's staircase) [14]. Fractal transformations have proved to be intriguing mathematical objects with potential application to image processing; see [5], [6], [7], [8], [9]. Hilbert's space filing curve can be constructed as such a transformation; see Example 6.4. A fractal Fourier theory can be based on particular fractal transformations [3]. The iPad app Frango Camera (Frango Studios Pty. Ltd) is a recreational application.

Given that a fractal transformation $T\colon A_{\mathcal{F}} \to A_{\mathcal{G}}$ takes one attractor to another, it is natural to ask whether there is a natural mapping defined from the ambient space of $\mathcal{F}$ to the ambient space of $\mathcal{G}$ for which T is just the restriction to the attractor. More precisely, let $\mathcal{F}$ and $\mathcal{G}$ be IFSs on complete metric spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively, and let $T\colon A_{\mathcal{F}} \to A_{\mathcal{G}}$ be a fractal transformation from an attractor with non-empty interior of $\mathcal{F}$ to the attractor of $\mathcal{G}$. The new perspective in this paper is of a *global fractal transformation* as a mapping $\mathbb{X} \to \mathbb{Y}$ on the ambient spaces. In fact, given $T\colon A_{\mathcal{F}} \to A_{\mathcal{G}}$, there are global fractal transformations $T_\theta\colon \mathbb{X} \to \mathbb{Y}$, where θ ranges over an infinite parameter set, each such transformation T_θ agreeing with T on $A_{\mathcal{F}}$. Figure 1 shows (part of) a checker board pattern on $\mathbb{R}^2$ and (part of) its image under two such global fractal transformations of the plane. Moreover, these particular fractal transformations are area preserving homeomorphisms (see Example 7.3).

Inexorably linked to the extension of a fractal transformation to a global fractal transformation is the extension of addressing from points of the attractor to points on the ambient space. Applications of addressing of points of the attractor, with respect to both the topology and dynamics of fractals, are numerous; see for example [2], [12], [25]. An application of this global addressing that is developed in this paper is the creation of tilings like those in Figures 9, 10, and 11.
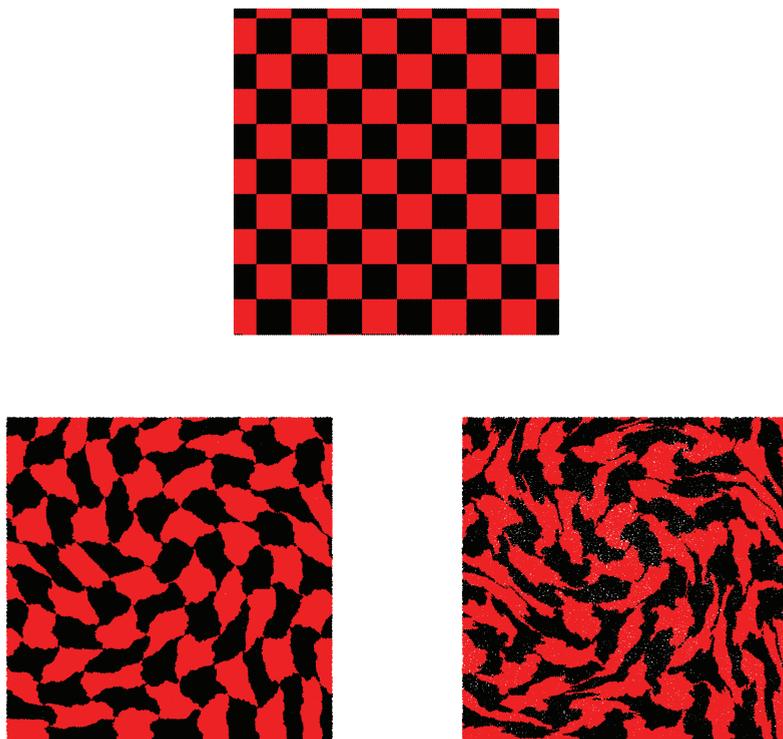
Figure 1. A checkerboard pattern in the plane and its image under two area preserving fractal homeomorphisms of the plane.

1.1. Organization of the paper. Definitions and notation related to attractors and their addressing are contained in Section 2. Masks and their corresponding sections are the subject of Section 3, and fractal transformations are covered Section 4. Precisely how a fractal transformation can be naturally extended from the attractor with non-empty interior to the ambient space is explained in Section 4 (Definition 4.1). Equivalent formulations of such a global fractal transformation are given by statement (2) of Theorem 5.3 and, in the case of a continuous fractal transformation, by Corollary 6.10.

Fundamental properties of fractal transformations in the large is the subjects of Sections 6 and 7. Section 6 concerns properties of continuous global fractal transformations. Theorem 6.2 provides sufficient conditions for a global fractal transformation to be continuous in terms of the address structure, and Theorem 6.12 provides conditions, in a special case of affine IFSs, in geometric terms. Theorem 6.7 concerns global fractal homeomorphisms. If the attractors of IFSs $\mathcal{F}$ and $\mathcal{G}$ are non-overlapping (defined in Sections 5), then the graph of a continuous

fractal transformation is itself the attractor of an appropriately defined IFS (Theorem 6.8), and the graph of a global fractal transformation has the simple form given by Theorem 6.11. Although a global fractal transformation T depends on a mask and on the parameter θ, whether or not T is continuous is independent of the mask and independent of θ in the non-overlapping case (Corollary 6.3).

In Section 7 it is shown that, in the case that $\mathbb{X} = \mathbb{Y} = \mathbb{R}^d$, under suitable conditions, a global fractal transformation is independent of the mask, is continuous and invertible almost everywhere, and is volume preserving (Theorem 7.1). This result is used to produce area preserving fractal homeomorphisms of the plane based on IFSs with triangular attractors (Corollary 7.2).

Global addressing is the subject of Section 5. This is used to create tilings of $\mathbb{X}$ based on addresses, a scheme that extends and simplifies previous tiling constructions. The basic idea, explained precisely by Theorem 8.3, is to define a tile as the set of all points of $\mathbb{X}$ whose address begins with a given prefix. Examples are given in Section 8.

2. Iterated function system, its attractor and coding map

In this paper an *iterated function system* (*IFS*) is denoted by

$$\mathcal{F} = \{\mathbb{X} : f_1, f_2, \dots, f_N\},$$

where $\mathbb{X}$ is a complete metric space and each $f_n : \mathbb{X} \to \mathbb{X}$, is a contraction *with a continuous inverse* for $n = 1, 2, 3, \dots N$. Let $\mathbb{H}$ be the collection of nonempty compact subsets of $\mathbb{X}$ and define $F : \mathbb{H} \to \mathbb{H}$ by

$$F(C) = \bigcup_{f \in \mathcal{F}} f(C) \tag{1}$$

for all $C \in \mathbb{H}$. Define $F^0(C) = C$ and let $F^k(C)$ denote the k-fold composition of F applied to C, namely, the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(C)$ over all finite words $i_1 i_2 \dots i_k$ of length k. It is a basic result due to Hutchinson [20] that, given such an IFS $\mathcal{F}$, there is a unique compact set A, called the *attractor* of $\mathcal{F}$, such that

(1) $F(A) = A$, and

(2) $\lim_{k \to \infty} F^k(C) = A$, for all $C \in \mathbb{H}$,

where the limit is with respect to the Hausdorff metric.

Let $[N] = \{1, 2, 3, \ldots, N\}$. Let

$$I = [N]^\infty$$

be the set of all infinite words $\sigma_1 \sigma_2 \sigma_3 \ldots$ over the alphabet $[N]$, and $[N]^*$ the set of all finite words over the alphabet $[N]$ (including the empty word). If $\sigma \in [N]^*$, then $|\sigma|$ denotes its length. To simplify the notation for composition of functions, for all $\sigma = \sigma_1 \sigma_2 \sigma_3 \ldots \in I$ and $k \in [N]$, let

$$
\begin{aligned}
\sigma|k &= \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_k & f_{\sigma|k} &= f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k} \\
\overleftarrow{\sigma|k} &= \sigma_k \sigma_{k-1} \ldots \sigma_1 & f_{\overleftarrow{\sigma|k}} &= f_{\sigma_k} \circ f_{\sigma_{k-1}} \circ \cdots \circ f_{\sigma_1}.
\end{aligned}
\tag{2}
$$

Let d be the metric on I defined by

$$
d(\sigma, \omega) =
\begin{cases}
2^{-\min\{k : \sigma_k \neq \omega_k\}} & \text{if } \sigma \neq \omega, \\
0 & \text{if } \sigma = \omega,
\end{cases}
\tag{3}
$$

for all $\sigma, \omega \in I$. In other words, σ and ω are close if they agree on a large initial segment. The *shift map* $S : I \to I$ defined by

$$S(\sigma_1 \sigma_2 \ldots) = \sigma_2 \sigma_3 \ldots$$

is continuous with respect to this metric. The metric space (I, d) is called the *code space* for the attractor A of $\mathcal{F}$, and the *coding map* $\pi : I \to A$ is defined by

$$\pi(\sigma) = \lim_{k \to \infty} f_{\sigma|k}(C). \tag{4}$$

It is well known that the limit is a single point in A independent of $C \in \mathbb{H}$, that π is continuous and surjective. The set-valued inverse $\pi^{-1}(a)$ comprises the *set of addresses* of the point $a \in A$.

3. Masks and sections

Sections and masks for an IFS $\mathcal{F}$ are prerequisite concepts in defining a fractal transformation. A *section* of the coding map $\pi : I \to A$ of $\mathcal{F}$ with attractor A is a map $\tau : A \to I$ such that $\pi \circ \tau = \mathrm{id}$, the identity on $\mathbb{X}$. A section τ thus assigns to each point of the attractor a unique address among its set of addresses. The set

$$\Omega_A := \tau(A)$$

will be referred to as the *address space* of $\mathcal{F}$ with respect to the section τ. It is natural to require that the shift of an address in Ω_A also be an address in Ω_A (see Example 3.1). More precisely, a section τ of the coding map is *shift invariant* if $S(\Omega_A) \subseteq \Omega_A$.

Example 3.1. Consider the IFS $\mathcal{F} = \{\mathbb{R}: f_0(x) = x/2,\, f_1(x) = x/2 + 1/2\}$. We use the indices 0 and 1 rather than 1 and 2 for reasons that will be clear below. The attractor is the interval $[0, 1]$ and

$$\pi(\sigma_1 \sigma_2 \ldots) = \frac{1}{2}\sigma_1 + \left(\frac{1}{2}\right)^2 \sigma_2 + \left(\frac{1}{2}\right)^3 \sigma_3 + \cdots$$

for all binary words $\sigma \in \{0, 1\}^\infty$. A section τ of π assigns to each $x \in [0, 1]$ a base 2 decimal expansion of x (omitting the decimal point). For example, the set of addresses of $x = 1/2$ is $\pi^{-1}(1/2) = \{0111\ldots, 1000\ldots\}$. If $\tau(1/2) = 0111\ldots$ and if τ is shift invariant, then $\tau(1/4) = 00111\ldots$ and $\tau(1/8) = 000111\ldots$, etc. And if $\tau(1/2) = 1000\ldots$ and if τ is shift invariant, then $\tau(1/4) = 01000\ldots$ and $\tau(1/8) = 001000\ldots$, etc.

