J. Fractal Geom. 8 (2021), 289–304
DOI 10.4171/JFG/107

Revolving fractals

Kiko Kawamura and Andrew Allen

Abstract. Davis and Knuth in 1970 introduced the notion of revolving sequences, as representations of a Gaussian integer. Later, Mizutani and Ito pointed out a close relationship between a set of points determined by all revolving sequences and a self-similar set, which is called the Dragon. We will show how their result can be generalized, giving new parametrized expressions for certain self-similar sets.

Mathematics Subject Classification (2020). Primary: 28A80; Secondary: 37B10.

Keywords. Revolving sequences, self-similar sets.

Introduction

In 1970, C. Davis and D. E. Knuth [1] introduced the notation of *revolving representations* of a Gaussian integer: for any $z = x + iy$ with $x, y \in \mathbb{Z}$, there exists a revolving sequence $(\delta_0, \delta_1, \ldots, \delta_n)$ such that

$$z = \sum_{k=0}^{n} \delta_{n-k}(1+i)^k,$$

where $\delta_k \in \{0, 1, -1, i, -i\}$ with the restriction that the non-zero values must follow the cyclic pattern from left to right:

$$\cdots \longrightarrow 1 \longrightarrow (-i) \longrightarrow (-1) \longrightarrow i \longrightarrow 1 \longrightarrow \cdots.$$

For instance, they gave the following example:

$$-5 + 33i = (1\ 0\ 0\ 0\ (-i)\ (-1)\ i\ \ 1\ 0\ (-i)\ 0)_{1+i}.$$

They also showed that each Gaussian integer has exactly four representations of this type: one each in which the right-most non-zero value takes on the values $1, -1, i, -i$.

Let W be the set of all revolving sequences, and define the set

$$X := \left\{ \sum_{n=1}^{\infty} \delta_n (1+i)^{-n} \colon (\delta_1, \delta_2, \delta_3, \dots) \in W \right\}.$$

Notice that each revolving sequence determines a complex number and X is a set of points in the complex plane. The set X is shown in the left half of Figure 1.

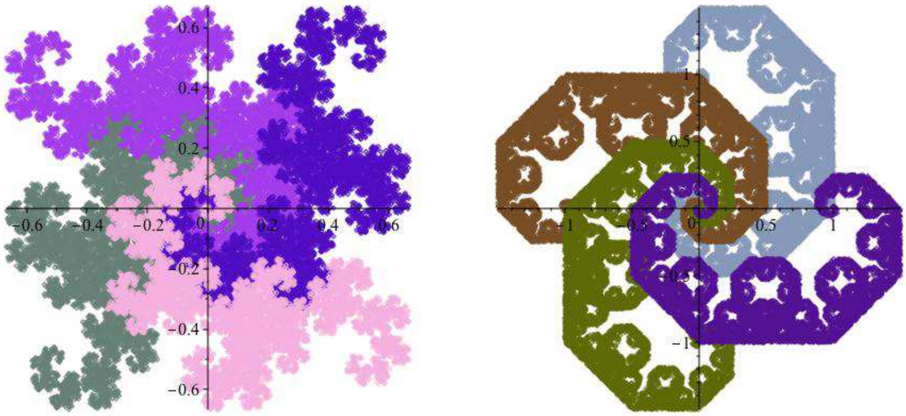

Figure 1. X (left) and X^* (right).

Mizutani and Ito [7] proved the following theorem using techniques from symbolic dynamics.

Theorem 0.1 (Mizutani–Ito, 1987). (i) *The set X is tiled by four Dragons $\{D_k, k = 0, 1, 2, 3\}$, that is*

$$X = \bigcup_{k=0}^{3} D_k = \bigcup_{k=0}^{3} i^k D,$$

where $D = \psi_1(D) \cup \psi_2(D)$ is the self-similar set generated by

$$\begin{cases} \psi_1(z) = \left(\dfrac{1-i}{2}\right) z, \\ \psi_2(z) = \left(\dfrac{-1-i}{2}\right) z + \dfrac{1-i}{2}. \end{cases}$$

(ii) *For each $k \neq k'$,*

$$\lambda(D_k \cap D_{k'}) = 0.$$

In the same paper, they mentioned an interesting question. Define another set X^* by

$$X^* := \left\{ \sum_{n=1}^{\infty} \overline{\delta_n}(1+i)^{-n} : (\delta_1, \delta_2, \delta_3, \dots) \in W \right\}.$$

Notice that $\overline{\delta_n}$ moves on the unit circle counterclockwise instead of clockwise. The set X^* is shown in the right half of Figure 1. Computer simulations suggested to Mizutani and Ito that X^* must be a union of four Lévy's curves; however, they could not give a mathematical proof.

