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Geodesic interpolation on Sierpiński gaskets

Caitlin M. Davis, Laura A. LeGare, Cory W. McCartan, and Luke G. Rogers

Abstract. We study the analogue of a convex interpolant of two sets on Sierpiński gaskets and an associated notion of measure transport. The structure of a natural family of interpolating measures is described and an interpolation inequality is established. A key tool is a good description of geodesics on these gaskets, some results on which have previously appeared in the literature [19, 17, 16, 11].

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Contents

Introduction

The notion of a convex interpolant $(1-t)A+tB = \{(1-t)a+tb\colon a \in A, b \in B\}$ for sets $A, B \subset \mathbb{R}^n$ and $t \in [0,1]$, and the Brunn–Minkowski inequality $|(1-t)A+tB|^{1/n} \geq (1-t)|A|^{1/n} + t|B|^{1/n}$ for the n-volume $|\cdot|$, have long had a central role in convex geometry. More recently, the class of functional inequalities that includes the Brunn–Minkowski inequality has been used to dramatically extend notions of curvature to more general settings, and a rich theory has developed around these advances [10, 3].

The study of functional inequalities in the setting of fractal metric-measure spaces is considerably less developed. One area in which there has been a great deal of work is in relating the variation with $\epsilon > 0$ of the volume of an ϵ-neighborhood of a set to the analytic and geometric properties of the set. For Euclidean sets with sufficiently smooth boundaries, such results can be obtained using the Steiner formula and inequalities of Brunn–Minkowski type. In the case of certain fractal sets and sets with fractal boundary in Euclidean space, one achievement of the theory developed by Lapidus and collaborators is a characterization of the volume of ϵ-neighborhoods using complex dimensions, which in turn are connected to analytic structure on the set through the zeta function of its Laplacian [14]. Functional inequalities classically associated with curvature are also beginning to be considered in fractal analytic settings [4, 1].

A feature of the preceding work is that it does not generally use convex interpolation. Indeed, we are not aware of previous work involving convex interpolation on fractal sets. The purpose of this paper is to consider the elementary notion of convex interpolant in the setting of one well-studied class of fractals, the Sierpiński gaskets S_n defined on regular n-simplices in $\mathbb{R}^n$. When endowed with the Euclidean metric restricted to the set, these examples are geodesic spaces. Following [6] we can therefore define a convex interpolant $\widetilde{Z}_t(A, B)$ which generalizes the Euclidean notion of $(1-t)A + tB$ by setting

$$\begin{aligned} \widetilde{Z}_t(a,b) &= \{x : d(a,x) = t d(a,b) \text{ and } d(x,b) = (1-t)d(a,b)\}, \\ \widetilde{Z}_t(A,B) &= \{\widetilde{Z}_t(a,b) : a \in A, b \in B\}. \end{aligned} \tag{0.1}$$

Our goal is to study some basic properties of this interpolating set and the naturally related notion of an interpolating measure on the sets S_n.

The study of $\widetilde{Z}_t(A, B)$ requires that we have a good understanding of geodesics in the Sierpiński gasket S_n. These have been studied, for example in [19] and more recently [17, 16, 11], but we make some explicit constructions and reprove some fundamental theorems by methods connected to the barycentric projection in [19] because they are essential in our treatment of $\widetilde{Z}_t$. The proofs are also, in our view, simpler than some of those in [17, 16, 11]. These results, including some which are new, are in Section 2. Our study of interpolation occupies Sections 3–5; we first treat $\widetilde{Z}_t(A, B)$ in the case where A is a cell and B a point and then when A and B are disjoint cells. It is easy to determine that no direct analogue of the Brunn–Minkowski inequality can hold, but we conclude with one possible interpolation inequality in Section 6.

We emphasize to the reader that this study is intentionally limited in scope. There are so many inequalities and applications of inequalities in this area that it is not practical to attempt an exhaustive treatment, even when we limit ourselves to such a simple class of examples. Moreover, the naturality of convex interpolation in our setting is not discussed, and in particular we do not consider whether the interpolation of measures is an optimizer of a transport problem (see [21]). No doubt each reader will notice problems they think should perhaps have been considered, and we hope they will be inspired to do so themselves.

1. Preliminaries

The *Sierpiński n-gasket* $S_n \subseteq \mathbb{R}^n$ is the unique nonempty compact attractor of the iterated function system (IFS)

$$F_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad F_i(x) = \frac{1}{2}(x + q_i),$$

where $i \in \{0, 1, \dots, n\}$, and where $\{q_0, q_1, \dots, q_n\}$ are the vertices of an n-simplex with sides of unit length. We begin with some essential definitions and properties of S_n, mostly following the conventions of Strichartz and coauthors in [2, 20].

An *m-level cell* of the Sierpiński n-gasket is a set of the form $F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}(S_n)$. We call the sequence of letters $w = w_1 w_2 \cdots w_m$ from the alphabet $\{0, \dots, n\}$ a *finite address* of length $|w| = m$. We identify finite addresses with cells and use the notation $\langle w \rangle := F_w(S_n)$. An $(m + k)$-level cell contained in a given m-level cell is called a *level k subcell* of the m-level cell. A given m-level cell $\langle w \rangle$ has $(n + 1)$ 1-level subcells, or *maximal subcells*: $\langle w0 \rangle, \langle w1 \rangle, \dots, \langle wn \rangle$. Since S_n is defined on an n-simplex with unit length sides, the side length of an m-level cell is $\left(\frac{1}{2}\right)^m$.

A strictly descending chain of cells $S_n \supseteq \langle w_1 \rangle \supseteq \langle w_1 w_2 \rangle \supseteq \cdots$ intersects to a point with *address* $w_1 w_2 \cdots$. As with cells, we identify infinite addresses with points by writing $\langle w_1 w_2 \cdots \rangle$. We use an overline to denote repeating characters in an address, so $20111 \cdots = 20\bar{1}$. In this notation, we have $q_j = \bar{j}$ for $j = 0, \dots, n$.

1.1. Vertices, address equivalence, and barycentric coordinates.

The *boundary points* of a cell $\langle w \rangle$ are the vertices $\langle w\bar{0} \rangle, \langle w\bar{1} \rangle, \dots, \langle w\bar{n} \rangle$. The *m-level vertex set* of the n-simplex, denoted V_n^m, is defined recursively by

$$V_n^0 = \{q_0, q_1, \dots, q_n\} \quad \text{and} \quad V_n^m = \bigcup_{i=0}^{n} F_i(V_n^{m-1}).$$

The set of all vertices of S_n, denoted V_n^*, is defined as $\bigcup_{m=0}^{\infty} V_n^m$; this set is dense in S_n.

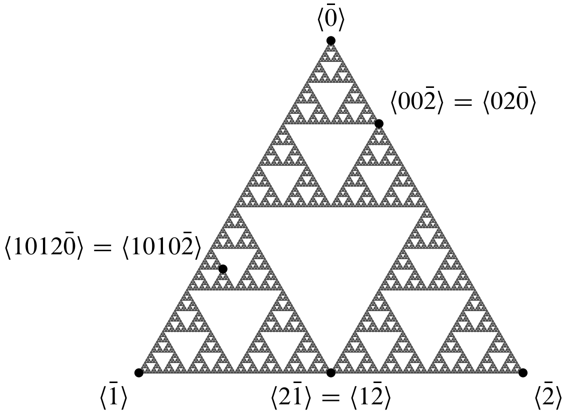


Figure 1. S_2, with several points labelled.

Every vertex in $V_n^* \setminus V_n^0$ sits at the intersection of two neighboring cells, and consequently can be described by exactly two addresses, which are readily seen to have the form $wj\bar{\imath}$ and $wi\bar{\jmath}$, where $\langle wi\rangle$ and $\langle wj\rangle$ are the intersecting cells. Figure 1 demonstrates this property in S_2 and illustrates the addressing scheme. Each point in $(S_n \setminus V_n^*) \cup V_n^0$ has a unique address.

In addition to point addresses, we make considerable use of the barycentric coordinate system on S_n. Recall that the convex hull of $\{q_0, q_1, \ldots, q_n\}$, which contains S_n, consists of the points

$$x = c_0 q_0 + c_1 q_1 + \cdots + c_n q_n, \tag{1.1}$$

in which each $c_j \geq 0$ and $c_0 + c_1 + \cdots + c_n = 1$. The c_j are called the *barycentric coordinates* of x, and we denote the i^{th} barycentric coordinate c_i of x by $[x]_i$.

It is useful to consider the dyadic expansions $[x]_i = \sum_{j=1}^{\infty} c_i^j 2^{-j}$ of the barycentric coordinates, because of the following easy result that is well known [9, p. 10] and will be used frequently throughout the present work.

Lemma 1.1. *A point x is in S_n if and only if there is a dyadic expansion of its barycentric coordinates with the property that for each j there is a unique $i \in \{0, \ldots, n\}$ so that $c_i^j = 1$. In fact, $x = \langle w_1 w_2 \cdots\rangle$ if and only if $c_i^j = 1$ precisely when $w_j = i$.*

Proof. Observe that points in the 1-cell $\langle i\rangle$ have $c_i^1 = 1$ and all other $c_k^1 = 0$. The result then follows by self-similarity and induction. $\square$

Remark. Points in V_n^* are those for which each c_i is a dyadic rational, and the two addresses for a vertex correspond to the two (nonterminating) binary representations of the vertex. For example, the vertex $\langle 1\bar{0}\rangle = \langle 0\bar{1}\rangle = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ in S_2 can be expressed in binary as either $(0.1, 0.0\bar{1}, 0)$ or $(0.0\bar{1}, 0.1, 0)$; it may also be represented as $(0.1, 0.1, 0)$, but this latter fails to satisfy the condition that exactly one of the $c_j^k = 1$ for a given value of k.

1.2. Self-similar measure. A natural class of measures on S_n are the *self-similar measures*. A measure μ_n of this type is a probability measure determined uniquely from a set of weights $\{\mu_n^i\}_{i=0}^n$, where each $\mu_n^i > 0$ and $\sum_i \mu_n^i = 1$, by the requirement that for any measurable $X \subseteq S_n$ one has the self-similar identity

$$\mu_n(X) = \sum_i \mu_n^i \mu_n(F_i^{-1}(X)).$$

The existence and uniqueness of such measures is due to Hutchinson [13].

The *standard measure* on S_n is self-similar with weights $\mu_n^i = \frac{1}{n+1}$ for all $i \in \{0, 1, \dots, n\}$. The standard measure of any m-level cell is $\left(\frac{1}{n+1}\right)^m$.

When we study interpolation in S_n, the interpolant sets will be determined by projections of the original sets. The measure on interpolant sets will be a projection of the original self-similar measure, and will therefore be self-similar itself. This is a consequence of the following results, which are well-known when each F_i is a homothety, as it is here.

Lemma 1.2. *Let $q_0, q_1, \dots, q_n \in \mathbb{R}^m$, and consider the IFS $\{F_i : \mathbb{R}^m \to \mathbb{R}^m\}$ with $F_i(x) = \frac{1}{2}(x + q_i)$. Fix $\vec{v}$ a unit vector and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by $x \mapsto \langle x, v\rangle$. Then, defining $\widetilde{F}_i := \phi \circ F_i \circ \phi^{-1}$, we have $\widetilde{F}_i(x) = \frac{1}{2}(x + \phi(q_i))$ for $x \in \mathbb{R}$.*

Proof. The map ϕ is linear, so if y satisfies $\phi(y) = x$, then

$$\phi \circ F_i(y) = \phi\Big(\frac{1}{2}(y + q_i)\Big) = \frac{1}{2}(\phi(y) + \phi(q_i)) = \frac{1}{2}(x\vec{v} + \phi(q_i)). \qquad \square$$

Proposition 1.3. *In the setting of Lemma 1.2, let K denote the attractor of the IFS and μ be the self-similar measure on K with weights μ^i. Then the pushforward measure $\phi_*\mu$ satisfies the self-similar identity*

$$\phi_*\mu(X) = \sum_i \mu^i \phi_*\mu(\widetilde{F}_i^{-1}(X))$$

for measurable $X \subseteq \phi(K)$.

Proof. By the definition of the pushforward measure and the self-similarity of μ,

$$\phi_*\mu(X) = \mu(\phi^{-1}(X)) = \sum_i \mu^i \mu(F_i^{-1}(\phi^{-1}(X))).$$

However $F_i^{-1} \circ \phi^{-1}(X) = \phi^{-1} \circ \widetilde{F}_i^{-1}(X)$, because, using Lemma 1.2,

$$\phi \circ F_i(y) = \phi(\frac{1}{2}(y + q_i)) = \frac{1}{2}(\phi(y) + \phi(q_i)) = \widetilde{F}_i \circ \phi(y).$$

Thus

$$\phi_*\mu(X) = \sum_i \mu^i\, \mu(\phi^{-1} \circ \widetilde{F}_i^{-1}(X)) = \sum_i \mu^i\, \phi_*\mu(\widetilde{F}_i^{-1}(X)). \qquad \square$$

2. Geodesics

Our goal in this section is to relate barycentric coordinates to distance, and to characterize nonuniqueness of geodesics in S_n. We begin by considering geodesics from a point to a boundary point, and then generalize to arbitrary points in S_n. We prove that there exist at most five distinct geodesics between any two points in S_2, and at most eight geodesics between any two points in S_n, $n \geq 3$.

