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Intermediate Assouad-like dimensions

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Abstract. We study a class of bi-Lipschitz-invariant dimensions that range between the box and Assouad dimensions. The quasi-Assouad dimensions and θ-Assouad spectrum are other special examples. These dimensions are localized, like Assouad dimensions, but vary in the depth of scale which is considered, thus they provide very refined geometric information. Our main focus is on the intermediate dimensions which range between the quasi-Assouad and Assouad dimensions, complementing the θ-Assouad spectrum which ranges between the box and quasi-Assouad dimensions.

We investigate the relationship between these and the familiar dimensions. We construct a Cantor set with a non-trivial interval of dimensions, the endpoints of this interval being given by the quasi-Assouad and Assouad dimensions of the set. We study stability and continuity-like properties of the dimensions. In contrast with the Assouad-type dimensions, we see that decreasing sets in $\mathbb{R}$ with decreasing gaps need not have dimension 0 or 1. As is the case for Hausdorff and Assouad dimensions, the Cantor set and the decreasing set have the extreme dimensions among all compact sets in $\mathbb{R}$ whose complementary set consists of open intervals of the same lengths.

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Contents

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1. Introduction and main results

1.1. Introduction. Over the years, many notions of dimension have been introduced to help understand the geometry of (often "small") subsets of metric spaces, such as subsets of $\mathbb{R}^n$ of Lebesgue measure zero. Hausdorff, box and packing dimensions are well known examples of such notions. More recently, the upper and lower Assouad dimensions of a set E, denoted $\dim_A E$ and $\dim_L E$ respectively, which quantify the "thickest" or "thinnest" part of the space, were introduced by Assouad in [1, 2] and Larman in [23]. Along with their less extreme versions, the upper and lower quasi-Assouad dimensions, $\dim_{qA} E$ and $\dim_{qL} E$, introduced in [4, 26], these dimensions have been extensively studied within the fractal geometry community; see for example, [5, 8, 11, 10, 12, 13, 14, 18, 22, 25, 27, 28] and the references cited therein. These dimensions can roughly be thought of as local refinements of the box-counting dimensions where one takes the most extreme local behaviour. The following relationships are known for all compact sets E:

$$\dim_L E \leq \dim_{qL} E \leq \dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E \leq \dim_{qA} E \leq \dim_A E,$$

where $\dim_H$, $\underline{\dim}_B$, $\overline{\dim}_B$ denote the Hausdorff, lower and upper box dimensions respectively. See [6, 8, 14, 26] for proofs.

In several recent papers, a range of intermediate dimensions have been introduced and studied. For instance, in [7], Falconer, Fraser and Kempton discussed a continuum of dimensions that lie between the Hausdorff and box dimensions. In [12, 13], Fraser and Yu focussed on the family of dimensions known as the upper (or lower) θ-Assouad spectrum, which lie between the upper (or lower) box and quasi-upper (resp., lower) Assouad dimensions. For further references, we refer the reader to Fraser's survey paper [9].

In this paper, we study a general class of intermediate dimensions, which we refer to as the upper and lower Φ-dimensions. These include the (quasi-) Assouad dimensions and θ-Assouad spectrum as special cases, with the box dimensions typically arising as a limit. More generally, the Φ-dimensions provide a range of bi-Lipschitz invariant dimensions between the box, quasi-Assouad and Assouad dimensions. As the box and (quasi-) Assouad dimensions for a given set can all be different, the intermediate Φ-dimensions provide more refined information about the local geometry of the set, such as detailed information about the scales at which one can observe extreme local behaviour.

Another motivation for us to investigate these intermediate dimensions was the classical problem of understanding the dimension of "rearrangements" of Cantor sets. This problem was first considered by Besicovitch and Taylor in [3] for the Hausdorff dimension in the deterministic case and later by Hawkes in [19] for the random situation. In [16] we prove that if $\Phi(x) \ll \log|\log x|/|\log x|$, then the upper and lower Φ-dimensions of almost all (in a natural probabilistic sense) rearrangements of a given Cantor set are 1 and 0 respectively, while if $\Phi(x) \gg \log|\log x|/|\log x|$, then almost all rearrangements have the same upper and lower Φ-dimensions as the original Cantor set. The first case includes the Assouad dimensions and the second, the quasi-Assouad. In [30], Troscheit obtained similar results for other random constructions.

1.2. Φ-dimensions. To explain these dimensions in more detail, we first recall that for the upper box dimension of a metric space E one considers the minimal number of balls of radius r that are required to cover the entire space E, say $N_r(E)$, and computes the infimal exponent s such that $N_r(E) \le r^{-s}$ as $r \to 0$. For the upper Assouad dimension of E one determines, instead, the infimal s such that

$$N_r(B(z,R) \cap E) \le (R/r)^s$$

for all $r \le R$ and all centres $z \in E$. The lower Assouad dimension is a similar local variation of the lower box dimension. The quasi-Assouad dimensions are less extreme versions of the Assouad dimensions, requiring only that the bounds hold for $r \le R^p$ where the exponent p decreases to 1.

Fraser and Yu observed in [12, Section 9] that a rich dimension theory can be developed by considering decreasing continuous functions $F(x) \le x$, choosing $r = F(R)$ (or $r \le F(R)$ in our modified case) and studying the corresponding Assouad-like dimensions. In [10, 12, 13], Fraser et al studied the special case of $F(x) = x^{1/\theta}$ for fixed $\theta \in (0,1)$, the so-called upper and lower θ-Assouad spectrum. As $\theta \to 1$, the upper (lower) θ-Assouad spectrum tends from below (resp., above) to the upper (resp., lower) quasi-Assouad dimension. As $\theta \to 0$, the upper θ-Assouad spectrum tends from above to the upper box dimension, while the lower θ-Assouad spectrum is dominated by the lower box dimension.

Motivated by the work of Fraser, we consider the quite general class of functions $F(x) = x^{1+\Phi(x)}$, requiring only that $F(x)$ decreases to 0 as $x \downarrow 0$. The *upper* and *lower Φ-dimensions* of E, denoted by $\overline{\dim}_\Phi E$, $\underline{\dim}_\Phi E$ respectively, arise by restricting to $r \le R^{1+\Phi(R)}$. We refer the reader to Definition 2.3 for the precise definitions.

When $\Phi = 0$ we recover the Assouad dimensions and when $\Phi = 1/\theta - 1$, we get the θ-Assouad spectrum. It will be shown that if $\Phi(x) \to \infty$ as $x \to 0$ (and $\underline{\dim}_\Phi E > 0$) the upper (lower) Φ-dimension is the upper (resp., lower) box dimension (Proposition 2.8), while if $\Phi(x) \to \delta \in (0,\infty)$, then the upper (lower) Φ-dimension coincides with the upper (resp., lower) θ-Assouad spectrum for $\theta = (1+\delta)^{-1}$ (Corollary 2.12). Thus our main interest in this paper is in the case that $\Phi \to 0$ (such as the function $\Phi(x) = \log|\log x|/|\log x|$, which appears naturally in the random problem) when we obtain a full range of intermediate dimensions between the quasi-Assouad and Assouad dimensions.

1.3. Summary of the main results. The primary purpose of this paper is to study the basic properties of these intermediate dimensions.

One easy property is that the upper Φ-dimension is finitely stable, but the lower Φ-dimension is not. See Proposition 2.5.

1.3.1. Relationship between dimensions. A natural question to ask is how the Φ-dimensions compare, both with each other and to the familiar dimensions. Clearly, they are naturally ordered: if $\Phi_1 \le \Phi_2$, then $\overline{\dim}_{\Phi_1}(E) \ge \overline{\dim}_{\Phi_2}(E)$ and vice versa for the lower Φ-dimensions. Thus the upper (lower) Φ-dimensions lie between the upper (lower) box and the upper (lower) Assouad dimensions.

Here is an overview of some of our main theoretical results on this question.

(1) If $\Phi(x) \le c/\left|\log x\right|$, then the upper and lower Φ-dimensions coincide with the upper and lower Assouad dimensions (respectively) for all sets E. See Proposition 2.10.

(2) The upper and lower quasi-Assouad dimensions are special examples of upper and lower Φ-dimensions, but the choice of dimension functions will depend on the underlying set E. See Proposition 2.15.

(3) If $\Phi_1/\Phi_2(x) \to 1$ as $x \to 0$, then the upper (and lower) Φ_1 and Φ_2-dimensions coincide for all sets E. See Proposition 2.11(i).

(4) If there is a constant $\xi > 0$ such that $\Phi_1 \ge (1+\xi)\Phi_2$, then there are sets E_1, E_2 with $\overline{\dim}_{\Phi_1}(E_1) \ne \overline{\dim}_{\Phi_2} E_1$ and $\underline{\dim}_{\Phi_1} E_2 \ne \underline{\dim}_{\Phi_2} E_2$. See Theorem 3.8.

(5) Given any family of decreasing dimension functions, $\{\Phi_p\}_{p\in(0,1)}$, with $\Phi_p \gg \Phi_q$ for $p > q$, and given any decreasing, continuous function $d\colon (0,1) \to [a,b] \subseteq (0,1)$, there is a set $E \subseteq \mathbb{R}$ with $\overline{\dim}_{\Phi_p}(E) = d(p)$ for all p. The analogous result holds for the lower dimensions. See Theorem 3.9. Hence there are subsets of $\mathbb{R}$ with a full (non-trivial) interval of dimensions whose endpoints are given by the quasi-Assouad and Assouad dimensions. See Corollary 3.11.

(6) It is known that the lower θ-Assouad spectrum of a set are not uniformly dominated above by the Hausdorff dimension. However $\underline{\dim}_{\Phi_\theta} E \le \frac{1}{\theta} \dim_H E$ for $\Phi_\theta = 1/\theta - 1$. See Proposition 2.16.

(7) In [5] and [13], it is shown that both the upper and lower θ-Assouad spectrum are continuous in the parameter θ. More generally, if $\Phi_t(x) = g(t)\Phi(x)$ where g is continuous and $g(t_0) \neq 0$, then $\overline{\dim}_{\Phi_t}(E) \to \overline{\dim}_{\Phi_{t_0}}(E)$ as $t \to t_0$, and similarly for the lower dimensions. But this need not be true when $g(t_0) = 0$. See Propositions 2.11(ii) and 3.6. This suggests that it may be difficult to find a one-parameter family of continuous dimension functions that interpolates precisely between the quasi-Assouad and Assouad dimensions.

1.3.2. Decreasing sets. In [15] it was shown that if $E = \{x_n\}_n \subseteq \mathbb{R}$ is a decreasing sequence with decreasing gaps, then the Assouad dimension is either 0 or 1. Likewise, the quasi-Assouad dimension is 0 if $\overline{\dim}_B E = 0$ and 1 otherwise. In contrast, in Example 2.18 we construct a decreasing set E with decreasing gaps and a dimension function $\Phi \to 0$ with $\dim_{qA} E = 0 < \overline{\dim}_\Phi E < \dim_A E = 1$.

1.3.3. Cantor sets and Rearrangements. We give formulas for the Φ-dimensions of Cantor-like sets, similar to those known for Hausdorff, box and Assouad dimensions, in Theorem 3.5. These are used in some of our constructions, such as in exhibiting sets with different values for various Φ-dimensions as mentioned in (4) and (5) above. This approach was taken in [14] and [26] to construct sets with different box and (quasi-) Assouad dimensions, but new ideas are required here.

In [3], Besicovitch and Taylor proved that if C is a Cantor-like set, then the interval $[0, \dim_H C]$ is precisely the set of Hausdorff dimensions of "rearrangements" of C, the sets whose complement consists of open intervals of the same lengths as the complementary intervals of C. This was extended in [15] where it was found that the set of attainable lower (upper) Assouad dimensions was $[0, \dim_L C]$ (resp., $[\dim_A C, 1]$), with 0 (resp., 1) being the lower (upper) Assouad dimension of the decreasing rearrangement. In Section 4, we study this problem for the Φ-dimensions and show that under natural assumptions, the Φ-dimensions of any rearrangement lies between the dimensions of the Cantor set and the decreasing rearrangement (whose upper Φ-dimension could be < 1). Further, for any $\Phi \to p \in [0, \infty]$ (including the quasi-Assouad dimensions), this full range of values can be attained. New construction techniques are needed to do this. In [16] some of these results are used in studying the Φ-dimensions of random rearrangements.

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2. Basic properties of the Φ-dimensions

2.1. Definitions. We begin with notation and definitions. Let X be a metric space.

Notation 2.1. We denote the closed ball centred at $z \in X$ with radius R by $B(z, R)$. For a bounded set $E \subseteq X$, the notation $N_r(E)$ will mean the least number of balls of radius r that cover E.

Definition 2.2. A map $\Phi\colon (0,1) \to \mathbb{R}^+$ is called a *dimension function* if $x^{1+\Phi(x)}$ decreases as x decreases to 0. We write $\mathcal{D}$ for the set of all dimension functions.

Of course, $R^{1+\Phi(R)} \leq R$, so $R^{1+\Phi(R)} \to 0$ as $R \to 0$ for any dimension function Φ. Special examples of dimension functions include the constant functions $\Phi(x) = \delta \geq 0$ and the function $\Phi(x) = 1/|\log x|$.

Definition 2.3. Let X be a metric space and $\Phi \in \mathcal{D}$. The *upper* and *lower* Φ*-dimensions* of $E \subseteq X$ are given by

$$\overline{\dim}_\Phi E = \inf\Big\{\alpha\colon (\exists c_1, c_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < c_1) \\ \sup_{z \in E} N_r(B(z,R) \cap E) \leq c_2\Big(\frac{R}{r}\Big)^\alpha\Big\}$$

and

$$\underline{\dim}_\Phi E = \sup\Big\{\alpha\colon (\exists c_1, c_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < c_1) \\ \inf_{z \in E} N_r(B(z,R) \cap E) \geq c_2\Big(\frac{R}{r}\Big)^\alpha\Big\}.$$

A standard argument based upon the relationship between balls in bi-Lipschitz spaces, shows that upper and lower Φ-dimensions are preserved under bi-Lipschitz maps. We also note that, as with the box dimensions, a set and its closure have the same upper or lower Φ-dimensions.

The upper and lower (quasi)-Assouad dimensions and spectrum can be expressed in terms of Φ-dimensions.

Remark 2.4. (i) The *upper Assouad* and *lower Assouad dimensions* of E, denoted $\dim_A E$ and $\dim_L E$ respectively, are the special cases of the upper and lower Φ-dimensions with $\Phi(x) = 0$ for all x.

(ii) The *upper* and *lower* θ*-Assouad spectrum*, $\overline{\dim}_A^\theta E$ and $\underline{\dim}_L^\theta E$, are the special cases of the upper and lower Φ_θ-dimensions with $\Phi_\theta(x) = 1/\theta - 1 > 0$ for all x. To be precise, the upper and lower θ-Assouad spectrum introduced in [12] only required consideration of $r = R^{1/\theta}$. However, it was shown in [10] that if we denote this dimension by $\overline{\dim}_A^{=\theta} E$, then $\overline{\dim}_A^\theta E = \sup_{\psi \le \theta} \overline{\dim}_A^{=\psi} E$, with the analogous statement proven in [5] (see also [18]) for the lower θ-Assouad spectrum.

(iii) The *upper quasi-Assouad* and *lower quasi-Assouad dimensions* are given by

$$\dim_{qA} E = \lim_{\theta \to 1} \overline{\dim}_{\Phi_\theta} E \quad \text{and} \quad \dim_{qL} E = \lim_{\theta \to 1} \underline{\dim}_{\Phi_\theta} E,$$

where, again, $\Phi_\theta(x) = 1/\theta - 1 > 0$. In Proposition 2.15 we will prove that the quasi-Assouad dimensions can also be obtained as Φ-dimensions, but the choice of Φ depends on the set E.