Recall that Lévy's curve is a continuous curve with positive area. It was introduced by Paul Lévy in 1939 [6]. Figure 2 shows the graph of Lévy's curve, which is a self-similar set $L = \phi_1(L) \cup \phi_2(L)$ generated by the similar contractions

$$\begin{cases} \phi_1(z) = \left(\frac{1+i}{2}\right) z, \\ \phi_2(z) = \left(\frac{1-i}{2}\right) z + \frac{1+i}{2}. \end{cases} \tag{0.1}$$

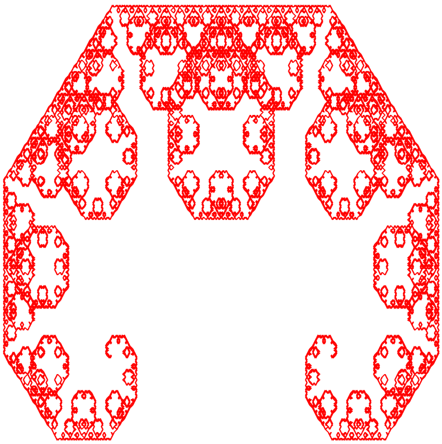

Figure 2. Lévy's curve.

Kawamura in 2002 [4] finally gave a proof of Mizutani and Ito's conjecture.

Theorem 0.2 (Kawamura, 2002). *The conjugate of X^* is a union of four copies of Lévy's curves L generated by* (0.1)*, that is*

$$\overline{X^*} = \bigcup_{k=0}^{3} i^k L.$$

It is worth mentioning that the proof is completely different from Mizutani and Ito's approach. Instead of using technique from symbolic dynamics, Kawamura considered the following functional equation

$$f_{\alpha,\gamma}(x) = \begin{cases} \alpha f_{\alpha,\gamma}(2x), & 0 \le x < 1/2, \\ \gamma f_{\alpha,\gamma}(2x-1) + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{0.2}$$

where α and γ are complex parameters satisfying $|\alpha| < 1$, $|\gamma| < 1$. She proved the existence of a unique bounded solution $f_{\alpha,\gamma}\colon [0,1] \to \mathbb{C}$ of (0.2) and gave the explicit expression. Observe that the closure of the image of this bounded solution $f_{\alpha,\gamma}([0,1])$ is a self-similar set generated by two contractions $\phi_1(z) = \alpha z$ and $\phi_2(z) = \gamma z + (1-\gamma)$. In particular, if $\alpha = (1+i)/2$ and $\gamma = (1-i)/2$, $f_{\alpha,\gamma}([0,1]) = L$.

Lévy's curve and Dragon are very different: one is a continuous curve while the other is a tiling fractal; however, both are self-similar sets. Thus, the following questions arise naturally.

(1) Is there a generalized relationship between sets of revolving sequences and self-similar sets? In particular, we are interested in describing self-similar sets which arise from more general revolving sequences, where the 90 degree angle of rotation is replaced with a more general angle.

(2) Is there a simpler way to prove both Mizutani and Ito's and Kawamura's theorems?

1. Generalized revolving sequences

Before stating our results, some notation need to be introduced. Let $\alpha \in \mathbb{C}$ denote a complex parameter satisfying $|\alpha| < 1$. Let θ be an angle with $-\pi < \theta \le \pi$ and a rational multiple of 2π. More precisely, there are $p \in \mathbb{N}, q \in \mathbb{N}_0$ such that $|\theta| = \frac{2\pi q}{p}$. Define

$$\Delta_\theta := \{0, 1, e^{i\theta}, e^{2i\theta}, \dots, e^{(p-1)i\theta}\}.$$

Definition 1.1. A sequence $(\delta_1, \delta_2, \dots) \in \Delta_\theta^{\mathbb{N}}$ satisfies the *Generalized Revolving Condition* (*GRC*), if the subsequence obtained after the removal of its zero elements is a (finite or infinite) truncation of the sequence $(e^{i\theta})$. More precisely, let $(n_i) := \{n\colon \delta_n \neq 0\}$. Then, $\delta_{n_{i+1}} = e^{i\theta}\delta_{n_i}$.