Let $x, y \in S_n$. A *path* from x to y in S_n is a continuous function $\gamma\colon [0, 1] \to S_n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We say that γ *passes through* a point z if for some $t \in (0, 1)$ we have $\gamma(t) = z$. The length of a path γ, given by $\mathcal{H}^1(\gamma([0, 1]))$, is denoted $|\gamma|$; a priori it may be infinite, but we will only be interested in finite paths. To avoid the usual problem of distinguishing γ from its image we will always assume γ is parametrized at constant speed, so $\left|\frac{\partial \mathcal{H}^1(\gamma([0,t]))}{\partial t}\right| = |\gamma|$ for a.e.-t, unless some other parametrization is specified.

It is easy to see that there is always a finite path between any x and y. We then define the *intrinsic metric*, $d\colon S_n \times S_n \to \mathbb{R}$ by

$$d(x, y) = \inf\{|\gamma|\colon \gamma \text{ is a path from } x \text{ to } y\}.$$

This was previously investigated for the case of S_2 in [19, Section 8], and later in [7, 16]. In particular, the question of the existence of minimizing paths, or *geodesics* has been considered in this case. Strichartz [19] used barycentric coordinates to give a simple construction of geodesics; we follow his method, but correct his statement that the maximum number of geodesics between an arbitrary pair of points is five rather than four. Saltan et al. [16] present a formula for d in terms of address representations of x and y, and obtain the correct value for the maximum number of geodesics between x and y, but their approach is somewhat complicated and is only done for $n = 2$. This result was generalized in [11] to the case $n \geq 3$. The rest of this section presents an alternative approach to these results for S_n, $n \geq 3$ and fixes notation that will be needed in our study of geodesic interpolation.

We begin by studying the problem of connecting a boundary point x of a cell to a point y in that cell by a geodesic. The following lemma proves that if such a geodesic exists it must lie in the cell.

Lemma 2.1. *Two boundary points of a cell $\langle w \rangle$ are joined by a unique geodesic, namely the line segment between them. If x is a boundary point of a cell $\langle w \rangle$ and $y \in \langle w \rangle$ then for any path between x and y that is not contained in $\langle w \rangle$ there is a strictly shorter path which is contained in $\langle w \rangle$.*

Proof. The first statement is an obvious consequence of the fact that the line segment between two boundary points of $\langle w \rangle$ is contained in $\langle w \rangle$ and is a Euclidean geodesic. The second uses the following observation: if γ is a path from x to y that exits $\langle w \rangle$ at a boundary point z then either γ re-enters at z in which case it can be shortened by removing the intervening component or it re-enters at another boundary point $\tilde{z}$, in which case it can be shortened by replacing the intervening component by the line segment from z to $\tilde{z}$, as the latter is geodesic. □

This lemma suggests a substantial reduction of the problem. We fix some notation.

Definition 2.2. For distinct points x and y, the unique smallest cell that contains both is called the *common cell* of x and y. If $\langle w \rangle$ is a cell, the intersection points of its $n+1$ maximal subcells, which have addresses $\langle wi\bar{j} \rangle$ for all pairs of distinct i, j, are called the *bridge points* of $\langle w \rangle$.

It is apparent that if x is the boundary point $\langle w\bar{\imath} \rangle$ of the common cell $\langle w \rangle$ of x and y then y is in a different maximal subcell $\langle wj \rangle$ than x, so $j \neq i$. Any path from x to y must pass through a boundary point of $\langle wj \rangle$, and such have the form $\langle wj\bar{k} \rangle$. The next lemma shows that we may assume $k = i$.

Lemma 2.3. *Let $x = \langle w\bar{\imath} \rangle$ and $y \in \langle wj \rangle$ with $j \neq i$. For any path γ from x to y there is a path of shorter or equal length that enters $\langle wj \rangle$ through the bridge point $\langle wi\bar{j} \rangle = \langle wj\bar{\imath} \rangle$.*

Proof. We know γ enters $\langle wj \rangle$ at a point $\langle wj\bar{k} \rangle$. Moreover, by modifying γ as in Lemma 2.1 to remove all excursions outside $\langle wj \rangle$, we obtain a (possibly shorter) path with the property that the only portion of the path that is outside $\langle wj\rangle$ is the initial segment from $\langle w\bar{i}\rangle$ to $\langle wj\bar{k}\rangle$. Let L denote the length of a side of a maximal subcell. If $k = i$ we are done, but if not then the point $\langle wj\bar{k}\rangle$ is not in $\langle wi\rangle$, so is distance at least L from $\langle wi\rangle$ and thus $2L$ from $\langle w\bar{i}\rangle$. However the line segments from $\langle wj\bar{k}\rangle$ to $\langle wj\bar{i}\rangle = \langle wi\bar{j}\rangle$ to $\langle w\bar{i}\rangle$ have length exactly $2L$, so using this as the initial segment of γ makes the path no longer and ensures it enters $\langle wj\rangle$ at the specified point. □

The preceding lemma tells us how to construct a geodesic from a boundary point of a cell to any point inside the cell. Moreover, it allows us to write the length of this geodesic in terms of the barycentric coordinates. The latter was previously noted in [19, 5].

Proposition 2.4. *Let $x = \langle w\bar{i}\rangle$ and $y \in \langle w\rangle$. Then $d(x, y) = [x]_i - [y]_i$ and there is a geodesic from x to y.*

Proof. It is sufficient to work on S_n, because $x, y \in \langle w\rangle$ implies the first $|w|$ binary terms of $[x]_i$ and $[y]_i$ are equal. Let w_m be the level m truncation of an (infinite) address for y and $x_m = \langle w_m\bar{i}\rangle$, so $x_0 = x$. Define a path γ on $[0, 1)$ by mapping $[1 - 2^{-k}, 1 - 2^{-(k+1)}]$ to the segment from x_k to x_{k+1}. Since $\{\bar{w_m}\}$ intersects to y we have $x_m \to y$ and thus may extend γ continuously to $[0, 1]$ by setting $\gamma(1) = y$. Lemma 2.3 ensures any path from x to y is at least as long as γ, so γ is a geodesic.

It remains to compute the length of γ. Observe that γ is constant on any segment where $x_k = x_{k+1}$, and otherwise has length $2^{-(k+1)}$. Moreover, $x_k = x_{k+1}$ if and only if the $(k+1)^{\text{th}}$ letter in the address of y is i. Now Lemma 1.1 says this occurs if and only if the $(k+1)^{\text{th}}$ term in the binary expansion of $[y]_i$ is 1, and that $x = \langle\bar{i}\rangle$ implies every term in the binary expansion of $[x]_i$ is 1. Thus the binary expansion of $[x]_i - [y]_i$ has coefficient 1 multiplying $2^{-(k+1)}$ precisely when $x_k \neq x_{k+1}$, proving the formula for $d(x, y)$. □

Remark. Note that the construction of γ terminates (i.e. $x_k = y$ for all sufficiently large k) if y is a vertex which has address $y = \langle w'\bar{i}\rangle$ for some word w'.

Corollary 2.5. *For any distinct pair of points x, y there is a geodesic from x to y. All geodesics are contained in the common cell of x and y.*

Proof. Take cells $\langle w_x\rangle$ containing x and $\langle w_y\rangle$ containing y with $|w_x| = |w_y| = m$ large enough that the cells do not intersect. There is a geodesic from each boundary point of $\langle w_x\rangle$ to x and from each boundary point of $\langle w_y\rangle$ to y. Moreover, between a boundary point of $\langle w_x \rangle$ and a boundary point of $\langle w_y \rangle$ any geodesic is composed of a finite union of edges in the graph at level m by Lemma 2.3. This reduces the problem of finding a geodesic to identifying the shortest curve in a finite collection, which may always be solved.

The fact that geodesics stay in the common cell is a consequence of Lemma 2.1, because a path that exits this cell may be written as the concatenation of a path from x to the cell boundary, a path between cell boundary points, and a path from the cell boundary to y, each of which can be strictly shortened if it exits the cell. □

Corollary 2.6. *Let $x = \langle \bar{\imath} \rangle$, and define a hyperplane $H = \{(y_0, y_1, \dots, y_n) \in S_n : y_i = a\}$ for some fixed a. Then $d(x, y)$ is constant for all $y \in H$.*

2.1. Uniqueness in S_2. We now turn to the question of geodesic uniqueness in the Sierpiński n-gasket. As shown in [17], there are at most five geodesics between two points in S_2. We provide an alternate, more geometrical proof, beginning with the case of geodesics between a boundary point of a cell and a point contained in the cell.

Proposition 2.7. *Let $x = \langle \bar{\imath} \rangle$ be a boundary point of S_n, and $y \neq x$. Then there is a unique geodesic between x and y unless $y = \langle wk\bar{\jmath} \rangle = \langle wj\bar{k} \rangle$ for some finite word w and some j, k such that i, j and k are distinct.*

Proof. Let γ_1 be a geodesic constructed as in Proposition 2.4 from a sequence of cells $\{\langle w_m \rangle\}$ that intersect to y and γ_2 be any other geodesic from x to y. Take the largest m such that γ_2 enters $\langle w_m \rangle$ through a vertex $z' = \langle w_m \bar{\jmath} \rangle$ with $j \neq i$, and write $z = \langle w_m \bar{\imath} \rangle$. Using Lemma 2.3 we can modify γ_2 to form $\tilde{\gamma}_2$ which passes through z and then z' and has the same length as γ_2. Then

$$\begin{aligned} d(x,z) + d(z,y) &= |\gamma_1| = |\tilde{\gamma}_2| \\ &= d(x,z) + d(z,z') + d(z',y) = d(x,z) + 2^{-m} + d(z',y). \end{aligned}$$

However Proposition 2.4 ensures $d(z, y) \leq 2^{-m} = d(z, z')$, so $d(z', y) = 0$ and $y = z'$ is a vertex. Moreover $\tilde{\gamma}_2 = \gamma_1$.

Now by our choice of m we know γ_1 and γ_2 coincide between x and $\langle w_{m-1} \bar{\imath} \rangle$ because they are built from the same bridge points. Thus the only difference between these geodesics occurs on $\langle w_{m-1} \rangle$, and they begin at $\langle w_{m-1} \bar{\imath} \rangle$ and end at $y = \langle w_m \bar{\jmath} \rangle$, where $w_m = w_{m-1} k$ for some k. Determining the possible paths in $\langle w_{m-1} \rangle$ is therefore the same as determining the geodesics in S_n from $\langle \bar{\imath} \rangle$ to $\langle k \bar{\jmath} \rangle = \langle j \bar{k} \rangle$ for some k and some $j \neq i$. This is an easy finite computation:

there is a unique such geodesic if $k = i$ or $k = j$ and exactly two geodesics if $k \neq i, j$, one through $\langle i\bar{k}\rangle = \langle k\bar{\imath}\rangle$ and one through $\langle i\bar{\jmath}\rangle = \langle j\bar{\imath}\rangle$. □

Remark. If y is a vertex of the form identified in the proposition then we may use either of its two addresses to construct a geodesic from x to y by the method of Proposition 2.4. The two addresses lead to the two distinct geodesics identified in Proposition 2.7.

Corollary 2.8. *Let $y \in \langle w\rangle \subseteq S_2$. If there are two distinct geodesics from y to a boundary point of $\langle w\rangle$ then there is only one geodesic from y to each of the other two boundary points of $\langle w\rangle$.*

Proof. There are two distinct geodesics from y to $\langle w\bar{\imath}\rangle$, so by Proposition 2.7 we have $y = \langle wj\bar{k}\rangle$ where j and k are distinct from each other and from i. The proposition also tells us that such a y has unique geodesics to the boundary points $\langle w\bar{\jmath}\rangle$ and $\langle w\bar{k}\rangle$, and since we are in S_2 with i, j, k distinct, this covers all boundary points. □

Our results on geodesics between a point in a cell and a boundary point of that cell have implications for geodesics between arbitrary points. Recall that the bridge points of S_n are the intersection points of maximal subcells. The following lemma gives a useful classification of geodesics between points in distinct maximal cells according to the number of bridge points they contain.