We refer the reader to the references cited in the introduction of this paper for further background information on these various Assouad-like dimensions.

It is easy to see that $\underline{\dim}_\Phi E = 0$ whenever E has an isolated point, thus we need not have $\underline{\dim}_\Phi E \le \underline{\dim}_\Phi F$ if $E \subseteq F$. This monotonicity property does, however, hold for the upper Φ-dimension, as does finite stability. We show this first, along with bounds on the lower Φ-dimension of finite unions.

Proposition 2.5. (i) *If* $E \subseteq F$, *then* $\overline{\dim}_\Phi E \le \overline{\dim}_\Phi F$. *Indeed, for* $E, F \subseteq X$,

$$\overline{\dim}_\Phi (E \cup F) = \max(\overline{\dim}_\Phi E, \overline{\dim}_\Phi F).$$

(ii) *For all* $E, F \subseteq X$,

$$\min(\underline{\dim}_\Phi E, \underline{\dim}_\Phi F) \le \underline{\dim}_\Phi (E \cup F) \le \max(\underline{\dim}_\Phi E, \overline{\dim}_\Phi F).$$

Proof. (i) The fact that $\overline{\dim}_\Phi (E \cup F) \ge \max(\overline{\dim}_\Phi E, \overline{\dim}_\Phi F)$ follows easily from the observation that if $z \in E \cup F$, say $z \in E$, then

$$N_r(B(z, R) \cap (E \cup F)) \ge N_r(B(z, R) \cap E).$$

To see the reverse inequality, first note that

$$N_r(B(z, R) \cap (E \cup F)) \le N_r(B(z, R) \cap E) + N_r(B(z, R) \cap F).$$

Fix $\varepsilon > 0$ and assume $z \in E$ and $r \le R^{1+\Phi(R)}$ for small R. Then there is a constant c such that $N_r(B(z, R) \cap E) \le c \left(\frac{R}{r}\right)^{d_1+\varepsilon}$ where $d_1 = \overline{\dim}_\Phi E$. If $B(z, R) \cap F$ is empty, then trivially

$$N_r(B(z,R)\cap(E\cup F))\leq c\Big(\frac{R}{r}\Big)^{d_1+\varepsilon}.$$

Otherwise, there is some $y\in B(z,R)\cap F$, and as $B(z,R)\cap F\subseteq B(y,2R)\cap F$ we have

$$N_r(B(z,R)\cap F)\leq N_r(B(y,2R)\cap F)\leq c'\Big(\frac{R}{r}\Big)^{d_2+\varepsilon}$$

for $d_2=\overline{\dim}_\Phi F$. In either case, there is a constant C such that

$$N_r(B(z,R)\cap(E\cup F))\leq C\Big(\frac{R}{r}\Big)^{d+\varepsilon}$$

for $d=\max(\overline{\dim}_\Phi E,\overline{\dim}_\Phi F)$ for all $z\in E\cup F$ and $r\leq R^{1+\Phi(R)}$, proving that $\overline{\dim}_\Phi(E\cup F)\leq d$.

(ii) The lower bound is obvious. For the upper bound, choose z_j and $r_j\leq R_j^{1+\Phi(R_j)}$ such that $N_{r_j}(B(z_j,R_j)\cap E)\leq c_1(\frac{R_j}{r_j})^{d_1+\varepsilon}$ where $d_1=\underline{\dim}_\Phi E$ and $\varepsilon>0$ is fixed. Choosing $y_j\in B(z_j,R_j)\cap F$ (if this set is non-empty), we have

$$\begin{aligned}N_{r_j}(B(z_j,R_j)\cap(E\cup F))&\leq N_{r_j}(B(z_j,R_j)\cap E)+N_{r_j}(B(y_j,2R_j)\cap F)\\&\leq c_1\Big(\frac{R_j}{r_j}\Big)^{d_1+\varepsilon}+c_2\Big(\frac{R_j}{r_j}\Big)^{d_2+\varepsilon}\leq C\Big(\frac{R_j}{r_j}\Big)^{d+\varepsilon}\end{aligned}$$

where $d_2=\overline{\dim}_\Phi F$ and $d=\max(d_1,d_2)$. □

Remark 2.6. We will always assume the underlying metric space X is *doubling*, which means that there is a constant $M\geq 1$ so that for any $R>0$, each ball of radius R can be covered with at most M balls of radius $R/2$. The least such M is called the *doubling constant* of the space. This condition is equivalent to saying the space has bounded upper Assouad dimension, [20]. For example, if $E\subseteq\mathbb{R}^d$, then $\dim_A E\leq d$.

The doubling assumption ensures that in the definitions of the Φ-dimensions the covering number $N_r(B(z,R)\cap E)$ could be replaced by the packing number $P_r(B(z,R)\cap E)$, the maximum number of disjoint balls of radius r, centred in $B(z,R)\cap E$. This is because the packing and covering numbers are comparable:

$$\frac{1}{M}N_r(F)\leq P_r(F)\leq M\cdot N_{r/2}(F)\quad\text{for any } F\subseteq X.$$

2.2. Relationships between dimensions. We begin by recalling the relationships between Assouad-like dimensions and various classical dimensions. We denote by $\dim_H$, $\underline{\dim}_B$ and $\overline{\dim}_B$ the Hausdorff, lower box and upper box dimensions respectively. The box dimensions are defined for bounded sets and satisfy

$$\dim_H E \le \underline{\dim}_B E \le \overline{\dim}_B E.$$

We refer to [6] for the definitions and basic properties of these dimensions.

Clearly we have the following relationships:

$$\dim_L E \le \dim_{qL} E \le \underline{\dim}_B E \le \overline{\dim}_B E \le \dim_{qA} E \le \dim_A E$$

and

$$\dim_L E \le \underline{\dim}_\Phi E \le \overline{\dim}_\Phi E \le \dim_A E.$$

Since $\underline{\dim}_B E \ge \dim_H E$, we obviously have

$$\dim_H E \le \dim_{qA} E$$

and for closed sets E it is also true that

$$\dim_{qL} E \le \dim_H E;$$

see [14] (and [24] for the earlier result, $\dim_L E \le \dim_H E$). This inequality is not true in general, for example, if $\mathbb{Q}$ denotes the set of rational numbers in $[0,1]$, then $\dim_H \mathbb{Q} = 0$, but $\dim_{qL} \mathbb{Q} = \dim_{qL} \mathbb{R} = 1$.

Obviously, if $\Phi(x) \le \Psi(x)$ for all $x \in (0,1)$, then

$$\underline{\dim}_\Phi E \le \underline{\dim}_\Psi E \quad \text{and} \quad \overline{\dim}_\Psi \le \overline{\dim}_\Phi E.$$

Consequently, if $\Phi(x) \to 0$ as $x \to 0$, then the Φ-dimensions give a range of dimensions between the Assouad and quasi-Assouad type dimensions:

$$\dim_L E \le \underline{\dim}_\Phi E \le \dim_{qL} E \le \dim_{qA} E \le \overline{\dim}_\Phi E \le \dim_A E\ .$$

Remark 2.7. We remark that in Section 3 of this paper many examples are constructed which demonstrate strictness in these inequalities. These are based on the formulas given in Theorem 3.5 for the Φ-dimensions of Cantor sets. The reader can also refer to [14, Ex. 16] or [26, Ex. 1.18] for similar constructions illustrating the strictness of the relationship between the quasi-Assouad and Assouad dimensions.

Here are some additional facts about the relationships between these dimensions.

Proposition 2.8. (i) *For any* $\Phi \in \mathcal{D}$,

$$\underline{\dim}_\Phi E \le \underline{\dim}_B E \le \overline{\dim}_B E \le \overline{\dim}_\Phi E.$$

(ii) *If* $\Phi(x) \to \infty$ *as* $x \to 0$, *then* $\overline{\dim}_\Phi E = \overline{\dim}_B E$. *If, in addition,* $\underline{\dim}_\Phi E > 0$, *then* $\underline{\dim}_\Phi E = \underline{\dim}_B E$.

Remark 2.9. Since $\underline{\dim}_\Phi E = 0$ if E has an isolated point, it need not be true that $\underline{\dim}_\Phi E = \underline{\dim}_B E$ under only the assumption that $\Phi(x) \to \infty$.

Proof. (i) As our metric space is assumed to be doubling, we even have that $\overline{\dim}_A E < \infty$. Thus, we may suppose $b = \overline{\dim}_B E$ and $b - \varepsilon = \overline{\dim}_\Phi E$ for some $\varepsilon > 0$. Given small r, choose R such that $r = R^{\beta(R)}$ where $\beta(R) = \max(1 + \Phi(R), 4)$. It is easy to see that

$$N_r(E) \le \sup_{z \in E} N_r\left(B(z, R) \cap E\right) \cdot N_R(E).$$

If R is small enough, then for some constant c,

$$N_r(E) \le c\left(\frac{R}{r}\right)^{b-\varepsilon/2} R^{-(b+\varepsilon)} = R^{-\beta(R)(b-\varepsilon/2+3\varepsilon/(2\beta(R))} \le c r^{-(b-\varepsilon/8)}.$$

This implies $\overline{\dim}_B E \le b - \varepsilon/8$, which is a contradiction.

The arguments are similar to show the lower box dimension is an upper bound on the lower Φ-dimensions, but using packing numbers instead of covering numbers, both for the lower Φ-dimensions and the lower box dimension.

(ii) If $\Phi(x) \to \infty$ as $x \to 0$, then $\Phi(x) > p$ for any p, provided x is sufficiently small. Thus the monotonicity of the Φ-dimensions implies that if $\theta = (1 + p)^{-1}$ and $\Psi(x) = p$, then $\overline{\dim}_B E \le \overline{\dim}_\Phi E \le \overline{\dim}_\Psi E = \overline{\dim}_A^\theta E$. The result for the upper dimension then follows since the upper θ-Assouad spectrum converges to $\overline{\dim}_B E$ as $p \to \infty$ ($\theta \to 0$), as a consequence of [10, Theorem 2.1] and [12, Proposition 3.1].

For the lower dimension case, suppose that $\underline{\dim}_\Phi E = d > 0$ and let $0 < \epsilon < d/2$ be given. Since $\Phi(x) \to \infty$, we have

$$\frac{R^{d/\epsilon-1}}{r} \ge \frac{R^{d/\epsilon-1}}{R^{1+\Phi(R)}} \longrightarrow \infty \implies \frac{\left(\frac{R}{r}\right)^{d-\epsilon}}{\left(\frac{1}{r}\right)^{d-2\epsilon}} \longrightarrow \infty$$

as $R \to 0$, provided $r \le R^{1+\Phi(R)}$. We note that for $r \le R^{1+\Phi(R)} \le \operatorname{diam}(E)$,

$$N_r(E) \ge N_r(B(z, R) \cap E) \ge C\left(\frac{R}{r}\right)^{d-\epsilon} \ge C r^{2\epsilon-d}$$

and thus $\underline{\dim}_B E \geq d - 2\epsilon$ for all such ϵ. This implies that $\underline{\dim}_B E \geq d$. Since we know that $\underline{\dim}_\Phi E \leq \underline{\dim}_B E$, we obtain equality. □

It is natural to ask when two dimension functions give rise to the same dimensions for all sets E.

Proposition 2.10. *If there is some constant c such that $\Phi(x) \leq c/|\log x|$, then* $\dim_L E = \underline{\dim}_\Phi E$ *and* $\dim_A E = \overline{\dim}_\Phi E$.

Proof. This is simply due to the fact that if $\Phi(x) \leq c/|\log x|$, then there are positive constants a, b such that $a < R^{\Phi(R)} < b$ for all $R \in (0,1)$. □

Proposition 2.11. (i) *Suppose $\Phi_1, \Phi_2 \in \mathcal{D}$ and $\Phi_1(x)/\Phi_2(x) \to 1$ as $x \to 0$. Then $\overline{\dim}_{\Phi_1} E = \overline{\dim}_{\Phi_2} E$ and $\underline{\dim}_{\Phi_1} E = \underline{\dim}_{\Phi_2} E$ for all sets E.*

(ii) *Assume $g\colon (0,1) \to \mathbb{R}^+$ is continuous at t_0 and $g(t_0) \neq 0$. Suppose $\Phi \in \mathcal{D}$ and put $\Phi_t(x) = g(t)\Phi(x)$. For any set E, $\overline{\dim}_{\Phi_t} E \to \overline{\dim}_{\Phi_{t_0}} E$ as $t \to t_0$. The same statement holds for the lower dimensions.*

From (i), we immediately deduce the following.

Corollary 2.12. *If $\Phi(x) \to \delta$ as $x \to 0$ for some $0 < \delta < \infty$, then for $\theta = (1+\delta)^{-1}$,*

$$\overline{\dim}_\Phi E = \overline{\dim}_A^\theta E \text{ and } \underline{\dim}_\Phi E = \underline{\dim}_L^\theta E.$$

The proof of the proposition is an easy consequence of the estimates in the Lemma below.

Lemma 2.13. *Let Φ and Ψ be dimension functions and assume that for some $\epsilon > 0$ there exists $x_0 > 0$ such that $|\Phi(x)/\Psi(x) - 1| \leq \epsilon$ for all $0 < x \leq x_0$.*

(i) *Choose c_1, c_2, depending on ϵ, such that*

$$N_r(B(z,R) \cap E) \leq c_2 \left(\frac{R}{r}\right)^{(\overline{\dim}_\Phi E + \epsilon)}$$

for all $z \in E$ and $r \leq R^{1+\Phi(R)} \leq R \leq c_1$. Then, there is a constant C such that for any $z \in E$ and $0 < r \leq R^{1+\Psi(R)} \leq R \leq c_1$ we have

$$N_r(B(z,R) \cap E) \leq C \left(\frac{R}{r}\right)^{(\overline{\dim}_\Phi E + \epsilon)(1+\epsilon)}. \tag{1}$$

(ii) *Analogously, chose* c_1, c_2 *such that*

$$N_r(B(z,R)\cap E) \geq c_2\Big(\frac{R}{r}\Big)^{(\overline{\dim}_\Psi E-\epsilon)}$$

for all $z\in E$ *and* $r\leq R^{1+\Psi(R)}\leq R\leq c_1$. *Then, there is some* $c>0$ *such that for any* $z\in E$ *and* $0<r\leq R^{1+\Phi(R)}\leq R\leq c_1$, *we have*

$$N_r(B(z,R)\cap E)\geq c\Big(\frac{R}{r}\Big)^{\underline{\dim}_\Psi E-\epsilon(1+\log_2 M)}, \tag{2}$$

where M *is the doubling constant of the space.*

Proof. (i) Let $d=\overline{\dim}_\Phi E$ and pick $0<r\leq R^{1+\Psi(R)}<R\leq c_1$. If $r\leq R^{1+\Phi(R)}$, then (1) follows by the definition of $\overline{\dim}_\Phi E$. Otherwise, $R^{1+\Phi(R)}<r\leq R^{1+\Psi(R)}$, hence $\Psi(R)<\Phi(R)$ and therefore $0<\Phi(R)-\Psi(R)\leq\epsilon\Psi(R)$. Consequently,

$$\begin{aligned}
N_r(B(z,R)\cap E) &\leq N_{R^{1+\Phi(R)}}(B(z,R)\cap E)\\
&\leq C\Big(\frac{R}{R^{1+\Phi(R)}}\Big)^{d+\epsilon}\\
&\leq C\Big(\frac{R}{r}\Big)^{d+\epsilon}R^{-(\Phi(R)-\Psi(R))(d+\epsilon)}\\
&\leq C\Big(\frac{R}{r}\Big)^{d+\epsilon}R^{-\epsilon\Psi(R)(d+\epsilon)}\\
&\leq C\Big(\frac{R}{r}\Big)^{(d+\epsilon)(1+\epsilon)}.
\end{aligned}$$