Notice that δ_n moves on the unit circle counterclockwise if $\theta > 0$, and clockwise if $\theta < 0$.

Define W_θ as the set of all generalized revolving sequences with parameter θ:

$$W_\theta := \{(\delta_1, \delta_2, \dots) \in \Delta_\theta^{\mathbb{N}} \colon (\delta_1, \delta_2, \dots) \text{ satisfies the GRC}\},$$

and for a given $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$, define

$$X_{\alpha,\theta} := \left\{ \sum_{n=1}^{\infty} \delta_n \alpha^n \colon (\delta_1, \delta_2, \delta_3, \dots) \in W_\theta \right\}.$$

Notice that each generalized revolving sequence determines a complex number and $X_{\alpha,\theta}$ is a set of points in the complex plane. Two examples of $X_{\alpha,\theta}$ are shown in Figure 3. It is not hard to imagine that $X_{\alpha,\theta}$ is a union of self-similar sets; however, it is not immediately clear which iterated function system generates these self-similar sets.

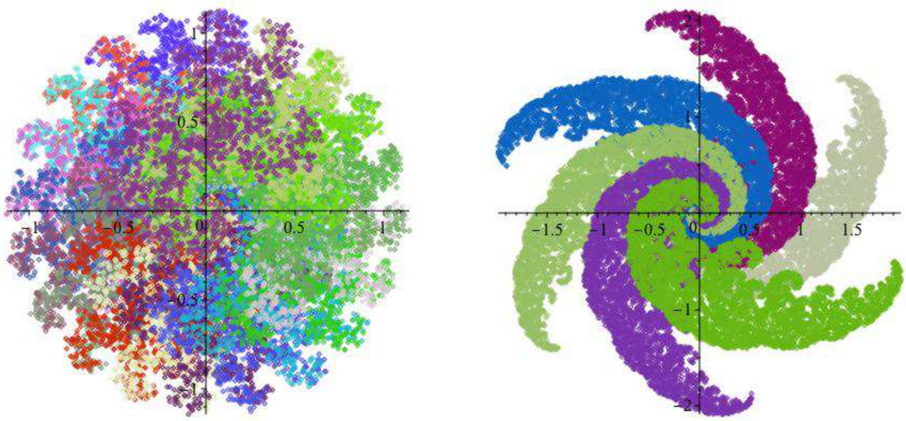

Figure 3. $X_{\alpha,\theta} \colon (\alpha, \theta) = \left(\frac{1+i}{2}, \frac{\pi}{10}\right)$ (left) and $(\alpha, \theta) = \left(\frac{1-i}{2}, \frac{\pi}{3}\right)$ (right).

Using a direct approach different from both [7] and [4], we obtain the following theorem.

Theorem 1.2. *$X_{\alpha,\theta}$ is a union of p copies of $K_{\alpha,\theta}$:*

$$X_{\alpha,\theta} = \bigcup_{l=0}^{p-1} (e^{i\theta})^l K_{\alpha,\theta},$$

where $K_{\alpha,\theta} = \psi_1(K_{\alpha,\theta}) \cup \psi_2(K_{\alpha,\theta})$ *is the self-similar set generated by the iterated function system* (*IFS*):

$$\begin{cases} \psi_1(z) = \alpha z, \\ \psi_2(z) = (\alpha e^{i\theta})z + \alpha. \end{cases} \tag{1.1}$$

Remark 1.3. Using a similar approach as in [4], Young [8] essentially found the result of Theorem 1.2 under the RTG Undergraduate Summer Research Program; however, his proof was incomplete.