A method for obtaining a shift invariant section for $\mathcal{F}$ is by way of a mask. A *mask* for the attractor A of an IFS $\mathcal{F} = \{\mathbb{X}: f_1, f_2, \ldots, f_N\}$ is a partition $\{M_1, M_2, \ldots, M_N\}$ of A such that $M_n \subseteq f_n(A)$ for all $n \in [N]$. A mask defines a dynamical system $\phi: A \to A$, called a *masked dynamical system*, according to

$$\phi(x) = f_n^{-1}(x) \quad \text{for } x \in M_n,$$

for all $n \in [N]$. The masked dynamical system determines a *section* $\tau: A \to I$ of the coding map $\pi: I \to A$ defined as follows:

$$(\tau(x))_k = j \quad \text{if } \phi^{k-1}(x) \in M_j.$$

A section τ of π constructed in this manner is called a *masked section*. The two sections in Example 3.1 are masked sections corresponding to the masks $\{[0, 1/2], (1/2, 1]\}$ and $\{[0, 1/2), [1/2, 1]\}$.

The particular mask $\{M_1, M_2, \ldots, M_N\}$, defined by

$$M_i = f_i(A) \setminus \bigcup_{j=1}^{i-1} M_j, \quad i = 1, 2, \ldots, N,$$

is called the *tops mask*. In fact, it was the tops mask that was used when the concept of fractal transformation was introduced [4]. The section τ corresponding to the tops mask is

$$\tau(x) = \max \pi^{-1}(x),$$

where the maximum is with respect to the lexicographic order on I with $1 > 2 > 3 > \cdots > N$.

In light of the following theorem, proved in [9], *all sections in this paper are assumed to be shift invariant.*

Theorem 3.2. *A section* $\tau\colon A \to I$ *is shift invariant if and only if it is a masked section.*

Let $\sigma \in [N]^*$. It will be useful in Section 8 to understand which are the points in the attractor A whose address in Ω_A begins with σ. The answer is provided by Proposition 3.3 below. Define a nested sequence of partitions of A starting with a mask M as follows. For each $k \geq 0$, define a partition $M^k = \{M_\sigma\colon |\sigma| = k\}$ of A recursively by letting $M^1 = M$ and

$$M^{k+1} = \{M_{\sigma j} = M_\sigma \cap f_\sigma(M_j)\colon |\sigma| = k, 1 \leq j \leq N\}. \tag{5}$$

Note that $M_\sigma = \emptyset$ is possible. The partition M^k will be called the kth *level mask.*

Proposition 3.3. *Given a mask M and the corresponding section τ, if $|\sigma| = k$, then*

$$M_\sigma = \{a \in A\colon \tau(a)|k = \sigma\}.$$

Proof. From the definition of section by way of the masked dynamical system, it readily follows that

$$\{a \in A\colon \tau(a)|k = \sigma\} = \bigcap_{j=1}^{k} f_{\sigma|j-1}(M_{\sigma_j}).$$

A straightforward induction using formula (5) suffices to verify the last equality above. $\qquad\qquad\square$

4. Global fractal transformations

Definition 4.1. Let $\mathcal{F} = \{\mathbb{X}\colon f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{\mathbb{Y}\colon g_1, g_2, \ldots, g_N\}$ be two IFSs with attractors $A_{\mathcal{F}}$ and $A_{\mathcal{G}}$, respectively. If $\tau_{\mathcal{F}}\colon A_{\mathcal{F}} \to I$ is a section for coding map $\pi_{\mathcal{F}}\colon I \to A_{\mathcal{F}}$, the *fractal transformation* associated with $\tau_{\mathcal{F}}$ is the map $T_{\mathcal{F}\mathcal{G}}\colon A_{\mathcal{F}} \to A_{\mathcal{G}}$ defined by

$$T_{\mathcal{F}\mathcal{G}} = \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}}.$$

Let $\theta \in I$ be fixed. Call θ *full* with respect to $\mathcal{F}$ if, for all $x \in \mathbb{X}$, there is a $k = k(x)$ such that

$$f_{\overleftarrow{\theta|k}}(x) = f_{\theta_k} \circ f_{\theta_{k-1}} \ldots \in f_{\theta_1}(x) \in A_{\mathcal{F}}.$$

Note that, for θ to be full with respect to $\mathcal{F}$, it is necessary for $A_{\mathcal{F}}$ to have nonempty interior; this follows from the Baire Category Theorem. Conversely, if $A_{\mathcal{F}}$ has nonempty interior, then θ is full for almost all $\theta \in I$ in the following two senses [10].

(1) The full words are dense in the coding space I. In particular, a word $\theta \in I$ is *disjunctive* if every finite word is a subword of θ. For instance, the binary Champernowne sequence $0100011011000001\ldots$, formed by concatenating all binary strings in lexicographic order, is disjunctive. The set of disjunctive words is dense in I, and every disjunctive word is full.

(2) Define a word $\theta \in [N]^\infty$ to be a *random word* if there is a $p > 0$ such that each $\theta_k, k = 1, 2, \ldots$, is selected at random from $\{1, 2, \ldots, N\}$ where the probability that $\theta_k = n, n \in [N]$, is greater than or equal to p, independent of the preceding outcomes. With probability 1 a random word θ is full.

It will be assumed henceforth that θ is full. Define

$$K = K(x) = \min\{k \colon f_{\overleftarrow{\theta|k}}(x) \in A_{\mathcal{F}}\}.$$

Now let $T_{\mathcal{F}\mathcal{G}} \colon A_{\mathcal{F}} \to A_{\mathcal{G}}$ be the fractal transformation associated with a section $\tau_{\mathcal{F}}$, and define

$$\widehat{T}_{\mathcal{F}\mathcal{G}} \colon \mathbb{X} \longrightarrow \mathbb{Y},$$

$$\widehat{T}_{\mathcal{F}\mathcal{G}}(x) = g_{-(\theta|K)} \circ T_{\mathcal{F}\mathcal{G}} \circ f_{\overleftarrow{\theta|K}}(x). \tag{6}$$

The map $\widehat{T}_{\mathcal{F}\mathcal{G}}$ will be called the *global fractal transformation* with respect to θ. Note that there is a global fractal transformation $\widehat{T}_{\mathcal{F}\mathcal{G}}$ for each $\theta \in I$ that is full with respect to $\mathcal{F}$.

5. Addressing points of $\mathbb{X}$

Fix $\theta = \theta_1\theta_2\theta_3 \ldots \in I$ throughout this section, and introduce the notation

$$-(\theta|k) = -(\theta_1\theta_2\theta_3 \ldots \theta_k) = (-\theta_1)(-\theta_2)(-\theta_3)\ldots(-\theta_k).$$

Define

$$\mathbb{I} = \{-(\theta|k)\sigma \colon k \geq 0, \sigma \in I, \theta_k \neq \sigma_1\}$$

which will be called the *global code space* or *θ-code space* to emphasis its dependence on θ. The global code space is a compact metric space with respect to

the metric given in (3). Extend the notation of Equation 2 as follows. For $j \in [N]$ and $\alpha \in \mathbb{I}$:

$$f_{-j} = (f_j)^{-1} \quad f_{\alpha|k} = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_k}$$

For example, $f_{-(\theta|k)} = f_{\theta_1}^{-1} \circ f_{\theta_2}^{-1} \circ \cdots \circ f_{\theta_k}^{-1}$.

Let $\mathcal{F}$ be an IFS with attractor A. In this section the domain of the coding map $\pi : I \to A$ of Equation (4) is extended from I to $\mathbb{I}$ and the domain of a section $\tau : A \to I$ is extended from A to $\mathbb{X}$. The notation $\hat{\pi}$ and $\hat{\tau}$ will be used for the extended versions. To do this define

$$\hat{\pi}(\alpha) = \lim_{k \to \infty} f_{\alpha|k}(C),$$

for all $\alpha \in \mathbb{I}$, where $C \in \mathbb{H}$. The limit exists and is independent of C because, by the definition of $\mathbb{I}$, there is a nonnegative integer j such that $S^j(\alpha) \in I$, which means that we can write

$$\hat{\pi}(\alpha) = f_{\alpha|j}(\pi(S^j(\alpha)).$$

The map

$$\hat{\pi} : \mathbb{I} \longrightarrow \mathbb{X}$$

will be called the *global coding map* of $\mathcal{F}$. If the attractor A of an IFS $\mathcal{F}$ has nonempty interior and θ is full, then the global coding map $\hat{\pi} : \mathbb{I} \to \mathbb{X}$ is continuous and surjective.

Definition 5.1. Let $\mathcal{F}$ be an IFS with attractor A. Fix a $\theta \in I$ that is full with respect to $\mathcal{F}$, and fix a mask for A with corresponding section $\tau : A \to I$. For $x \in \mathbb{X}$, recall the notation in Definition 4.1:

$$K = K(x) = \min\{k : a := f_{\overleftarrow{\theta|k}}(x) \in A\}.$$

Define $\hat{\tau} : \mathbb{X} \to \mathbb{I}$ by the following concatenation:

$$\hat{\tau}(x) = -(\theta|K)\tau(a).$$

Clearly $\hat{\tau}$ is an extension of τ, i.e., $\hat{\tau}(x) = \tau(x)$ for $x \in A$. It is easy to check that $\hat{\tau}(x) \in \mathbb{I}$ for all $x \in \mathbb{X}$. Thus $\hat{\tau}$ provides a unique address in $\mathbb{I}$ to each point of $\mathbb{X}$. The map $\hat{\tau}$ will be called a *global section* or θ-*section* of the global coding map $\hat{\pi} : \mathbb{I} \to \mathbb{X}$. The set

$$\Omega_{\mathbb{X}} = \hat{\tau}(\mathbb{X})$$

will be called the *address space* of $\mathcal{F}$ with respect to $\hat{\tau}$.

Example 5.2. Consider the IFS $\mathcal{F} = \{\mathbb{R}: f_0(x) = x/2, f_1(x) = x/2 + 1/2\}$ of Example 3.1, whose attractor is the interval $[0, 1]$. As discussed in Example 3.1, a section $\tau: [0, 1] \rightarrow I$, in this case, is obtained by letting $\tau(x)$ be the binary decimal expansion of x (without the decimal point) for all $x \in [0, 1]$. Actually there are two shift invariant sections, depending on whether $\tau(1/2) = 1000\ldots$ or $\tau(1/2) = 0111\ldots$. For this example let $\theta = 010101\ldots \in I$, which is full with respect to $\mathcal{F}$. Using Definition 5.1 and a little calculation, a global section (θ-section) $\hat{\tau}: \mathbb{R} \rightarrow \mathbb{I}$ can be obtained:

$$\hat{\tau}(x) = \begin{cases} (-0-1)^n \sigma_1 \\ \qquad \text{if } x \in \left[-\frac{2}{3}(4^n - 1), -\frac{2}{3}(4^{n-1} - 1) \right), n = 1, 2, 3, \ldots, \\ \tau(x) \\ \qquad \text{if } x \in [0, 1], \\ (-0-1)^n (-0)\sigma_2 \\ \qquad \text{if } x \in \left(\frac{1}{3}(4^n + 2), \frac{1}{3}(4^{n+1} + 2) \right], n = 0, 1, 2, \ldots, \end{cases}$$

where

$$\sigma_1 = \tau\left(\frac{1}{4^n}\left(x - \frac{2}{3}\right) + \frac{2}{3} \right) \quad \text{and} \quad \sigma_2 = \tau\left(\frac{1}{4^n}\left(\frac{x}{2} - \frac{1}{3}\right) + \frac{1}{3} \right).$$

If e is the base of the natural logarithm, for example, then

$$\hat{\tau}(e) = (-0)(-1)(-0)\tau((e + 2)/8)$$
$$= (-0)(-1)(-0)110101101111110\ldots.$$

The following theorem shows that the global fractal transformation $\hat{T}: \mathbb{X} \rightarrow \mathbb{Y}$ satisfies the same property that defines the fractal transformation $T: A_{\mathcal{F}} \rightarrow A_{\mathcal{G}}$.