(ii) Now let $d=\underline{\dim}_\Psi E$ and pick $0<r\leq R^{1+\Phi(R)}<R\leq c_1$. Again, if $r\leq R^{1+\Psi(R)}$, then (2) follows by the definition of $\underline{\dim}_\Psi E$. Otherwise, $R^{1+\Psi(R)}<r\leq R^{1+\Phi(R)}$, hence $\Phi(R)<\Psi(R)$ so $\Psi(R)(1-\epsilon)\leq\Phi(R)$. Then,

$$N_r(B(z,R)\cap E)\geq N_{R^{1+\Phi(R)}}(B(z,R)\cap E)\geq N_{R^{1+(1-\epsilon)\Psi(R)}}(B(z,R)\cap E).$$

But $R^{1+(1-\epsilon)\Psi(R)}=R^{1+\Psi(R)}2^T$ with $T=\epsilon\Psi(R)\log_2 R^{-1}$, and since the metric space is doubling with doubling constant M, iterating this definition we have that each ball of radius $R^{1+(1-\epsilon)\Psi(R)}$ can be covered by at most $M^{\lceil T\rceil}$ balls of radius $R^{1+\Psi(R)}$. Therefore,

$$\begin{aligned}
N_{R^{1+(1-\epsilon)\Psi(R)}}(B(z,R)\cap E) &\geq M^{-\lceil T\rceil}N_{R^{1+\Psi(R)}}(B(z,R)\cap E)\\
&\geq cM^{-\lceil T\rceil}\Big(\frac{R}{R^{1+\Psi(R)}}\Big)^{d-\epsilon}\\
&= cR^{\epsilon\Psi(R)\log_2 M}R^{-\Psi(R)(d-\epsilon)}\\
&= cR^{-\Psi(R)(d-\epsilon(1+\log_2 M))}\\
&\geq c\Big(\frac{R}{r}\Big)^{d-\epsilon(1+\log_2 M)}.
\end{aligned}$$

Here the last inequality holds since $R^{1+\Psi(R)}<r$. □

Proof of Proposition 2.11. (i) The assumption $\Phi_1(x)/\Phi_2(x) \to 1$ as $x \to 0$ ensures that for each $\epsilon > 0$ there is some $c_0 = c_0(\epsilon)$ such that

$$\left|\frac{\Phi_1(R)}{\Phi_2(R)} - 1\right| \le \epsilon \quad \text{for } R \le c_0.$$

Thus (1) holds for all $R \le \min(c_0, c_1)$ (where c_1 was introduced in Lemma 2.13) and that implies $\overline{\dim}_{\Phi_2} E \le (\overline{\dim}_{\Phi_1} E + \epsilon)(1+\epsilon)$ for any $\epsilon > 0$. Hence $\overline{\dim}_{\Phi_2} E \le \overline{\dim}_{\Phi_1} E$. Since the roles of Φ_1 and Φ_2 can be interchanged because the condition is symmetric, the opposite inequality also holds.

The statement for the lower dimensions follows in the same way. Observe that by symmetry there is no need to consider separately the hypothetical case when one of the dimensions is zero.

(ii) Given $\epsilon > 0$ choose $\delta > 0$ such that

$$\max(|g(t_0)/g(t) - 1|, |g(t)/g(t_0) - 1|) \le \epsilon$$

for any $|t - t_0| < \delta$. For each such t, apply Lemma 2.13 to Φ_t and Φ_{t_0}. □

In Proposition 3.6 we will see that the convergence of (ii) need not hold if $g(t_0) = 0$.

Remark 2.14. In order to apply Lemma 2.13 to prove the "continuity" property of Prop. 2.11(ii) , it was necessary that the convergence of $\Phi_t(x)/\Phi_{t_0}(x)$ was uniform in x. For instance, Lemma 2.13 cannot be applied to families such as $\Phi_t(x) = |\log x|^{-t}$ with $t \in (0, 1]$. We do not know if there is a one-parameter family of dimension functions which range continuously from the quasi-Assouad to the Assouad dimensions.

The quasi-Assouad dimensions can also be understood as special cases of Φ-dimensions, but the functions Φ need to be tailored for the specific set.

Proposition 2.15. *For any $E \subseteq X$, there are dimension functions Φ_1, Φ_2 (depending on E), which tend to* 0 *and satisfying* $\dim_{qA} E = \overline{\dim}_{\Phi_1} E$ *and* $\dim_{qL} E = \underline{\dim}_{\Phi_2} E$.

Proof. Since our choice of dimension functions will satisfy $\Phi_i \to 0$, it will automatically be true that $\overline{\dim}_{\Phi_1} E \ge \dim_{qA} E$ and $\underline{\dim}_{\Phi_2} E \le \dim_{qL} E$. Thus we need only check the opposite inequalities.

First, consider the quasi-Assouad dimension. Put $d = \dim_{qA} E$. By definition, for each $n \in \mathbb{N}$, there are $\delta_n \downarrow 0$ and $\rho_n \downarrow 0$ such that for all $r \le R^{1+\delta_n} \le R \le \rho_n$ and $z \in E$, we have

$$N_r(B(z, R) \cap E) \le \left(\frac{R}{r}\right)^{d+1/n}.$$

Put $n_1 = 1$ and inductively define a subsequence $\{n_j\}$ so that $p_j^{1+\varepsilon_{j-1}} = p_{j+1}^{1+\varepsilon_j}$ where, for notational convenience, we put $p_j = \rho_{n_j}$ and $\varepsilon_j = \delta_{n_j}$. Since $\varepsilon_{j-1} > \varepsilon_j$, we also have $p_{j+1}^{1+\varepsilon_{j-1}} < p_{j+1}^{1+\varepsilon_j}$, hence there is some $q_j \in (p_{j+1}, p_j]$ with $q_j^{1+\varepsilon_{j-1}} = p_{j+1}^{1+\varepsilon_j}$.

We are now ready to define Φ_1. For $R \in (q_j, p_j]$ we put $\Phi_1(R) = \varepsilon_{j-1}$, while for $R \in (p_{j+1}, q_j]$ we define $\Phi_1(R)$ by the rule that $R^{1+\Phi_1(R)} = p_{j+1}^{1+\varepsilon_j}$. Observe that in either case, $\Phi_1(R) \geq \varepsilon_j$. It is straight forward to verify that the function $R^{1+\Phi(R)}$ decreases to 0 as R decreases to 0, and therefore Φ_1 is a dimension function.

If $R \in (p_{j+1}, q_j]$, then $R^{1+\varepsilon_{j-1}} \leq q_j^{1+\varepsilon_{j-1}} = R^{1+\Phi_1(R)}$ and thus $\Phi_1(R) \leq \varepsilon_{j-1}$. This shows that Φ_1 tends to 0.

We will check that $\overline{\dim}_{\Phi_1} E \leq d + 1/n_N$ for any given N. To do this, choose any $R \leq p_N$, $r \leq R^{1+\Phi_1(R)}$ and $z \in E$. Find $k \geq N$ so $R \in (p_{k+1}, p_k]$ so that $\Phi_1(R) \geq \varepsilon_k$. Thus $r \leq R^{1+\varepsilon_k} = R^{1+\delta_{n_k}}$ and consequently, since $n_k \leq n_N$,

$$N_r(B(z, R) \cap E) \leq \left(\frac{R}{r}\right)^{d+1/n_k} \leq \left(\frac{R}{r}\right)^{d+1/n_N},$$

completing the verification.

The argument for the quasi-lower Assouad dimension is the same, building Φ_2 using the fact that for each $n \in \mathbb{N}$, there are $\delta_n \downarrow 0$ and $\rho_n \downarrow 0$ such that for all $r \leq R^{1+\delta_n} \leq R \leq \rho_n$ and $z \in E$, we have

$$N_r(B(z, R) \cap E) \geq \left(\frac{R}{r}\right)^{d+1/n},$$

and the proposition follows. □

Although the lower θ-Assouad spectrum is bounded above by the lower box dimension, it is not always bounded above by the Hausdorff dimension. In fact, $\sup_\theta \underline{\dim}_L^\theta E$ is not even bounded above by a uniform multiple of the Hausdorff dimension. This is a consequence of [13, Theorem 3.3] and [5]; see particularly the comments in [13] following the statement of the theorem.

However, we have the following relationship between the lower θ-Assouad spectrum and the Hausdorff dimension that does not seem to have been previously observed. This follows easily from [14] where it was shown that $\dim_{qL} E \leq \dim_H E$ for closed subsets $E \subseteq \mathbb{R}^d$, but the same proof is valid for closed subsets in a doubling metric space.

Proposition 2.16. *If E is a closed subset of X, then $\underline{\dim}_L^\theta E \leq \frac{1}{\theta} \dim_H E$ for any $\theta \in (0, 1)$.*

Proof. Fix $\theta \in (0,1)$ and recall that $\underline{\dim}_{\Phi_\theta} E = \underline{\dim}_L^\theta E$ for $\Phi_\theta = 1/\theta - 1$. If $\underline{\dim}_{\Phi_\theta} E = 0$ there is nothing to prove, so assume $\alpha < \underline{\dim}_{\Phi_\theta} E$ for some $\alpha > 0$.

The doubling property implies that the covering numbers, N_r, can be replaced by the packing numbers, P_r, in the definition of the Φ-dimensions. Thus we can pick $\rho = \rho(\theta, \alpha) > 0$ such that for any $z \in E$ and any $r \leq 2R^{1/\theta} \leq R \leq \rho$,

$$P_r(B(z,R) \cap E) \geq 2^\alpha \left(\frac{R}{r}\right)^\alpha.$$

In particular, $P_{2R^{1/\theta}}(B(z,R) \cap E) \geq R^{(1-1/\theta)\alpha}$.

In [14, Proposition 10], it is shown that under this assumption, there is a probability measure μ, supported on E, and a constant A such that

$$\mu(U) \leq A(\operatorname{diam}(U))^{\alpha\theta}$$

for all Borel sets U. But then the mass distribution principle (see [6, Proposition 2.1]) implies $\theta\alpha \leq \dim_H E$. As this is true for all $\alpha < \underline{\dim}_{\Phi_\theta} E$, it must be that $\theta \underline{\dim}_{\Phi_\theta} E \leq \dim_H E$. □

Corollary 2.17. *If E is a closed set and $\Phi(x) \to \delta > 0$ as $x \to 0$, then $\underline{\dim}_\Phi E \leq (\delta + 1) \dim_H E$.*

2.3. Dimensions of decreasing sequences. In [14] and [15] it was shown that if $E = \{x_n\} \subseteq \mathbb{R}^+$ is a decreasing sequence with the sequence of "gaps", $\{x_n - x_{n+1}\}$, also decreasing, then both the upper Assouad and upper quasi-Assouad dimensions of E are either 0 or 1. The upper Assouad dimension of such a set is 0 if and only if the sequence of gaps is lacunary. Likewise, the upper quasi-Assouad dimension is 0 if and only if $\overline{\dim}_B E = 0$.

This dichotomy fails for the upper Φ-dimensions. Indeed, if we choose $\Phi(x) = \delta > 0$ for all x, it follows from [10] and [12, Theorem 6.2], that

$$\overline{\dim}_\Phi E = \min\{(1 + \delta^{-1})\overline{\dim}_B E, 1\}.$$

Therefore, $\overline{\dim}_\Phi E \in (0,1)$ if $0 < \overline{\dim}_B E < \delta/(1+\delta)$. However, in this case the upper Φ-dimension is bounded above by the upper quasi-Assouad dimension, and necessarily $\dim_{qA} E = \dim_A E = 1$.

More interestingly, the dichotomy fails also for upper Φ-dimensions that lie between the upper quasi-Assouad and upper Assouad dimensions. Indeed, we have the following.

Example 2.18. There is a decreasing set E, with decreasing gaps, and a dimension function $\Phi \to 0$, with $\dim_{qA} E = 0$ and $\dim_A E = 1$, but with $0 < \overline{\dim}_\Phi E < 1$.

Construction. Let $E = \{x_n\}_{n=1}^{\infty}$ where $x_n = n^{-\log n}$. Define Φ by the rule $x_n^{1+\Phi(x_n)} = 2(4n)^{-(1+\log 4n)}\log(4n)$ and extend Φ to $\mathbb{R}$ by setting $\Phi(x) = \Phi(x_n)$ if $x \in (x_{n+1}, x_n)$. We will verify that E and Φ have the stated properties. Of course, if $\Phi(x) \to 0$ as $x \to 0$, then we must have $\dim_{qA} E \leq \overline{\dim}_{\Phi} E \leq \dim_A E$ and thus the properties $\dim_{qA} E = 0$ and $\dim_A E = 1$ will follow once we have shown that Φ tends to 0 and $0 < \overline{\dim}_{\Phi} E < 1$.

The fact that E is a decreasing set follows from the fact that the function $f(z) = z^{-\log z}$ has negative derivative. Similarly, $x^{1+\Phi(x)}$ can be seen to be decreasing by checking the function $g(z) = z^{-(1+\log z)}\log z$ has negative derivative for large z. Thus Φ is a dimension function. One can directly calculate Φ and see that $\Phi(x_n) \sim 1/\log n$. That shows $\Phi(x) \to 0$ as $x \to 0$.

From the derivative of the function $h(x) = x^{-\log x} - (x+1)^{-\log(x+1)}$ one can also confirm that the sequence $\{x_n - x_{n+1}\}$ is decreasing. An application of the mean value theorem shows that $x_n - x_{n+1} = -f'(\xi_n)$ for some $\xi_n \in [n, n+1]$ and thus

$$2(n+1)^{-\log(n+1)}\log(n+1)/(n+1) \leq x_n - x_{n+1} \leq 2n^{-\log n}\log n/n.$$

This shows that if we take $R = x_k$ and put

$$r = R^{1+\Phi(R)} = 2(4k)^{-(1+\log 4k)}\log(4k),$$

then

$$x_{4k} - x_{4k+1} \leq r \leq x_{4k-1} - x_{4k}.$$

Because the gaps are decreasing in length, $x_i - x_{i+1} \geq r$ whenever $i = k, \ldots, 4k-1$, and $x_i - x_{i+1} \leq r$ whenever $i \geq 4k$. Consequently,

$$N_{r/2}(B(0,R) \cap E) = 3k + \frac{x_{4k}}{r} = 3k + \frac{(4k)^{-\log 4k}k}{2(4k)^{-\log 4k}\log 4k}$$

and hence for large enough k,

$$3k \leq N_{r/2}(B(0,R) \cap E) \leq 4k$$

Since

$$\frac{R}{r} = \frac{k^{1+2\log 4}}{2\log 4k}4^{1+\log 4},$$

we deduce that $\overline{\dim}_{\Phi} E \geq 1/(1+2\log 4)$.

A similar statement holds for any r with $x_{4k} - x_{4k+1} \leq r \leq x_{4k-1} - x_{4k}$. More generally, there are constants $c_1, c_2, c_3 > 0$ such that if

$$x_{4n} - x_{4n+1} \leq r \leq x_{4n-1} - x_{4n}$$

where $n = Lk + j$ with $0 \leq j < k$ and $L \geq 4$, then

$$N_{r/2}(B(0,R) \cap E) = n - k + \frac{x_n}{r} \leq c_1\Big(n - k + \frac{n}{\log n}\Big) \leq c_2 n,$$

and

$$\frac{R}{r} \geq c_3 \frac{n^{1+\log n}}{k^{\log k} \log n}.$$

Thus, for $t = (1 + \log 3)^{-1}$, we get

$$\Big(\frac{R}{r}\Big)^t \geq n\Big(c_3 \frac{n^{(\log n - \log 3)}}{k^{\log k} \log n}\Big)^t \geq n\Big(c_3 \frac{L^{\log Lk - \log 3} k^{\log L - \log 3}}{\log((L+1)k)}\Big)^t$$

and the last quotient is bounded away from 0.

