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Fourier multipliers and transfer operators

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Abstract. A recent paper of Chen and Volkmer estimated a quantity related to the spectral radius of a transfer operator and with significance in the study of Fourier multipliers. We provide a rigorous proof of their conjectured numerical value.

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1. Introduction

In an interesting recent paper, Chen and Volkmer considered a family of operators with trigonometric weights and estimated their spectral radii. In addition to their intrinsic interest, their study is motivated by their connections to problems in the theory of Fourier Multipliers (see Application 1.3 below).

As a simple example, consider the bounded linear operator

$$\mathcal{L}\colon C^0([0,1]) \longrightarrow C^0([0,1])$$

defined by

$$(\mathcal{L}u)(t) = \frac{1}{3}\sum_{i=0}^{3}\Big|\sin\Big(\frac{\pi(t+i)}{3}\Big)\Big| u\Big(\frac{t+i}{3}\Big).$$

In [1] they have established rigorous bounds on the spectral radius

$$c := \lim_{n\to+\infty} \|\mathcal{L}^n\|^{\frac{1}{n}}$$

of the form

$$0.643815 \le c \le 0.649415$$

and they then conjecture:

Conjecture 1.1 (Chen–Volkmer). $c = 648314\ldots.$

We have the following rigorous estimate.

Theorem 1.2. *We can write*

$$c = 0.648314752798325682324771447\ldots \pm 10^{-27}.$$

We will first prove an easier preliminary estimate in Theorem 3.1 which is accurate to 20 decimal places. This only requires the ideas from the original article of Ruelle [4]. The method of proof of the stronger Theorem 1.2 is based on the approach that was used in [3]. In the interests of simplicity, we have not implemented the full machinery from [3], which would have lead to an even more accurate estimate for c.

Part of the interest in the estimate in Theorem 1.2 is that it can be applied directly to Bochner–Riesz type Fourier multipliers, relative to the middle third Cantor set, as explained in [1] and as we briefly recall below.

Application 1.3. Let μ be the usual natural measure supported on the middle third cantor set and let $\alpha = \frac{\log 2}{\log 3}$ be its dimension. A function $m(\cdot)$ is an L^1 Fourier multiplier if the linear operator $T\colon L^1(dx) \to L^1(dx)$ defined by $\widehat{Tf}(\xi) = m(\cdot)\hat{f}$ is bounded. To define Bochner–Riesz type Fourier multipliers, assume $\chi(x)$ is a compactly supported bump function with $\hat{\chi} \geq 0$ and $\delta > 0$, then let

$$m_\delta(\xi) = \int \frac{\chi(\xi-\eta)}{|\xi-\eta|^{\alpha-\delta}} d\mu(\eta)$$

By Theorem 6.1 in [1], this is an L^1-Fourier multiplier if and only if

$$\delta > \alpha + \frac{\log c}{\log 3} = 0.236451214234647382935\ldots.$$

Application 1.4. Let $f(t) = |\sin(\pi t)|$. We can write

$$\int_0^1 \prod_{k=0}^{n-1} f(d^k x)dx \sim c^n, \qquad \text{as } n \to +\infty.$$

In particular, where $d = 3$ we can use the numerical estimates for c in Theorem 1.2. Quantities of this form appear in the work of Fan and Lau [2], for example.

I am very grateful to the referee for his careful reading of this note.

2. Estimating the spectral radius

To estimate the value of c we first introduce the following complex function.

Definition 2.1. *We can formally define a complex function*

$$d(z) := \exp\Big(- \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{3^n - 1} \sum_{j=0}^{3^n-1} \prod_{k=0}^{n-1} \sin\Big(\frac{3^k j\pi}{3^n - 1}\Big)\Big)$$

for $z \in \mathbb{C}$.

There is a natural dynamical interpretation for $d(z)$ whereby the values $\frac{3^k j}{3^n-1}$ (mod 1) correspond to periodic points for the classical trebling map

$$x \longmapsto 3x \pmod 1$$

of the unit circle.

One easily sees that $d(z)$ converges to an analytic function for $|z| < 1$. The connection between $d(z)$ and c is given by the following proposition.

Proposition 2.2 (after Ruelle). *The function* $d(z)$ *extends analytically to* $\mathbb{C}$. *The smallest positive zero* $\alpha > 0$ *is the reciprocal of the spectral radius* c, *i.e.,* $c = \frac{1}{\alpha}$.

This is easily deduced from [4] (see also [3]).