Theorem 5.3. *Let $\mathcal{F}$ be an IFS with global coding map $\hat{\pi}_{\mathcal{F}}: \mathbb{I} \rightarrow \mathbb{X}$, and global section $\hat{\tau}_{\mathcal{F}}: \mathbb{X} \rightarrow \mathbb{I}$. Let $\mathcal{G}$ be an IFS with the same number of functions as $\mathcal{F}$ and with global coding map $\hat{\pi}_{\mathcal{G}}: \mathbb{I} \rightarrow \mathbb{Y}$. If $T_{\mathcal{F}\mathcal{G}}$ is a corresponding global fractal transformation, then*

(1) $\hat{\pi}_{\mathcal{F}} \circ \hat{\tau}_{\mathcal{F}} = \text{id}$, *and*

(2) $\hat{T}_{\mathcal{F}\mathcal{G}} = \hat{\pi}_{\mathcal{G}} \circ \hat{\tau}_{\mathcal{F}}$.

Proof. Concerning statement (1), assume that $x \in \mathbb{X}$ and that $a \in A_{\mathcal{F}}$ is the point in Definition 5.1. Further let $\tau_{\mathcal{F}}(a) = \sigma$. We have

$$\hat{\pi}_{\mathcal{F}} \circ \hat{\tau}_{\mathcal{F}}(x) = \lim_{j \to \infty} f_{-(\theta|K)(\tau_{\mathcal{F}}(a)|j)}(A)$$

$$= f_{-(\theta|K)} \lim_{j \to \infty} f_{\tau_{\mathcal{F}}(a)|j}(A)$$

$$= f_{-(\theta|K)} \circ \pi_{\mathcal{F}} \circ \tau_{\mathcal{F}}(a)$$

$$= f_{-(\theta|K)}(a)$$

$$= f_{-(\theta|K)} \circ f_{\overleftarrow{\theta|K}}(x)$$

$$= x.$$

Concerning statement (2), for all $x \in \mathbb{X}$ we have

$$\hat{T}(x) = g_{-(\theta|K)} \circ T \circ f_{\overleftarrow{\theta|K}}(x)$$

$$= g_{-(\theta|K)} \circ \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ f_{\overleftarrow{\theta|K}}(x)$$

$$= g_{-(\theta|K)} \circ \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}}(a)$$

$$= \hat{\pi}_{\mathcal{G}}(-(\theta|K)\tau_{\mathcal{F}}(a))$$

$$= \hat{\pi}_{\mathcal{G}} \circ \hat{\tau}_{\mathcal{F}}(x). \qquad \square$$

The following lemma will be helpful in the next section. For a set B, its closure is denoted by $\bar{B}$ and its interior by B^o. The attractor A of $\mathcal{F}$ is *non-overlapping* if $(f(A) \cap f'(A))^o = \emptyset$ for all distinct $f, f' \in \mathcal{F}$.

Lemma 5.4. *If the attractor A of an IFS is non-overlapping and $\Omega_{\mathbb{X}}$ is the address space with respect to a θ-section, then $\overline{\Omega_{\mathbb{X}}} = \mathbb{I}$.*

Proof. It is proved in [3] that, if A is non-overlapping and if D is the set of words $\sigma \in I$ such that $\pi(\sigma)$ has a unique address, the D is dense in I. Assume that $\alpha = -(\theta|k)\omega \in \mathbb{I}$. To show that $\alpha \in \overline{\Omega_{\mathbb{X}}}$, let $\sigma^1, \sigma^2, \sigma^2, \ldots$ be a sequence of words in D such that $\lim_{n \to \infty} \sigma_n = \omega$. Then $\alpha^n = -(\theta|k)\sigma^n \to \alpha$ as $n \to \infty$. Since $\omega_1 \neq \theta_k$, it is also the case that $\sigma_1^n \neq \theta_k$ for n sufficiently large. If it can be shown that $\alpha^n \in \Omega_{\mathbb{X}}$, then $\alpha \in \overline{\Omega_{\mathbb{X}}}$, which completes the proof.

To show that $\alpha^n \in \Omega_{\mathbb{X}}$, let $a_n = \pi(\sigma^n)$ and $x_n = f_{-(\theta|k)}(a_n)$. By the definition of the global section, $\hat{\tau}(x_n) = \alpha^n$ unless $f_{\overleftarrow{\theta|k-1}}(x_n) \in A$. But in this case $a_n \in f_{\theta_k}(A)$, which implies that a_n has an address that begins with θ_k. On the other hand, for n sufficiently large, σ^n is an address of a_n that does not begin with θ_k. This contradicts the fact that $\sigma_n \in D$ for all n. $\qquad \square$

6. Continuity of global fractal transformations

Let $\mathcal{F} = \{\mathbb{X}: f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{\mathbb{Y}: g_1, g_2, \ldots, g_N\}$. It is assumed through-out this section that the word $\theta \in I$ is full with respect to $\mathcal{F}$, that $\tau_{\mathcal{F}}$ is a section of $\mathcal{F}$, and that $\hat{\tau}_{\mathcal{F}}$ is a corresponding global θ-section. Let $T_{\mathcal{F}\mathcal{G}}: A_{\mathcal{F}} \to A_{\mathcal{G}}$ be the fractal transformation associated with $\tau_{\mathcal{F}}$ and $\widehat{T}_{\mathcal{F}\mathcal{G}}: \mathbb{X} \to \mathbb{Y}$ the global fractal transformation associated with $\hat{\tau}_{\mathcal{F}}$.

For an IFS $\mathcal{F}$, let $P_{\mathcal{F}} = \{\pi_{\mathcal{F}}^{-1}(x): x \in A_{\mathcal{F}}\}$, which is a partition of the code space I. Similarly, let $\widehat{P}_{\mathcal{F}} = \{\hat{\pi}_{\mathcal{F}}^{-1}(x): x \in \mathbb{X}\}$, which is a partition of the global code space $\mathbb{I}$. A partition P is *finer* than a partition Q if for each part X in P there is a part Y in Q such that $X \subseteq Y$.

Lemma 6.1. *If $P_{\mathcal{F}}$ is a finer partition than $P_{\mathcal{G}}$, then $\widehat{P}_{\mathcal{F}}$ is a finer partition than $\widehat{P}_{\mathcal{G}}$.*

Proof. Assume that $\hat{\pi}_{\mathcal{F}}(\alpha) = \hat{\pi}_{\mathcal{F}}(\beta)$. Letting $\alpha = -(\theta|k)\sigma$ and $\beta = -(\theta|j)\omega$, we have $x := \hat{\pi}_{\mathcal{F}}(-(\theta|k)\sigma) = \hat{\pi}_{\mathcal{F}}(-(\theta|j)\omega)$, where $j \leq k$ and $\sigma, \omega \in I$. Then

$$
\begin{aligned}
\pi_{\mathcal{F}}(\sigma) &= f_{\overleftarrow{\theta|k}}(x) \\
&= f_{\theta_k} \circ \cdots \circ f_{\theta_{j+1}} \circ f_{\overleftarrow{\theta|j}}(x) \\
&= f_{\theta_k} \circ \cdots \circ f_{\theta_{j+1}}(\pi_{\mathcal{F}}(\omega)) \\
&= \pi_{\mathcal{F}}(\theta_k \ldots \theta_{j+1}\omega).
\end{aligned}
$$

Because σ and $\theta_k \ldots \theta_{j+1}\omega$ are in the same set of partition $P_{\mathcal{F}}$, they are also in the same set of $P_{\mathcal{G}}$. Therefore

$$
\begin{aligned}
\hat{\pi}_{\mathcal{G}}(\alpha) &= \hat{\pi}_{\mathcal{G}}(-(\theta|k)\sigma) \\
&= g_{-(\theta|k)}(\pi_{\mathcal{G}}(\sigma)) \\
&= g_{-(\theta|k)}(\pi_{\mathcal{G}}(\theta_k \ldots \theta_{j+1}\omega)) \\
&= \hat{\pi}_{\mathcal{G}}(-(\theta|j)\omega) \\
&= \hat{\pi}_{\mathcal{G}}(\beta). \qquad \square
\end{aligned}
$$

Theorem 6.2. (1) *If $P_{\mathcal{F}}$ is a finer partition than $P_{\mathcal{G}}$, then the global fractal transformation $\widehat{T}_{\mathcal{F}\mathcal{G}}: \mathbb{X} \to \mathbb{Y}$ is continuous.*

(2) *If $T_{\mathcal{F}\mathcal{G}}: A_{\mathcal{F}} \to A_{\mathcal{G}}$ is continuous and the attractor of $\mathcal{F}$ is non-overlapping, then $P_{\mathcal{F}}$ is a finer partition than $P_{\mathcal{G}}$.*

Proof. The proof of statement (1) follows along similar lines as [4, Theorem 1]. Assume that $P_{\mathcal{F}}$ is a finer partition than $P_{\mathcal{G}}$. By Lemma 6.1, $\hat{P}_{\mathcal{F}}$ is a finer partition than $\hat{P}_{\mathcal{G}}$. Consider the following maps:

$$\mathbb{I} \xrightarrow{\hat{\pi}_{\mathcal{F}}} \mathbb{X} \xrightarrow{\hat{T}} \mathbb{Y}.$$

Because σ and $\hat{\tau}_{\mathcal{F}} \circ \hat{\pi}_{\mathcal{F}}(\sigma)$ belong to the same set in $P_{\mathcal{F}}$, they also belong to the same set in $P_{\mathcal{G}}$. Therefore $\hat{T} \circ \hat{\pi}_{\mathcal{F}}(\alpha) = \hat{\pi}_{\mathcal{G}} \circ \hat{\tau}_{\mathcal{F}} \circ \hat{\pi}_{\mathcal{F}}(\alpha) = \hat{\pi}_{\mathcal{G}}(\alpha)$ for all $\alpha \in \mathbb{I}$. Since $\hat{\pi}_{\mathcal{G}}$ is continuous, so is $\hat{T} \circ \hat{\pi}_{\mathcal{F}}$. From the facts that $\mathbb{I}$ is compact, $\hat{\pi}_{\mathcal{F}}$ is continuous, and $\hat{T} \circ \hat{\pi}_{\mathcal{F}}$ is continuous, it follows by standard topological arguments that $\hat{T}$ is continuous.