If $R \in (x_{k+1}, x_k)$, then since $\Phi(R) = \Phi(x_k)$ we make a similar argument. Finally, we note that if $z > 0$, then the decreasingness of the gaps means

$$N_{r/2}\,(B(z,R) \cap E) \leq N_{r/2}(B(0,R)).$$

Thus $0 < 1/(1 + 2\log 4) \leq \overline{\dim}_\Phi E \leq 1/(1 + \log 3) < 1$. □

Remark 2.19. It is an open problem to characterize the dimension functions Φ for which the 0, 1 dichotomy holds.

3. Examples of Φ-dimensions

In this section we will construct various examples. These will show the sharpness of some of the basic properties, such as Proposition 2.11, as well as illustrating their distinctness. In particular, we will give an example of a set with specified values for a countable family of Φ-dimensions and whose set of all dimensions between quasi-Assouad and Assouad is an interval.

In all these examples, the set E will be a Cantor set, by which we mean a perfect subset of $[0, 1]$ of Lebesgue measure zero, that has a construction as outlined below. We begin this section by determining a formula for the Φ-dimension of Cantor sets. It will be convenient to make use of the following notation.

Notation 3.1. We write $f \sim g$ if there are positive constants c_1, c_2 such that $c_1 f(x) \leq g(x) \leq c_2 f(x)$ for all x. The symbols $\gtrsim$ and $\lesssim$ are defined similarly.

3.1. Φ-dimensions of Cantor sets. Given a decreasing, summable sequence, $a = \{a_j\}$ with $\sum_j a_j = 1$, by the *Cantor set associated with* a, denoted by C_a, we mean the compact subset of [0, 1] constructed as follows. In the first step, we remove from [0, 1] an open interval of length a_1, resulting in two closed intervals I_1^1 and I_2^1. Having constructed the k-th step, we obtain the closed intervals $I_1^k, \dots, I_{2^k}^k$ contained in $[0, 1]$. The intervals I_j^k, $j = 1, \dots, 2^k$, are called the Cantor intervals of step k. The next step consists in removing from each I_j^k an open interval of length a_{2^k+j-1}, obtaining the closed intervals I_{2j-1}^{k+1} and I_{2j}^{k+1}. We define

$$C_a := \bigcap_{k\geq 1} \bigcup_{j=1}^{2^k} I_j^k.$$

This construction uniquely determines the set because the lengths of the removed intervals on each side of a given gap are known. The classical middle-third Cantor set is the Cantor set associated with the sequence $\{a_i\}$ where $a_i = 3^{-n}$ if $2^{n-1} \leq i \leq 2^n - 1$. All associated Cantor sets are uncountable, compact, totally disconnected and, in fact, are all homeomorphic.

If we put

$$s_n = 2^{-n} \sum_{j\geq 2^n} a_j,$$

then s_n is the average length of the Cantor intervals of step n. The decreasing property of the sequence $\{a_j\}$ ensures that all the intervals of step n have lengths satisfying

$$s_{n+1} \leq \operatorname{length}(I_j^n) \leq s_{n-1}$$

and that $s_n \geq a_{2^{n+1}}$. Of course, always $s_{n+1} \leq s_n/2$.

When the gap sizes $a_{2^n} = \cdots = a_{2^{n+1}-1}$ for all n, the intervals at step n all have the same length (namely s_n), and the Cantor set is sometimes called a central Cantor set. The classical middle-third Cantor set is such an example. In this case, the ratio s_{j+1}/s_j is referred to as the ratio of dissection at step (or level) j.

We will assume the sequence $\{a_j\}$ is *doubling*, meaning there is a constant κ such that $a_n \leq \kappa a_{2n}$ for all n. This ensures that

$$\tau = \inf s_{n+1}/s_n > 0$$

since

$$s_n \leq \frac{1}{2^n}\Big(\sum_{j=2^{n+1}}^{\infty} a_j + a_{2^n} 2^n \Big) \leq 2s_{n+1} + \kappa^2 a_{2^{n+2}} \leq (2+\kappa^2) s_{n+1}.$$

Thus under the doubling assumption we have $s_j \sim s_{j+1}$. It is easily seen that such a Cantor set is uniformly perfect, where we recall that a set E is called uniformly perfect if there is a constant $c > 0$ so that for every $z \in E$ and $r > 0$ we have $B(z,r) \setminus B(z,cr) \neq \emptyset$ whenever $E \setminus B(z,r) \neq \emptyset$. A set E is uniformly perfect if and only if $\underline{\dim}_L E > 0$ [21], and consequently, $\underline{\dim}_\Phi C_a > 0$ whenever a is a doubling sequence.

For Cantor sets, it is helpful to understand the comparison $r \leq R^{1+\Phi(R)}$ in terms of the sequence $\{s_n\}$. For this we introduce the following notation.

Notation 3.2. Given $\Phi \in \mathcal{D}$ and a doubling, decreasing, summable sequence $a = \{a_j\}$, define the associated *depth function* $\phi\colon \mathbb{N} \to \mathbb{N}$ by the rule that $\phi(n)$ is the minimal integer j such that $s_{n+j} \leq s_n^{1+\Phi(s_n)}$.

In other words, $\phi(n)$ is the minimal integer with $s_{n+\phi(n)}/s_n \leq s_n^{\Phi(s_n)}$. We remark that ϕ depends on both Φ and the underlying Cantor set (or, equivalently, the sequence $\{a_j\}$). We will frequently refer to Φ/ϕ as a *dimension/depth function pair* associated with the Cantor set.

If ϕ is bounded, then the sequence $\{s_n^{\Phi(s_n)}\}$ is bounded away from 0. The decreasingness of the function $R^{1+\Phi(R)}$ implies that if $s_n \leq R \leq s_{n-1}$, then

$$\tau s_n^{\Phi(s_n)} \leq R^{\Phi(R)} \leq \frac{1}{\tau} s_{n-1}^{\Phi(s_{n-1})}.$$

Hence if ϕ is bounded, then Proposition 2.10 implies the upper (or lower) Φ-dimension coincides with the upper (resp., lower) Assouad dimension.

A very useful observation for constructing examples is to note that if E is *any* Cantor set with $\tau = \inf s_{j+1}/s_j$ and $\rho = \sup s_{j+1}/s_j \leq 1/2$, and Φ is a dimension function with associated depth function ϕ with respect to E, then we have

$$\frac{(\phi(n)-1)}{n} \frac{\log \rho}{\log \tau} \leq \Phi(s_n) \leq \frac{\phi(n)}{n} \frac{\log \tau}{\log \rho}. \tag{3}$$

This is because the doubling property ensures

$$\tau^{\phi(n)} \leq \frac{s_{n+\phi(n)}}{s_n} \leq s_n^{\Phi(s_n)} \leq \rho^{n\Phi(s_n)}$$

and

$$\tau^{n\Phi(s_n)} \leq s_n^{\Phi(s_n)} \leq \frac{s_{n+\phi(n)-1}}{s_n} \leq \rho^{\phi(n)-1}.$$

If, in addition, $\phi(n) \geq 2$ (as is typically the case in interesting examples), then we see that $\phi(n)$ is comparable to $n\Phi(s_n)$ with constants depending only on τ, ρ because

$$\frac{\phi(n)}{n}\frac{\log\rho}{2\log\tau} \le \Phi(s_n) \le \frac{\phi(n)}{n}\frac{\log\tau}{\log\rho} \tag{4}$$

Remark 3.3. Notice that if we are given an increasing function $\phi\colon \mathbb{N} \to \mathbb{N}$, and a Cantor set C_a, we can define a function Φ by the rule $R^{1+\Phi(R)} = s_{n+\phi(n)}$ if $R \in (s_{n+1}, s_n]$. If $R_1 \le R_2$ with $R_1 \in (s_{n+1}, s_n]$ and $R_2 \in (s_{k+1}, s_k]$, then $n \ge k$, so $\phi(n) \ge \phi(k)$ and hence $s_{n+\phi(n)} \le s_{k+\phi(k)}$. Consequently, $R_1^{1+\Phi(R_1)} = s_{n+\phi(n)} \le s_{k+\phi(k)} \le R_2^{1+\Phi(R_2)}$. Furthermore, $R^{1+\Phi(R)} = s_{n+\phi(n)} \to 0$ as $n \to \infty$ and hence as $R \to 0$. Thus Φ is a dimension function with associated depth function ϕ.

Corollary 3.4. (i) *If* $\phi(n)/n \to \infty$, *then* $\overline{\dim}_\Phi C_a = \overline{\dim}_B C_a$ *and* $\underline{\dim}_\Phi C_a = \underline{\dim}_B C_a$.

(ii) *The quasi-Assouad dimensions are obtained by taking* $\phi(n) = \delta n$ *and letting* $\delta \to 0$.

Proof. These follow from the fact that $\phi(n)/n \sim \Phi(s_n)$. The statement in (i) about the lower Φ dimension follows from Proposition 2.8 (ii), since $\underline{\dim}_\Phi C_a > 0$ because C_a is uniformly perfect. □

More generally, we have the following formulas for the Φ-dimensions of Cantor sets.

Theorem 3.5. *Let* a *be a decreasing, summable, doubling sequence and* C_a *the associated Cantor set. The upper and lower* Φ*-dimensions of* C_a *are given by*

$$\overline{\dim}_\Phi C_a = \inf\left\{\beta\colon (\exists\, k_0, c_0 > 0)\ (\forall k \ge k_0,\, n \ge \phi(k))\left(\frac{s_k}{s_{k+n}}\right)^{\beta} \ge c_0 2^n\right\} \tag{5}$$

and

$$\underline{\dim}_\Phi C_a = \sup\left\{\beta\colon (\exists\, k_0, c_0 > 0)\ (\forall k \ge k_0,\, n \ge \phi(k))\left(\frac{s_k}{s_{k+n}}\right)^{\beta} \le c_0 2^n\right\}. \tag{6}$$

The proof is omitted as the arguments are similar to those given in [15] for Assouad dimensions and in [5] and [26] for the quasi-Assouad dimensions.

3.2. Basic properties revisited.

With the formulas for the Φ-dimensions of Cantor sets, it is easy to give examples of sets with any specified Φ-dimension in $(0, 1)$. The key idea is that if E is a central Cantor set with ratios of dissection r_k at step k, and there is an increasing sequence of integers $\{n_j\}$ (possibly even very sparse) such that $r_k = \rho$ for all $k = n_j + 1, \ldots, n_j + \phi(n_j)$ and $r_k = \tau \leq \rho$ otherwise, then $\overline{\dim}_\Phi E = \log 2 / |\log \rho|$, where Φ/ϕ is a dimension/depth function pair associated with E. A similar idea can be applied for the lower Φ-dimension.

In this subsection we will use this principle to obtain (partial) converses to Proposition 2.11. First, we will show that the continuity properties described in Proposition 2.11(ii) can fail when $g(t_0) = 0$.

Proposition 3.6. *Suppose ϕ is an increasing depth function tending to infinity, but with $\phi(n)/n \to 0$ as $n \to \infty$. There is a central Cantor set E such that if Φ is the dimension function associated with the depth function ϕ (and Cantor set E) and $\Phi_t = t\Phi$, then $\dim_{\Phi_t} E \in [\dim_{qA} E, \dim_A E]$, but $\lim_{t\to 0} \overline{\dim}_{\Phi_t} E < \dim_A E$.*

Proof. Choose $A, B > 0$ from (3) such that if E is a Cantor set with $\inf s_{j+1}/s_j \geq 1/27$ and Ψ/ψ is any dimension/depth function pair associated with E, then

$$A\frac{\psi(n) - 1}{n} \leq \Psi(s_n) \leq B\frac{\psi(n)}{n} \quad \text{for all } n.$$

In particular, this holds for the depth function ϕ and any associated dimension function Φ, and also for the depth function ϕ_t associated with $t\Phi$. Without loss of generality we can assume $\phi(n) \geq 2$ for all n and therefore for all $k \in \mathbb{N}$

$$A\frac{\phi(n)}{2kn} \leq \frac{1}{k}\Phi(s_n) = \Phi_{1/k}(s_n) \leq B\frac{\phi_{1/k}(n)}{n}.$$

That shows that for each k there is some N_k such that if $n \geq N_k$, then $\phi_{1/k}(n) \geq 2$. Thus we also have $A\phi_{1/k}(n)/(2n) \leq \Phi_{1/k}(s_n)$ for all $n \geq N_k$ and therefore with the constant $C = A/(2B)$ (and any such Cantor set E) we have

$$\frac{C}{k}\phi(n) \leq \phi_{1/k}(n) \leq \frac{1}{Ck}\phi(n) \quad \text{for all } k \text{ and } n \geq N_k.$$

To construct the Cantor set E, we will first choose an integer-valued function $f(n) \to \infty$ with $f(n)/\phi(n) \to 0$ as $n \to \infty$. Then choose an increasing sequence of integers $\{n_k\}$ with $n_k \geq \max(2N_k, 8n_{k-1})$ and satisfying

$$f(n_k) \leq \min\left(\frac{n_k}{8}, \frac{C}{2k}\phi(n_k - f(n_k))\right).$$

The Cantor set will be defined by setting the ratios of dissection to be $1/3$ on steps $n_j + 1, \ldots, n_j + f(n_j)$ for all $j = 1, 2, \ldots$ and equal to $1/27$ on all other levels. Certainly, $\dim_A E = \log 2/\log 3$.

Let r_i denote the ratio of dissection at step i. Our choice of n_j ensures that if $n \in \{n_j - f(n_j) + 1, \ldots, n_j + f(n_j)\}$ and $m \geq 2f(n_j)$, then at least as many

$r_i = 1/27$ as are equal to $1/3$ for i ranging over $\{n+1, \dots, n+m\}$. Hence the geometric mean of these ratios is at most $1/9$. The same conclusion clearly also holds if $n \notin \{n_j - f(n_j) + 1, \dots, n_j + f(n_j)\}$.

In order to bound $\overline{\dim}_{\Phi_{1/k}} E$ we use formula (5), noting first that it suffices to consider $(s_n/s_{n+m})^{1/m}$ where $n \geq n_k$ and $m \geq \phi_{1/k}(n)$. If $n \in \{n_j - f(n_j) + 1, \dots, n_j + f(n_j)\}$ for some $j \geq k$, then as ϕ is increasing and $n \geq N_k$,

$$m \geq \phi_{1/k}(n) \geq \frac{C}{k}\phi(n) \geq \frac{C}{k}\phi(n_j - f(n_j)) \geq \frac{C}{j}\phi(n_j - f(n_j)) > 2f(n_j).$$

By our previous remark, $(s_{n+m}/s_n)^{1/m} \leq 1/9$. The same bound clearly holds if $n \geq n_k$ does not belong to any such interval. Consequently, (5) implies $\overline{\dim}_{\Phi_{1/k}} E \leq \log 2/\log 9$. By monotonicity, $\overline{\dim}_{\Phi_t} E \leq \log 2/\log 9$ for all $t > 0$. □

Remark 3.7. We remark that a similar argument could be used to prove that there is a central Cantor set E and dimension function Φ so that $\lim_{t\to 0} \underline{\dim}_{\Phi_t} E > \dim_L E$. One could also similarly arrange for $\overline{\dim}_{\Phi_t} E \in [\dim_{qA} E, \dim_A E]$, but $\lim_{t\to\infty} \overline{\dim}_{\Phi_t} E > \dim_{qA} E$ and likewise for the quasi-lower Assouad dimension.

We will use a similar technique to obtain a partial converse to Proposition 2.11.