We can therefore consider the expansion

$$d(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

where $b_n \in \mathbb{C}$, $n \geq 1$. To estimate α, and thus c, we can consider the sequence of truncations

$$d_N(z) = 1 + \sum_{n=1}^{N} b_n z^n$$

and consider their smallest zeros $z_N > 0$. These give approximations $c_N = \frac{1}{z_N}$ to c which can be efficiently computed to high accuracy.

The larger we choose N, the better the approximation. As we can see from Table 1, a good approximation to z_N based on $N = 15$ is

$$z_{15} = 1.54246065770321086721156153797824343314677299\ldots.$$

which corresponds to an approximation c_{15} to c of

$$c_{15} = 0.64831475279832568232477144769601239350930429\ldots.$$

Table 1. The smallest positive zeros z_N for the approximations $d_N(z)$ to $d(z)$.

N	z_N
5	1.542460746170081385780221528370497949793310989
6	1.54246065795434503327103554155227221319368967
7	1.54246065770349592275701430307427675571759810
8	1.54246065770321099322481446600053607984440150
9	1.54246065770321086723283011105355390760844307
10	1.54246065770321086721156288770669047295526711
11	1.42460657703210867211561538010061072043216200
12	1.54246065770321086721156153797824370901762535
13	1.54246065770321086721156153797824343314764551
14	1.54246065770321086721156153797824343314677299
15	1.54246065770321086721156153797824343314677299

However, it remains to get rigorous error bounds on these approximations. We will prove a preliminary error bound (Theorem 3.1) in §3 and then, after a little more work, prove a better error bound (Theorem 1.2) in §4.

3. A preliminary result

In the next two sections we will explain how to obtain basic rigorous bounds on the approximation of c by c_N, or equivalently, of α by z_N. This based on using bounds on $\sum_{n=16}^{\infty} |b_n|(z_N \pm \epsilon)^n$.

In this section, we begin by describing a basic method which already gives quite good bounds. This establishes the following preliminary version of Theorem 1.2 with a more modest error estimate.

Theorem 3.1. *We can estimate*

$$c = 0.6483147527983256823247 \pm 10^{-21}.$$

In the next section, we will apply a more sophisticated approach which improves these initial estimates. Both methods are based on studying the operator $\mathcal{L}$ acting on a smaller space of functions. However, this does not effect the spectral radius.

3.1. Space of analytic functions. We can use estimates based on the operator $\mathcal{L}$ restricted to a Hilbert space of analytic functions on the disk $D(r) = \{z \in \mathbb{C}: |z - \frac{1}{2}| < r\}$ of radius $r > \frac{1}{2}$.

Definition 3.2. *We can let $\mathcal{H}$ be the space of analytic functions $f: D(r) \to \mathbb{C}$ with norm $\|\cdot\|^2$ given by*

$$\|f\|^2 = \frac{1}{r^2} \sup_{0<\rho<r} \int_0^1 \left| f\left(\frac{1}{2} + \rho e^{2\pi i\theta}\right)\right|^2 d\theta.$$

We can define a natural basis $q_n(z) = \frac{1}{r^n}\left(z - \frac{1}{2}\right)^n$, $n \geq 1$, for $\mathcal{H}$. Moreover, since $\cup_{i=0}^2 T_i D(z, r) \subset D\left(\frac{r+1}{3}\right)$ where $T_i(z) = \frac{z+i}{3}$, for $i = 0, 1, 2$, we see that the operator $\mathcal{L}$ also preserves $\mathcal{H}$.

3.2. First bounds on coefficients.

We can obtain easy bounds on the coefficients by using simple "Euler bounds" (using the terminology from [3]).

Lemma 3.3. *We can bound*

$$|b_n| \leq C^n \frac{\theta^{n(n+1)/2}}{\prod_{k=0}^\infty (1-\theta^k)}, \quad n \geq 1, \tag{3.1}$$

where $\theta = \theta(r) = \frac{r+1}{3r}$ and $C = C(r) = (e^{\pi\theta r} + 1)/2$.

These standard bounds follow easily as in [4] and [3].

We have the freedom to change $r > 1/2$ in order to find the best possible bound. The following example illustrates this.