Lemma 2.9. *Any geodesic between two points in S_n passes through at most two bridge points.*

Proof. Let $x \in \langle i\rangle$ and $y \in \langle j\rangle$. We may construct a path between them by concatenating a geodesic from x to $\langle i\bar{\jmath}\rangle$ and a geodesic from $\langle i\bar{\jmath}\rangle$ to y. By Proposition 2.4 each such geodesic has length at most $\frac{1}{2}$, so $d(x, y) \leq 1$. However, bridge points are separated by distance $\frac{1}{2}$, so there can be at most two on a geodesic. □

We note that if the geodesic from x to y passes through only one bridge point it must be the intersection point of the maximal cells containing them.

Definition 2.10. Let $x \in \langle i\rangle$ and $y \in \langle j\rangle$, $i \neq j$. The geodesic γ from x to y is a P_1 *geodesic* if it passes through the bridge point $\langle i\bar{\jmath}\rangle = \langle i\rangle \cap \langle j\rangle$ and a P_2 *geodesic* if it passes through two bridge points, $\langle i\bar{k}\rangle$ and $\langle j\bar{k}\rangle$, where $k \neq i, j$.

Cristea and Steinsky [7] provide geometric criteria for S_2 and S_3 that allow one to determine whether one or both types of geodesics exist between some pair of points. The proof of the following theorem, which gives a sharp bound on the number of geodesics between x and y in S_2, recovers their results for S_2. The sharp bound was previously proved in [17] by a different method.

Theorem 2.11. *There are at most five distinct geodesics between any two points in S_2, and this bound is sharp.*

Proof. Fix x and y. Corollary 2.5 tells us that all geodesics between x and y lie in their common cell, so we may assume the common cell is S_2. Thus $x \in \langle i \rangle$ and $y \in \langle j \rangle$ with $i \neq j$, and the geodesics between them are either P_1 geodesics through $\langle i\bar{j} \rangle$ or P_2 geodesics through $\langle i\bar{k} \rangle$ and $\langle j\bar{k} \rangle$ where $k \neq i, j$.

If γ is a geodesic from x to y then its restriction to $\langle i \rangle$ is a geodesic from x to either $\langle i\bar{j} \rangle$ or $\langle i\bar{k} \rangle$. Proposition 2.7 says there are at most two geodesics to either of these points and Corollary 2.8 says that if there are two to one such point then there is only one to the other, so there are at most three options for the restriction of γ to $\langle i \rangle$. Similarly there are at most three options for the restriction of γ to $\langle j \rangle$.

We now consider how the pieces of geodesic previously described may be combined. As our goal is to upper bound the number of geodesics, it suffices to consider only the cases where all possible options for the restriction of γ to $\langle i \rangle$ and $\langle j \rangle$ are part of geodesics from x to y.

Case 1 There are two distinct geodesics between x and $\langle i\bar{j} \rangle$ and two between y and $\langle i\bar{j} \rangle$, providing four P_1 paths. In this case the geodesics from x to $\langle i\bar{k} \rangle$ and y to $\langle j\bar{k} \rangle$ are unique, as is that between these bridge points, so there is one P_2 path. If the P_1 and P_2 geodesics are the same length then there are five geodesics in total, otherwise there are four or one. Figure 2 shows that five may be achieved.

Case 2 There are two distinct geodesics between x and $\langle i\bar{j} \rangle$ and two between y and $\langle j\bar{k} \rangle$. Then there is one from y to $\langle i\bar{j} \rangle$ so there are two P_1 paths, and there is one from x to $\langle i\bar{k} \rangle$, so there are two P_2 paths. If all of these have the same length there are four geodesics, otherwise there are two. □

2.2. Uniqueness in S_n. The proof of Theorem 2.11 relies on two facts about S_2: there are only three bridge points, so there is at most one pair of bridge points through which a P_2 geodesic can pass, and nonuniqueness of a geodesic to one bridge point implies uniqueness to the other two bridge points (Corollary 2.8). Neither of these arguments is directly applicable to S_n, $n \geq 3$. However we can obtain a sharp bound in this more general setting by making a more detailed analysis of P_2 geodesics.

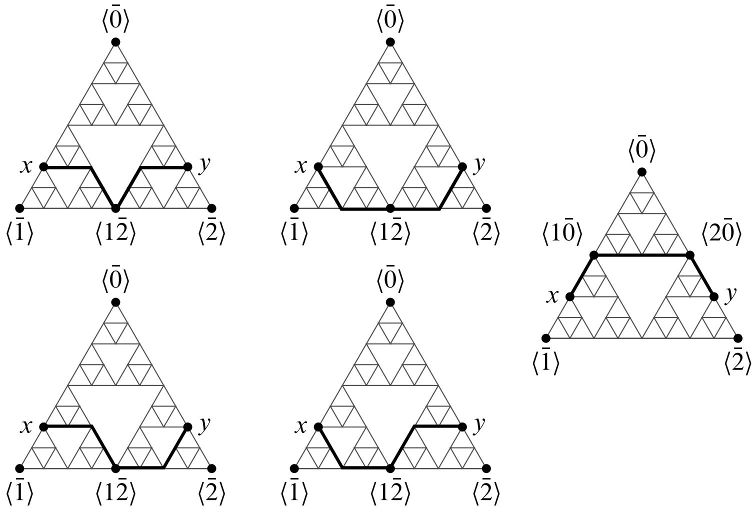


Figure 2. Points x and y that are connected by five distinct geodesics: four P_1 geodesics (left) and one P_2 geodesic (right).

Lemma 2.12. *Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$, where $i \neq j$. Then there exist P_2 geodesics between x and y passing through at most two distinct pairs of bridge points.*

Proof. Suppose there is a P_2 geodesic γ from x to y through $\langle i\bar{k}\rangle$ and $\langle j\bar{k}\rangle$. Applying Proposition 2.4 we have $d(x, \langle i\bar{k}\rangle) = [\langle i\bar{k}\rangle]_k - [x]_k = \frac{1}{2} - [x]_k$ and similarly for $d(y, \langle j\bar{k}\rangle)$, so

$$d(x, y) = |\gamma| = \frac{1}{2} + d(x, \langle i\bar{k}\rangle) + d(y, \langle j\bar{k}\rangle) = \frac{3}{2} - [x]_k - [y]_k.$$

However in the proof of Lemma 2.9 we saw that $d(x, y) \leq 1$, so $[x]_k + [y]_k \geq \frac{1}{2}$.

Now $\sum_{k=0}^{n} [x]_k + [y]_k = 2$ from the definition of the barycentric coordinates, and $x \in \langle i \rangle$, $y \in \langle j \rangle$ implies $[x]_i \geq \frac{1}{2}$ and $[y]_j \geq \frac{1}{2}$, so the number of k for which $[x]_k + [y]_k \geq \frac{1}{2}$ is at most 2, which implies there are at most two values of k for which there is a P_2 geodesic through the bridge points $\langle i\bar{k}\rangle$ and $\langle j\bar{k}\rangle$. □

It is apparent in the preceding proof that the locations of x and y are tightly constrained when they admit geodesics through two distinct pairs of bridge points. The following result makes this precise.

Lemma 2.13. *Let $x \in \langle i \rangle$ and $y \in \langle j \rangle$ with $i \neq j$. If there are P_2 geodesics from x to y through two distinct pairs of bridge points, then there are exactly two P_2 geodesics from x to y.*

Proof. Let γ_1 be a P_2 geodesic passing through $\langle i\bar{k}\rangle$ and $\langle j\bar{k}\rangle$, and let γ_2 be a P_2 geodesic passing through $\langle i\bar{l}\rangle$ and $\langle j\bar{l}\rangle$. It follows from the proof of Lemma 2.9 that $|\gamma_1| = |\gamma_2| \leq 1$, so $|\gamma_1| + |\gamma_2| \leq 2$. Now by the triangle inequality

$$\begin{aligned} 2 &= 1 + d(\langle i\bar{k}\rangle, \langle i\bar{l}\rangle) + d(\langle j\bar{k}\rangle, \langle j\bar{l}\rangle) \\ &\leq \frac{1}{2} + d(\langle i\bar{k}\rangle, x) + d(x, \langle i\bar{l}\rangle) + \frac{1}{2} + d(\langle j\bar{k}\rangle, y) + d(y, \langle j\bar{l}\rangle) \\ &= |\gamma_1| + |\gamma_2| \leq 2. \end{aligned}$$

Thus equality holds in the triangle inequality and x lies on the geodesic connecting $\langle i\bar{l}\rangle$ to $\langle i\bar{k}\rangle$, and y lies on the geodesic connecting $\langle j\bar{l}\rangle$ to $\langle j\bar{k}\rangle$, both of which are lines in $\mathbb{R}^n$. This shows the geodesics from x to $\langle i\bar{k}\rangle$ and to $\langle i\bar{l}\rangle$ are unique and similarly for y; the result then follows from Lemma 2.12 □

Lemmas 2.12 and 2.13 provide us with sufficient restrictions on the number of geodesics to prove our main result on geodesics in S_n, $n \geq 3$.

Theorem 2.14. *There exist at most eight distinct geodesics between any two points in S_n, $n \geq 3$ and this bound is sharp.*

Proof. Let $x \in \langle i\rangle$ and $y \in \langle j\rangle$. As in the proof of Theorem 2.11 we may assume $i \neq j$. By Proposition 2.7, there exist at most two geodesics from x to $\langle i\bar{j}\rangle$ and two geodesics from $\langle i\bar{j}\rangle$ to y. Concatenations of these pairs of geodesics yield a maximum of four P_1 geodesics.

By Lemma 2.12, there exist P_2 geodesics through at most two distinct pairs of bridge points. If there are P_2 geodesics through two distinct pairs of bridge points, then by Lemma 2.13, there are exactly two P_2 geodesics between x and y. If, in addition, there exist P_1 geodesics, then there are at most six total geodesics.

If, instead, there are P_2 geodesics only through a single pair of bridge points $\langle i\bar{k}\rangle$, $\langle j\bar{k}\rangle$, then they are obtained by concatenating geodesics from x to $\langle i\bar{k}\rangle$ (of which there are at most two by Proposition 2.7) and y to $\langle j\bar{k}\rangle$ (of which there are again at most two) with the interval from $\langle i\bar{k}\rangle$ to $\langle j\bar{k}\rangle$. This yields at most four P_2 geodesics between x and y. If, in addition, there exist P_1 geodesics, then there are at most eight geodesics in total. Sharpness is demonstrated by Example 2.15, which is written in S_3 and embeds in S_n for all $n \geq 3$. □

Example 2.15. Let $x = \langle 202\bar{1}\rangle = \langle 201\bar{2}\rangle$ and $y = \langle 303\bar{1}\rangle = \langle 301\bar{3}\rangle$. Then the following are the geodesics from x to y:

$$
\begin{aligned}
\gamma_1 &: x \longrightarrow \langle 202\bar{3}\rangle \longrightarrow \langle 20\bar{3}\rangle \longrightarrow \langle 2\bar{3}\rangle = \langle 3\bar{2}\rangle \longrightarrow \langle 30\bar{2}\rangle \longrightarrow \langle 303\bar{2}\rangle \longrightarrow y,\\
\gamma_2 &: x \longrightarrow \langle 202\bar{3}\rangle \longrightarrow \langle 20\bar{3}\rangle \longrightarrow \langle 2\bar{3}\rangle = \langle 3\bar{2}\rangle \longrightarrow \langle 30\bar{2}\rangle \longrightarrow \langle 301\bar{2}\rangle \longrightarrow y,\\
\gamma_3 &: x \longrightarrow \langle 201\bar{3}\rangle \longrightarrow \langle 20\bar{3}\rangle \longrightarrow \langle 2\bar{3}\rangle = \langle 3\bar{2}\rangle \longrightarrow \langle 30\bar{2}\rangle \longrightarrow \langle 303\bar{2}\rangle \longrightarrow y,\\
\gamma_4 &: x \longrightarrow \langle 201\bar{3}\rangle \longrightarrow \langle 20\bar{3}\rangle \longrightarrow \langle 2\bar{3}\rangle = \langle 3\bar{2}\rangle \longrightarrow \langle 30\bar{2}\rangle \longrightarrow \langle 301\bar{2}\rangle \longrightarrow y,\\
\gamma_5 &: x \longrightarrow \langle 202\bar{0}\rangle \longrightarrow \langle 20\bar{0}\rangle = \langle 2\bar{0}\rangle \longrightarrow \langle 3\bar{0}\rangle = \langle 30\bar{0}\rangle \longrightarrow \langle 303\bar{0}\rangle \longrightarrow y,\\
\gamma_6 &: x \longrightarrow \langle 202\bar{0}\rangle \longrightarrow \langle 20\bar{0}\rangle = \langle 2\bar{0}\rangle \longrightarrow \langle 3\bar{0}\rangle = \langle 30\bar{0}\rangle \longrightarrow \langle 301\bar{0}\rangle \longrightarrow y,\\
\gamma_7 &: x \longrightarrow \langle 201\bar{0}\rangle \longrightarrow \langle 20\bar{0}\rangle = \langle 2\bar{0}\rangle \longrightarrow \langle 3\bar{0}\rangle = \langle 30\bar{0}\rangle \longrightarrow \langle 303\bar{0}\rangle \longrightarrow y,\\
\gamma_8 &: x \longrightarrow \langle 201\bar{0}\rangle \longrightarrow \langle 20\bar{0}\rangle = \langle 2\bar{0}\rangle \longrightarrow \langle 3\bar{0}\rangle = \langle 30\bar{0}\rangle \longrightarrow \langle 301\bar{0}\rangle \longrightarrow y,
\end{aligned}
$$

where each geodesic is composed of the edges in S_n joining each pair of consecutive points.