Example 1.4. Both Mizutani and Ito's and Kawamura's results are included in this setting as special cases. It is clear that $X_{\alpha,\theta}$ is a union of Dragons if $\alpha = \frac{1-i}{2}$ and $\theta = -\pi/2$. If $\alpha = \frac{1-i}{2}$ and $\theta = \pi/2$, then $X_{\alpha,\theta}$ is a union of Lévy's curves generated by

$$\begin{cases} \psi_1(z) = \left(\dfrac{1-i}{2}\right)z, \\ \psi_2(z) = \left(\dfrac{1+i}{2}\right)z + \dfrac{1-i}{2}. \end{cases} \tag{1.2}$$

Notice that (1.2) is different from (0.1). Let P be the self-similar set $P = \psi_1(P) \cup \psi_2(P)$ generated by (1.2). It is clear that $\overline{L} = P$ since

$$\overline{L} = \overline{\phi_1(L)} \cup \overline{\phi_2(L)} = \psi_1(\overline{L}) \cup \psi_2(\overline{L}).$$

Recall the celebrated theorem of Hutchinson [3]: for any finite family of similar contractions $\psi_1, \psi_2, \ldots, \psi_m$ on $\mathbb{R}^n$, there exists a unique self-similar set $X \subset \mathbb{R}^n$, which is a unique non-empty compact solution of the set equation $X = \psi_1(X) \cup \psi_2(X) \cup \cdots \cup \psi_m(X)$. However, the converse is not true. In fact, a self-similar set can be constructed by many different families of similar contractions.

One of the challenges of this type of question is to find a suitable pair of contractions which matches the position of the set $X_{\alpha,\theta}$ exactly. Once the suitable pair of contractions is found, a more direct proof is possible, using the following lemma and proposition.

Define a subset of $X_{\alpha,\theta}$ as follows.

$$X_{1,\alpha,\theta} := \left\{ \sum_{n=1}^{\infty} \delta_n \alpha^n : \delta_{j_1} = 1, (\delta_1, \delta_2, \delta_3, \ldots) \in W_\theta \right\}, \tag{1.3}$$

where $j_1 := \min\{j : \delta_j \neq 0\}$.

Lemma 1.5. *$X_{1,\alpha,\theta}$ is a closed set.*

Proof. Let (x_k) be a sequence in $X_{1,\alpha,\theta}$, converging to some point x. For each $k \in \mathbb{N}$, there exists a sequence $(\delta_n^k) \in W_\theta$ such that $x_k = \sum_{n=1}^\infty \delta_n^k \alpha^n$.

It suffices to construct a sequence (δ_n) such that for each $n \geq 1$, there exists $k \in \mathbb{N}$ such that (δ_n) and (δ_n^k) start with the same initial word of length n. Obviously, then $x = \sum_{n=1}^\infty \delta_n \alpha^n \in X_{1,\alpha,\theta}$.

A suitable sequence (δ_n) can be constructed by induction as follows. First, choose $\delta_1 \in \{0, 1\}$ so that there exist infinitely many $k \in \mathbb{N}$ such that $\delta_1^k = \delta_1$. Next, suppose an initial word $\delta_1, \delta_2, \dots, \delta_n$ has been defined for some $n \geq 1$ so that $\delta_1^k \delta_2^k \dots \delta_n^k = \delta_1 \delta_2 \dots \delta_n$ for infinitely many $k \in \mathbb{N}$. Then choose $\delta_{n+1} \in \{0, e^{i\theta}\delta_{j_0(n)}\}$, where $j_0(n) := \max\{j \leq n : \delta_j \neq 0\}$, in such a way that there exist infinitely many $k \in \mathbb{N}$ such that

$$\delta_1^k \delta_2^k \dots \delta_n^k \delta_{n+1}^k = \delta_1 \delta_2 \dots \delta_n \delta_{n+1}.$$

This gives the desired sequence (δ_n). □

Corollary 1.6. *$X_{\alpha,\theta}$ is a closed set.*

Proof. Notice that

$$X_{\alpha,\theta} = \bigcup_{l=0}^{p-1} (e^{i\theta})^l X_{1,\alpha,\theta}.$$

Using the fact that the union of finitely many closed set is closed, $X_{\alpha,\theta}$ is closed. □

Proposition 1.7. *$X_{1,\alpha,\theta}$ satisfies the set equation*:

$$X_{1,\alpha,\theta} = \psi_1(X_{1,\alpha,\theta}) \cup \psi_2(X_{1,\alpha,\theta}),$$

where $\{\psi_1, \psi_2\}$ is the IFS from (1.1).