Example 3.4 ($r = \frac{5}{2}$). *If we take the value $r = \frac{5}{2}$ then we have $\theta = \frac{7}{12}$ and $C = 20.0318\cdots$.*

Given z_N and $\delta > 0$, to see that there is a zero $z \in [z_N - \delta, z_N + \delta]$ for $d(z)$ it suffices to show rigorously that

$$\begin{aligned} &|d_N(z_N - \delta) - d(z_N - \delta)| < \epsilon, \\ &|d_N(z_N + \delta) - d(z_N + \delta)| < \epsilon, \\ &d_N(z_N - \delta) \leq -\epsilon < 0 < \epsilon \leq d_N(z_N + \delta). \end{aligned} \tag{3.2}$$

Then by the triangle inequality and the intermediate value theorem the zero z lies between $z_N - \delta$ and $z_N + \delta$. To show the first two inequalities in (3.2) we can bound

$$\begin{aligned} |d_N(z_N \pm \delta) - d(z_N \pm \delta)| &\leq \left| \sum_{n=N+1}^\infty (z_N \pm \delta)^n b_n \right| \\ &\leq \sum_{n=N+1}^\infty (z_N \pm \delta)^n |b_n| \end{aligned} \tag{3.3}$$

and we can use the bound (3.1) in Lemma 3.3 for individual terms. This gives a basic, but still respectable, upper bound on (3.3) of the following form

$$\begin{aligned}\sum_{n=N}^{\infty} C^n \theta^{n(n+1)/2}(z_N \pm \delta)^n &\le (C(z_N \pm \delta))^N \theta^{N(N+1)/2} \sum_{k=0}^{\infty}(C(z_N \pm \delta)\theta^N)^k \theta^{k/2} \\ &\le \frac{(C(z_N \pm \delta))^N \theta^{N(N+1)/2}}{1-(C(z_N \pm \delta)\theta^N)}.\end{aligned} \tag{3.4}$$

Letting $N = 15$ and $\delta = 10^{-21}$ we take ϵ to be the resulting numerical value of the upper bound in (3.4), which is approximately $6.66711 \cdots \times 10^{-22}$. Finally, we can numerically compute (with an accuracy far in excess of 22 decimal places) the values

$$d_N(z_N + \delta) = -8.09\ldots \times 10^{-22},$$

$$d_N(z_N - \delta) = 8.09\ldots \times 10^{-22}$$

to establish the final inequality in (3.2) with with $\epsilon = 10^{-21}$, say. This establishes Theorem 3.1 (with a more modest error estimate than in Theorem 1.2).

4. Proof of Theorem 1.2

We next describe a method for improving the error terms in order to establish Theorem 1.2.

4.1. Better bounds on the coefficients. The advantage of working with the Hilbert space $\mathcal{H}$ of analytic functions rather than, say, Hölder continuous functions is that we obtain good bounds on the coefficients c_n in terms of the norm of the operator acting on the functions $\{q_n\}$. We can write

$$\|\mathcal{L}q_n\|^2 = \frac{1}{r^{2n}} \int_0^1 \left|(\mathcal{L}q_n)\left(\frac{1}{2} + re^{2\pi i n\theta}\right)\right|^2 d\theta. \tag{4.1}$$

We then denote

$$\alpha_n := \left(\sum_{k=n-1}^{\infty} \|\mathcal{L}q_k\|^2\right), \quad n \ge 1.$$

This leads to the following bound on $|b_n|$, which helps to improve on the preliminary bounds (3.1).

Lemma 4.1. *Let $M \geq N$. We can bound the coefficients $|b_n|$ for $N+1 \leq n \leq M$ by*

$$|b_n| \leq \sum_{k=1}^{\min\{M,n\}} Q_{k,M} C^{(n-k)} \Big(\frac{\Theta^{M(n-k)+(n-k)(n-k+1)/2}}{\prod_{i=1}^{n-k}(1-\Theta^i)} \Big),$$

where

$$\Theta = \frac{r+1}{3r} < 1, \quad C = \frac{e^{\pi\Theta r}+1}{\Theta\sqrt{1-\Theta^2}},$$

and

$$Q_{k,M} = \sum_{m_1<\cdots<m_k\leq M} \alpha_{m_1}\alpha_{m_2}\cdots\alpha_{m_k}.$$

Proof. We will sketch the proof, using the bounds established in [3]. In particular, by Lemma 2 in [3] we can bound

$$|b_n| \leq \sum_{i_1<\cdots<i_n} \prod_{j=1}^{n} s_{i_j}(\mathcal{L}), \tag{4.2}$$

where

$$s_i(\mathcal{L}) = \inf\{\|\mathcal{L}-K\| : \operatorname{rank} K \leq i-1\}, \quad i \geq 1,$$

are the approximation numbers. (The inequality (4.2) has its origins in the Weyl inequality.) Moreover, by Proposition 1 in [3] $s_n(\mathcal{L}) \leq z_n$ for $n \geq 1$. Therefore, we can bound the right hand side of (4.2) by

$$\sum_{k=1}^{\min\{M,n\}} Q_{k,M} \Big(\sum_{M<m_{k+1}<\cdots<m_n} \alpha_{m_{k+1}}\cdots\alpha_{m_n} \Big). \tag{4.3}$$