Theorem 3.8. *Suppose Φ_1, Φ_2 are dimension functions decreasing to 0 as $x \to 0$ with $|\log x|\,\Phi_2(x) \to \infty$ as $x \to 0$. Assume there is some $\xi > 0$ such that $\Phi_1(x) \geq (1+\xi)\Phi_2(x)$ for all x sufficiently small. Then there is a Cantor set E such that $\overline{\dim}_{\Phi_1} E < \overline{\dim}_{\Phi_2} E$ and a Cantor set F with $\underline{\dim}_{\Phi_1} F > \underline{\dim}_{\Phi_2} F$.*

Proof. We will give the proof for the upper Φ-dimension. The lower Φ-dimension case is similar.

The monotonicity property of the Φ-dimensions implies that $\overline{\dim}_{\Phi_1} E \leq \overline{\dim}_{\Phi_2} E$ for all sets E. It is the strictness of the inequality that we need to verify for an appropriate choice of E.

The strategy of the proof will be to build a central Cantor set by inductively specifying the ratios of dissection at each level. For most levels, the ratio will be a fixed small number, say τ. However, we will specify the ratios to be a fixed number $\rho > \tau$ on the levels $n_j + 1, \dots, n_j + \phi_2(n_j)$, where ϕ_2 is the depth function associated with Φ_2 and the Cantor set, and $\{n_j\}$ is a sparse set. By consideration of $(s_{n_j+\phi_2(n_j)}/s_{n_j})^{1/\phi_2(n_j)}$ (the geometric mean of the ratios at levels $n_j + 1, \dots, n_j + \phi_2(n_j)$) and the formula for the Φ-dimensions of Cantor sets from (5), we have $\overline{\dim}_{\Phi_2} E = \log 2/|\log \rho|$. However, these depths will be too shallow to give the Φ_1-dimension and consequently we will be able to conclude that $\overline{\dim}_{\Phi_1} E < \log 2/|\log \rho|$.

One complication with this strategy is that the depth functions depend on the construction of the Cantor set. However, our construction of the Cantor set depends (at least, to some extent) on the depth functions. Fortunately, we do have enough control on the depth functions to overcome this complication. We address this issue first.

Fix small $\varepsilon > 0$ such that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 (1+\xi) \geq (1+\xi/2).$$

Choose $0 < \tau < \rho < 1/2$ with $|\log \tau / \log \rho| \leq 1+\varepsilon$. It follows from (3) that if E is any Cantor set with all ratios between τ and ρ, and Φ/ϕ any dimension/depth function pair associated with E, then

$$\frac{(\phi(n)-1)}{n}\frac{\log \rho}{\log \tau} \leq \Phi(s_n) \leq \frac{\phi(n)}{n}\frac{\log \tau}{\log \rho} \leq (1+\varepsilon)\frac{\phi(n)}{n}. \tag{7}$$

By assumption, given any $C > 0$, there is some $x_0 = x_0(C)$ such that if $x \leq x_0$, then $|\log x|\, \Phi_i(x) \geq C$. Choose N_0 such that $\tau^{N_0} \leq x_0$. Since the functions Φ_i are decreasing as $x \to 0$, it follows that if $n \geq N_0$ and E is a Cantor set with all ratios of dissection at least τ, then $\Phi_i(s_n) \geq \Phi_i(\tau^n) \geq C/\,|\log \tau^n|$ and hence $n\Phi_i(s_n) \geq (1+\varepsilon)/\varepsilon$ if we take a suitable choice for C, depending on ε and τ. Coupled with the right hand side of (7), this shows that for all $n \geq N_0$,

$$\phi_i(n) \geq \frac{n\Phi_i(s_n)}{1+\varepsilon} \geq \frac{1}{\varepsilon}$$

and hence $\phi_i(n) - 1 \geq (1-\varepsilon)\phi_i(n)$ for $i = 1, 2$. Consequently, using the left hand side of (7) we also have

$$\frac{\phi_i(n)}{n}\frac{(1-\varepsilon)}{(1+\varepsilon)} \leq \Phi_i(s_n) \quad \text{for all } n \geq N_0.$$

As $\Phi_i \downarrow 0$, this further ensures that there exists N_1 such that

$$\phi_i(n) \leq \varepsilon n \quad \text{for } n \geq N_1.$$

We remind the reader that having fixed ε, τ and ρ, these inequalities and the choices of N_0 and N_1 depend only Φ_1 and Φ_2 for *any* choice of Cantor set, provided the ratios of dissection are chosen from $[\tau, \rho]$. As we will see, these relationships give us enough control on the depth functions.

Construction of the Cantor set. We will continue to use the notation from above. Let $n_1 \geq \max(8N_0, 8N_1)$ and choose $n_{j+1} \geq 16n_j$. We will inductively define a central Cantor set by specifying the ratios of dissection r_k at each level k. To begin, we put $r_k = \tau$ for $k = 1, \dots, n_1$. Thus $s_{n_1} = \tau^{n_1}$. Define ℓ_1 to be the least integer with $\rho^{\ell_1} \leq s_{n_1}^{\Phi_2(s_{n_1})}$ and let $r_k = \rho$ for $k = n_1 + 1, \dots, n_1 + \ell_1$. Notice that this construction means $\ell_1 = \phi_2(n_1) \leq \varepsilon n_1$, thus $n_1 + \ell_1 < n_2$. We put $r_k = \tau$ for $k = n_1 + \ell_1 + 1, \dots, n_2$.

Now we proceed inductively. We assume integers $\ell_1, \dots, \ell_{j-1}$ have been chosen in the same way and we have put $r_k = \rho$ if $k = n_i + 1, \dots, n_i + \ell_i$ for $i = 1, \dots, j-1$, and $r_k = \tau$ otherwise on $\{1, \dots, n_j\}$. Thus s_{n_j} is determined. Define ℓ_j to be the least integer satisfying $\rho^{\ell_j} \leq s_{n_j}^{\Phi_2(s_{n_j})}$. We will put $r_k = \rho$ if $k = n_j + 1, \dots, n_j + \ell_j$ and $r_k = \tau$ on $\{n_j + \ell_j + 1, \dots, n_{j+1}\}$. Again $\ell_j = \phi_2(n_j) \leq \varepsilon n_j$. This completes the construction of E.

Verification of the Φ_i-dimensions. The fact that the ratios equal ρ on the consecutive levels $n_j + 1, \dots, n_j + \phi_2(n_j)$ for all j and are equal to τ otherwise, certainly means $\overline{\dim}_{\Phi_2} E = \log 2 / |\log \rho|$.

Since $\phi_i(n_j) \leq \varepsilon n_j$ and Φ_1 is decreasing, the choice of ε gives that for each j and $n \in \{n_j - \ell_j, \dots, n_j\}$,

$$\begin{aligned}
\phi_1(n) &\geq \frac{n}{1+\varepsilon}\Phi_1(s_n) \\
&\geq \frac{n_j - \ell_j}{1+\varepsilon}\Phi_1(s_{n_j}) \\
&\geq \frac{(1-\varepsilon)n_j}{1+\varepsilon}\Phi_1(s_{n_j}) \\
&\geq \frac{1-\varepsilon}{1+\varepsilon}(1+\xi)n_j\Phi_2(s_{n_j}) \\
&\geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2(1+\xi)\phi_2(n_j) \\
&\geq (1+\xi/2)\phi_2(n_j).
\end{aligned} \tag{8}$$

Since the sequence $\{s_n^{1+\Phi(s_n)}\}$ is decreasing (for any dimension function Φ), for any $n, m \geq 1$ and associated depth function ϕ we have

$$s_{n+m+\phi(n+m)} \leq s_{n+m}^{1+\Phi(s_{n+m})} \leq s_n^{1+\Phi(s_n)} < s_{n+\phi(n)-1}$$

by the definition of ϕ. That means $n + m + \phi(n+m) > n + \phi(n) - 1$, and as these are integers this implies, in particular, that for $i = 1, 2$,

$$n_j + m + \phi_i(n_j + m) \geq n_j + \phi_i(n_j) \tag{9}$$

for all $m \geq 1$. As (8) holds for $n = n_j$, this gives

$$\begin{aligned} n_j + m + \phi_1(n_j + m) - (n_j + \phi_2(n_j)) &\geq n_j + \phi_1(n_j) - (n_j + \phi_2(n_j)) \\ &\geq (\xi/2)\phi_2(n_j). \end{aligned} \tag{10}$$

Since $\phi_i(n) \leq \varepsilon n$ for all $n \geq N_1$ and $\ell_j = \phi_2(n_j)$, we also know that

$$\begin{aligned} n_j + \phi_2(n_j) + \max_{n \in [n_j - \ell_j, n_j + \ell_j]} \phi_1(n) &\leq n_j + \varepsilon n_j + \varepsilon(n_j + \ell_j) \\ &\leq (1+\varepsilon)^2 n_j \\ &\leq n_{j+1}/4 \\ &< (n_{j+1} - \phi_2(n_{j+1}))/2. \end{aligned}$$

In particular, this guarantees that if $n \in \{n_j - \ell_j + 1, \dots, n_j + \ell_j\}$, then $n + \phi_1(n) < (n_{j+1} - \phi_2(n_{j+1}))/2$. Together with (10), it follows that for such n there are at least $(\xi/2)\,\phi_2(n_j)$ ratios equal to τ and at most $\phi_2(n_j) = \ell_j$ ratios equal to ρ on the levels $n+1, \dots, n+\phi_1(n)$. Hence the geometric mean of these ratios is dominated by

$$(\rho^{\ell_j} \tau^{\xi \ell_j/2})^{1/((1+\xi/2)\ell_j)} = \rho^{1/(1+\xi/2)} \tau^{\xi/(2+\xi)} := \sigma < \rho.$$

If $m \geq \phi_1(n)$, the choice of ratios ensures that there could only be an even greater proportion of the ratios on the levels $n+1, \dots, n+m$ having value τ. Thus we can conclude that the geometric mean of the ratios from the levels $n+1, \dots, n+m$ is also dominated by σ whenever $m \geq \phi_1(n)$ and $n \in \{n_j - \ell_j + 1, \dots, n_j + \ell_j\}$.

If $n \notin \{n_j - \ell_j + 1, \dots, n_j + \ell_j\}$ for any j, then it is obvious from the construction that, on the levels $n+1, \dots, n+m$ (for any $m \geq 1$), there are at least as many ratios equal to τ as equal to ρ, and hence the geometric mean is even smaller.

We deduce that

$$\overline{\dim}_{\Phi_1} E \leq \frac{\log 2}{|\log \sigma|} < \frac{\log 2}{|\log \rho|} = \overline{\dim}_{\Phi_2} E,$$

which concludes the proof. □

A modification of this argument would allow us to show that given $0 < a < b < 1/2$ there is an example of a Cantor set E where

$$\overline{\dim}_{\Phi_1} E = \frac{\log 2}{|\log a|} < \frac{\log 2}{|\log b|} = \overline{\dim}_{\Phi_2} E.$$

To do this, we will choose $0 < c < a$. Then, instead of assigning ratio ρ on the levels $n_j + 1, \dots, n_j + \phi_2(n_j)$ and τ otherwise, we will put ratios b on levels

$n_{2j}+1,\ldots,n_{2j}+\phi_2(n_{2j})$, ratios a on levels $n_{2j+1}+1,\ldots,n_{2j+1}+\phi_1(n_{2j+1})$ and ratio c elsewhere. The choice of sequence $\{n_j\}$ may need to be even more sparse to ensure that $\phi_1(n_j)$ is sufficiently large in comparison with $\phi_2(n_j)$ to guarantee that the geometric mean of ratios from any $\phi_1(n)$ consecutive levels beginning at n is at most a. The fact that the ratios at levels $n_{2j+1}+1,\ldots,n_{2j+1}+\phi_1(n_{2j})$ are equal to a implies that $\overline{\dim}_{\Phi_1} E = \frac{\log 2}{|\log a|}$. From their values on levels $n_{2j}+1$, $\ldots, n_{2j}+\phi_2(n_{2j})$ one can deduce that $\overline{\dim}_{\Phi_2} E = \frac{\log 2}{|\log b|}$. The details are left for the reader.

A further modification of the argument would also enable us to construct a (single) Cantor set E with both $\overline{\dim}_{\Phi_1} E < \overline{\dim}_{\Phi_2} E$ and $\underline{\dim}_{\Phi_1} E > \underline{\dim}_{\Phi_2} E$.

3.3. Continuum of Φ-dimensions. In the next result we use the method described in the previous remark to show that we can construct a Cantor set with countably many specified values for Φ-dimensions. Furthermore, there is a Cantor set with a continuum of Φ-dimensions between the quasi-Assouad and Assouad dimensions.

Theorem 3.9. *Assume that for each $p \in (0,1)$, Φ_p are dimension functions decreasing to 0 as $x \to 0$ and satisfying $|\log x|\Phi_p(x) \to \infty$ as $x \to 0$. Assume, also, that*

$$\Phi_p(x)/\Phi_q(x) \longrightarrow \infty \quad \textit{as } x \to 0 \textit{ whenever } p > q.$$

Choose any $0 < \alpha < \beta < 1$ and suppose $d\colon (0,1) \to [\alpha,\beta]$ is monotonically decreasing and continuous. Then there is a central Cantor set E with

$$\overline{\dim}_{\Phi_p} E = d(p) \quad \textit{for each } p \in (0,1).$$

The analogous result holds for the lower Φ-dimensions.

Remark 3.10. An example of a class of dimension functions that satisfy the conditions of the theorem are the functions $\Phi_p(x) = |\log x|^{p-1}$.

Proof. We will construct a central Cantor set E with the property that if the map $f\colon (0,1)\bigcap \mathbb{Q} \to [a,b]$ is monotonically decreasing, then $\overline{\dim}_{\Phi_p} E = \log 2/|\log f(p)|$ for every rational $p \in (0,1)$. To obtain the theorem, put $a = 2^{-1/\alpha}$, $b = 2^{-1/\beta}$ and define the decreasing continuous function $f\colon (0,1) \to [a,b]$ by $f(x) = 2^{-1/d(x)}$. The proof follows from this property using the monotonicity of the functions $p \to \overline{\dim}_{\Phi_p} E$ and the fact that the function d of the theorem is assumed to be continuous and decreasing.

As in the proof of the previous theorem our strategy will be to inductively define the ratios of dissection of the Cantor set. These ratios will lie in $[a^2, b]$ and so by (3), with $c = \log b/(2\log a)$ we have

$$c(\phi(n) - 1) \leq n\Phi(s_n) \leq \frac{1}{c}\phi(n) \quad \text{for all } n,$$

for any dimension function Φ and corresponding depth function ϕ associated with such a Cantor set.

Since $|\log x|\,\Phi_p(x) \to \infty$ for each p, there is a choice of $I_p \in \mathbb{N}$ such that if $n \geq I_p$ and $x \leq a^{2n}$, then $\Phi_p(x) \geq C/|\log x|$ for a suitable constant C. Consequently, as $s_n \geq a^{2n}$, we will have $\phi_p(n) \geq cn\Phi_p(a^{2n}) \geq 2$ for all $n \geq I_p$, (whatever the choice of E, as long as the ratios lie between a^2 and b). Thus with $A = 2/c$ and $B = c$,

$$Bn\Phi_p(s_n) \leq \phi_p(n) \leq An\Phi_p(s_n) \quad \text{for all } n \geq I_p. \tag{11}$$

As Φ_p decreases to 0, there is also an index $J_p \in \mathbb{N}$ such that

$$\Phi_p(b^n) \leq 1/(8A) \quad \text{for all } n \geq J_p. \tag{12}$$

As in the proof of Theorem 3.8, we will pick a sparse sequence $\{n_j\}$ and assign ratios a^2 except on the levels $\{n_j + 1, \ldots, n_j + \phi_{r_j}(n_j)\}$ where the ratios will be $f(r_j)$. Each p must occur as an r_j infinitely often so that we will have $\overline{\dim}_{\Phi_p} E \geq \log 2/|\log f(p)|$. The numbers n_j will need to be sufficiently sparse so that if $q > p$, this length of levels (where the ratio exceeds $f(q)$) is too short to influence the $\overline{\dim}_{\Phi_q} E$ calculation.