Note that each portion of these connecting x or y to a bridge point is geodesic because it is constructed by the algorithm in Proposition 2.4. The first four are P_1 paths, with length $2\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{4}\right) = 1$. The second four are P_2 paths, with length $2\left(\frac{1}{8} + \frac{1}{8}\right) + \frac{1}{2} = 1$. We have constructed all the candidates to be geodesic from x to y (as described in Corollary 2.5), so the fact that they are all the same length ensures all are geodesics.

For use in later sections of this paper, it is convenient to note the following consequence of our analysis of the structure of geodesics.

Theorem 2.16. *The set of pairs of points $(x, y) \in S_n \times S_n$ such that there is more than one geodesic from x to y has zero $\mu_n \times \mu_n$-measure.*

Proof. First observe that sets of the form $\{a\} \times S_n$ or $S_n \times \{b\}$ are null for $\mu_n \times \mu_n$. Taking the countable union over $a \in V_n^*$ and over $b \in V_n^*$ gives a null set. Now observe from Proposition 2.7 that x is connected to any boundary point of a cell containing x by more than one geodesic then (x, y) is in one of these null sets and similarly for y. Accordingly, we can assume that there is a unique geodesic from x to any boundary point of a cell containing x, and similarly for y.

In this circumstance the only way x and y can be joined by more than one geodesic involves at least one P_2 geodesic. Precisely, there is a cell $\langle w_x\rangle$ containing x and a cell $\langle w_y\rangle$ containing y, these cells are joined by distinct geodesics γ and γ' such that γ enters $\langle w_x\rangle$ at a_x and $\langle w_y\rangle$ at a_y, and γ' enters $\langle w_x\rangle$ at a'_x and $\langle w_y\rangle$ at a'_y. Moreover the fact that these have equal length may be written as $d(x, a_x) + d(y, a_y) + |\gamma| = d(x, a'_x) + d(y, a'_y) + |\gamma'|$. There are countably many choices of pairs $\langle w_x \rangle$, $\langle w_y \rangle$ and, for each pair, finitely many possibilities for γ and γ', so to prove the set of pairs (x, y) joined by non-unique geodesics of this type is null we need only prove that for such a pair of cells, geodesics and boundary points, one has

$$(\mu_n \times \mu_n)(\{(x, y) \colon d(x, a_x) - d(x, a'_x) + d(y, a_y) - d(y, a'_y) = |\gamma'| - |\gamma|\}) = 0. \tag{2.1}$$

Moreover, since $\mu_n \times \mu_n$ is a product measure, by Fubini's theorem it is sufficient that this set has zero μ_n measure for each fixed y. More precisely, since fixing y fixes the value $d(y, a_y) - d(y, a'_y)$, it is enough that for any s, $\mu_n(\{x : d(x, a_x) - d(x, a'_x) = s\}) = 0$. Clearly the question of whether this set is null is invariant under rescaling the cell $\langle w_x \rangle$ to be S_n, and by symmetry we may assume $a_x = q_0$, $a'_x = q_1$; we assume this is the case.

Now we use Proposition 2.4 to write $d(x, q_0)$ as the projection on the barycentric coordinate corresponding to q_0, and writing $d(x, q_1)$ in the same manner we find that $d(x, q_0) - d(x, q_1)$ is the projection of $x - q_0$ on the unit vector $q_1 - q_0$ which is parallel to an edge of the simplex. Parametrizing the position along the line from q_0 to q_1 by $[0, 1]$ and writing $\pi \colon S_n \to [0, 1]$ for the projection on $q_1 - q_0$ in this parametrization, we see that the measure $\mu_n(\{x \colon d(x, q_0) - d(x, q_1) \in E\}) = \mu_n \circ \pi^{-1}(E)$ is the pushforward measure $\pi_* \mu_n$ of μ_n under π. However this is a self-similar measure on $[0, 1]$ by Proposition 1.3, with the self-similarity relation

$$\pi_* \mu_n(s) = \frac{1}{n+1}\Big(\pi_* \mu_n(2s) + \pi_* \mu_n(2s - 1) + (n-1)\pi_* \mu_n\Big(2s - \frac{1}{2}\Big)\Big).$$

This measure is non-atomic. To see this, suppose the contrary. It is a probability measure, so there is an atom which attains the maximal mass among atoms; we let s_0 be the location of such an atom. Then the self-similarity relation says

$$(n+1)\pi_* \mu_n(\{s_0\}) = \pi_* \mu_n(\{2s_0\}) + \pi_* \mu_n(\{2s_0 - 1\}) + (n-1)\pi_* \mu_n\Big(\Big\{2s_0 - \frac{1}{2}\Big\}\Big)$$

but at most two of the points $2s_0$, $(2s_0 - 1)$ and $(2s_0 - \frac{1}{2})$ are in $[0, 1]$, and the atoms at these points have mass not exceeding $\mu_n(\{s_0\})$, so that

$$(n+1)\pi_* \mu_n(\{s_0\}) \le n\pi_* \mu_n(\{s_0\})$$

and thus $\pi_* \mu_n(\{s_0\}) = 0$.

The fact that $\pi_* \mu_n$ is non-atomic says precisely that

$$\mu_n(\{x \colon d(x, q_0) - d(x, q_1) = s\}) = 0$$

for every choice of s. As previously noted, this ensures the measure of the set in (2.1) is zero, and by taking the union over the countably many possible cells and geodesics connecting their boundary points we complete the proof. □

Remark. The paper [11] states Theorem 2.16 as their Theorem 1.3, but it appears to us that something is missing in the proof. Specifically, the authors reduce to the situation where, in our notation, $x \in \langle 0\rangle$, $y \in \langle 1\rangle$ and there are a P_1 and a P_2 geodesic between these points (see the reasoning following their Lemma 4.3, where they say that there are vertices from our V_1 which they call b_1, b_i and b_i', such that

$$d(x, b_1) + d(b_1, y) = d(x, b_i) + d(b_i, b_i') + d(b_i, y)$$

with $d(b_i, b_i') = \frac{1}{2}$.) Using their Lemmas 4.4–4.7 they appear to be saying, in the proof of Proposition 4.8, that then x and y are points of V_*. (They write this as $x = \sigma 1^\infty$, $y = \sigma' 0^\infty$.) Yet we can give an example of points x and y in S_2 that are as described above but are not from V_*, as follows. Consider the three line segments forming a triangle around the central hole of the gasket S_2. These have vertices with addresses $\langle 0\bar{1}\rangle$, $\langle 0\bar{2}\rangle$ and $\langle 1\bar{2}\rangle$. Take a point x in $\langle 0\rangle$ at distance s from $\langle 0\bar{2}\rangle$ where s is not a dyadic rational (so $x \notin V_*$) and $s < \frac{1}{4}$. Take y in $\langle 1\rangle$ at distance $\frac{1}{4} - s$ from $\langle 1\bar{2}\rangle$. Evidently, any geodesic between these points lies on the three line segments. Now the distance from x to y through the points $\langle 0\bar{2}\rangle = \langle 2\bar{0}\rangle$ and $\langle 2\bar{1}\rangle = \langle 1\bar{2}\rangle$ is $s + \frac{1}{2} + \left(\frac{1}{4} - s\right) = \frac{3}{4}$, because it includes the edge through the cell $\bar{2}$. It is equally apparent that the distance via $\langle 0\bar{1}\rangle = \langle 1\bar{0}\rangle$ is $\frac{1}{2} - s + \frac{1}{2} - \left(\frac{1}{4} - s\right) = \frac{3}{4}$. So there are two geodesics joining these points but neither point is in V_*.

3. Interpolation

In Euclidean space the barycenter of sets A and B is $(1-t)A + tB = \{(1-t)a + tb \colon a \in A, b \in B\}$. In a geodesic space the natural analogue, introduced in [6], is the set defined by

$$\begin{aligned} \widetilde{Z}_t(a,b) &= \{x \colon d(a,x) = t d(a,b) \text{ and } d(x,b) = (1-t) d(a,b)\}, \\ \widetilde{Z}_t(A,B) &= \{\widetilde{Z}_t(a,b) \colon a \in A, b \in B\}. \end{aligned} \tag{3.1}$$

From our results on geodesics we know that $\widetilde{Z}_t(a,b)$ is a single point for almost all a and b, but in any case contains at most 8 points.

The classical Brunn–Minkowski inequality in $\mathbb{R}^n$ says $\operatorname{vol}(\widetilde{Z}_t(A,B))^{1/n} \geq (1-t)\operatorname{vol}(A)^{1/n} + t\operatorname{vol}(B)^{1/n}$. This convexity result has many applications, for which we refer to the survey [10]. Our first result makes it clear that no such result can be true for the self-similar measure μ_n on S_n.

Proposition 3.1. *For $A, B \subset S_n$ the set $\bigcup_{t\in(0,1)} \widetilde{Z}_t(A, B)$ has Hausdorff dimension at most 1 and hence μ_n-measure zero.*

Proof. We have shown that all geodesics are constructed as in Corollary 2.5 using the method from the proof of Proposition 2.4. In that argument, the set $\bigcup_{t\in(0,1)} \widetilde{Z}_t(a, b)$ lies entirely on the countable collection of Euclidean line segments joining vertices from V_n^*, no matter what a and b are. Hence the union $\bigcup_{t\in(0,1)} \widetilde{Z}_t(A, B)$ also lies in this countable collection of Euclidean line segments, which is a set of Hausdorff dimension 1. □

We note in passing an amusing consequence of the preceding proof which emphasizes the difference with the Euclidean case. We call a set A convex when $a, b \in A$ implies $\widetilde{Z}_t(a, b) \subset A$ for all t, and observe that the intersection of convex sets is convex, so A has a smallest closed convex superset, called its *convex hull.*

Corollary 3.2. *The convex hull of a closed set A has the same μ_n measure as A.*

Proof. The essential idea of the proof is to take a closed convex set by adjoining to A a portion of the union of line segments in Proposition 3.1 which accumulates only at A.

Given A, let $V = V_n^* \cap \bigcup_{t\in(0,1)} \widetilde{Z}_t(A, A)$ be the set of vertices on geodesics between points of A. Observe that if $x \in V$ and $d(x, A) \geq \delta > 0$ then from the construction of all geodesics in Corollary 2.5 and Proposition 2.4 it must be that x lies on the edge of a cell which intersects A and has size at least δ. Let B consist of all geodesics between all pairs of points in V. Note that if $x \in B$ and $d(x, A) \geq \delta$ it lies on one of the finitely many edges of cells of size at least δ, and thus $\{b \in B : d(b, A) \geq \delta\}$ is closed. It follows that any accumulation point of B that is at a positive distance from A is in B, and thus that $A \cup B$ is closed.

Let us consider the geodesics between points of $A \cup B$. If $a, b \in B$ then the geodesic from a to b is in B because it is a subset of a geodesic between points of V. If $a, b \in A$ then the geodesic between them is the increasing union of geodesics between points of V with ends that accumulate to a and b, so is also in $A \cup B$. Similarly, if $a \in A$ and $b \in B$ the geodesic between them is a union of this type except that one part of the geodesic between points of V is terminated at b. So $A \cup B$ is convex.

Since $A \cup B$ is closed and convex it must contain the convex hull of A. However, B is a countable union of line segments, so it is one-dimensional and $\mu_n(A \cup B) = \mu_n(A)$. □

Proposition 3.1 tells us that there is no hope that a power of $\mu_n(\widetilde{Z}_t(A,B))$ is convex in t, but it remains possible that the measure-theoretic properties of $\widetilde{Z}_t(A,B)$ reflect some aspects of the geometric structure of the Sierpiński gasket S_n. We record some definitions and basic notions that are useful in investigating this question.