Concerning statement (2), assume that $T = T_{\mathcal{F}\mathcal{G}}$ is continuous and that the attractor of $\mathcal{F}$ is non-overlapping. We first show that $T \circ \pi_{\mathcal{F}}(\sigma) = \pi_{\mathcal{G}}(\sigma)$ for $\sigma \in \Omega_{A_{\mathcal{F}}}$. Let $\sigma \in \Omega_{A_{\mathcal{F}}}$, say $\sigma = \tau_{\mathcal{F}}(a)$ where $a \in A_{\mathcal{F}}$. Then

$$T \circ \pi_{\mathcal{F}}(\sigma) = \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ \pi_{\mathcal{F}} \circ \tau_{\mathcal{F}}(a) = \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}}(a) = \pi_{\mathcal{G}}(\sigma).$$

But since the attractor of $\mathcal{F}$ is non-overlapping, we have $\overline{\Omega_{\mathbb{X}}} = \mathbb{I}$ by Lemma 5.4; in particular $\overline{\Omega_{A_{\mathcal{F}}}} = I$. Since the two continuous functions $T \circ \pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ agree on $\Omega_{A_{\mathcal{F}}}$, they also agree on I. Now if $\pi_{\mathcal{F}}(\sigma) = \pi_{\mathcal{F}}(\omega)$, then $\pi_{\mathcal{G}}(\sigma) = T \circ \pi_{\mathcal{F}}(\sigma) = T \circ \pi_{\mathcal{F}}(\omega) = \pi_{\mathcal{G}}(\omega)$. $\square$

Corollary 6.3. *If the attractor of $\mathcal{F}$ is non-overlapping, then the following hold.*

(1) *A global fractal transformation $\hat{T}_{\mathcal{F}\mathcal{G}} \colon \mathbb{X} \to \mathbb{Y}$ is continuous if and only if $P_{\mathcal{F}}$ is a finer partition than $P_{\mathcal{G}}$.*

(2) *Whether or not $\hat{T}_{\mathcal{F}\mathcal{G}}$ is continuous is independent of the mask and independent of θ.*

Proof. Statement (1) comes directly from Theorem 6.2, and statement (2) follows from statement (1) because the partitions $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ do not depend on the mask nor on θ. $\square$

Example 6.4 (a continuous map from $\mathbb{R}$ onto $\mathbb{R}^2$). Let

$$\mathcal{F} = \left\{ \mathbb{R} \colon f_i(x) = \frac{x + i - 1}{4}, i = 1, 2, 3, 4 \right\},$$

$$\mathcal{G} = \{ \mathbb{R}^2 \colon g_i, i = 1, 2, 3, 4 \}.$$

where the four functions in $\mathcal{G}$ are the similitudes taking the square $ABCD$ in Figure 2 to the four smaller squares $abcd$. The attractor of $\mathcal{F}$ is the unit interval $[0, 1]$ and the attractor of $\mathcal{G}$ is the unit square $[0, 1]^2$. In [3, Example 3.5] it is shown that the fractal transformation $T_{\mathcal{F}\mathcal{G}}$ is the Hilbert space filling curve (shown on the left in Figure 2), i.e. Hilbert's continuous function h from the interval $[0, 1]$ onto $[0, 1] \times [0, 1]$. For any $\theta \in I$ that is full with respect to $\mathcal{F}$, the corresponding global fractal transformation $\widehat{T}_{\mathcal{F}\mathcal{G}}$ is a continuous map from $\mathbb{R}$ onto $\mathbb{R}^2$.
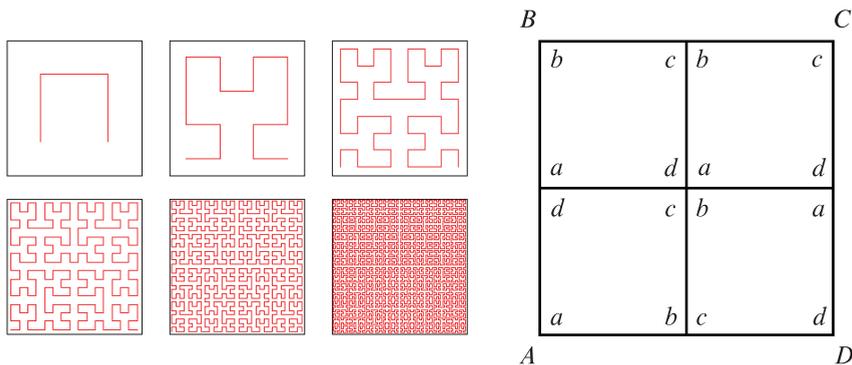
Figure 2. The Hilbert space filling curve is a fractal transformation.

Theorem 6.5. *If $\mathcal{F}$ and $\mathcal{G}$ have non-overlapping attractors and $T_{\mathcal{F}\mathcal{G}}$ is continuous, then $T = T_{\mathcal{F}\mathcal{G}}$ satisfies the following commutative diagram for all $n \in [N]$:*

$$
\begin{array}{ccc}
A_{\mathcal{F}} & \xrightarrow{f_n} & A_{\mathcal{F}} \\
{\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\
A_{\mathcal{G}} & \xrightarrow{g_n} & A_{\mathcal{G}}.
\end{array} \tag{7}
$$

Proof. Let $x \in A_{\mathcal{F}}$ and $\sigma = \tau_{\mathcal{F}}(x)$ and $\omega = \tau_{\mathcal{F}}(f_n(x))$. Since

$$\pi_{\mathcal{F}}(n\sigma) = f_n(\pi_{\mathcal{F}}(\sigma)) = f_n(x) = \pi_{\mathcal{F}}(\omega)$$

the words $n\sigma$ and ω lie in the same part of the partition $P_{\mathcal{F}}$. By Theorem 6.2 they also lie in the same part of the partition $P_{\mathcal{G}}$. Therefore

$$
\begin{aligned}
T \circ f_n(x) &= \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ f_n(x) \\
&= \pi_{\mathcal{G}}(\omega) \\
&= \pi_{\mathcal{G}}(n\sigma) \\
&= g_n \circ \pi_{\mathcal{G}}(\sigma)
\end{aligned}
$$

$$= g_n \circ \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}}(x)$$

$$= g_n \circ T(x). \qquad \square$$

Lemma 6.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be IFSs with the same number N of functions. Assume that $T_{\mathcal{F}\mathcal{G}}: A_{\mathcal{F}} \to A_{\mathcal{G}}$ is bijective and continuous. Let $a_F \in A_{\mathcal{F}}$ and $a_G = T_{\mathcal{F}\mathcal{G}}(a_F)$. For all $n \in [N]$, if $g_n^{-1}(a_G) \in A_{\mathcal{G}}$, then $f_n^{-1}(a_F) \in A_{\mathcal{F}}$.*

Proof. Let $b_G = g_n^{-1}(a_G)$ and, since $T_{\mathcal{F}\mathcal{G}}$ is surjective, let $b_F \in A_{\mathcal{F}}$ such that $T_{\mathcal{F}\mathcal{G}}(b_F) = b_G$. If $c_F = f_n(b_F)$, then by diagram (7) we have

$$T_{\mathcal{F}\mathcal{G}}(c_F) = g_n \circ T_{\mathcal{F}\mathcal{G}} \circ f_n^{-1}(c_F)$$

$$= g_n \circ T_{\mathcal{F}\mathcal{G}}(b_F)$$

$$= g_n(b_G)$$

$$= a_G$$

$$= T_{\mathcal{F}\mathcal{G}}(a_F).$$

Since $T_{\mathcal{F}\mathcal{G}}$ is injective, $c_F = a_F$ and hence $f_n^{-1}(a_F) = f_n^{-1}(c_F) = b_F \in A_{\mathcal{F}}$. $\square$

Theorem 6.7. *Let $\widehat{T}_{\mathcal{F}\mathcal{G}}$ be a global fractal transformation associated with the θ-section $\hat{\tau}_{\mathcal{F}}$. If $P_{\mathcal{F}} = P_{\mathcal{G}}$, then $\widehat{T}_{\mathcal{F}\mathcal{G}}: \mathbb{X} \to \mathbb{Y}$ is a homeomorphism and there is a θ-section $\hat{\tau}_{\mathcal{G}}$ with respect to which*

$$(\widehat{T}_{\mathcal{F}\mathcal{G}})^{-1} = \widehat{T}_{\mathcal{G}\mathcal{F}}.$$

Proof. By Theorem 6.2, the transformation $T_{\mathcal{F}\mathcal{G}}$ is continuous, and since $P_{\mathcal{F}} = P_{\mathcal{G}}$, the fractal transformation $T_{\mathcal{F}\mathcal{G}}: A_{\mathcal{F}} \to A_{\mathcal{G}}$ is a bijection. If $T_{\mathcal{F}\mathcal{G}}$ is the fractal transformation corresponding to shift invariant section τ_F of $\mathcal{F}$, then $\tau_{\mathcal{G}} = \tau_{\mathcal{F}} \circ (T_{\mathcal{F}\mathcal{G}})^{-1}$ is a shift invariant section of $\mathcal{G}$. Let $\hat{\tau}_{\mathcal{G}}$ be the associated θ-section of $\mathcal{G}$. It follows from Lemma 6.6 that $\hat{\tau}_{\mathcal{F}}(x) = \hat{\tau}_{\mathcal{G}}(\widehat{T}_{\mathcal{F}\mathcal{G}}(x))$ for all $x \in \mathbb{X}$. Therefore

$$\widehat{T}_{\mathcal{G}\mathcal{F}} \circ \widehat{T}_{\mathcal{F}\mathcal{G}}(x) = \hat{\pi}_{\mathcal{F}} \circ \hat{\tau}_{\mathcal{G}}(\widehat{T}_{\mathcal{F}\mathcal{G}}(x)) = \hat{\pi}_{\mathcal{F}} \circ \hat{\tau}_{\mathcal{F}}(x) = x,$$

which implies that $(T_{\mathcal{F}\mathcal{G}})^{-1} = T_{\mathcal{G}\mathcal{F}}$. By Theorem 6.2, $\widehat{T}_{\mathcal{G}\mathcal{F}}$ is also continuous and hence $\widehat{T}_{\mathcal{F}\mathcal{G}}$ is a homeomorphism. $\square$

Theorem 6.8. *If the fractal transformation $T_{\mathcal{F}\mathcal{G}}: A_{\mathcal{F}} \to A_{\mathcal{G}}$ is continuous and the attractor of $\mathcal{F}$ is non-overlapping, then the graph $G \subset A_{\mathcal{F}} \times A_{\mathcal{G}}$ of $T_{\mathcal{F}\mathcal{G}}$ is the attractor of the IFS $\mathcal{H} = \{\mathbb{X} \times \mathbb{Y} : h_1, h_2, \ldots, h_N\}$, where $h_i(x, y) = (f_i(x), g_i(y))$ for $i = 1, 2, \ldots, N$.*

Proof. Denote $T_{\mathcal{F}\mathcal{G}}$ by T, and let $(x, T(x))$, $x \in A_{\mathcal{F}}$ be an arbitrary point of G. If $\tau_{\mathcal{F}}$ is the section of $\pi_{\mathcal{F}}$ for T and $\sigma = \tau_{\mathcal{F}}(x)$, then

$$(x, T(x)) = (\pi_{\mathcal{F}} \circ \tau_{\mathcal{F}}(x), \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}}(x)) = (\pi_{\mathcal{F}}(\sigma), \pi_{\mathcal{G}}(\sigma)) = \pi_{\mathcal{H}}(\sigma) \in A_{\mathcal{H}}.$$

The last equality follows from

$$\begin{aligned}
\pi_{\mathcal{H}}(\sigma) &= \lim_{k \to \infty} h_{\sigma|k}(B) \\
&= \lim_{k \to \infty} (f_{\sigma|k}(B), g_{\sigma|k}(B)) \\
&= (\pi_{\mathcal{F}}(\sigma), \pi_{\mathcal{G}}(\sigma)),
\end{aligned}$$

for any nonempty compact set B.