Construction of the Cantor set. To begin, we list $(0, 1) \cap \mathbb{Q}$ as $\{r_i\}_{i=1}^{\infty}$ where each rational number is repeated infinitely often in $\{r_i\}$. To start the construction of E, pick $n_1 \geq \max(I_{r_1}, 8J_{r_1})$. We will set the ratios of dissection to be a^2 on the levels $\{1, \ldots, n_1\}$. Choose the minimal integer ℓ_1 such that $f(r_1)^{\ell_1} \leq s_{n_1}^{\Phi_{r_1}(s_{n_1})}$ and put $m_1 = 4(n_1 + \ell_1)$. Set the ratios equal to $f(r_1)$ on the levels $\{n_1 + 1, \ldots, n_1 + \ell_1\}$ and a^2 on the levels $\{n_1 + \ell_1 + 1, \ldots, m_1\}$.

Notice that $\ell_1 = \phi_{r_1}(n_1)$ and the choice of n_1 ensures that

$$\ell_1 \leq An_1\Phi_{r_1}(s_{n_1}) \leq An_1\Phi_{r_1}(b^{n_1}) \leq n_1/8$$

by (11) and (12).

We proceed inductively and suppose we have chosen n_i, ℓ_i, m_i for $i = 1, \ldots, j - 1$, (with the properties described below) and have specified that the ratios of dissection on levels $\{1,\dots,m_{j-1}\}$ should be a^2 except on the levels $\{n_i+1,\dots,n_i+\ell_i\}$, for $i=1,\dots,j-1$, when they will be $f(r_i)$.

Now pick n_j large enough to satisfy the following conditions:

(i) $n_j \ge 8\max(I_{r_j}, J_{r_j}, m_{j-1}\}$;

(ii) if $i<j$ and $r_i>r_j$, then

$$\Phi_{r_i}(s_{m_{j-1}}a^{2(n_j-m_{j-1})}) \ge \frac{8A}{B}\Phi_{r_j}(s_{m_{j-1}}a^{2(n_j-m_{j-1})}),$$

which can be done since $\Phi_{r_i}(x)/\Phi_{r_j}(x)\to\infty$ as $x\to 0$.

We will assign ratio a^2 on levels $\{m_{j-1}+1,\dots,n_j\}$, so $s_{m_{j-1}}a^{2(n_j-m_{j-1})} = s_{n_j}$ and that means (ii) actually says

$$\Phi_{r_i}(s_{n_j}) \ge \frac{8A}{B}\Phi_{r_j}(s_{n_j}) \quad \text{whenever } i<j \text{ and } r_i>r_j. \tag{13}$$

Choose the minimal integer ℓ_j such that $f(r_j)^{\ell_j} \le S_{n_j}^{\Phi_{r_j}(n_j)}$, put $m_j = 4(n_j+\ell_j)$ and assign the ratios on levels $\{n_j+1,\dots,n_j+\ell_j\}$ to be $f(r_j)$ and the ratios on the levels $\{n_j+\ell_j+1,..,m_j\}$ to be a^2.

Note that $\ell_j = \phi_{r_j}(n_j)$ and property (i) in the definition of n_j, together with (11) and (12), ensures $\ell_j \le n_j/8$. In particular, $n_j-\ell_j \ge 7/8n_j \ge 7m_{j-1}$ and $n_j+\ell_j = m_j/4$.

This completes the construction of E.

Verification of the Φ-dimensions. We now need to verify that we obtain the desired value for each $\overline{\dim}_{\Phi_q} E$. We can easily see that $\overline{\dim}_{\Phi_q} E \ge \log 2/|\log f(q)|$ by noting that

$$\left(\frac{s_{n_j+\phi_{r_j}(n_j)}}{s_{n_j}}\right)^{1/\phi_{r_j}(n_j)} = f(r_j)$$

for the infinitely many choices of $r_j = q$. So we only need to prove the other inequality.

Assume the first occurrence of q in $\{r_i\}$ is with $i=j_0$. It will be enough to show that $(s_{k+m}/s_k)^{1/m} \le f(q)$ whenever $k \ge n_{j_0}$ and $m \ge \phi_q(k)$. In other words, we want to prove that the geometric mean of the ratios $r_{k+1},\dots,r_{k+m}$ is at most $f(q)$ for all $m\ge\phi_q(k)$ and $k\ge n_{j_0}$. A key point to observe is that the geometric mean of any collection of ratios where there are at least as many ratios equal to a^2 as otherwise, is at most $a \le f(q)$ for any q.

Given $k \ge N_{j_0}$, choose $j \ge j_0$ such that $k\in\{m_{j-1}+1,\dots,m_j\} := B_j$. If either $k \le n_j-\ell_j$ or $k > n_j+\ell_j$, then this is the situation with respect to the ratios $r_{k+1}, \ldots, r_{k+m}$ (regardless of the size of m), so the geometric mean is suitably small.

Thus we can assume $k \in \{n_j - \ell_j + 1, n_j + \ell_j\}$. If $r_j \geq q = r_{j_0}$, then $f(r_j) \leq f(q)$ and hence all ratios from B_j are at most $f(q)$. In this case it is clear that the geometric mean of the collection $r_{k+1}, \ldots, r_J$, where $J = \min(k+m, m_j)$, is at most $f(q)$. If $k + m > m_j$, then the set of ratios $\{r_{m_j+1}, \ldots, r_{k+m}\}$ contains more ratios equal to a^2 than otherwise, so its geometric mean is even at most a and thus the geometric mean of the full collection $\{r_{k+1}, \ldots, r_{k+m}\}$ is at most $f(q)$.

The last case to consider is that for this choice of j (which we remind the reader is $\geq j_0$), we have $r_j < q = r_{j_0}$ and therefore $f(r_j) > f(q)$. From (13), we note that

$$\Phi_q(s_{n_j}) = \Phi_{r_{j_0}}(s_{n_j}) \geq \frac{8A}{B} \Phi_{r_j}(s_{n_j}).$$

The remaining arguments are now similar to the proof of Theorem 3.8. Recall that $k \in \{n_j - \ell_j + 1, n_j + \ell_j\}$. If $n_j - \ell_j < k \leq n_j$, then $k \geq I_{r_j}$, so

$$\begin{aligned} \phi_q(k) &\geq Bk\Phi_q(s_k) \\ &\geq B(n_j - \ell_j)\Phi_q(s_{n_j}) \\ &\geq \frac{7}{8} B n_j \Phi_q(s_{n_j}) \\ &\geq 7A n_j \Phi_{r_j}(s_{n_j}) \\ &\geq 7\phi_{r_j}(n_j) = 7\ell_j. \end{aligned} \tag{14}$$

The fact that $n_j \geq 8J_q$ also guarantees that $\phi_q(k) \leq Ak\Phi_q(s_k) \leq k/8$, so $k + \phi_q(k) < m_j/2$. Thus the collection $\{r_{k+1}, \ldots, r_{k+\phi_q(k)}\}$ contains at most ℓ_j terms of ratio $f(r_j)$ and at least $6\ell_j$ terms of ratio a^2, and therefore has geometric mean at most a.

If, instead, $n_j < k \leq n_j + \ell_j$, then, as in the proof of Theorem 3.8 (see particularly (9)),

$$k + \phi_q(k) - (n_j + \ell_j) \geq \phi_q(n_j) - \phi_{r_j}(n_j) \geq 6\ell_j$$

where the final inequality comes from applying (14) with $k = n_j$. Again,

$$k + \phi_q(k) \leq 9k/8 < m_j/2$$

and thus again we deduce that the geometric mean of $\{r_{k+1}, \ldots, r_{k+\phi_q(k)}\}$ is at most a.

For either choice of k, if $m > \phi_q(k)$, then since $n_j + \ell_j < k + \phi_q(k) \leq m_j/2$ the collection of ratios $\{r_{k+\phi_q(k)+1}, \ldots, r_{k+m}\}$ has more that are value a^2 than otherwise, and hence has geometric mean at most a, as well. Thus we conclude $(s_k/s_{k+m})^{1/m} \le a$ in this (final) case.

This completes the proof. □

Corollary 3.11. *Given* $0 < \alpha < \beta < 1$, *there is a set* $E \subseteq [0,1]$ *such that*

$$\{\overline{\dim}_\Phi E \colon \Phi \in \mathcal{D},\ \lim_{x\to 0} \Phi(x) = 0\} = [\alpha, \beta] = [\dim_{qA} E, \dim_A E].$$

Proof. Let $D(E) = \{\overline{\dim}_\Phi E \colon \Phi \in \mathcal{D}, \lim_{x\to 0} \Phi(x) = 0\}$. We will let $\Phi_p(x) = |\log x|^{p-1}$ for $p \in (0,1)$. Clearly,

$$\{\overline{\dim}_{\Phi_p} E \colon p \in (0,1)\} \subseteq D(E) \subseteq [\dim_{qA} E, \dim_A E],$$

so it will be sufficient to construct a set E with $\{\overline{\dim}_{\Phi_p} E \colon p \in (0,1)\} = (\alpha, \beta)$ and $\dim_{qA} E = \alpha$, $\dim_A E = \beta$. The previous theorem would permit us to construct such a set satisfying the first property and would also have $\dim_A E = \beta$. However, its quasi-Assouad dimension is a^2, so we need to modify the construction slightly.

We can do this by requiring the sequence $\{n_j\}$ to grow so rapidly that in addition to the requirements from before, we can also have k_j much greater than m_j and n_{j+1} much greater than $2k_j$. On the levels $k_j + 1, \dots, 2k_j$ we will set the ratios to equal to $a = 2^{-1/\alpha}$ (rather than a^2). One can see that $\dim_{qA} E = \log 2/|\log a| = \alpha$ by considering the terms s_{k_j}/s_{2k_j}. The sparseness of the $\{k_j\}$ will ensure that the other dimensions are not affected by this change. We leave the technical details to the reader. □

4. Φ-dimensions of complementary sets in $\mathbb{R}$

4.1. Bounds for Φ-dimensions of complementary sets

4.1.1. Complementary sets. Every closed subset of the interval $[0,1]$ of Lebesgue measure zero is of the form $E = [0,1] \setminus \bigcup_{j=1}^{\infty} U_j$ where $\{U_j\}$ is a disjoint family of open subintervals of $[0,1]$ whose lengths sum to one. We will let $a = \{a_j\}_{j=1}^{\infty}$ where a_j is the length of U_j. Of course, $\sum_j a_j = 1$ and without loss of generality we can assume $a_{j+1} \le a_j$. We will denote by $\mathcal{C}_a$ the collection of all such closed sets E. These are called the *complementary sets of* a.

One example of a complementary set is the Cantor set associated with a, denoted C_a. Another is the countable set, D_a, called the decreasing rearrangement, defined as

$$D_a = \Big\{ \sum_{i \ge k} a_i \colon k = 1, 2, \dots \Big\} = \{1, 1 - a_1, 1 - a_1 - a_2, \dots\}.$$

As is well known, all complementary sets of a given sequence a have the same upper and lower box dimensions [6, Section 3.2], but, of course, this need not be true for other dimensions. For instance, the Hausdorff dimension of the decreasing rearrangement is 0, but this need not be true for the Cantor set. In [3], Besicovitch and Taylor proved that the Cantor set C_a had the maximum Hausdorff dimension of any set in $\mathcal{C}_a$. Further, they showed given any $s \leq \dim_H C_a$ there is some set $E \in \mathcal{C}_a$ with $\dim_H E = s$. The same result was shown to be true with the Hausdorff dimension replaced by the packing dimension in [17]. In [15], it was shown that the Cantor set and the decreasing set also have the extremal Assouad dimensions (under natural assumptions on the gap sequence a). But unlike the situation for Hausdorff, packing and lower Assouad dimensions, $\dim_A C_a$ is minimal among the sets in $\mathcal{C}_a$ and $\dim_A D_a$ is maximal (and equals 1 for such a). Again, it was shown that the full range of possible dimensions is attained, namely $\{\dim_L E : E \in \mathcal{C}_a\} = [0, \dim_L C]$ and $\{\dim_A E : E \in \mathcal{C}_a\} = [\dim_A C, 1]$.

In this section, we will prove analogous results for the Φ-dimensions, although some proofs are necessarily quite different.

4.1.2. Decreasing rearrangement. We first prove that the decreasing rearrangement is always one of the extreme values of the Φ-dimension over the class $\mathcal{C}_a$. This requires a proof in the case of the upper Φ-dimension as this dimension need not be one, c.f., Example 2.18. To begin, we first point out the following elementary result which essentially can be found in [6].

Lemma 4.1. *Suppose F, G are two compact sets in $\mathcal{C}_a$ for some decreasing, summable sequence $a = (a_n)$. For any $r > 0$,*

$$\frac{1}{16} \leq \frac{N_r(F)}{N_r(G)} \leq 16.$$

Proposition 4.2. *If a is any decreasing, summable sequence, then $\overline{\dim}_\Phi E \leq \overline{\dim}_\Phi D_a$ and $\underline{\dim}_\Phi E \geq \underline{\dim}_\Phi D_a = 0$ for all $E \in \mathcal{C}_a$.*

Proof. As D_a has isolated points, $\underline{\dim}_\Phi D_a = 0$ for all dimensions functions Φ and hence is the minimal lower Φ-dimension.

To prove that $\overline{\dim}_\Phi D_a$ is the maximal upper Φ-dimension we will simply show that

$$N_r(E \cap B(z, R)) \leq 64 N_r(D_a \cap B(0, R))$$

for all $z \in E$ and $r \leq R$.

To see this, let $\{a_{j_i}\}$ be the set of gap lengths that are completely contained in $E \cap B(z, R)$ and let $R' = \sum a_{j_i}$. Then $N_r(E \cap B(z, R)) = N_r(E \cap I)$ for a suitable interval $I \subseteq B(z, R)$ of length $R' \leq 2R$. Let E' denote the set formed by removing from $[0, R']$ the gaps of lengths $\{a_{j_i}\}$ in decreasing order (from right to left). By Lemma 4.1, $N_r(E \cap I) \leq 16 N_r(E' \cap [0, R'])$.

Choose n such that $a_n \leq 2r < a_{n-1}$ and suppose that

$$\sum_{i=m+1}^{\infty} a_i < R' \leq \sum_{i=m}^{\infty} a_i$$

(we will say that R' belongs to gap a_m in the set D_a). If $n \leq m$, then all gaps in the construction of D_a intersecting $[0, R']$ have length at most $2r$. Thus $N_r(D_a \cap [0, R']) = \lceil \frac{R'}{2r} \rceil$ is the maximum possible value and hence it dominates $N_r(E' \cap [0, R'])$.

So assume $n > m$ and let $A = \sum_{i=n}^{\infty} a_i < R'$. As $a_j \leq 2r$ for $j \geq n$ and $a_j > 2r$ for $j \leq n-1$,

$$\begin{aligned} N_r(D_a \cap [0, R']) &\geq N_r(D_a \cap [0, A]) + N_r(D_a \cap [A + a_{n-1}, R']) \\ &\geq \left\lceil \frac{A}{2r} \right\rceil + \max(n - m - 2, 0) \end{aligned}$$

(where $[A + a_{n-1}, R']$ is empty if $R' < A + a_{n-1}$).