Proof. Let $x = \sum_{n=1}^\infty \delta_n \alpha^n \in X_{1,\alpha,\theta}$. If $\delta_1 = 0$, set $\delta_j' := \delta_{j+1}$ for $j = 1, 2, \dots$. Then (δ_j') satisfies the generalized revolving condition with its first nonzero digit equal to 1, so

$$x = \alpha \sum_{j=1}^\infty \delta_j' \alpha^j \in \psi_1(X_{1,\alpha,\theta}).$$

If $\delta_1 = 1$, set $\delta_j' := e^{-i\theta}\delta_{j+1}$ for $j = 1, 2, \dots$. Since the second nonzero digit of (δ_n) is $e^{i\theta}$, the sequence (δ_j') satisfies the generalized revolving condition with its first nonzero digit equal to 1, so

$$x = \alpha + \alpha e^{i\theta} \sum_{j=1}^{\infty} \delta'_j \alpha^j \in \psi_2(X_{1,\alpha,\theta}).$$

Thus $X_{1,\alpha,\theta} \subset \psi_1(X_{1,\alpha,\theta}) \cup \psi_2(X_{1,\alpha,\theta})$.

The reverse inclusion follows analogously. Let $x \in \psi_1(X_{1,\alpha,\theta}) \cup \psi_2(X_{1,\alpha,\theta})$. If $x = \alpha \sum_{n=1}^{\infty} \delta_n \alpha^n \in \psi_1(X_{1,\alpha,\theta})$, set

$$\delta'_j := \begin{cases} 0 & \text{if } j = 1, \\ \delta_{j-1} & \text{if } j \geq 2. \end{cases}$$

Then (δ'_j) satisfies the generalized revolving condition with its first nonzero digit equal to 1, so

$$x = \sum_{n=2}^{\infty} \delta_{n-1} \alpha^n = \sum_{j=1}^{\infty} \delta'_j \alpha^j \in X_{1,\alpha,\theta}.$$

If $x = (\alpha e^{i\theta}) \sum_{n=1}^{\infty} \delta_n \alpha^n + \alpha \in \psi_2(X_{1,\alpha,\theta})$, set

$$\delta'_j := \begin{cases} 1 & \text{if } j = 1, \\ e^{i\theta} \delta_{j-1} & \text{if } j \geq 2. \end{cases}$$

Since the second nonzero digit of (δ'_j) is $e^{i\theta}$, the sequence (δ'_j) satisfies the generalized revolving condition with its first nonzero digit equal to 1, so

$$x = \sum_{n=2}^{\infty} (e^{i\theta} \delta_{n-1}) \alpha^n + \alpha = \sum_{j=1}^{\infty} \delta'_j \alpha^j \in X_{1,\alpha,\theta}.$$

Thus, $\psi_1(X_{1,\alpha,\theta}) \cup \psi_2(X_{1,\alpha,\theta}) \subset X_{1,\alpha,\theta}$. □

Proof of Theorem 1.2. Since the set equation $X = \psi_1(X) \cup \psi_2(X)$ has a unique nonempty compact solution, Theorem 1.2 follows immediately from Lemma 1.5 and Proposition 1.7. □

2. Signed revolving sequences

Theorem 1.2 shows a direct relationship between generalized revolving sequences and self-similar sets generated by the IFS from (1.1):

$$\begin{cases} \psi_1(z) = \alpha z, \\ \psi_2(z) = (\alpha e^{i\theta}) z + \alpha. \end{cases}$$

Many interesting self-similar sets are generated by (1.1); however, Koch's curve, a famous self-similar set, is not generated by (1.1) but by a different pair of two similar contractions:

$$\begin{cases} \psi_1(z) = \alpha \bar{z}, \\ \psi_2(z) = (\alpha e^{i\theta})\bar{z} + \alpha. \end{cases} \tag{2.1}$$

In particular, if $\alpha = 1/2 + (\sqrt{3}/6)i, \theta = -\pi/3$, the IFS (2.1) generates Koch's curve.

A reversed question arises naturally: what kind of revolving sequences are related to self-similar sets generated by the IFS (2.1)? More precisely, given the attractor $K^2_{\alpha,\theta}$ of the IFS (2.1), we want to find a suitable set of "revolving" sequences such that the analog of the set $X_{\alpha,\theta}$ from Section 1 is

$$\bigcup_{l=0}^{p-1} (e^{i\theta})^l K^2_{\alpha,\theta}.$$

Recall that $\alpha \in \mathbb{C}$ is a complex parameter satisfying $|\alpha| < 1$ and θ be an angle with $|\theta| = \frac{2\pi q}{p}$ where $p \in \mathbb{N}, q \in \mathbb{N}_0$. The generalized revolving sequences from Section 1 always follow a fixed direction on the unit circle, depending on the given θ. How does the introduction of complex conjugates in the IFS influence the corresponding type of revolving sequences?