Definition 3.3. Let $A, B \subseteq S_n$. A *common path* $\hat{\gamma}\colon [0,1] \to S_n$ from A to B is a finite length path such that for each $a \in A$ and $b \in B$ there is a geodesic $\hat{\gamma}_{a\to b}$ and $t_1, t_2 \in [0,1]$, called the *entry* and *exit times,* with $\hat{\gamma} = \hat{\gamma}_{a\to b}([t_1,t_2])$. The *initial* and *final entry times* of $\hat{\gamma}$ are, respectively, the infimum t_1^i and supremum t_1^f of the set of entry times over $a \in A$ and $b \in B$. The *initial* and *final exit times* t_2^i and t_2^f are similarly defined from the set of exit times.

If $\hat{\gamma}$ is a maximal common path under inclusion and $t_1^f < t_2^i$, we call $\hat{\gamma}$ a *regular common path*.

For arbitrary A and B there need not be a regular common path, but in many simple cases either there is such a path or there is a natural way to decompose A and B so as to obtain such paths between components. Indeed, it is easy to see that there is a regular common path between disjoint cells that are sufficiently small compared to their separation. Using this, if A and B are disjoint one may take a union of cells covering A and another union of cells covering B so that any pair of cells, one from the first union and the other from the second, admits a regular common path. Of equal importance is the fact that understanding regular common paths is sufficient for studying some aspects of the transport of measure via the set $\widetilde{Z}_t(A,B)$, at least for fairly simple choices of A and B. One way to see this is as follows. Begin by deleting a nullset of $X \times X$ from Theorem 2.16 so that geodesics are unique, then observe that if A and B are separated by a distance $3\epsilon > 0$ then these geodesics must each contain one of finitely many edges of size bounded below by ϵ. It follows that $A \times B$ can be decomposed into finitely many sets $A_j \times B_j$ so that geodesics from A_j to B_j have some piece of common path. See also the remark following Definition 4.1.

When there is a common path it is natural to consider only that part of $\widetilde{Z}_t$ that lies on the common path.

Definition 3.4. For A and B that admit a common path and $a \in A$, $b \in B$, define a modified interpolant $Z_t(a,b) = \hat{\gamma}_{a\to b}(t)$, i.e., the point in $\widetilde{Z}_t$ which lies on the geodesic included in the common path.

This modified interpolant is indeed a function, and for all $t \in [t_1^f, t_2^i]$ we have $\widetilde{Z}_t(A,B) \subset \hat{\gamma}$, as described in the following result.

Proposition 3.5. *Let A, B be connected subsets of S_n for which $\hat{\gamma}$ is a regular common path. For each $t \in [t_1^f, t_2^i]$ there exists an interval $I_t \subseteq [0, 1]$ such that $Z_t(A, B) = \hat{\gamma}(I_t)$.*

Proof. Fix $t \in [t_1^f, t_2^i]$. Continuity of $d(x, y)$ implies $Z_t(a, b)$ is continuous on the connected set $A \times B$ and thus $Z_t(A, B)$ is a connected subset of $\hat{\gamma}$. Such subsets have the stated form. □

Definition 3.6. In the circumstances of Proposition 3.5, let

$$H_t\colon (0, 1) \longrightarrow Z_t(A, B) = \hat{\gamma}(I_t)$$

be the parametrization obtained from the increasing linear surjection $(0, 1) \to I_t$, which may also be defined at 0 or 1, followed by $\hat{\gamma}$.

Motivated by the Brunn–Minkowski inequality, our basic object of study will be the measure on $\hat{\gamma}$ that is induced by the natural measures on A and B via the interpolant Z_t. In the next two sections we consider two basic cases: when A is a cell with measure μ_n and $B = \{b\}$ is a point with Dirac mass, and when A and B are both cells with measure μ_n.

4. Cell-to-point interpolation of measure

In this section we consider $Z_t(A, b)$, where A is a cell not containing b, for which we use the notation $Z_{t,b}(a) = Z_t(a, b)$. We assume that all geodesics from points $a \in A = \langle w \rangle$ to b pass through a single boundary point $\dot{a}$, which we call the *entry point* of A and that the geodesic $\hat{\gamma}$ from $\dot{a}$ to b is unique. Then $\hat{\gamma}$ is a common path from A to b, and we note that in this situation all exit times coincide, $t_2^i = t_2^f = 1$, so $\hat{\gamma}$ is regular. Then for $t \in [t_1^f, 1]$ we know from Proposition 3.5 that $Z_t(A, b)$ is an interval and $Z_{t,b}$ is a function, permitting us to study interpolation by considering the pushforward of μ_n under $Z_{t,b}$, as in the following definition. Note, too, that $t_1^f = (1 + 2^{|w|} d(\dot{a}, b))^{-1}$.

Definition 4.1. For $t \in [0, 1]$ let $\eta_t(X) = \mu_n(Z_{t,b}^{-1}(X))$, for all Borel sets $X \subseteq \hat{\gamma}$.

Remark. The definition of η_t depends on our assumptions regarding the common path, and one might think this could be avoided by instead studying something like $\eta'_t(X) = \mu_n\bigl(\{a\colon Z_t(a, b) \cap X \neq \emptyset\}\bigr)$. However, a little thought shows that there is not much loss of generality in studying the simpler quantity η_t instead. We made two assumptions: that all geodesics from points $a \in A = \langle w \rangle$ to b pass through a single boundary point $\dot{a}$ of A, and that the geodesic from $\dot{a}$ to b is unique. The latter can fail only if b is from the easily described subset of V_* for which the path $\dot{a}$ to b is non-unique, but in this case η'_t is a sum of copies of η_t is duplicated on each path, so it is enough to understand η_t. To achieve the former we can decompose A into subsets. From the proof of Theorem 2.16 the set of points in A which are equidistant from b along paths through distinct boundary points lies on a hyperplane orthogonal to the edge between these boundary points. This is a measure zero set so can be deleted without affecting η_t (or η'_t). Repeating this for each pair of boundary points we are left with finitely many open subsets of A, each of which is then a countable union of cells. Each cell obtained in this manner has both of our assumed properties, so η'_t can be written as a countable (and locally finite) sum of measures of the type η_t.

Lemma 4.2. *Under the above assumptions, for $t \in [0, 1]$,*

$$Z_{t,b}^{-1}\big(\hat{\gamma}(s)\big) = \Big\{a \in A\colon \frac{d(a,\dot{a})}{d(\dot{a},b)} = \frac{1-s}{1-t} - 1\Big\},$$

which is non-empty when $1 - \frac{\operatorname{diam}(A)}{d(\dot{a},b)} \leq \frac{1-s}{1-t} \leq 1$. This is illustrated in Figure 3.

Proof. Recall that $\hat{\gamma}$ has constant speed parametrization, so $x = \hat{\gamma}(s)$ implies $d(\dot{a},x) = sd(\dot{a},b)$ and $d(x,b) = (1-s)d(\dot{a},b)$. From (3.1) the set $Z_{t,b}^{-1}(x)$ consists of those $a \in A$ so $x \in Z_t(a,b)$, which means $d(a,x) = td(a,b)$ and $d(x,b) = (1-t)d(a,b)$. However, $\dot{a}$ is on the geodesic from a to b and the geodesic from a to x, so we have both $d(a,b) = d(a,\dot{a}) + d(\dot{a},b)$ and $d(a,x) = d(a,\dot{a}) + d(\dot{a},x)$. From this

$$d(a,\dot{a})+sd(\dot{a},b) = d(a,\dot{a})+d(\dot{a},x) = d(a,x) = td(a,b) = td(a,\dot{a})+td(\dot{a},b)$$

which may be rearranged to obtain

$$d(a,\dot{a}) = \Big(\frac{1-s}{1-t} - 1\Big)d(\dot{a},b)$$

and therefore the desired expression for $Z_{t,b}^{-1}(\hat{\gamma}(s))$. The condition for the set to be non-empty is a consequence of there being points $a \in A$ with $0 \leq d(a,\dot{a}) \leq \operatorname{diam}(A)$. ☐

From Proposition 2.4 the set of points in A at a prescribed distance from $\dot{a}$ is a level set of the barycentric coordinate corresponding to $\dot{a}$, see Figure 3. We define an associated projection.

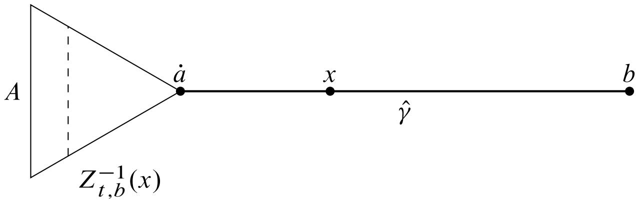


Figure 3. The preimage of x under $Z_{t,b}$ consists of points equidistant from $\dot{a}$.

Definition 4.3. If $A = \langle w \rangle$ and $\dot{a} = \langle w\bar{\imath} \rangle$, let $\varphi_{\dot{a}}(y) = [F_w^{-1}(y)]_i$, so $\varphi_{\dot{a}} \colon \langle w \rangle \to [0,1]$ is the projection of A on the scaled barycentric coordinate with $\varphi_{\dot{a}}(\dot{a}) = 1$ and $\varphi_{\dot{a}} = 0$ at the other boundary points of A. Note that $d(a,\dot{a}) = 2^{-|w|}(1-\varphi_{\dot{a}}(a))$ for $a \in A$.

In particular, the projection allows us to use the parametrization H_t from Definition 3.6 to give a more convenient version of Lemma 4.2 when $t \in [t_1^f, 1]$.

Lemma 4.4. *For $t \in [t_1^f, 1]$, we have $Z_{t,b}^{-1}(s) = \varphi_{\dot{a}}^{-1} \circ H_t^{-1}$.*

Proof. Since both $Z_{t,b}$ and $H_t \circ \varphi_{\dot{a}}$ map $A \to Z_t(A,b)$, are constant on level sets of $\varphi_{\dot{a}}$ and linear with respect to distance, they are equal. □

These considerations further suggest we consider a pushforward measure under the scaled barycentric projection.

Definition 4.5. Let ν_n be the pushforward measure $\nu_n(X) = (\varphi_{\langle\bar{0}\rangle})_* \mu_n(X) = \mu_n \circ \varphi_{\langle\bar{0}\rangle}^{-1}(X)$ on Borel subsets of $[0,1]$.

As S_n is rotationally symmetric, we could have defined ν_n using any boundary point map $\varphi_{\langle\bar{\imath}\rangle}$, and obtained the same measure. Moreover, the fact that $\varphi_{\dot{a}}^{-1} = F_w \circ \varphi_{\langle\bar{\imath}\rangle}^{-1}$ implies that $\mu_n \circ \varphi_{\dot{a}}^{-1} = (n+1)^{-|w|}\nu_n$. It is equally important that ν_n satisfies a simple self-similarity condition.

Lemma 4.6. *If $\widetilde{F}_i = \varphi \circ F_i \circ \varphi^{-1}$ then $\nu_n = \frac{1}{n+1}\nu_n \circ \widetilde{F}_0^{-1} + \frac{n}{n+1}\nu_n \circ \widetilde{F}_1^{-1}$.*

Proof. Recall that $\varphi(q_0) = 1$ and $\varphi(q_j) = 0$, for $j \neq 0$, while from Lemma 1.2 we have $\widetilde{F}_j(x) = \frac{1}{2}(x + \varphi(q_j))$. Thus $\widetilde{F}_0(x) = \frac{1}{2}(x+1)$ and $\widetilde{F}_j(x) = \frac{1}{2}x$ if $j \neq 0$. Proposition 1.3 says that ν_n is self-similar under the IFS $\{\widetilde{F}_i\}$ with equal weights, and the result follows from the fact that n of these maps are the same. □

See Figure 4 for an approximating histogram of ν_2, where the weights are $\frac{1}{3}$ and $\frac{2}{3}$.

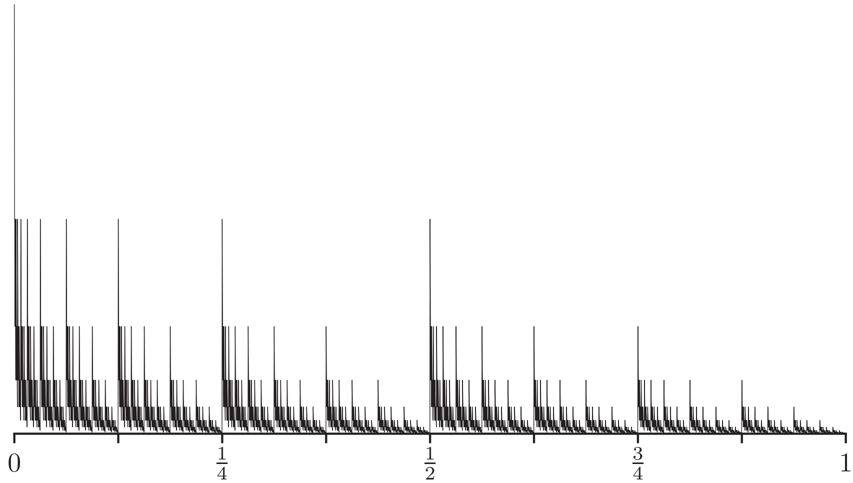


Figure 4. Approximating histogram of the self-similar measure ν_2.