Assume that the number A belongs to gap a_{j_s} in the set E', so

$$N_r(E' \cap [0, R']) \leq \left\lceil \frac{A}{2r} \right\rceil + s.$$

Since $\sum_{i=n}^{\infty} a_i = A \geq \sum_{i=s+1}^{\infty} a_{j_i}$, it follows that $j_s \leq n-1$ and consequently, $a_{j_{s-k}} \geq a_{n-1-k}$ for all $k = 0, \ldots, s-1$. Thus if $n - s + 1 \leq m$, then

$$\begin{aligned} R' - A &> a_{j_{s-1}} + a_{j_{s-2}} + \cdots + a_{j_2} + a_{j_1} \\ &\geq a_{n-2} + a_{n-3} + \cdots + a_{n-s+1} + a_{n-s} \\ &\geq a_{n-2} + a_{n-3} + \cdots + a_{n-s+1} + a_{n-1} \\ &\geq R' - A. \end{aligned}$$

This contradiction proves $s \leq n - m$. Thus

$$N_r(D_a \cap [0, R']) \geq \left\lceil \frac{A}{2r} \right\rceil + \max(s - 2, 0),$$

from which it is easy to check that

$$\begin{aligned} N_r(D_a \cap [0, R]) &\geq N_r(D_a \cap [0, R']) \\ &\geq \frac{1}{4}\left(\left\lceil \frac{A}{2r} \right\rceil + s\right) \\ &\geq \frac{1}{4} N_r(E' \cap [0, R']) \\ &\geq \frac{1}{64} N_r(E \cap B(z, R)), \end{aligned}$$

and the proposition follows. □

4.1.3. Cantor Sets. We now focus our attention on decreasing, summable sequences a with the property that there are constants τ and λ with

$$0 < \tau \leq s_{j+1}/s_j \leq \lambda < 1/2. \tag{15}$$

Here, as before, $s_j = 2^{-j} \sum_{i \geq 2^j} a_i$. We will call a sequence $\{a_j\}$ with this property *level comparable*. Of course, the doubling assumption automatically gives the left hand inequality and central Cantor sets have the level comparable property precisely when their ratios of dissection are bounded away from 0 and $1/2$.

The level comparable assumption is very useful as it ensures that $s_k \sim a_{2^k}$ since $s_k \geq a_{2^{k+1}} \gtrsim a_{2^k}$ and

$$(1 - 2\lambda)s_k \leq s_k - 2s_{k+1} \leq a_{2^k} . \tag{16}$$

We remind the reader that the symbols $\sim$ and $\gtrsim$ were defined in Notation 3.1.

For level comparable sequences, the Cantor set has the other extreme value for the Φ-dimensions.

Theorem 4.3. *If $a = \{a_j\}$ is a level comparable sequence and Φ is a dimension function, then for all $E \in \mathcal{C}_a$ we have $\overline{\dim}_\Phi E \geq \overline{\dim}_\Phi C_a$ and $\underline{\dim}_\Phi E \leq \underline{\dim}_\Phi C_a$.*

Proof. We begin with the upper Φ-dimension. Observe that if ϕ is bounded, then the upper Φ-dimension is the Assouad dimension and the result is already known in that case, see [15, Thm. 3.5]. So assume otherwise. Some modifications to the proof of Theorem 3.5 in [15] are required.

Let $d = \overline{\dim}_\Phi C_a$. From the formula for the upper Φ-dimension of C_a, Theorem 3.5, we know there must exist κ_0, c_0 and indices $k \geq \kappa_0$ and $n \geq \phi(k)$ such that

$$c_0 2^n \geq \left(\frac{s_k}{s_{k+n}}\right)^{d-\varepsilon} \geq 2^{n\varepsilon}\left(\frac{s_k}{s_{k+n}}\right)^{d-2\varepsilon},$$

where the latter inequality holds because $s_k/s_{k+1} \geq 2$ for all k.

We will refer to the complementary gaps of lengths $a_{2^{k-1}}, \dots, a_{2^k-1}$ as the gaps of level k.

Remove from $[0, \sum a_j]$ the complementary gaps of levels $1, \dots, k$ to obtain the set $J_1 \cup \cdots \cup J_{M_k} \cup \{\text{singletons}\}$ where J_i are non-trivial, closed, disjoint intervals, $M_k \le 2^k$ and $\sum_i |J_i| = 2^k s_k$. Let b_i denote the number of gaps of step $k+n$ contained in J_i and put $r = a_{2^{k+n}}/2$. If we let x_i be an endpoint of J_i, then as the gaps of step $k+n$ are at least $2r$ in length, $N_r(B(x_i, |J_i|) \cap E) \ge b_i$. Since $\sum_i b_i = 2^{k+n}$,

$$\sum_i N_r(B(x_i, |J_i|) \cap E) \ge 2^{k+n} \ge \frac{2^k 2^{n\varepsilon}}{c_0} \Big(\frac{s_k}{s_{k+n}}\Big)^{d-2\varepsilon}.$$

Let

$$\mathcal{I} = \{i \in \{1, \dots, M_k\} : |J_i| \le s_k\}.$$

If there is some index $i \in I$ with $N_r(B(x_i, |J_i|) \cap E) \ge (s_k/s_{k+n})^{d-2\varepsilon}$, then

$$N_r(B(x_i, s_k) \cap E) \ge N_r(B(x_i, |J_i|) \cap E) \ge \Big(\frac{s_k}{s_{k+n}}\Big)^{d-2\varepsilon}. \tag{17}$$

Otherwise,

$$\begin{aligned}
\sum_{i \notin \mathcal{I}} N_r(B(x_i, |J_i|) \cap E) &= \sum_i N_r(B(x_i, |J_i|) \cap E) - \sum_{i \in \mathcal{I}} N_r(B(x_i, |J_i|) \cap E) \\
&\ge 2^{k+n} - |\mathcal{I}| \max_{i \in \mathcal{I}} N_r(B(x_i, |J_i|) \cap E) \\
&\ge \frac{2^k 2^{n\varepsilon}}{c_0} \Big(\frac{s_k}{s_{k+n}}\Big)^{d-2\varepsilon} - 2^k \Big(\frac{s_k}{s_{k+n}}\Big)^{d-2\varepsilon} \\
&\ge 2^k \Big(\frac{s_k}{s_{k+n}}\Big)^{d-2\varepsilon} \Big(\frac{2^{n\varepsilon}}{c_0} - 1\Big).
\end{aligned}$$

Since $n \ge \phi(k) \to \infty$, we can assume $2^{n\varepsilon}/c_0 - 1 \gtrsim 2^{n\varepsilon}$. Recall that $\sum_i |J_i| = 2^k s_k$, thus

$$\sum_{i \notin \mathcal{I}} N_r(B(x_i, |J_i|) \cap E) \gtrsim 2^k \Big(\frac{s_k}{s_{k+n}}\Big)^{d-2\varepsilon} 2^{n\varepsilon} \gtrsim 2^k 2^{n\varepsilon} \Big(\frac{2^{-k} \sum_{i \notin \mathcal{I}} |J_i|}{a_{2^{k+n}}}\Big)^{d-2\varepsilon}.$$

An application of Holder's inequality gives

$$\begin{aligned}
\sum_{i \notin \mathcal{I}} N_r(B(x_i, |J_i|) \cap E) &\gtrsim 2^k 2^{n\varepsilon} \Big(\frac{2^{-k}}{a_{2^{k+n}}}\Big)^{d-2\varepsilon} \sum_{i \notin \mathcal{I}} |J_i|^{d-2\varepsilon} |\mathcal{I}^c|^{-(1-d+2\varepsilon)} \\
&\gtrsim 2^{n\varepsilon} \Big(\frac{2^k}{|\mathcal{I}^c|}\Big)^{1-d+2\varepsilon} \frac{\sum_{i \notin \mathcal{I}} |J_i|^{d-2\varepsilon}}{r^{d-2\varepsilon}} \\
&\gtrsim 2^{n\varepsilon} \frac{\sum_{i \notin \mathcal{I}} |J_i|^{d-2\varepsilon}}{r^{d-2\varepsilon}},
\end{aligned}$$

with the final inequality arising because $|\mathcal{J}^c| \leq M_k \leq 2^k$ and $d \leq 1$. It follows that in this case there must be some choice of $i \notin \mathcal{J}$ such that

$$N_r\left(B(x_i, |J_i|) \cap E\right) \geq c2^{n\varepsilon}\left(\frac{|J_i|}{r}\right)^{d-2\varepsilon}. \tag{18}$$

By definition, $i \notin \mathcal{J}$ implies $|J_i| \geq s_k$ and thus

$$|J_i|^{1+\Phi(|J_i|)} \geq s_k^{1+\Phi(s_k)} \geq s_{k+n} \sim r.$$

As either (17) or (18) must hold, we deduce that $\overline{\dim}_\Phi E \geq d - 2\varepsilon$ and that gives the desired result.

The proof for the lower Φ-dimension is a straightforward modification of Theorem 4.1 of [15]. □

Combining Proposition 4.2 and Theorem 4.3 gives the following statement.

Corollary 4.4. *If a is any level comparable sequence, then for all $E \in \mathcal{C}_a$ we have $\overline{\dim}_\Phi E \in [\overline{\dim}_\Phi C_a, \overline{\dim}_\Phi D_a]$ and $\underline{\dim}_\Phi E \in [0, \underline{\dim}_\Phi C_a]$. In particular, these statements are true for the quasi-Assouad dimensions.*

4.2. An interval of Φ-dimensions for complementary sets. In [15] it was shown that if a is any level comparable sequence, then for every $c \in [0, \dim_L C_a]$ and $d \in [\dim_A C_a, 1]$ there are sets $E_c, E_d \in \mathcal{C}_a$ with[1]

$$\dim_L E_c = c \quad \text{and} \quad \dim_A E_d = d.$$

These results continue to be true for the quasi-Assouad and Φ-dimensions when $\Phi \to \delta$, with $\delta \in [0, \infty]$. For the lower Φ-dimensions essentially the same proof as given in [15] for the lower Assouad dimension works. We give a brief sketch of the main idea at the beginning of the proof of Theorem 4.5.

For the upper Φ-dimension, note that the case $\delta = \infty$ is trivial since we recover the upper box dimension and all complementary sets of a given sequence have the same upper box dimension. Different proofs are required for the cases $\Phi \to \delta$ for $\delta = 0$ or $\delta > 0$, and these are necessarily different from the proof given for the Assouad dimension in [15] as the set constructed there only exhibits large local "thickness" on scales r that are nearly as large as R, and hence are not suitable for use in obtaining these other dimensions.

Theorem 4.5. *Suppose a is a level comparable sequence and Φ is a dimension function with $\Phi(x) \to \delta$, for some $\delta \in [0, \infty]$. Then for every $c \in [0, \underline{\dim}_\Phi C_a]$ and $d \in [\overline{\dim}_\Phi C_a, \overline{\dim}_\Phi D_a]$, there are sets $E_c, E_d \in \mathcal{C}_a$ with $\underline{\dim}_\Phi E_c = c$ and $\overline{\dim}_\Phi E_d = d$. A similar statement holds with $\underline{\dim}_\Phi C_a$ replaced by $\dim_{qL} C_a$ and $\overline{\dim}_\Phi C_a$ replaced by $\dim_{qA} C_a$.*

[1] Actually, the assumption that a is doubling suffices for the upper Assouad dimension.

Remark 4.6. We remind the reader that for any doubling sequence a (and hence any level comparable sequence) and any dimension function $\Phi \to 0$, we have $\overline{\dim}_B D_a > 0$ and thus $\overline{\dim}_\Phi D_a \geq \dim_{qA} D_a = 1$ by [14].

Combining this result with Theorem 4.3 gives the following.

Corollary 4.7. *Suppose a is any level comparable sequence and Φ is a dimension function with $\Phi(x) \to \delta$. Then*

$$\{\overline{\dim}_\Phi E : E \in \mathcal{C}_a\} = [\overline{\dim}_\Phi C_a, \overline{\dim}_\Phi D_a]$$

and

$$\{\underline{\dim}_\Phi E : E \in \mathcal{C}_a\} = [\underline{\dim}_\Phi D_a, \underline{\dim}_\Phi C_a] = [0, \underline{\dim}_\Phi C_a].$$

Proof of Theorem 4.5. For the lower dimension case, the same proof given in [15, Theorem 4.3] for the lower Assouad dimension, with the obvious modifications, works for the lower Φ-dimensions and the lower quasi-Assouad dimension. A sketch of the proof is that for $0 < \alpha < \underline{\dim}_\Phi C_a$, it is possible to find a subsequence of a whose Cantor rearrangement is an α-Ahlfors regular set and such that the Cantor rearrangement of the remaining gaps has lower Φ-dimension equal to $\underline{\dim}_\Phi C_a$. This gives a complementary set E with $\underline{\dim}_\Phi E = \alpha$.

For the upper Φ-dimension problem, we first remark that if $\Phi \to \infty$, then $\overline{\dim}_\Phi E = \overline{\dim}_B E$ and all sets $E \in \mathcal{C}_a$ have the same upper box dimension.

Thus it remains to study the upper Φ-dimension problem when $\Phi \to \delta \in [0, \infty)$. We will first give the proof for the case $\Phi \to \delta \neq 0$, where we can take advantage of an explicit formula for the Φ-dimension of the decreasing rearrangement. The harder case, $\delta = 0$, will be done second.

Case $\Phi \to \delta \in (0, \infty)$. According to Corollary 2.12, $\overline{\dim}_\Phi E = \overline{\dim}_A^\theta E$ for all E, where $\theta = (1 + \delta)^{-1}$. Observe that for any decreasing set D,

$$\overline{\dim}_A^\theta D = \min\left(\frac{\overline{\dim}_B D}{1 - \theta}, 1\right). \tag{19}$$

This follows from [12, Theorem 6.2] and [10, Theorem 2.1].

Given any $0 < d < \overline{\dim}_A^\theta D_a$, we will use the above formula to construct a subsequence b of a such that if $\tilde{a}$ is the subsequence obtained after removing b from a, then $\overline{\dim}_A^\theta D_b = d$ and $\overline{\dim}_A^\theta C_{\tilde{a}} = \overline{\dim}_A^\theta C_a$. The set $E_d = D_b \cup C_{\tilde{a}}$ will belong to $\mathcal{C}_a$ and by the union property, its upper θ-dimension will be given by

$$\overline{\dim}_A^\theta E_d = \max(d, \overline{\dim}_A^\theta C_a),$$

which will prove the statement of the theorem.

Let $\sigma := \overline{\dim}_B D_a$. By [29, Section 3.4] we have

$$\sigma = \limsup_{n\to\infty} \frac{\log n}{-\log a_n} = \lim_{k\to\infty} \frac{\log n_k}{-\log a_{n_k}},$$

where $\{n_k\}$ is chosen to be a suitable sequence, say $n_{k+1} \geq 2^{n_k}$. Choose γ such that $\gamma\sigma = d(1-\theta)$ and define the subsequence b by

$$b_m = \begin{cases} a_{n_k}, & \lfloor m^{1/\gamma} \rfloor \leq n_k < \lfloor (m+1)^{1/\gamma} \rfloor, \\ a_{\lfloor m^{1/\gamma} \rfloor}, & \text{otherwise.} \end{cases}$$

Note that for the integers m_k where $b_{m_k} = a_{n_k}$ we have $m_k \sim n_k^{\gamma}$, so

$$\lim_k \frac{\log m_k}{-\log b_{m_k}} = \lim_k \frac{\log n_k^{\gamma}}{-\log a_{n_k}} = \sigma\gamma.$$

Moreover, for $\epsilon > 0$ we have $\log n/(-\log a_n) < \sigma + \epsilon$ for all n large enough, so for large m, with $m \neq m_k$,

$$\frac{\log m}{-\log b_m} = \frac{\gamma \log m^{1/\gamma}}{-\log a_{\lfloor m^{1/\gamma} \rfloor}} \leq \gamma(\sigma + 2\epsilon).$$

Therefore, $\overline{\dim}_B D_b = \sigma\gamma$, and by (19) $\overline{\dim}_A^{\theta} D_b = d$.