Definition 2.1. A sequence $(\delta_1, \delta_2, \dots) \in \Delta_\theta^{\mathbb{N}}$ satisfies the *Signed Revolving Condition* (*SRC*), if

(1) δ_1 is free to choose;

(2) if $\delta_1 = \delta_2 = \cdots = \delta_k = 0$, then δ_{k+1} is free to choose;

(3) otherwise, $\delta_{k+1} = 0$ or

$$\delta_{k+1} = \begin{cases} (e^{+i\theta})\delta_{j_0(k)} & \text{if } j_0(k) \text{ is odd}, \\ (e^{-i\theta})\delta_{j_0(k)} & \text{if } j_0(k) \text{ is even}, \end{cases}$$

where $j_0(k) := \max\{j \le k : \delta_j \neq 0\}$.

Roughly speaking, $j_0(k)$ is the last time before time k that δ_j is on the unit circle.

Notice that δ_n is either zero or lies on the unit circle, and its direction of motion (that is, where it moves to at time n) depends on the last time $j < n$ when δ_j is on the unit circle. If the last visit to the unit circle happened at an even time, then δ_n moves clockwise along the circle. On the other hand, if the last visit to the unit circle happened at an even time, then δ_n moves counterclockwise along the circle. For example,

$$0 \longrightarrow 1 \longrightarrow e^{-i\theta} \longrightarrow 0 \longrightarrow 1 \longrightarrow e^{i\theta} \longrightarrow 0 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots .$$

Compared to the generalized revolving sequences from Section 1, which always move in the same direction, we see that the direction of movement of the sequence (δ_n) depends on its past.

Define $W_\theta^{\pm}$ as the set of all signed revolving sequences with parameter θ:

$$W_\theta^{\pm} := \{(\delta_1, \delta_2, \dots) \in \Delta_\theta^{\mathbb{N}} : (\delta_1, \delta_2, \dots \delta_k, \dots) \text{ satisfies the SRC}\},$$

and for a given $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$, define

$$X_{\alpha,\theta}^2 := \left\{ \sum_{n=1}^{\infty} \delta_n \prod_{j=1}^{n} \eta_j : (\delta_1, \delta_2, \dots) \in W_\theta^{\pm} \right\},$$

where $\eta_1 = \alpha$ and $\eta_{j+1} = \overline{\eta_j}$ for $j = 1, 2, \dots$. Four examples of $X_{\alpha,\theta}^2$ are shown in Figure 4.

Let $j_1 := \min\{j : \delta_j \neq 0\}$ and define a subset of $X_{\alpha,\theta}^2$ as follows:

$$X_{1,\alpha,\theta}^2 = \left\{ \sum_{n=1}^{\infty} \delta_n \prod_{j=1}^{n} \eta_j : \delta_{j_1} = 1, (\delta_1, \delta_2, \dots) \in W_\theta^{\pm} \right\}. \tag{2.2}$$

A straightforward modification of the proof of Lemma 1.5 gives the following Lemma 2.2 and Corollary 2.3.

Lemma 2.2. *$X_{1,\alpha,\theta}^2$ is a closed set.*

Corollary 2.3. *$X_{\alpha,\theta}^2$ is a closed set.*

Proposition 2.4. *$X_{1,\alpha,\theta}^2$ satisfies the set equation*

$$X_{1,\alpha,\theta}^2 = \psi_1(X_{1,\alpha,\theta}^2) \cup \psi_2(X_{1,\alpha,\theta}^2),$$

where $\{\psi_1, \psi_2\}$ is the IFS from (2.1).