Since the IFS $\{\widetilde{F}_0, \widetilde{F}_1\}$ satisfies the open set condition it is fairly elementary to compute the Hausdorff dimension of ν_n, e.g. using the approach in Chapter 5.2 of [8]. One expression for this dimension is $\inf\{\dim_{\mathrm{Hausd}}(E) : \nu_n(E) > 0\}$.

Proposition 4.7. *The Hausdorff dimension of ν_n is $\frac{(n+1)\log(n+1) - n\log n}{(n+1)\log 2}$, which is less than 1 if $n > 1$. In particular it is singular with respect to Lebesgue measure on* $[0, 1]$.

With the pushforward measure ν_n in hand, we can give an elementary and concise description of the common path measure η_t using Lemma 4.4; it is the main result of this section.

Theorem 4.8. *Let $A = \langle w \rangle$ be a cell and $B = \{b\}$ with $b \notin A$. If $t \in [t_1^f, 1]$ then $\eta_t = (n+1)^{-|w|}\nu_n \circ H_t^{-1}$, so is singular with respect to arc length and has dimension as in Proposition 4.7.*

Proof. We have $\eta_t = \mu_n \circ Z_{t,b}^{-1} = \mu_n \circ \varphi_{\bar{a}}^{-1} \circ H_t^{-1} = (n+1)^{-|w|}\nu_n \circ H_t^{-1}$. □

It should be remarked that we could have described η_t for $t \in [0, 1]$ rather than only $t \in [t_1^f, 1]$ by using Lemma 4.2 instead of Lemma 4.4, but the notation is considerably less elementary and the gain is minimal because in this case one can instead compute η_t for the largest subcell $A' \subset A$ such that $t > t_1^f$ for A'.

5. Interpolation of measures

The general interpolation problem involves understanding $\{(a,b)\colon Z_t(a,b)=x\}\subset A\times B$ and its product measure. We slightly abuse notation by calling this measure η_t, as we did in the case $B=\{b\}$

Definition 5.1. Let A and B be sets of nonzero μ_n-measure, and suppose there is a regular common path $\hat{\gamma}$ between them. Define a measure η_t on $\hat{\gamma}$ to be the pushforward of $\mu_n\times\mu_n$ on $A\times B$, so that for each $t\in[0,1]$ and Borel set X,

$$\eta_t(X)=(\mu_n\times\mu_n)\circ Z_t^{-1}(X).$$

As we did in the case of interpolation between a cell and a point, we take the viewpoint that interpolation between sets A and B should be understood as a superposition of interpolation between pairs of cells. This is by no means always possible, but it is possible for a large class of sets; for example, it is true when A and B are both open. Using the same considerations made when discussing point to set interpolation, we further note that from the proof of Theorem 2.16 the product $A\times B$ may be decomposed into a $\mu_n\times\mu_n$-nullset, which is obtained as a finite union of sets of the type in (2.1), and a countable union $A_j\times B_j$ in which A_j and B_j are disjoint cells joined by a unique common path. Accordingly, we focus our investigation on η_t when A and B are as in Definition 5.1.

We conclude with a discussion of interpolation when μ_n is replaced with an unequally distributed self-similar measure.

5.1. Cell-to-cell interpolation. Let A be a k-level cell and B an m-level cell for which there is a common path $\hat{\gamma}$ which is the unique geodesic joining the boundary points $\dot{a}\in A$ and $\dot{b}\in B$. From Lemma 4.2 we know that $Z_t(a,b)=Z_t(a',b)$ if $\varphi_{\dot{a}}(a)=\varphi_{\dot{a}}(a')$ and similarly for the second coordinate using $\varphi_{\dot{b}}$, so it is natural to write $Z_t^{-1}(x)$ using these barycentric coordinates. Note that they are scaled differently on A and B, as in the following definition.

Definition 5.2. Define $\psi_t\colon[0,1]\times[0,1]\to[0,1]$ by

$$\psi_t(s,r)=\frac{2^{-k}(1-t)s+2^{-m}t(1-r)}{2^{-k}(1-t)+2^{-m}t}.$$

The following result is similar to Lemma 4.4 and is illustrated in Figure 5. Recall from Definition 3.6 that H_t parametrizes $Z_t(A,B)$ when the latter is contained in $\hat{\gamma}$.

Lemma 5.3. *For all* $t \in [t_1^f, t_2^i]$, $Z_t(a,b) = H_t \circ \psi_t(\varphi_{\dot a}(a), \varphi_{\dot b}(b))$.

Proof. Recall that $Z_t(a,b) = x$ means $d(a,x) = td(a,b)$. Suppose now that $x = H_t \circ \psi_t \circ (\varphi_{\dot a}(a), \varphi_{\dot b}(b))$. We establish several points that together show $d(a,x) = td(a,b)$, proving the result.

Recall from Definition 4.3 that $d(a,\dot a) = 2^{-k}(1-\varphi_{\dot a}(a))$ and $d(\dot b, b) = 2^{-m}(1-\varphi_{\dot b}(b))$. Substituting into ψ_t gives

$$\psi_t\big(\varphi_{\dot a}(a), \varphi_{\dot b}(b)\big) = \frac{(1-t)(2^{-k} - d(a,\dot a)) + td(\dot b, b)}{2^{-k}(1-t) + 2^{-m}t}. \tag{5.1}$$

To proceed we need more information about H_t, the parametrization of $Z_t(A,B)$. Using Lemma 4.2 we find that the extreme points of $Z_t(A,B)$ are x_1 and x_2 satisfying $d(\bar a, x_1) = td(\bar a, \dot b)$ and $d(x_2, \bar b) = (1-t)d(\dot a, \bar b)$, where $\bar a \neq \dot a$ is a boundary point of A and $\bar b \neq \dot b$ is a boundary point of B. Since $d(\bar a, \dot a) = 2^{-k}$ and $d(\dot b, \bar b) = 2^{-m}$ this yields $d(\bar a, x_1) = t2^{-k} + td(\dot a, \dot b)$ and $d(x_2, \bar b) = (1-t)2^{-m} + (1-t)d(\dot a, \dot b)$. Moreover $t \in [t_1^f, t_2^i]$ implies that for any $a \in A$ and $b \in B$ the geodesic from a to b contains the following points in order: $a, \dot a, x_1, x, x_2, \dot b, b$. We use this and the side lengths of the cells A and B to determine that

$$\begin{aligned} d(x_1, x_2) &= d(\bar a, \bar b) - d(\bar a, x_1) - d(x_2, \bar b) \\ &= 2^{-k} + 2^{-m} + d(\dot a, \dot b) - td(\bar a, \dot b) - (1-t)d(\dot a, \bar b) \\ &= 2^{-k} + 2^{-m} + d(\dot a, \dot b) - t2^{-k} - td(\dot a, \dot b) \\ &\quad - (1-t)2^{-m} - (1-t)d(\dot a, \dot b) \\ &= 2^{-k}(1-t) + 2^{-m}t \end{aligned}$$

which is the denominator in ψ_t.

Now H_t is the linear parametrization of the path from x_1 to x_2, so $x = H_t(q)$ means $d(x_1, x) = qd(x_1, x_2)$. Substituting $q = \psi_t\big(\varphi_{\dot a}(a), \varphi_{\dot b}(b)\big)$ from (5.1) we have

$$d(x_1, x) = d(x_1, x_2)\psi_t(\varphi_{\dot a}(a), \varphi_{\dot b}(b)) = (1-t)(2^{-k} - d(a, \dot a)) + td(\dot b, b).$$

We can then compute $d(a,x)$ as follows, using $2^{-k} + d(\dot a, x_1) = d(\bar a, x_1) = t2^{-k} + td(\dot a, \dot b)$.

$$\begin{aligned} d(a,x) &= d(a,\dot a) + d(\dot a, x_1) + d(x_1, x) \\ &= d(a, \dot a) + d(\dot a, x_1) + (1-t)(2^{-k} - d(a,\dot a)) + td(\dot b, b) \\ &= td(\dot a, \dot b) + td(a, \dot a) + td(\dot b, b) = td(a,b) \end{aligned}$$

from which $x = Z_t(a,b)$ as required. □

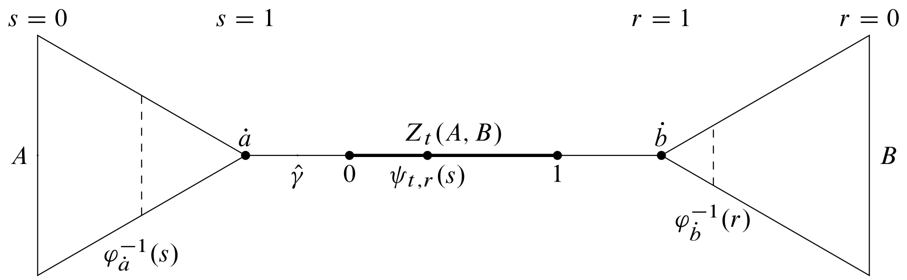


Figure 5. A schematic of cell-to-cell interpolation on a common path $\hat{\gamma}$. The function $\psi_{t,r}$ (Definition 5.2) describes at a given t where in the interval $Z_t(A, B)$ a point lying on the line $\varphi_{\dot{a}}^{-1}(s)$ is as it is interpolated to a point lying on the line $\varphi_{\dot{b}}^{-1}(r)$.

We can now prove an analogue of Theorem 4.8 to characterize the transported measure during cell-to-cell interpolation. We follow this result with two others characterizing the measure as not only self-similar and but also as a projection of a self-similar measure, or as a kind of convolution.

Theorem 5.4. *If A is a k-level cell and B an m-level cell that are joined by a regular common path $\hat{\gamma}$ that is the unique geodesic between boundary points $\dot{a} \in A$ and $\dot{b} \in B$, then for all $t \in [t_1^f, t_2^i]$*

$$\eta_t = (n+1)^{-k-m}(\nu_n \times \nu_n) \circ \psi_t^{-1} \circ H_t^{-1}.$$

Proof. This is an immediate consequence of Lemma 5.3 applied to the definition of η_t, because the functions $\varphi_{\dot{a}}^{-1}$ and $\varphi_{\dot{b}}^{-1}$ may be pulled into the product measure as follows:

$$\begin{aligned}\eta_t &= (\mu_n \times \mu_n) \circ Z_t^{-1} \\ &= (\mu_n \times \mu_n) \circ (\varphi_{\dot{a}}, \varphi_{\dot{b}})^{-1} \circ \psi_t^{-1} \circ H_t^{-1} \\ &= (\mu_n \circ \varphi_{\dot{a}}^{-1} \times \mu_n \circ \varphi_{\dot{b}}^{-1}) \circ \psi_t^{-1} \circ H_t^{-1}\end{aligned}$$

so we can use $\mu_n \circ \varphi_{\dot{a}}^{-1} = (n+1)^{-k}\nu_n$ and similarly $\mu_n \circ \varphi_{\dot{b}}^{-1} = (n+1)^{-m}\nu_n$. □

Since it is a product of self-similar measures, the measure $\nu_n \times \nu_n$ is self-similar. This is recorded in Proposition 5.6 after defining notation for the two-dimensional IFS. It is illustrated in Figure 6.

Definition 5.5. Let $q_{00} = (0,0), q_{01} = (0,1), q_{10} = (1,0)$, and $q_{11} = (1,1)$. For $i, j \in \{0,1\}$, define $G_{ij} : [0,1]^2 \to [0,1]^2$ by

$$G_{ij}(x) = \frac{1}{2}(x + q_{ij}),$$

and fix weights $w_{ij} = w_i w_j$, where $w_0 = \frac{n}{n+1}$ and $w_1 = \frac{1}{n+1}$.

The functions G_{ij} are an IFS generating the unit square and are related to the functions $\widetilde{F}_i$ by

$$G_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \widetilde{F}_{1-i}(x) \\ \widetilde{F}_{1-j}(y) \end{pmatrix}.$$

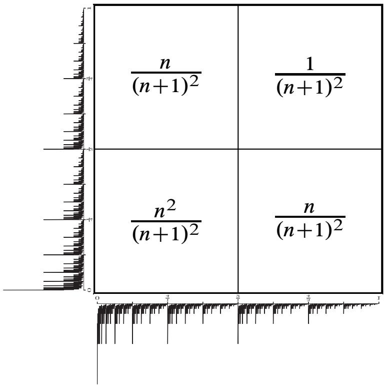


Figure 6. A diagram of the product measure $\nu_n \times \nu_n$. The IFS weights are indicated in each cell.