Finally, note that $\lfloor (m+1)^{1/\gamma} \rfloor - \lfloor m^{1/\gamma} \rfloor \to \infty$ as m increases. As the original sequence was doubling, this ensures that the sequence $\tilde{a}$ consisting of the remaining gaps is comparable to the original sequence a. In consequence, $\overline{\dim}_A^{\theta} C_{\tilde{a}} = \overline{\dim}_A^{\theta} C_a$ and, as we noted above, that completes the proof in this case.

Case $\Phi \to 0$. We will give a detailed proof for the quasi-Assouad dimension. It will be clear that the same arguments will work for the upper Φ-dimension with $\Phi \to 0$. Our proof is constructive. The set $E = E_d \in \mathcal{C}_a$ will again have the form $E = \mathcal{A} \cup \mathcal{B}$, with $\dim_{qA} \mathcal{A}$ equal to the desired in-between value d and $\dim_{qA} \mathcal{B} = \dim_{qA} C_a$. The union property for the quasi-Assouad dimension will ensure that E has the desired quasi-Assouad dimension.

If $b = \{b_j\}$ is the sequence with $b_{2^j+t} = a_{2^j}$ for $t = 0, \ldots, 2^j - 1$, then a, b are comparable sequences and if E is the set formed with some rearrangement of a and F is the corresponding rearrangement of b, then E and F are bi-Lipschitz equivalent. So without loss of generality we will assume a is constant along diadic blocks. Moreover, a level comparable sequence $\{a_j\}$ has the property that there are constants u, v such that

$$1 > u \geq \frac{a_{2^j}}{a_{2^{j-1}}} \geq v > 0 \quad \text{for all } j.$$

If $\dim_{qA} C_a = 1$, there is nothing to do. So assume $1 > d \geq \dim_{qA} C_a$, say $d = \log 2/|\log \beta|$ where $\beta < 1/2$.

To simplify the notation, we will let $\alpha_j = a_{2^j}$. Temporarily fix m. Given $j \geq 1$, choose the minimal index $i(j) \geq 1$ such that $\alpha_{m+i(j)}/\alpha_m \leq \beta^j$ and choose the maximal integer $J_j \geq 1$ such that

$$J_j \frac{\alpha_{m+i(j)}}{\alpha_m} \leq \beta^j.$$

The minimality of $i(j)$ ensures that

$$J_j \frac{\alpha_{m+i(j)}}{\alpha_m} \leq \beta^j < \frac{\alpha_{m+i(j)-1}}{\alpha_m}$$

which implies

$$J_j \leq \frac{\alpha_{m+i(j)-1}}{\alpha_{m+i(j)}} \leq \frac{1}{v}.$$

Similarly, the maximality of J_j means that

$$(J_j + 1)\frac{\alpha_{m+i(j)}}{\alpha_m} > \beta^j,$$

so

$$\alpha_{m+i(j)} \geq \frac{\alpha_m \beta^j}{1 + 1/v} = c_1 \alpha_m \beta^j$$

where $c_1 > 0$ is independent of m and j. Moreover, the fact that $v^i \leq \alpha_{m+i}/\alpha_m \leq u^i$, coupled with the definition of $i(j)$, implies

$$jc_3 \leq j\frac{\log \beta}{\log v} + 1 \leq i(j) \leq j\frac{\log \beta}{\log v} = jc_2$$

where we again note that c_2, c_3 are positive constants, independent of m and j.

Construction of the set E. We now form a Cantor-tree like arrangement with blocks of gaps. The first block will consist of J_1 gaps of length $\alpha_{m+i(1)}$ placed adjacently. The blocks of level 2 will each consist of J_2 gaps of length $\alpha_{m+i(2)}$ placed adjacently and there will be two blocks of level 2, one to the left and the other to the right of the block of level 1. In general, there will be 2^{j-1} blocks of level j, each consisting of J_j gaps of length $\alpha_{m+i(j)}$ placed in a Cantor-like arrangement. If we do this for $j = 1, \ldots, n$, we will call the resulting finite set $X_{m,n}$. Note that the length of any block of level j in $X_{m,n}$ is equal to $J_j \alpha_{m+i(j)}$ and satisfies

$$c_1 \alpha_m \beta^j \leq J_j \alpha_{m+i(j)} \leq \alpha_m \beta^j. \tag{20}$$

Hence the diameter of $X_{m,n}$ is at least the length of block 1 which is $\geq c_1\alpha_m\beta$, and the diameter of $X_{m,n}$ is at most

$$\sum_{j=1}^{n} 2^{j-1}\alpha_m\beta^j \leq \alpha_m\frac{\beta}{1-2\beta} = c_4\alpha_m\beta. \tag{21}$$

Since $i(j) \geq c_2 j$, for each k the number of gaps of length α_{m+k} that we will require is

$$\sum_{\substack{j\in\{1,\dots,n\}\\ i(j)=k}} J_j 2^{j-1} \leq \sum_{j=1}^{k/c_2} J_j 2^{j-1} \leq \frac{1}{v}2^{k/c_2}.$$

As $j \in \{1,\dots,n\}$ and $i(j) \leq c_3 j$, we have $k \leq c_3 n$. Of course, for each k there are a total of 2^{m+k} gaps of this size available in the sequence a, so we have enough gaps, even twice as many as we need, provided

$$\frac{1}{v}2^{k/c_2} \leq 2^{m+k-1} \quad \text{for each } k = 1,\dots,c_3 n.$$

Hence there is some $c_5 > 0$ (and independent of m) such that if $n \leq c_5 m$, then there will be enough gaps to carry out this construction.

Lastly, we will select a rapidly growing sequence of integers $\{m_k\}$ and let $n_k = [c_5 m_k]$. We will set $A_k = X_{m_k,n_k}$. We will want m_{k+1} to be much larger than $m_k + c_3 n_k$, so that we will not use any gaps from the same diadic blocks in two different sets A_j. Also, we will want to choose m_k increasing so rapidly that the diameter of A_{k+1} is at most $1/2$ diameter of A_k.

We will position the sets A_k adjacent to each other in decreasing order and let

$$\mathcal{A} = \bigcup_{k=1}^{\infty} A_k.$$

The gaps of the sequence $\{a_j\}$ that were not used in the construction of the sets A_k will be then placed to form a Cantor set $\mathcal{B}$ to the left of A_1. This completes the construction of the set $E = \mathcal{A} \cup \mathcal{B} \in C_a$.

Computation of $\dim_{qA} E$. Since there are at least half the gaps a_j left in each diadic block, the decreasing sequence consisting of the remaining gaps is comparable to the original sequence. Hence $\dim_{qA}\mathcal{B} = \dim_{qA} C_a \leq d$. Thus, to see that the rearranged set $\mathcal{A} \cup \mathcal{B}$ has quasi-Assouad dimension d, it will be enough to prove $\dim_{qA}\mathcal{A} = d$.

(a) Lower bound for $\dim_{qA}\mathcal{A}$. We will let $|Y|$ denote the diameter of a set $Y \subseteq \mathbb{R}$.

To see that $\dim_{qA}\mathcal{A} \geq d$, consider $R = |A_k| \sim \alpha_{m_k}\beta$ (by (21)) and $r = \frac{1}{2}$length of block of level n_k in A_k, so that $r \sim \alpha_{m_k}\beta^{n_k}$ (by (20)). Notice that if $\delta > 0$ (independent of k) is chosen such that $\beta^{c_5}v^{-\delta} < 1$, then as $a_{2^n} \geq v^n$, for sufficiently large k,

$$\frac{r}{R^{1+\delta}} \leq c\frac{\alpha_{m_k}\beta^{n_k}}{(\alpha_{m_k}\beta)^{1+\delta}} = c\frac{\beta^{n_k}}{\beta^{1+\delta}\alpha_{m_k}^{\delta}} \leq \frac{c\beta^{-1}}{\beta^{1+\delta}}\Big(\frac{\beta^{c_5}}{v^{\delta}}\Big)^{m_k} < 1.$$

If we let $z \in A_k$, then $N_r(B(z,R)\cap A_k) \geq 2^{n_k-1}$ since the blocks of level n_k are separated by at least r, while $(R/r)^d \sim \beta^{-dn_k} = 2^{n_k}$. In order for there to be a constant C such that $N_r(B(z,R)\cap A_k) \leq C(R/r)^t$ for all k, we must have $t \geq \log 2/|\log\beta| = d$. This shows $\dim_{qA}\bigcup A_k \geq d$.

(b) Upper bound for $\dim_{qA}\mathcal{A}$. For this, we will prove the following claim.

Claim. *There is a constant C, independent of k, such that*

$$N_r(B(z,R)\cap A_k) \leq C\Big(\frac{\min(|A_k|,R)}{r}\Big)^d \tag{22}$$

for all $r < \min(|A_k|,R)$ and all $z \in \mathcal{A}$.

Assuming the claim, we can even prove that $\dim_A\mathcal{A} \leq d$. Take $R \leq |A_1|/2$. Suppose $r < R$ and that

$$|A_{k+1}|/2 < R \leq |A_k|/2.$$

Then $B(z,R)$ can intersect at most two (consecutive) sets A_i for $i \leq k$, (say $i = m, m+1$), as well as possibly $\bigcup_{i=k+1}^{\infty} A_i$. Assume

$$|A_j| \leq r < |A_{j-1}|$$

where, of course, $j \geq k+1$. Since $\sum_{i=j}^{\infty}|A_i| \leq 2|A_j|$, one ball of radius r will cover $\bigcup_{i=j}^{\infty} A_i$. As $r \leq |A_k| \leq |A_{m+1}|$, from (22) we have

$$\begin{aligned} N_r(B(z,R)\cap\mathcal{A}) &\leq N_r(B(z,R)\cap(A_m\cup A_{m+1})) + N_r\Big(B(z,R)\cap\bigcup_{i=k+1}^{\infty}A_i\Big) \\ &\leq 2C\Big(\frac{R}{r}\Big)^d + \sum_{i=k+1}^{j-1} N_r(B(z,R)\cap A_i) + 1 \end{aligned}$$

(where the sum is empty if $j-1 < k+1$). Since $r < |A_i|$ for $i = k+1, \dots, j-1$, from (22) we again see that

$$N_r(B(z,R) \cap \mathcal{A}) \leq 2C\left(\frac{R}{r}\right)^d + C\sum_{i=k+1}^{j-1}\left(\frac{|A_i|}{r}\right)^d + 1$$
$$\leq C'\left(\frac{R}{r}\right)^d + C'\left(\frac{|A_{k+1}|}{r}\right)^d$$
$$\leq C''\left(\frac{R}{r}\right)^d.$$

That proves that $\dim_A \mathcal{A} \leq d$ and hence $\dim_{qA} \mathcal{A} = d$.

Proof of the claim. Choose $\gamma \leq 1$ such that the diameter of $A_k \geq \gamma\alpha_{m_k}\beta$ for all k. Temporarily fix k. Choose R and $r < \min(|A_k|, R)$.

First, suppose there is some $j \in \mathbb{N}$ such that

$$\gamma\alpha_{m_k}\beta^{j+1}/4 < R \leq \gamma\alpha_{m_k}\beta^j/4$$

(in particular, $R < |A_k|$). If $j > n_k$, then $2R$ is smaller than the smallest block in A_k and thus $B(z,R)$ can intersect at most two blocks in A_k. As there are at most $1/v$ gaps in each block,

$$N_r(B(z,R) \cap A_k) \leq \frac{2}{v} \leq C\left(\frac{R}{r}\right)^d.$$

Hence assume $j \leq n_k$. Then $2R$ is less than the length of any block of level $\leq j$ and thus $B(z,R) \cap A_k$ can intersect at most two (consecutive) blocks of level $\leq j$, as well as the interval I in-between (where an in-between interval could mean the interval between the left or right-most block of level j and the endpoint of the set A_k). The points in $\mathcal{A}$ from the two blocks of level at most j can be covered by $2/v$ balls of radius r, hence

$$N_r(B(z,R) \cap A_k) \leq \frac{2}{v} + N_r(B(z,R) \cap I).$$

Notice that the interval I will contain (at most) 2^{n-j} blocks of level $n \geq j+1$. Also, observe that the interval between two consecutive blocks of level n (should it exist in A_k) has length at most

$$\sum_{i=n+1}^{\infty} 2^{i-(n+1)}\alpha_{m_k}\beta^i \leq \alpha_{m_k}\frac{\beta^{n+1}}{1-2\beta}.$$

Thus if

$$\alpha_{m_k}\frac{\beta^{n+1}}{1-2\beta} < r \leq \alpha_{m_k}\frac{\beta^n}{1-2\beta} \quad \text{for some } n \geq j+1, \tag{23}$$

then each such subinterval can be covered by one ball of radius r. There are at most 2^{n-j} such subintervals contained in I. Additionally, the points in $\mathcal{A}$ from each of the blocks of levels $j+1,\ldots,n$ contained in I can be covered by $1/v$ balls of radius r and there are $\leq 2^{n-j}$ such blocks. So

$$\begin{aligned} N_r(B(z,R)\cap I) &\leq 2^{n-j}+2^{n-j}/v \\ &\leq C'2^{n-j} \\ &= C'(\beta^{j-n})^d \\ &\leq C''(R/r)^d. \end{aligned}$$

Thus for such r we certainly have

$$N_r(B(z,R)\cap A_k) \leq \frac{2}{v}+C''\Big(\frac{R}{r}\Big)^d \leq C\Big(\frac{\min(|A_k|,R)}{r}\Big)^d$$

for a suitable constant C (recalling that $R<|A_k|$ and $R/r\geq 1$).

If (23) does not hold, we must have

$$\alpha_{m_k}\frac{\beta^{j+1}}{1-2\beta}<r\leq R\leq \gamma\alpha_{m_k}\beta^j/4.$$

Then $B(z,R)$ is covered by a bounded number (independent of j,k) of balls of radius r and that also suffices to prove

$$N_r(B(z,R)\cap A_k)\leq C\Big(\frac{R}{r}\Big)^d \leq C\Big(\frac{\min(|A_k|,R)}{r}\Big)^d$$

for these r.

Otherwise, $R>\gamma\alpha_{m_k}\beta/4$. If (still) $R\leq |A_k|$, then we argue similarly, taking as I the full set A_k. Finally, suppose $R>|A_k|$. Then

$$N_r(B(z,R)\cap A_k)\leq N_r(B(z',|A_k|)\cap A_k)$$

where $z'\in A_k$. As $r<|A_k|$, the previous work shows

$$N_r(B(z',|A_k|)\cap A_k)\leq C\Big(\frac{|A_k|}{r}\Big)^d \leq C\Big(\frac{\min(|A_k|,R)}{r}\Big)^d.$$

This completes the proof of the claim. △

Conclusion of the proof for general case of $\Phi\to 0$. Lastly, we remark that the same arguments show that if $\Phi\to 0$, then for each $d\in[\overline{\dim}_\Phi C_a,1)$ there is some $E=\mathcal{A}\cup\mathcal{B}\in\mathcal{C}_a$ with $\dim_{qA}\mathcal{A}=\dim_A\mathcal{A}$, so that also $\overline{\dim}_\Phi\mathcal{A}=d$. Further, $\overline{\dim}_\Phi\mathcal{B}=\overline{\dim}_\Phi C_a$ and thus $\overline{\dim}_\Phi E=d$ by the union result, Proposition 2.5. Since we have $1=\dim_{qA}D_a=\overline{\dim}_\Phi D_a$ the proof is complete when $\Phi\to 0$. □

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