Conversely, let $(\pi_{\mathcal{F}}(\sigma), \pi_{\mathcal{G}}(\sigma))$ be an arbitrary point in $A_{\mathcal{H}}$, i.e., $\sigma \in I$ is arbitrary. If $x = \pi_{\mathcal{F}}(\sigma)$, then $T(x) = \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ \pi_{\mathcal{F}}(\sigma)$. It now suffices to show that $\pi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ \pi_{\mathcal{F}} = \pi_{\mathcal{G}}$, so that $(\pi_{\mathcal{F}}(\sigma), \pi_{\mathcal{G}}(\sigma)) = (x, T(x))$ is a point of G. Since T is assumed continuous and the attractor of $\mathcal{F}$ is non-overlapping $\mathcal{P}_{\mathcal{F}}$ is a finer partition than $\mathcal{P}_{\mathcal{G}}$ by statement (2) of Theorem 6.2. Since σ and $\tau_{\mathcal{F}}(x)$ are in the same part of $\mathcal{P}_{\mathcal{F}}$, they are also in the same part of $\mathcal{P}_{\mathcal{G}}$. Therefore

$$\pi_{\mathcal{G}}(\sigma) = \pi_{\mathcal{G}}(\tau_{\mathcal{F}}(x)) = \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}} \circ \pi_{\mathcal{F}}(\sigma). \qquad \square$$

Example 6.9 (Graphs of Fractal Transformations). For

$$\mathcal{F} = \{\mathbb{R}: x/3, x/3 + 1/3, x/3 + 2/3\}$$

and

$$g = \{\mathbb{R}: x/4, x/2 + 1/4, x/4 + 3/4\},$$

the fractal transformation $T_{\mathcal{F}\mathcal{G}}$ is a homeomorphism by Theorem 6.7, and by Theorem 6.8 its graph is the attractor of the IFS

$$\mathcal{H} = \{\mathbb{R}^2: (x/3, y/4), (x/3 + 1/3, y/2 + 1/4), (x/3 + 2/3, y/4 + 3/4)\}.$$

The graph of $T_{\mathcal{F}\mathcal{G}}$, i.e., the attractor $A_{\mathcal{H}}$, is shown in Figure 3.

For $\mathcal{F} = \{\mathbb{R}: x/2, x/2 + 1/2\}$ and $g_r = \{\mathbb{R}: rx, rx + (1 - r)\}$, where $0 < r < 1, r \neq 1/2$, the fractal transformation $T_{\mathcal{F}\mathcal{G}}$ is not continuous. For $r < 1/2$, the attractor of $\mathcal{G}$ is totally disconnected, and for $r > 1/2$ the attractor of $\mathcal{G}$ is overlapping. These two cases are shown in Figure 4 for $r = 1/3$ and $r = 2/3$.

According to the next result, Definition 4.1 of global fractal transformation can be simplified in the continuous case; it is not necessary to take the *minimum k* such that $f_{\overleftarrow{\theta|k}}(x) \in A_{\mathcal{F}}$.
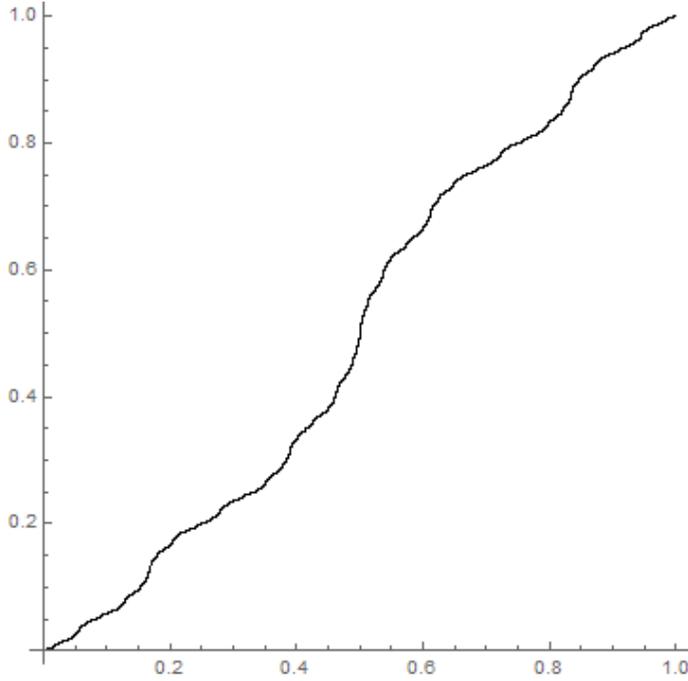
Figure 3. The graph of a fractal homeomorphism on $[0, 1]$.

Corollary 6.10. *If $\widehat{T}_{\mathcal{F}\mathcal{G}}$ is continuous and the attractor of $\mathcal{F}$ is non-overlapping, then the global fractal transformation $\widehat{T}_{\mathcal{F}\mathcal{G}}\colon \mathbb{X} \to \mathbb{Y}$ is given by*

$$\widehat{T}_{\mathcal{F}\mathcal{G}}(x) = g_{-(\theta|k)} \circ T_{\mathcal{F}\mathcal{G}} \circ f_{\overleftarrow{\theta|k}}(x)$$

for any $k = k(x)$ such that $f_{\overleftarrow{\theta|k}}(x) \in A_{\mathcal{F}}$.

Proof. That the definition holds for any $k \geq K$ follows from

$$g_{-(\theta|k+1)} \circ T_{\mathcal{F}\mathcal{G}} \circ f_{\overleftarrow{\theta|k+1}} = g_{-(\theta|k)} \circ g_{\theta_{k+1}}^{-1} \circ T_{\mathcal{F}\mathcal{G}} \circ f_{\theta_{k+1}} \circ f_{\overleftarrow{\theta|k}}$$

$$= g_{-(\theta|k)} \circ T_{\mathcal{F}\mathcal{G}} \circ f_{\overleftarrow{\theta|k}},$$

the last equality following from diagram (7). $\qquad\square$

Theorem 6.11. *If the attractor of $\mathcal{F}$ is non-overlapping and $\widehat{T}_{\mathcal{F}\mathcal{G}}\colon \mathbb{X} \to \mathbb{Y}$ is continuous, then the graph of $\widehat{T}_{\mathcal{F}\mathcal{G}}$ is $\{(\hat{\pi}_{\mathcal{F}}(\sigma), \hat{\pi}_{\mathcal{G}}(\sigma))\colon \sigma \in \mathbb{I}\}$.*
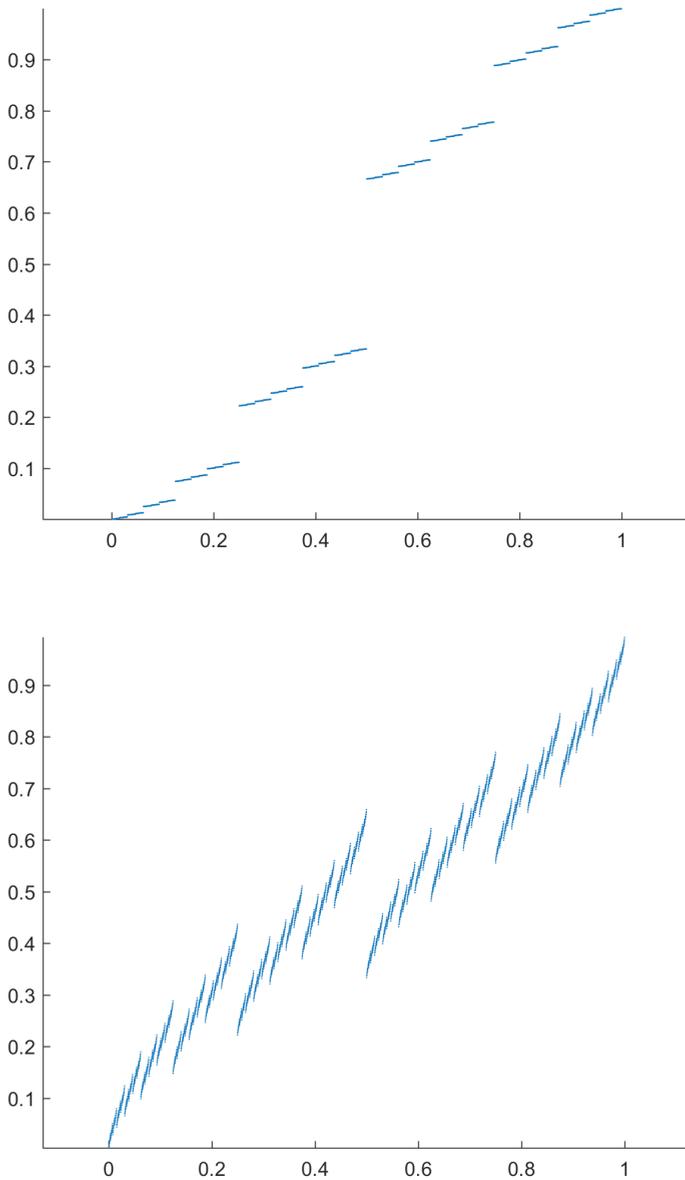
Figure 4. A fractal transformation from the interval to the Cantor set (top) and a fractal transformation from the interval to an overlapping attractor (bottom).

Proof. Let G be the graph of $\widehat{T} = \widehat{T}_{\mathcal{F}\mathcal{G}}$, and assume that $(x, y) \in G$. If $\alpha = \hat{\tau}_{\mathcal{F}}(x)$, then by Theorem 5.3 we have

$$x = \hat{\pi}_{\mathcal{F}} \circ \hat{\tau}_{\mathcal{F}}(x) = \hat{\pi}_{\mathcal{F}}(\alpha),$$

$$y = \widehat{T}(x) = \hat{\pi}_{\mathcal{G}} \circ \hat{\tau}_{\mathcal{F}}(x) = \hat{\pi}_{\mathcal{G}}(\alpha).$$

Therefore $G \subseteq \{(\hat{\pi}_{\mathcal{F}}(\sigma), \hat{\pi}_{\mathcal{G}}(\sigma)) : \sigma \in \Omega_{\mathbb{X}} \subset \mathbb{I}\}$.