Proposition 5.6. *The measure $\nu_n \times \nu_n$ satisfies the self-similar relation*

$$\nu_n \times \nu_n = \sum_{i,j} w_{ij}(\nu_n \times \nu_n) \circ G_{ij}^{-1}.$$

Proof. We compute from the self-similarity of ν_n that

$$\begin{aligned}
\nu_n \times \nu_n &= \Big(\sum_i w_i \nu_n \circ \widetilde{F}_{1-i}^{-1}\Big)\Big(\sum_j w_j \nu_n \circ \widetilde{F}_{1-j}^{-1}\Big) \\
&= \sum_{i,j} w_i w_j (\nu_n \times \nu_n) \circ (\widetilde{F}_{1-i}^{-1} \times \widetilde{F}_{1-j}^{-1}) \\
&= \sum_{i,j} w_{ij}(\nu_n \times \nu_n) \circ G_{ij}^{-1}.
\end{aligned}$$ □

Theorem 5.4 establishes that η_t depends only on the linear parametrization H_t of $Z_t(A, B)$ and the pushforward measure

$$\tilde{\nu}_n^t = (\nu_n \times \nu_n) \circ \psi_t^{-1}. \tag{5.2}$$

This measure has a simple geometric meaning. Observe that ψ_t is a scaled projection from the unit square to the unit interval along lines of slope $2^{k-m}\big(\frac{t}{1-t}\big)$.

The corresponding pushforward is then a generalization of a convolution; the usual convolution $\nu_n * \nu_n$ occurs when the lines have slope -1. From Proposition 1.3 we also find that $\tilde{\nu}_n^t$ is self-similar.

Theorem 5.7. *For $t \in [0,1]$ let $\widetilde{G}_{ij} = \psi_t \circ G_{ij} \circ \psi_t^{-1} \colon [0,1] \to [0,1]$. Then*

$$\tilde{\nu}_n^t = \sum_{i,j} w_{ij} \tilde{\nu}_n^t \circ \widetilde{G}_{ij}^{-1}.$$

The maps in the IFS $\{\widetilde{G}_{ij}\}$ take $[0,1]$ to overlapping segments in $[0,1]$, with overlaps that depend on t. Figure 7 shows these overlapping segments, along with their corresponding weights, for one choice of t, and Figure 8 shows approximating histograms of $\tilde{\nu}_n^t$ for several t values.

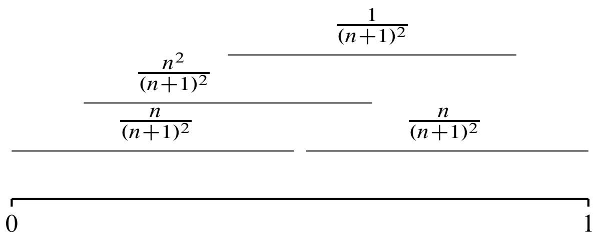


Figure 7. Distribution of self-similar weights of $\tilde{\nu}_n^t$ for some t.

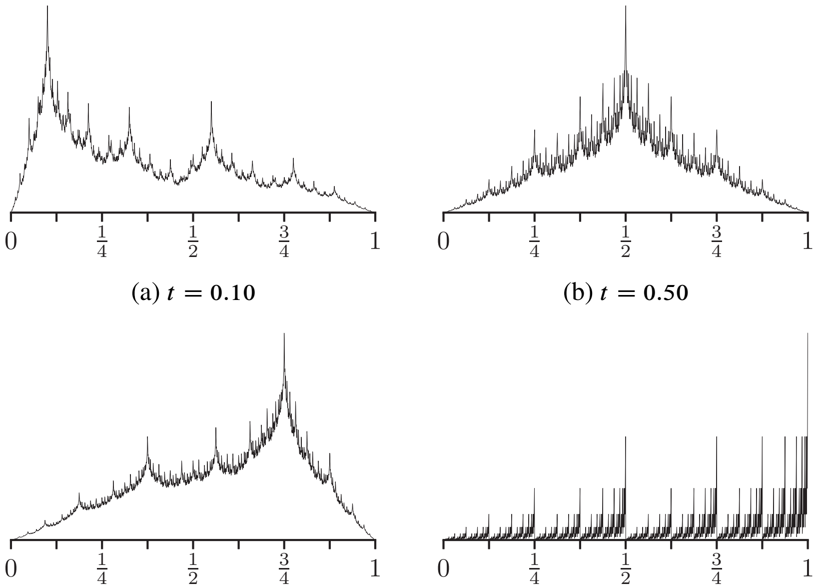
It is generally difficult to compute the dimensions of measures from overlapping IFS, but we may deduce some results from the Marstrand projection theorem [15]. First note that the dimension of $\nu_n \times \nu_n$ is twice that of ν_n, and is given by the formula

$$2\frac{(n+1)\log(n+1) - n\log n}{(n+1)\log 2}. \tag{5.3}$$

This expression is decreasing with limit zero as n increases. In particular, it is less that 1 for $n \geq 9$ and greater than 1 for $2 \leq n \leq 8$, from which we deduce the following using Theorems 6.1 and 6.3 in [12].

Theorem 5.8. *If $2 \leq n \leq 8$ then for almost all $t \in [0,1]$ the measure $\tilde{\nu}_n^t$ is absolutely continuous with respect to Lebesgue measure on $[0,1]$. For $n \geq 9$, it is singular with respect to Lebesgue measure, and in fact has lower Hausdorff dimension given by* (5.3).

We note that recent results of Shmerkin and Solomyak [18] show the set of exceptional t in this theorem is not just zero measure but zero Hausdorff dimension.



(a) $t = 0.10$ (b) $t = 0.50$

(c) $t = 0.75$ (d) $t = 1.00$

Figure 8. Approximating histograms of the convolution measure $\tilde{\nu}_n^t$ at various values of t.

5.2. Alternate weightings of self-similar measures on the gasket. We can generalize our previous results to self-similar measures other than the standard measure on S_n. Consider a self-similar measure μ_n' on S_n given by weights $\{\mu_n^i\}_{i=0}^n$. In this case, the pushforward measure $\nu_n' = \varphi_* \mu_n'$ is self-similar, but has weights dependent on the reference point of the projection φ. In particular, if we consider the projection with respect to a vertex $\langle w\bar{\imath}\rangle$, the self-similarity relation is given by

$$\nu_n'(X) = \mu_n^i \nu_n' \circ \widetilde{F}_1^{-1}(X) + \Big(\sum_{j \neq i} \mu_n^j\Big) \nu_n' \circ \widetilde{F}_0^{-1}(X).$$

This follows from Lemma 1.2 and Proposition 1.3. Self-similarity also carries over to $\tilde{\nu}_n^t$, but the self-similarity weights depend on orientations of both the starting and ending cells with respect to the common path; if $\langle w\bar{\imath}\rangle$ is the entry point and $\langle v\bar{\jmath}\rangle$ the exit point of a common path, then $\tilde{\nu}_n^t$ has self-similarity relations as in Theorem 5.7, but with

$$w_{00} = \mu_n^i \mu_n^j, \qquad w_{01} = \mu_n^i \Big(\sum_{k \neq j} \mu_n^k\Big),$$
$$w_{10} = \Big(\sum_{k \neq i} \mu_n^k\Big) \mu_n^j, \quad w_{11} = \Big(\sum_{k \neq i} \mu_n^k\Big)\Big(\sum_{k \neq j} \mu_n^k\Big).$$

6. An interpolation inequality

The model for an inequality involving the interpolant set Z_t is the classical Brunn–Minkowski inequality, which says that for sets in $\mathbb{R}^n$ the Euclidean volume $|\cdot|$ satisfies $\big|Z_t(A,B)|^{1/n} \geq (1-t)|A|^{1/n} + t|B|^{1/n}$. We have already noted that this inequality cannot be valid for μ_n because $Z_t(A,B)$ for $t \in (0,1)$ has Hausdorff dimension at most 1, and thus μ_n-measure zero. When seeking alternative inequalities it is not entirely clear which measures to use: μ_n is natural for A and B, and is equivalent to ν_n for the barycentric projection of these sets, at least under the conditions considered in the previous sections, but ν_n is not natural for $Z_t(A,B)$ because it is defined on [0, 1], not on the common path. Since $\bigcup_{t\in(0,1)} Z_t$ is at most one-dimensional (from Proposition 3.1), the geometrically defined natural measures to consider would seem to be Hausdorff measures of dimension at most one. In light of the work done in the previous sections, natural choices of dimension are that of ν_n, that of $\nu_n \times \nu_n$, and 1. The first is likely to be uninteresting, because if $\nu_n(A)\nu_n(B) > 0$ then $Z_t(A,B)$ has dimension at least $\min\{1, 2\dim(\nu_n)\}$ as seen in Theorem 5.8. But the others also present some issues with optimality, validity or both: for example, if A and B are connected then Z_t contains an interval and therefore has infinite measure in dimensions less than one, so any inequality for dimension less than one will be trivially true on connected sets, while Theorem 5.8 ensures that for $n \geq 9$ one could have $\nu_n(A) = \nu_n(B) = 1$ and yet $Z_t(A,B)$ has dimension less than one, so no inequality for Hausdorff 1-measure can be true for general A and B in this case.

In this section we derive an inequality in the case where A and B are connected sets, so $Z_t(A,B)$ contains an interval and therefore the correct measure to use for Z_t is the one-dimensional Hausdorff measure $\mathcal{H}^1$. There is an easy bound if A and B are cells.

Proposition 6.1. *Suppose $A = \langle v\rangle$ and $B = \langle w\rangle$ are disjoint cells. Then we have the sharp inequality*

$$\mathcal{H}^1(Z_t(A,B)) \geq (1-t)\mu_n(A)^{\log 2/\log(n+1)} + t\mu_n(B)^{\log 2/\log(n+1)}.$$

Proof. Take $a \in A$ and $b \in B$ so that $d(a,b)$ is maximal. The geodesic from a to b passes through boundary points $\dot{a} \in A$ and $\dot{b} \in B$, with $d(a,\dot{a}) = 2^{-|v|}$ and $d(b,\dot{b}) = 2^{-|w|}$. Then $Z_t(A,B)$ contains an interval along this geodesic, and it is easy to compute a lower bound for its length, which gives

$$\mathcal{H}_1(Z_t(A,B)) \geq (1-t)2^{-|v|} + t2^{-|w|}.$$

However $\mu_n(A) = (n+1)^{-|v|}$ and $\mu_n(B) = (n+1)^{-|w|}$, from which the assertion is immediate. Sharpness occurs when Z_t is equal to this interval, which is true provided $t_1^f < t < t_2^i$; this can be arranged by suitably choosing A and B. □

If A and B are connected but are not cells then we can take minimal cells $\langle v\rangle \supset A$ and $\langle w\rangle \supset B$. Provided $\langle v\rangle$ and $\langle w\rangle$ are disjoint and joined by a common path from $\dot{a}$ to $\dot{b}$ our reasoning from the the proof of Proposition 6.1 is still useful, but the lower bound for the $\mathcal{H}_1$ measure must now also involve the sizes of the intervals $\varphi_{\dot{a}}(A)$ and $\varphi_{\dot{b}}(B)$ obtained by barycentric projection. (Note that these are intervals because A and B are connected.) Indeed, the geodesic between the maximally separated points $a \in A$ and $b \in B$ begins with a path of length at least $2^{-|v|}\mathcal{H}_1(\varphi_{\dot{a}}(A))$ in A and ends with one of length at least $2^{-|w|}\mathcal{H}_1(\varphi(B))$ in B, so that

$$\mathcal{H}^1(Z_t(A,B)) \geq (1-t)2^{-|v|}\mathcal{H}_1(\varphi_{\dot{a}}(A)) + t2^{-|w|}\mathcal{H}_1(\varphi_{\dot{b}}(B)) \tag{6.1}$$

and in order to proceed we must bound $\mathcal{H}_1(\varphi_{\dot{a}}(A))$ from below using $\mu_n(F_v^{-1}A)$. It is obvious that $\mu_n(F_v^{-1}(A)) \leq \nu_n(\varphi_{\dot{a}}(A))$. To compare $\mathcal{H}_1(\varphi_{\dot{a}}(A))$ and $\nu_n(\varphi_{\dot{a}}(A))$ we establish some lemmas; the conclusion of our reasoning regarding a lower bound for $\mathcal{H}_1(Z_t(A,B))$ when A and B are connected is in Theorem 6.6.

Lemma 6.2. *If* $[a, a+x] \subseteq [0,1]$ *then* $\nu_n([0,x]) \geq \nu_n([a,a+x]) \geq \nu_n([1-x,1])$.

Proof. As ν_n is non-atomic, $\nu_n([a, a+x])$ is continuous in a and x, and it suffices to consider dyadic rationals of arbitrary scale m, so $a = \sum_{i=1}^m a_i 2^{-i}$ and $x = \sum_{i=1}^m x_i 2^{-i}$.