Proof. Let $x = \sum_{n=1}^{\infty} \delta_n \prod_{l=1}^{n} \eta_l \in X_{1,\alpha,\theta}^2$. If $\delta_1 = 0$, set $\delta'_j := \overline{\delta_{j+1}}$ for $j = 1, 2, \dots$. Then (δ'_j) satisfies the signed revolving condition with its first

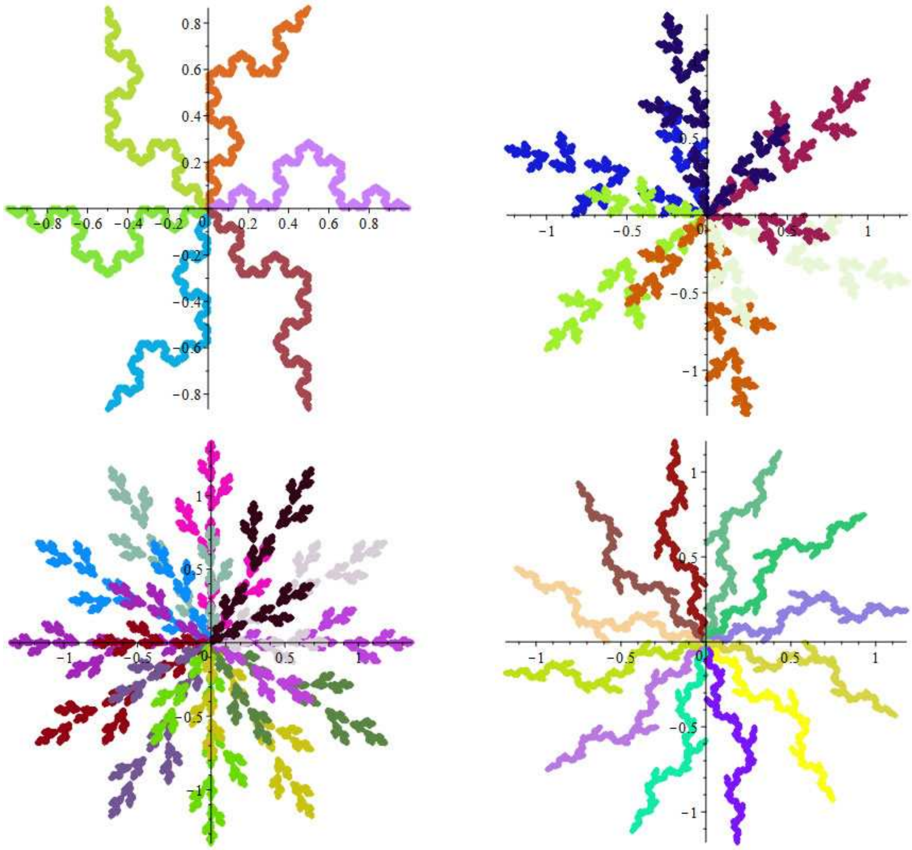

Figure 4. $X^2_{\alpha,\theta}$: $(\alpha,\theta) = \big(\frac{1}{2} + \frac{\sqrt{3}i}{6}, -\frac{\pi}{3}\big)$ (top left), $(\alpha,\theta) = \big(\frac{1}{2} + \frac{\sqrt{3}i}{6}, \frac{\pi}{3}\big)$ (top right), $(\alpha,\theta) = \big(\frac{1}{2} + \frac{\sqrt{3}i}{6}, \frac{\pi}{6}\big)$ (bottom left), $(\alpha,\theta) = \big(\frac{1}{2} + \frac{\sqrt{3}i}{6}, -\frac{\pi}{6}\big)$ (bottom right).

nonzero digit equal to 1, so

$$x = \alpha \sum_{j=1}^{\infty} \overline{\delta'_j} \prod_{l=1}^{j} \overline{\eta_l} = \alpha \overline{\sum_{j=1}^{\infty} \delta'_j \prod_{l=1}^{j} \eta_l} \in \psi_1(X^2_{1,\alpha,\theta}).$$

If $\delta_1 = 1$, set $\delta'_j := e^{i\theta}\overline{\delta_{j+1}}$ for $j = 1, 2, \ldots$. Since the second nonzero digit of (δ_n) is $e^{i\theta}$, the sequence (δ'_j) satisfies the signed revolving condition with its first nonzero digit equal to 1, so

$$x = \alpha + \alpha e^{i\theta} \sum_{j=1}^{\infty} \overline{\delta'_j} \prod_{l=1}^{j} \overline{\eta_l} \in \psi_2(X^2_{1,\alpha,\theta}).$$

Thus $X^2_{1,\alpha,\theta} \subset \psi_1(X^2_{1,\alpha,\theta}) \cup \psi_2(X^2_{1,\alpha,\theta})$. The reverse inclusion follows analogously. □

Since the set equation $X = \psi_1(X) \cup \psi_2(X)$ has a unique nonempty compact solution, Theorem 2.5 follows immediately from Lemma 2.2 and Proposition 2.4.