In the other direction, let $x = \hat{\pi}_{\mathcal{F}}(\alpha)$ and $y = \hat{\pi}_{\mathcal{G}}(\alpha)$ for some $\alpha \in \mathbb{I}$. Then

$$\widehat{T}(\hat{\pi}_{\mathcal{F}}(\alpha)) = \hat{\pi}_{\mathcal{G}} \circ \hat{\tau}_{\mathcal{F}} \circ \hat{\pi}_{\mathcal{F}}(\alpha) = \hat{\pi}_{\mathcal{G}}(\alpha),$$

the last equality for the following reason. Because $\widehat{T}$ is continuous, its restriction T to $A_{\mathcal{F}}$ is also continuous and therefore, by statement (2) of Theorem 6.2, $P_{\mathcal{F}}$ is a finer partition than $P_{\mathcal{G}}$. This implies, by Lemma 6.1, that $\widehat{P}_{\mathcal{F}}$ is a finer partition than $\widehat{P}_{\mathcal{G}}$. Hence, $\hat{\pi}_{\mathcal{F}}(\hat{\tau}_{\mathcal{F}} \circ \hat{\pi}_{\mathcal{F}}(\alpha)) = \hat{\pi}_{\mathcal{F}}(\alpha)$ implies $\hat{\pi}_{\mathcal{G}}(\hat{\tau}_{\mathcal{F}} \circ \hat{\pi}_{\mathcal{F}}(\alpha)) = \hat{\pi}_{\mathcal{G}}(\alpha)$. $\quad\square$

6.1. Fractal homeomorphisms based on an IFS with triangular attractor.

It would be convenient if the criterion of Theorem 6.2 for continuity in terms of the code space structure could be replaced by a geometric criterion. This is done below for the special case where both IFSs are affine, i.e. the functions are affine functions.

Let ABC be a triangle and Δ a triangulation of ABC. This means that the intersection of any two triangles in Δ is either empty, a single point, or a side of both triangles. Let V denote the set of all vertices of Δ. (There may be vertices that lie on the sides of ABC.) Call the triangulation Δ *colored* if there is a *coloring* $c : V \to \{1, 2, 3\}$ such that vertices that are adjacent in Δ receive different colors. In other words, the vertices in any triangle in Δ receives all three colors. Furthermore, it is required that $c(A) = 1, c(B) = 2, c(C) = 3$.

If Δ is a colored triangulation of ABC and Δ' is a colored triangulation of triangle $A'B'C'$, then Δ and Δ' are said to be *combinatorially equivalent*, denoted $\Delta' \approx \Delta$, if there is a bijection $\phi : V \to V'$ from the vertex set V of Δ to the vertex set V' of Δ' that preserves colors, edges, and triangles. Thus ϕ can be thought of as acting on the set of vertices, the set of edges, and the set of triangles of the triangulation. The colored triangulations (although the colors are not shown) in Figure 5 are combinatorially equivalent triangulations.

Denote the set of triangles of triangulation Δ by $\{t_1, t_2, \ldots, t_N\}$. Given a colored triangulation Δ of ABC, define an IFS $F_{\Delta} = \{f_1, f_2, \ldots, f_N\}$, where $f_n, n = 1, 2, \ldots, N$, is the unique affine function that takes ABC onto triangle t_n, preserving the colors of the three vertices.
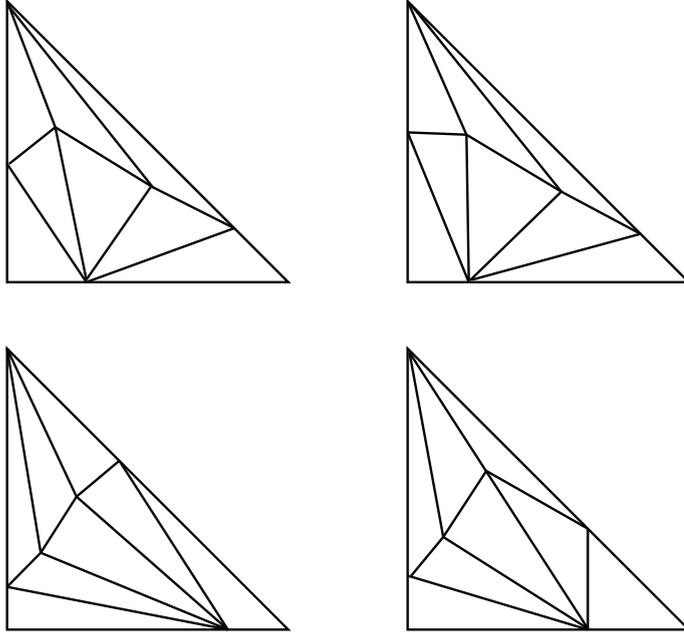
Figure 5. Four combinatorially and area equivalent triangulations.

Theorem 6.12. *Assume that $\triangle$ and $\triangle'$ are colored triangulations of ABC and $A'B'C'$, respectively and $\triangle \approx \triangle'$. If $\mathcal{F}_\triangle$ and $\mathcal{F}_{\triangle'}$ are the corresponding IFSs as described above, where each triangle $t_n \in \{t_1, t_2, \ldots, t_N\}$. corresponds to triangle t'_n in $\{t'_1, t'_2, \ldots, t'_N\}$ under the combinatorial equivalence, then the fractal transformation $\widehat{T}_{\mathcal{F}_\triangle \mathcal{F}_{\triangle'}}: \mathbb{R}^2 \to \mathbb{R}^2$ is a fractal homeomorphism.*

Proof. If the boundary of the triangle ABC is denoted by $\partial\triangle$, then using the Hutchinson operator F defined by Equation (1), the points, sides, and triangles determined by $F(\partial\triangle)$ is the original colored triangulation $\triangle$. Moreover, $\partial\triangle, F(\partial\triangle), F^2(\partial\triangle), \ldots$ is an infinite sequence of colored triangulations of ABC, each a subdivision of the previous. Similarly $\partial\triangle, G(\partial\triangle), G^2(\partial\triangle), \ldots$ is an infinite sequence of colored triangulations of $A'B'C'$, and it is clear that $F^n(\partial\triangle) \approx G^n(\partial\triangle)$ for $n = 0, 1, 2, \ldots$. Therefore, if ABC is denoted $\blacktriangle$ and $A'B'C'$ is denoted $\blacktriangle'$ and if $\sigma, \omega \in I$, then $f_{\sigma|k}(\blacktriangle) \cap f_{\omega|k}(\blacktriangle) \neq \emptyset$ if and only if $g_{\sigma|k}(\blacktriangle') \cap g_{\omega|k}(\blacktriangle') \neq \emptyset$. Hence $\bigcap_{k=1}^\infty f_{\sigma|k}(\blacktriangle) \cap \bigcap_{k=1}^\infty f_{\omega|k}(\blacktriangle) \neq \emptyset$ if and only if $\bigcap_{k=1}^\infty g_{\sigma|k}(\blacktriangle') \cap \bigcap_{k=1}^\infty g_{\omega|k}(\blacktriangle') \neq \emptyset$. But $\pi_\mathcal{F}(\sigma) = \bigcap_{k=1}^\infty f_{\sigma|k}(\blacktriangle)$, and likewise for $\pi_\mathcal{F}(\omega), \pi_\mathcal{G}(\sigma)$ and $\pi_\mathcal{G}(\omega)$. This implies that $\pi_\mathcal{F}(\sigma) = \pi_\mathcal{F}(\omega)$ if and only if $\pi_\mathcal{G}(\sigma) = \pi_\mathcal{G}(\omega)$. By Theorem 6.7, the global fractal transformation $\widehat{T}_{\mathcal{F}_\triangle \mathcal{F}_{\triangle'}}$ is a fractal homeomorphism. $\square$

Example 6.13. Consider the four combinatorially equivalent triangulations appearing in Figure 5. Taking distinct pairs, there are six corresponding non-identity global fractal transformations of the form $\widehat{T}_{\mathcal{F}_\Delta \mathcal{F}_{\Delta'}}$. It is a consequence of Theorem 6.12 that each of these six is a homeomorphism; three are inverses of the other three.

7. Volume preserving fractal transformations

All IFSs in this section are on $\mathbb{R}^d$. Let μ denote Lebesgue measure on $\mathbb{R}^d$, i.e., volume. Two transformations on $\mathbb{R}^d$ will, in this section, be consider the same if they coincide except on a set of measure 0. Let $\mathcal{F} = \{\mathbb{R}^d : f_1, f_2, \ldots, f_N\}$ and $\mathcal{G} = \{\mathbb{R}^d : g_1, g_2, \ldots, g_N\}$. Throughout this section it is assumed that $\tau_{\mathcal{F}}$ is a section of $\mathcal{F}$, that $T_{\mathcal{F}\mathcal{G}} : A_{\mathcal{F}} \to A_{\mathcal{G}}$ is the fractal transformation associated with $\tau_{\mathcal{F}}$, that $\theta \in I$ is full with respect to $\mathcal{F}$ and $\mathcal{G}$, and that $\widehat{T}_{\mathcal{F}\mathcal{G}} : \mathbb{R}^d \to \mathbb{R}^d$ is the global fractal transformation associated with θ. In addition it is assumed that

(1) the attractors of $\mathcal{F}$ and $\mathcal{G}$ are non-overlapping;

(2) $\mu(A_{\mathcal{F}}) = \mu(A_{\mathcal{G}})$;

(3) for every $n \in [N]$, the functions $f_n(x) = L_n(x) + a_n$ and $g_n(x) = L'_n(x) + a'_n$ are affine such that the linear parts satisfy $|\det L_n| = |\det L'_n|$.

Theorem 7.1. *Under the assumptions above,*

(1) $\widehat{T}_{\mathcal{F}\mathcal{G}} : \mathbb{R}^d \to \mathbb{R}^d$ *is independent of the particular mask* (*equivalently the corresponding section*) *used*;

(2) $\widehat{T}_{\mathcal{F}\mathcal{G}}$ *is continuous almost everywhere*;

(3) $\widehat{T}_{\mathcal{F}\mathcal{G}}$ *is bijective almost everywhere and* $(\widehat{T}_{\mathcal{F}\mathcal{G}})^{-1} = \widehat{T}_{\mathcal{G}\mathcal{F}}$;

(4) $\widehat{T}_{\mathcal{F}\mathcal{G}}$ *is volume preserving, i.e.,* $\mu(\widehat{T}_{\mathcal{G}\mathcal{F}}(X)) = \mu(X)$ *for all measurable sets* $X \subseteq \mathbb{R}^d$.

Proof. We first show that Lebesgue measure is the unique normalized invariant measure of $\mathcal{F}$. This means that, for any measurable set B,

$$\mu(B) = \sum_{n=1}^{N} p_n \mu(f_n^{-1}(B)),$$

where the domain of each f_n is restricted to $A_{\mathcal{F}}$ and $p_n = |\det L_n|$. If $\{M_1, \ldots, M_N\}$ is the mask, then

$$\sum_{n=1}^{N} p_n \mu(f_n^{-1}(B)) = \sum_{n=1}^{N} p_n \mu\left(f_n^{-1}\left(\bigcup_{i=1}^{N}(B \cap M_i)\right)\right)$$

$$= \sum_{n=1}^{N} \sum_{i=1}^{N} p_n \mu(f_n^{-1}(B \cap M_i))$$

$$= \sum_{n=1}^{N} p_n \mu(f_n^{-1}(B \cap M_n))$$

$$= \sum_{n=1}^{N} \mu(f_n(f_n^{-1}(B \cap M_n)))$$

$$= \sum_{n=1}^{N} \mu(B \cap M_n)$$

$$= \mu(B).$$

The fact that $\mu(f_n^{-1}(M_i)) = 0$ for $n \neq i$ in a non-overlapping attractor is used to justify the first equality on the second line above. Likewise, the Lebesgue measure is the unique normalized invariant measure of $\mathcal{G}$ with the same values of the p_n.