Observe that we can assume $x \leq a$ and $a + x \leq 1 - x$, because if the intervals intersect then it suffices to prove the inequality for the complement of the intersection (for example, if $x > a$ the first inequality may be proved by showing $\nu_n([0,a]) \geq \nu_n([x, a+x])$ because then $\nu_n([0,x]) = \nu_n([0,a]) + \nu_n([a,x]) \geq \nu_n([a,x]) + \nu_n([x,a+x]) = \nu_n([a,a+x])$). Note in particular that this assumption provides $x \leq \frac{1}{2}$.

We induct on m, with the easily verifiable base case $m = 1$. Supposing it is true to scale $m-1$, take a and x at dyadic scale m and use the self-similarity of ν_n from Lemma 4.6. If both a and $a+x$ are in $[0, \frac{1}{2}]$ then the scaling map is $F_0^{-1}(y) = 2y$ and thus $\nu_n([a, a+x]) = \frac{n}{n+1}\nu_n([2a, 2a+2x)])$. Both $2a$ and $2a+2x$ are dyadic of scale $m-1$, so that $\nu_n([0,2x]) \geq \nu_n([2a, 2a+2x]) \geq \nu_n([1-2x, 1])$ from the inductive assumption. We can then use the self-similarity a second time, in the reverse direction, to obtain the desired inequality. (This latter uses $x \leq \frac{1}{2}$.) The proof if both a and $a+x$ are in $[\frac{1}{2}, 1]$ follows the same reasoning but uses $F_1^{-1}(y) = 2y - 1$ on $y \in \left[\frac{1}{2}, 1\right]$.

For the remaining case we have $x \le a < \frac{1}{2} < a + x \le 1 - x$, so we separate at $\frac{1}{2}$ and use the self-similarity to write

$$\begin{aligned} \nu_n([a, a + x]) &= \nu_n\Big(\Big[a, \frac{1}{2}\Big]\Big) + \nu_n\Big(\Big[\frac{1}{2}, a + x\Big]\Big) \\ &= \frac{n}{n+1}\nu_n([2a, 1]) + \frac{1}{n+1}\nu_n([0, 2a + 2x - 1]). \end{aligned} \tag{6.2}$$

Since $2a$ and $2a + 2x - 1$ are dyadic of scale $m - 1$ we can apply the inductive assumption to obtain $\nu_n([2a, 1]) \le \nu_n([2a+2x-1, 2x])$ and $\nu_n([0, 2a+2x-1]) \ge \nu_n([1 - 2x, 2a])$. Thus

$$\begin{aligned} \nu_n([0, x]) &= \frac{n}{n+1}\nu_n([0, 2x]) \\ &= \frac{n}{n+1}(\nu_n([2a + 2x - 1, 2x]) + \nu_n([0, 2a + 2x - 1])) \\ &\ge \frac{n}{n+1}\nu_n([2a, 1]) + \frac{1}{n+1}\nu_n([0, 2a + 2x - 1]) \\ &\ge \frac{1}{n+1}(\nu_n([2a, 1]) + \nu_n([1 - 2x, 2a])) \\ &= \frac{1}{n+1}\nu_n([1 - 2x, 1]) \\ &= \nu_n([1 - x, 1]), \end{aligned}$$

where the beginning and end inequalities again use the self-similarity in reverse, which uses the earlier established fact that $x \le \frac{1}{2}$. Comparing the middle term to (6.2) establishes the desired inequality. □

Having determined that $\nu_n([a, a + x]) \le \nu_n([0, x])$, we next look for a minimal concave bounding function having the form found in the classical Brunn–Minkowski inequality. The fact that the following function bounds $\nu_n([0, x])$ is proved in Corollary 6.5 and illustrated in Figure 9. Note that Figure 9 makes it clear this is not the minimal concave bounding function, but only the minimal one having the classical Brunn–Minkowski form.

Definition 6.3. Let $d_n = \frac{\log 2}{\log \frac{n+1}{n}}$, and $\Phi_n(x) = (1 - (1 - x)^{d_n})^{1/d_n}$.

Lemma 6.4. *We have*

$$n\Phi_n(2x) \le (n + 1)\Phi_n(x) \qquad \text{if } x \in [0, 1/2], \tag{6.3}$$

$$\Phi_n(2x - 1) \le (n + 1)\Phi_n(x) - n \qquad \text{if } x \in [1/2, 1]. \tag{6.4}$$

Proof. Dividing both sides of (6.3) and taking the d_n power we find it is equivalent to

$$A_1(x) = 1-(1-2x)^{d_n} \le 2(1-(1-x)^{d_n}) = A_2(x).$$

We have $A_1(0) = 0 = A_2(0)$. Moreover

$$A_1'(x) = 2d_n(1-2x)^{d_n-1} \le 2d_n(1-x)^{d_n-1} = A_2'(x)$$

because $0 \le 1-2x \le 1-x$ and $d_n - 1 \ge 0$. The inequality (6.3) follows.

The inequality (6.4) is equivalent to

$$A_3(1-x) = (1-(2(1-x))^{d_n})^{1/d_n} \le (n+1)(1-(1-x)^{d_n})^{1/d_n} - n = A_4(1-x)$$

for $y = 1-x \in [0, 1/2]$. We have $A_3(0) = 1 = A_4(0)$ and compare derivatives as follows:

$$A_3'(y) = (1-(2y)^{d_n})^{\frac{1}{d_n}-1} 2^{d_n} y^{d_n-1} \le (n+1)(1-y^{d_n})^{\frac{1}{d_n}-1} y^{d_n-1} = A_4'(y)$$

because $2^{d_n} \le (n+1)$ and $d_n \ge 1$ gives both $0 \le 1-y^{d_n} \le 1-(2y)^{d_n}$ on $[0, 1/2]$ and $\frac{1}{d_n} - 1 \le 0$. The former is easily checked using the fact that $2^{-d_n}(n+1)$ is decreasing in n and equal to 1 when $n = 1$. Thus $A_3(y) \le A_4(y)$ on $[0, 1/2]$ and this establishes (6.4). □

Corollary 6.5. $\nu_n([0, x]) \le \Phi_n(x) = (1-(1-x)^{d_n})^{1/d_n}$ *on* $[0, 1]$.

Proof. As in the proof of Lemma 6.2 it is sufficient (by continuity of the functions) to prove this for dyadic rational x, which we do by induction on the degree m of the dyadic rational. The base case is $m = 0$ where the equalities $\nu_n(\{0\}) = 0 = \Phi_n(0)$ and $\nu_n([0, 1]) = 1 = \Phi_n(1)$ are immediate. If $x = k2^{-m}$ is a dyadic rational we use the self-similarity of ν_n from Lemma 4.6, then the fact that $2x$ and $2x-1$ are dyadic rationals of lower degree so satisfy the inequality by induction, and finally Lemma 6.4, to compute

$$\nu_n([0, x]) = \begin{cases} \dfrac{n}{n+1}\nu_n([0, 2x]) \le \dfrac{n}{n+1}\Phi_n(2x) \le \Phi_n(x) & \text{if } x \le \dfrac{1}{2}, \\ \dfrac{n}{n+1} + \dfrac{1}{n+1}\nu_n([0, 2x-1]) & \\ \quad \le \dfrac{n}{n+1} + \dfrac{1}{n+1}\Phi_n(2x-1) \le \Phi_n(x) & \text{if } x > \dfrac{1}{2}. \end{cases}$$

□

Theorem 6.6. *Let* $A, B \subset S_n$ *be connected sets contained in minimal disjoint cells,* $A \subset \langle v \rangle$, $B \subset \langle w \rangle$, *and suppose there is a common path between boundary points* $\dot{a} \in \langle v \rangle$ *and* $\dot{b} \in \langle w \rangle$. *For all* $t \in (0, 1)$,

$$1 - (1 - \mathcal{H}_1(Z_t(A, B)))^{d_n} \ge (1-t)\mu_n(A)^{\log 2/\log(n+1)} + t\mu_n(B)^{\log 2/\log(n+1)}.$$

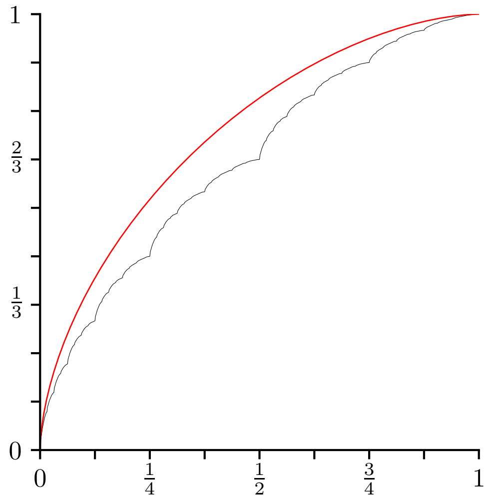


Figure 9. The cumulative measure $\nu_2([0,x])$ (in black) is bounded by $\Phi_2(x)$ (in red).

Proof. The function $\Phi_n^{d_n}$ is concave, so applying it to both sides of (6.1) gives

$$\Phi_n^{d_n}(\mathcal{H}_1(Z_t(A,B))) \geq (1-t)\Phi_n^{d_n}(2^{-|v|}\mathcal{H}_1(\varphi_{\dot a}(A))) + t\,\Phi_n^{d_n}(2^{-|w|}\mathcal{H}_1(\varphi_{\dot b}(B))).$$

However we saw in Corollary 6.5 that $\Phi_n(x)$ bounds $\nu_n([0,x])$ and in Lemma 6.2 that $\nu_n([0,x])$ bounds ν_n of any interval of this length. The latter bound applies to $\varphi_{\dot a}(A)$, which is an interval by the connectedness of A. Also using that $\nu_n([0,2^{-m}x]) = \big(\frac{n}{n+1}\big)^m \nu_n([0,x])$ from the self-similarity of ν_n, we have

$$\Phi_n(2^{-|v|}\mathcal{H}_1(\varphi_{\dot a}(A))) \geq \nu_n([0,2^{-|v|}\mathcal{H}_1(\varphi_{\dot a}(A))]) \geq \Big(\frac{n}{n+1}\Big)^{|v|}\nu_n(\varphi_{\dot a}(A)).$$

But we also know $\nu_n(\varphi_{\dot a}(A)) \geq (n+1)^{|v|}\mu_n(A) = \mu_n(A)/\mu_n(\langle v\rangle)$ because the discussion following Definition 4.5 showed $\nu_n = (n+1)^{|v|}\mu_n \circ \varphi_{\dot a}^{-1}$. Using this and the definition of d_n we obtain

$$\Phi_n^{d_n}\big(2^{-|v|}\mathcal{H}_1(\varphi_{\dot a}(A))\big) \geq \Big(\frac{n}{n+1}\Big)^{|v|d_n}\Big(\frac{\mu_n(A)}{\mu_n(\langle v\rangle)}\Big)^{d_n} = 2^{-|v|}\Big(\frac{\mu_n(A)}{\mu_n(\langle v\rangle)}\Big)^{d_n}.$$

A similar bound applies for B, so we have

$$\begin{aligned}\Phi_n^{d_n}\big(\mathcal{H}_1(Z_t(A,B))\big) &\geq (1-t)2^{-|v|}\Big(\frac{\mu_n(A)}{\mu_n(\langle v\rangle)}\Big)^{d_n} + t2^{-|w|}\Big(\frac{\mu_n(B)}{\mu_n(\langle w\rangle)}\Big)^{d_n}\\ &= (1-t)\mu(\langle v\rangle)^{d_n'-d_n}\mu_n(A)^{d_n} + t\mu_n(\langle w\rangle)^{d_n'-d_n}\mu_n(B)^{d_n}\end{aligned}$$

with

$$d_n' = \frac{\log 2}{\log(n+1)}.$$

The fact that

$$d'_n - d_n = \frac{\log 2 \log n}{\log\left(\frac{n+1}{n}\right)\log(n+1)} > 0$$

and both $\mu_n(\langle v \rangle) \geq \mu_n(A)$ and $\mu_n(\langle w \rangle) \geq \mu_n(B)$ then gives the result. □

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Caitlin M. Davis, Department of Mathematics, University of Wisconsin-Madison,
480 Lincoln Dr, Madison, WI 53706, USA

e-mail: cmdavis22@wisc.edu

Laura A. LeGare, Department of Mathematics, University of Notre Dame,
255 Hurley Bldg, Notre Dame, IN 46556, USA

e-mail: llegare@nd.edu

Cory W. McCartan, Department of Statistics, Harvard University, One Oxford Street, Suite 400, Cambridge, MA 02138-2901, USA

e-mail: cmccartan@g.harvard.edu

Luke G. Rogers, Department of Mathematics, University of Connecticut, 341 Mansfield Rd, Storrs, CT 06269, USA

e-mail: luke.rogers@uconn.edu