Moreover, we can bound $\alpha_n \leq C\Theta^n$, for $n > M$, (by Corollary 1 in [3]) and thus we can bound the inner summation in (4.3) by

$$C^{(n-k)} \sum_{M<m_{k+1}<\cdots<m_n} \Theta^{m_{k+1}+\cdots+m_n} = C^{n-k} \frac{\Theta^{M(n-k)+(n-k)(n-k+1)/2}}{\prod_{i=1}^{n-k}(1-\Theta^i)},$$

which can be compared with Lemma 3.3. This completes the proof. □

In this lemma, the improvement on the bounds comes from the terms $Q_{k,M}$ being relatively small for small k. In order to make this into a bound which can be numerically estimated we need to replace α_n by the upper bound

$$\widetilde{\alpha_n} := \Big(\sum_{k=n-1}^{L} \|\mathcal{L}g_k\|^2 + \frac{C^2\theta^{2L}}{1-\Theta^2} \Big)^{1/2}$$

where $L \geq M$ following [3]. Correspondingly, we can bound $Q_{k,M}$ by

$$\widetilde{Q_{k,M}} = \sum_{m_1 < \cdots < m_k \leq M} \widetilde{\alpha_{m_1}} \widetilde{\alpha_{m_2}} \cdots \widetilde{\alpha_{m_k}}.$$

In particular, we can bound the contribution from the terms for $N+1 \leq n \leq M$ by

$$\begin{aligned}
&\sum_{n=N+1}^{M} (z_N \pm \delta)^n |b_n| \\
&\quad \leq \sum_{n=N+1}^{M} (z_N \pm \delta)^n \sum_{k=1}^{\min\{M,n\}} \widetilde{Q_{k,M}} C^{(n-k)} \left(\frac{\Theta^{M(n-k)+(n-k)(n-k+1)/2}}{\prod_{i=1}^{n-k} (1-\Theta^i)} \right).
\end{aligned}$$

Finally, for the terms with $n \geq M+1$ we have an upper bound

$$\begin{aligned}
&\sum_{n=M+1}^{\infty} (z_N \pm \delta)^n |b_n| \\
&\quad \leq \sum_{k=1}^{M} \widetilde{Q_{k,M}} \left(\sum_{n=M+1}^{\infty} (z_N \pm \delta)^n C^{(n-k)} \left(\frac{\Theta^{M(n-k)+(n-k)(n-k+1)/2}}{\prod_{i=1}^{n-k} (1-\Theta^i)} \right) \right) \\
&\quad \leq \frac{1}{\prod_{i=1}^{\infty} (1-\Theta^i)} \\
&\qquad \times \sum_{k=1}^{M} \widetilde{Q_{k,M}} \left(\sum_{n=M+1}^{\infty} (z_N \pm \delta)^n C^{(n-k)} \left(\Theta^{M(n-k)+(n-k)(n-k+1)/2} \right) \right) \\
&\quad \leq \frac{1}{\prod_{i=1}^{\infty} (1-\Theta^i)} \sum_{k=1}^{M} \widetilde{Q_{k,M}} (\alpha + \epsilon)^k \frac{(C\theta^M (z_N \pm \delta))^{(M+1-k)}}{1 - C\theta^M (z_N \pm \delta)}.
\end{aligned}$$

It now only remains to make judicious choices for the various arbitrary values, which we do in the next subsection.

4.2. Error term bounds. For practical considerations (based on computation time) we can choose $N = 15$, $M = 100$ and $L = 500$. We can let $r = 2$ then $\Theta = \frac{1}{2}$ and $e^{\pi} + 1 = 13.4187\ldots$. We make no claim that these choices are optimal, but with these choices we obtain concrete bounds on the contribution coming from the terms corresponding to $n \geq 16$.

We now take $\delta = 10^{-26}$, and with this choice we have the bound

$$\sum_{n=16}^{\infty} |b_n| (z_n + \delta)^n \leq 3.63 \cdots \times 10^{-27},$$

which can be compared with the bound in (3.3). In particular, since we can compute

$$d_N(z_N + \delta) = -8.09\ldots \times 10^{-27},$$

$$d_N(z_N - \delta) = 8.09\ldots \times 10^{-27}.$$

we can use the corresponding inequalities in (3.2) to complete the proof of Theorem 1.2.