For $k \geq 1$, let

$$X_k = f_{-(\theta|k)}(A_{\mathcal{F}}) \setminus f_{-(\theta|k-1)}(A_{\mathcal{F}}) \quad \text{and} \quad Y_k = g_{-(\theta|k)}(A_{\mathcal{G}}) \setminus g_{-(\theta|k-1)}(A_{\mathcal{G}}).$$

Since θ is full with respect to both $\mathcal{F}$ and $\mathcal{G}$, we have $\bigcup_{k \geq 1} X_k = \bigcup_{k \geq 1} Y_k = \mathbb{R}^d$. Also, for $x \in X_k$,

$$T_{\mathcal{FG}}(x) = g_{-(\theta|k)} \circ T_{\mathcal{FG}} \circ f_{\overleftarrow{\theta|k}}(x).$$

By [3, Proposition 2.5 and Theorem 2.4], the fractal transformation $T_{\mathcal{FG}} \colon A_{\mathcal{F}} \to A_{\mathcal{G}}$ satisfies statements (1–4) of the theorem, statement (4) for the invariant measures on $\mathcal{F}$ and $\mathcal{G}$, which in this case is Lebesgue measure. Since corresponding functions f_n and g_n have the same determinant, they scale volume identically; therefore the global fractal transformation $\hat{T}_{\mathcal{FG}}$ also satisfies statements (1–4). $\square$

7.1. Area preserving homeomorphisms of a triangle to itself.

This continues the discussion in Subsection 6.1. If two colored triangulations Δ and Δ' of the same triangle ABC are combinatorially equivalent and pairs of triangles that correspond under the combinatorial equivalence have equal area, then call Δ and Δ' *area equivalent*, denoted $\Delta \equiv \Delta'$. The result below follows immediately from Theorems 6.12 and 7.1.

Corollary 7.2. *If* $\Delta \equiv \Delta'$, *then* $\widehat{T}_{\mathcal{F}_\Delta \mathcal{F}_{\Delta'}}$ *is an area preserving homeomorphism.*

Example 7.3 (An Infinite Family of Area Preserving Fractal Homeomorphisms). Figure 6 shows an infinite family Δ_r, Δ'_r of area equivalent triangulations, where the positive real r is as shown in the figure. If we start with a checkerboard pattern on the plane, part of which is shown in the top panel of Figure 1, then its image under the fractal transformation $\widehat{T}_{\mathcal{F}_\Delta \mathcal{F}_{\Delta'}}$ is illustrated in the bottom panel, with $r = 0.55$ in the left figure and $r = 0.6$ in the right figure.
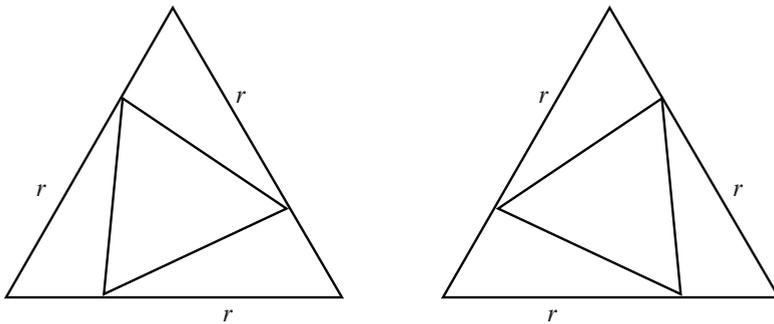
Figure 6. Area equivalent triangulations.

Example 7.4 (another family of area preserving fractal homeomorphisms). Figure 5 shows four area equivalent colored triangulations. Three area preserving fractal homeomorphisms $T : ABC \rightarrow ABC$ (and three inverses) can be produced from these 4 triangulations according to Corollary 7.2, and from each of these three, an area preserving fractal homeomorphism $\widehat{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for each $\theta \in I$ that is full with respect to $\mathcal{F}$ and $\mathcal{G}$. The four area equivalent colored triangulations were found by solving a system of 7 quadratic equations in 7 unknowns, as explained in the paragraph below.

In general, there is no loss of generality is assuming that ABC is an equilateral triangle or an isosceles right triangle (as in Figures 5 and 6) because, for any triangle T, there is an area preserving affine map taking T onto an equilateral triangle (isosceles right triangle). Given a triangulation Δ, it is possible to search algebraically for possible triangulations Δ' such that $\Delta' \equiv \Delta$ by solving a system of quadratic equations where the unknowns are the coordinates of the vertices of Δ' and the equations are obtained by setting the area of a triangle in Δ' equal to the area of its corresponding triangle in Δ. It is routine to show that the number of equations equals the number of unknowns. Therefore, according to Bezout's theorem, unless there are infinitely many solutions, there are at most 2^{N-1} possible Δ' such that $\Delta' \equiv \Delta$, where N is the number of triangles in the triangulation.

Questions arise concerning the prevalence of pairs Δ, Δ' such that $\Delta \equiv \Delta'$. Although the computing time quickly becomes prohibitive with increasing N, a few examples appear to indicate that, for many triangulations Δ, there is at least one triangulation $\Delta' \neq \Delta$ such that $\Delta' \equiv \Delta$, and often several distinct Δ' such that $\Delta' \equiv \Delta$.

Conjecture 7.5. *There is no triangulation Δ such that there are infinitely many distinct Δ' with $\Delta' \equiv \Delta$.*

8. Using global addressing to create tilings

Let $\mathbb{X}$ be a complete metric space. A *tile* is a compact subset of $\mathbb{X}$. A *tiling* $\mathbb{T}$ of $\mathbb{X}$ is a countable set of tiles with the property that $(t \cap t')^o = \emptyset$ for all $t \neq t' \in \mathbb{T}$ and $\cup\{t : t \in \mathbb{T}\} = \mathbb{X}$. Tilings of $\mathbb{X}$ will be constructed from an IFS by basically defining a tile as the set of points whose global address begins with a given initial segment.

For $j \in [N]$, define a *j-tree* as a rooted tree such that

(1) every non-leaf node except the root has N children, and these children are labeled bijectively with $[N]$, and

(2) the root has $N - 1$ children, and these children are labeled bijectively with $[N] \setminus \{j\}$.

Given $\theta \in I$, define $\mathcal{W} = (W_0, W_1, W_2, \dots)$ to be a θ-*sequence of trees* if W_0 is the tree consisting of a single node, the root, and $W_k, k \geq 1$, is a θ_k-tree. Figure 7 shows the first four terms W_1, W_2, W_3, W_4 of a $(2121\dots)$-sequence of trees (the trivial tree W_0 is omitted).
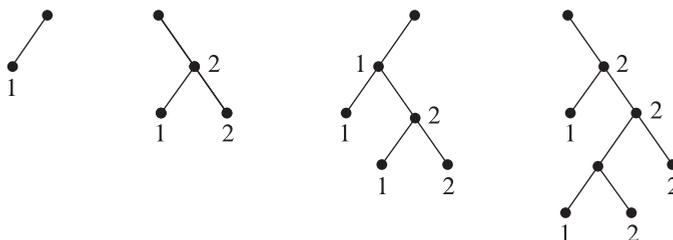
Figure 7. The first four terms W_1, W_2, W_3, W_4 of a θ-sequence of trees, where $\theta = 2121\dots$.

Example 8.1. The *depth* of a leaf in a rooted tree is the length (number of edges) of the unique path from the root to the leaf. Given $\theta \in I$, let $\mathcal{W} = (W_0, W_1, W_2, \ldots)$ be the θ-sequence of trees with the property that the depth of every leaf of W_k is exactly k. This sequence will be referred to as the *standard θ-sequence*.

Fix $\theta \in I$ and fix a θ-sequence $\mathcal{W}$ of trees. For any $k \geq 0$ and any node u of W_k, we will use the following notation:

$$\sigma(u) = \sigma_1 \sigma_2 \ldots \sigma_j,$$

$$\alpha(u) = (-\theta|k)\sigma(u),$$

where $\sigma_1 \sigma_2 \ldots \sigma_j \in [N]^*$ is obtained by listing, in order, the labels on the nodes on the unique path from the root of W_k to u. For the rightmost leaf u in the rightmost tree in Figure 7, for example, $\sigma(u) = 222$ and $\alpha(u) = (-2)(-1)(-2)(-1)222$.

Now fix an IFS $\mathcal{F}$ with attractor A and section τ. Let $\hat{\tau}$ be the corresponding θ-section, where θ is full with respect to $\mathcal{F}$. For each θ-sequence $\mathcal{W}$ of rooted trees, a tiling $\mathbb{T} = \mathbb{T}(\theta, \mathcal{W})$ of $\mathbb{X}$ can be constructed as follows. Let L denote the set of all leaves of all the trees in $\mathcal{W}$, and for $u \in L$ let

$$S(u) = \{\alpha(u)\omega : \omega \in I\}$$

In other words, $S(u)$ is the set of all words in the global code space that begin with $\alpha(u)$, and $t(u)$, defined formally below, is the set of all points in $\mathbb{X}$ whose global address begins with $\alpha(u)$.

Definition 8.2. Let

$$t(u) = \{x : \hat{\tau}(x) \in S(u)\},$$

$$\mathbb{T}(\theta, \mathcal{W}) = \{\overline{t(u)} : u \in L\},$$

Given a mask $\{M_1, M_2, \ldots, M_N\}$ for the attractor A of $\mathcal{F}$, let

$$M^j = \{M_\sigma : |\sigma| = j\}$$

be the jth level mask as defined in Section 3. If u is a leaf of W_k, define

$$t'(u) = f_{-(\theta|k)}(M_{\sigma(u)}) \setminus f_{-(\theta|k-1)}(A),$$

$$\mathbb{T}'(\theta, \mathcal{W}) = \{\overline{t'(u)} : u \in L\}.$$

Call $\mathbb{T}(\theta, \mathcal{W})$ and $\mathbb{T}'(\theta, \mathcal{W})$ both $(\theta, \mathcal{W})$-*tilings*. The same terminology is used for both because, by the following theorem, they are the same tiling.

Theorem 8.3. *If a mask* $\{M_1, M_2, \ldots, M_N\}$ *for the attractor of an IFS* $\mathcal{F}$ *is chosen so that* $(\overline{M_i} \cap \overline{M_j})^o = \emptyset$ *for all* $i \neq j$, *then* $\mathbb{T}(\theta, \mathcal{W})$ *is a tiling of* $\mathbb{X}$ *for any* θ *that is full with respect to* $\mathcal{F}$ *and for any* θ-*sequence* $\mathcal{W}$ *of trees. Moreover* $\mathbb{T}(\theta, \mathcal{W}) = \mathbb{T}'(\theta, \mathcal{W})$.

Proof. Each tile $\overline{t(u)}$ is clearly compact.

To show that $\cup\{t(u): u \in L\} = \mathbb{X}$, let $x \in \mathbb{X}$. Let $P = \{S(u): u \in L\}$. From the definition of a θ-sequence of trees, it follows that P is a partition of $\mathbb{I}$. Therefore there is a leaf u such that $\hat{\tau}(x) \in S(u)$; hence $x \in t(u)$.

It remains to show that distinct tiles do not overlap (intersection has empty interior). Again because P is a partition of $\mathbb{I}$, the sets in $\{t(u): u \in L\}$ are pairwise disjoint. It is therefore sufficient to show that the closures of two distinct such sets do not overlap. By Proposition 3.3, if u is a leaf of W_k, then

$$t(u) = f_{-(\theta|k)}(M_{\sigma(u)}) \setminus Z = f_{-(\theta|k)}(M_{\sigma(u)}) \setminus f_{\theta|k-1}(A),$$

where Z is the set of points $z \in f_{-(\theta|k)}(M_{\sigma(u)})$ such that $\hat{\tau}(z)$ does not begin with $-(\theta|k)$. The last equality above is a consequence of Definition 5.1 of the global section. This proves that $\mathbb{T}(\theta, \mathcal{W}) = \mathbb{T}'(\theta, \mathcal{W})$ and also that the closure of $t(u)$, and the closure of $t(u')$ do not overlap if $u \neq u'$. This is because $t(u) \cap t(u') = \emptyset$ and the closures overlapping would contradict the assumption that distinct sets in the mask closures $\{\overline{M_1}, \overline{M_2}, \ldots, \overline{M_N}\}$ do not overlap. $\square$

Corollary 8.4. *If the attractor* A *of an IFS* $\mathcal{F}$ *is non-overlapping, then the tiling* $\mathbb{T}(\theta, \mathcal{W})$ *can be expressed in the following simplified form, independent of the mask*:

$$t(u) = f_{\alpha(u)}(A),$$

$$\mathbb{T} = \{t(u): u \in L\}.$$

Most of the fractal tilings that appear in textbooks and recreational mathematics websites, although not defined that way, are $(\theta, \mathcal{W})$-tilings.

Example 8.5 (Rauzy tribonacci tilings). Let $\mathcal{F} = \{\mathbb{C}: f_1(z), f_2(z), f_3(z)\}$ where

$$f_1(z) = \beta z \quad f_2(z) = \beta^2 z + \beta \quad f_3(z) = \beta^3 z + \beta^2 + \beta,$$

where β is a complex root of $z^3 - z^2 - z - 1$. The attractor A of $\mathcal{F}$ is the well studied Rauzy tribonacci fractal [13], [22]. The images of A under the maps f_1, f_2, f_3 are compact sets similar to A with scaling ratios β, β^2, β^3, respectively. Since A is non-overlapping, Corollary 8.4 can be used to construct tilings. Take

a θ-sequence $\mathcal{W}$ of rooted trees defined as follows. For a leaf u, let $e(u)$ denote the sum of the labels on the unique path between the root and u, and let $e^-(u)$ denote the same sum minus the label at u. For $k \geq 1$, let $e(k)$ be the sum of the first k terms of θ. A rooted labeled tree W_k is uniquely determined by the following requirement: u is a leaf of W_k if and only if $e(u) \geq e(k) > e^-(u)$. As an example, take $\theta = 123123\ldots$. The terms W_1, W_2 in $\mathcal{W}$ are shown in Figure 8. In W_2, consider the second leaf from the left u. For that leaf we have $k = 2, e(k) = 3, e(u) = 4, e^-(u) = 2$, and clearly $e(u) \geq e(k) > e^-(u)$. A portion of the resulting $(\theta, \mathcal{W})$-Rauzy tiling $\mathbb{T}(\theta, \mathcal{W})$ is illustrated in Figure 9. There are exactly three tiles in $\mathbb{T}(\theta, \mathcal{W})$ up to congruence.
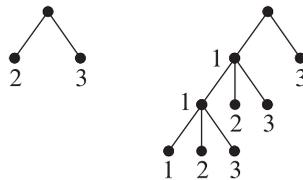
Figure 8. The terms W_1 and W_2 in the $(123123\ldots)$-sequence of trees in the construction of the Rauzy tiling in Example 8.5.

Figure 9. A portion of a Rauzy tiling of the plane in Example 8.5.

Example 8.6 (self-similar, quasiperiodic polygonal tilings). The $(\theta, \mathcal{W})$-tiling in Figure 10 was obtained using a θ-sequence $\mathcal{W}$ constructed in a somewhat similar manner to that of Example 8.5; see [11] for the precise construction. Using this method it is possible to construct many tilings $\mathbb{T}$, like the one in Figure 10, with the following properties.

(1) There are finitely many polygonal tiles in $\mathbb{T}$ up to congruence.

(2) The tiling $\mathbb{T}$ is self-similar: there is a similitude ϕ of the plane with scaling ratio greater than 1 such that, for every $t \in \mathbb{T}$, its inflated image $\phi(t)$ is the union of tiles in $\mathbb{T}$.

(3) The tiling is quasiperiodic (or repetitive): for any patch U of tiles in $\mathbb{T}$, there is a number R such that any disk of radius R contains, up to congruence, a copy of U.

This example and the others in [11] are a generalization of tilings by *rep-tiles*, the term coined by S. W. Golomb [18] and popularized by M. Gardner [17]. The subject of self-similarity and quasiperiodicity of tilings gained impetus with the discovery of quasicrystals [24] for which D. Shechtman was awarded the Nobel Prize for chemistry in 2011; see [23] for a history and mathematical exposition.
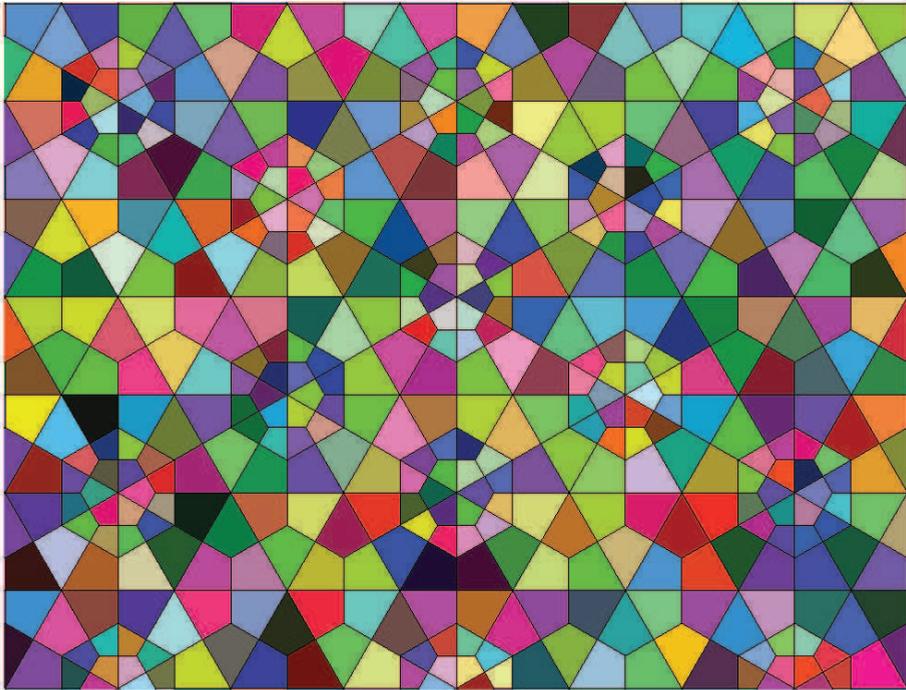
Figure 10. A self-similar polygonal $(\theta, \mathcal{W})$-tiling.

Example 8.7 (tiling by Robinson triangles). In Examples 8.5 and 8.6, the attractors are non-overlapping. This is an example of a tiling from an overlapping attractor. Let A be the isosceles triangle on the left in Figure 11, with sides of length 1, R, R, where $R = (1 + \sqrt{5})/2$ (the golden ratio), and angles $\pi/5, 2\pi/5, 2\pi/5$. The tile A and the obtuse triangle B in the tiling on the right in Figure 11, with sides of length 1, 1, R and angles $\pi/5, \pi/5, 3\pi/5$, are due to R. M. Robinson and are cited in [19].
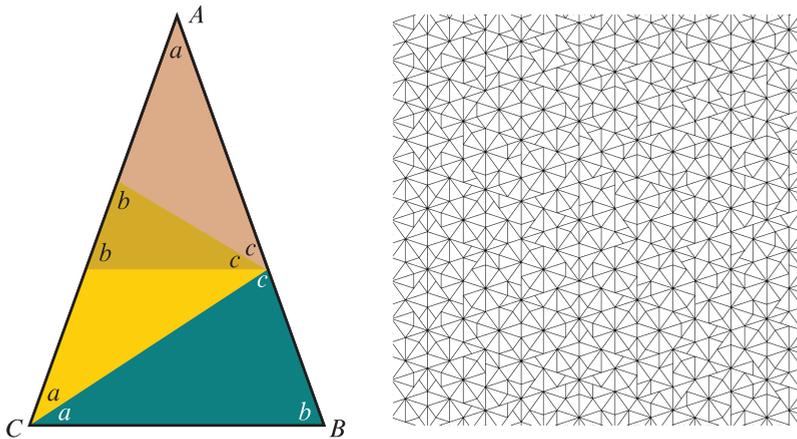
Figure 11. A $(\theta, \mathcal{W})$-tiling from an overlapping IFS; see Example 8.7.

Consider the IFS $\mathcal{F} = \{\mathbb{C}: f_1, f_2, f_3\}$ where

$$f_1(z) = r e^{3i\phi} \bar{z} + e^{-i\phi/2} \quad f_2(z) = r e^{4i\phi} z + e^{-i\phi/2} \quad f_3(z) = r\bar{z},$$

and $r = (\sqrt{5} - 1)/2$ (the reciprocal of the golden ratio) and $\phi = \pi/5$. The three functions f_1, f_2, f_3 are similitudes with the same scaling ratio r. The attractor of $\mathcal{F}$ is A, the triangle shown at the left in Figure 11. The first similitude takes the triangle A onto the green triangle; the second takes A onto the triangle that is the union of the two yellow tones; the third takes A onto the triangle that is the union of the two brown tones. To be exact, the first and third functions are orientation reversing, while the second is orientation preserving. Notice that the intersection of the second and third small isosceles triangles has nonempty interior; the attractor is overlapping. Let $M = \{M_1, M_2, M_3\}$ be the tops mask. More precisely, M_1 and M_2 are the green and yellowish acute triangles, respectively, and M_3 is the obtuse brown triangle at the top in the figure. The θ-sequence $\mathcal{W}$ of trees used in the $(\theta, \mathcal{W})$-tiling at the right in Figure 11, is the standard sequence (see Example 8.1). The second formulation of Definition 8.2 was used to create this picture. There are

tilings by copies of the Robinson triangles *A* and *B* that are equivalent to Penrose tilings by kites and darts [19], [21], but this is not one of them. In a Penrose tiling by Robinson tiles, the intersection of any two abutting triangles is a common edge, which is not the case in this example. We would expect that much of the theory in this paper can be extended, although notationally cumbersome, to graph IFSs. In this context, the Penrose tilings would be an example.

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