Table 2. The first few values of $\|\mathcal{L}q_k\|$ and $\widetilde{\alpha_n}$

k	$\|\mathcal{L}q_k\|$	n	$\widetilde{\alpha_n}$
1	5.086023222	1	5.220281249
2	1.113848198	2	1.176309529
3	0.317490130	3	0.378214620
4	0.178212967	4	0.205539086
5	0.091573051	5	0.102403389
6	0.041201463	6	0.045834817
7	0.017851464	7	0.020081582
8	0.008068454	8	0.009197562
9	0.003857587	9	0.004415337
10	0.001881168	10	0.002148074
11	0.000910710	11	0.001037028
12	0.000435966	12	0.000236576
13	0.000207753	13	0.000113166
14	0.000099228	14	0.000054408
15	0.000047642	15	0.000026275

5. Generalizations

The method we have described above for estimating the spectral radius of $\mathcal{L}$ applies more generally. For example, one can consider transfer operators

$$\mathcal{L}_{d,q}: C^0([0,1]) \longrightarrow C^0([0,1])$$

(where $d \geq 2$ and $q \geq 1$) defined by

$$(\mathcal{L}_{d,q}u)(t) = \frac{1}{d}\sum_{i=0}^{d} f_q\Big(\frac{t+i}{d}\Big)u\Big(\frac{t+i}{d}\Big),$$

where either $f_q(t) = |\cos(\pi t)|^q$ or $f_q(t) = |\sin(\pi t)|^q$.

Table 3. The first few values of $\widetilde{Q_{100,k}}$ and the corresponding bounds on b_n.

k	$\widetilde{Q_{100,k}}$
1	7.166413516
2	11.262024489
3	6.009094824
4	1.345702822
5	0.135230283
6	0.006173517
7	0.000128659
8	$1.241338291 \times 10^{-6}$
9	$5.638788290 \times 10^{-9}$
10	$1.219753034 \times 10^{-11}$
11	$1.262398454 \times 10^{-14}$
12	$6.257402794 \times 10^{-18}$
13	$1.486214073 \times 10^{-21}$
14	$1.694078690 \times 10^{-25}$
15	$9.291306353 \times 10^{-30}$

n	bounds on b_n
1	$5.176742012 \times 10^{-52}$
2	$1.526929255 \times 10^{-48}$
3	$2.251911477 \times 10^{-45}$
4	$1.660556730 \times 10^{-42}$
5	$6.122462367 \times 10^{-40}$
6	$1.128674038 \times 10^{-37}$
7	$1.040353545 \times 10^{-35}$
8	$4.794721336 \times 10^{-34}$
9	$1.104881739 \times 10^{-32}$
10	$1.273028787 \times 10^{-31}$
11	$7.333826946 \times 10^{-31}$
12	$2.112482380 \times 10^{-30}$
13	$3.042464622 \times 10^{-30}$
14	$2.190927381 \times 10^{-30}$
15	$7.888609052 \times 10^{-31}$

Definition 5.1. *We let $c_d(q)$ be the spectral radius of $\mathcal{L}_{d,q}$.*

There are particular cases where $c_d(q)$ is easily computed. In the case that $d = 2$ one can explicitly compute $c_2(q) = 2^{-\min\{q,1\}}$ $(q > 0)$. Furthermore, when $q = 2$ and $q = 4$ then one can explicitly compute $c(2) = \frac{1}{3}$ and $c(4) = \frac{3}{8}$, see [5]. In general, when q is even then the values $c_d(q)$ can be easily expressed in terms of finite matrices.

For completeness, we briefly describe the modifications for the more general setting. In the general case, we can introduce the following complex function, generalizing Definition 2.1.

Definition 5.2. *We can formally define*

$$d_{d,q}(z) := \exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{d^n - 1} \sum_{j=0}^{d^n-1} \prod_{k=0}^{d-1} f_q\Big(\frac{2^k j}{d^n - 1}\Big)\Big)$$

for $z \in \mathbb{C}$.

Since $\|f_q\|_\infty \leq 1$ we see that $d(z)$ converges to an analytic function for $|z| < 1$. We can therefore consider the expansion

$$d_{d,q}(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

where $b_n \in \mathbb{C}$, $n \geq 1$. One can show that the smallest positive zero $\alpha = \alpha_{d,q} > 0$ is the reciprocal of the spectral radius c, i.e., $c = c_{d,q} = \frac{1}{\alpha}$ of $\mathcal{L}_{d,q}$. To estimate α, and thus c, we can consider the sequence of truncations

$$d_N(z) = 1 + \sum_{n=1}^{N} b_n z^n$$

and consider their smallest zeros $z_N > 0$. These give approximations $c_N = \frac{1}{z_N}$ to c.

We can choose small complex neighbourhoods $U_i \supset [i/q, (i+1)/q]$ and consider analytic functions on $\coprod_i U_i$. The operator is nuclear. In particular, this allows computation of the values $c = c_{d,q}$ much as before.

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