Theorem 2.5. *Let $K^2_{\alpha,\theta}$ be the self-similar set generated by the IFS from (2.1). Then,*

$$\bigcup_{k=0}^{p-1} (e^{i\theta})^k K^2_{\alpha,\theta} = X^2_{\alpha,\theta}.$$

Remark 2.6. It is interesting to note that, while $K^2_{\alpha,\theta}$ is the attractor of an autonomous IFS (where the maps applied at each step do not change), its representation by a set of revolving sequences involves a rule that is past-dependent.

3. Alternating sequences

Both Propositions 1.7 and 2.4 gave more explicit description of certain self-similar sets generated by the IFS from (1.1) and (2.1) respectively. In these two iterated function systems, either both maps or neither involve a reflection. But what happens if exactly one of the maps includes a reflection? For example, what kind of revolving sequences are related to self-similar sets generated by the IFS

$$\begin{cases} \psi_1(z) = \alpha z, \\ \psi_2(z) = (\alpha e^{i\theta})\bar{z} + \alpha, \end{cases} \tag{3.1}$$

where $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$ and $|\theta| = \left|\frac{2\pi q}{p}\right| \leq \pi$?

(Notice that the self-similar sets generated by

$$\begin{cases} \psi_1(z) = \alpha \bar{z}, \\ \psi_2(z) = (\alpha e^{i\theta}) z + \alpha, \end{cases}$$

are essentially the same as those generated by (3.1), so this fourth case does not require separate treatment.)

As in Section 2, we want to find a suitable set of "revolving" sequences $X^3_{1,\alpha,\theta}$ satisfying the set equation

$$X^3_{1,\alpha,\theta} = \psi_1(X^3_{1,\alpha,\theta}) \cup \psi_2(X^3_{1,\alpha,\theta}),$$

where ψ_1 and ψ_2 are the maps in (3.1).

Surprisingly, $X^3_{1,\alpha,\theta}$ is not parametrized by a set of "revolving" sequences but by what we call "alternating" sequences.

Definition 3.1. A sequence $(\delta_1, \delta_2, \dots) \in \Delta_\theta^{\mathbb{N}}$ satisfies *the Alternating Condition* (*AC*), if

(1) δ_1 is free to choose;

(2) if $\delta_1 = \delta_2 = \cdots = \delta_k = 0$, then δ_{k+1} is free to choose;

(3) otherwise, $\delta_{k+1} = 0$ or

$$\delta_{k+1} = \begin{cases} (e^{+i\theta})\delta_{j_0(k)}, & \text{if } N_{j_0(k)} \text{ is odd,} \\ (e^{-i\theta})\delta_{j_0(k)}, & \text{if } N_{j_0(k)} \text{ is even,} \end{cases}$$

where $j_0(k) := \max\{j \le k : \delta_j \neq 0\}$ and $N_{j_0(k)} := \#\{j \le j_0(k) : \delta_j \neq 0\}$.

Roughly speaking, $N_{j_0}(k)$ is the number of times until $j_0(k)$ that δ_j is on the unit circle. Notice that any $\delta_k \neq 0$ must alternate between two values on the unit circle. For example, the following sequence satisfies the AC:

$$0 \longrightarrow 0 \longrightarrow 1 \longrightarrow e^{i\theta} \longrightarrow 0 \longrightarrow 1 \longrightarrow 0 \longrightarrow e^{i\theta} \longrightarrow 0 \longrightarrow \cdots .$$

Define W_θ^A as the set of all alternating sequences with parameter θ:

$$W_\theta^A := \{(\delta_1, \delta_2, \dots) \in \Delta_\theta^{\mathbb{N}} : (\delta_1, \delta_2, \dots) \text{ satisfies the AC}\},$$

and for a given $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$, define

$$X^3_{\alpha,\theta} := \Big\{ \sum_{n=1}^{\infty} \delta_n \prod_{j=1}^{n} \xi_j : (\delta_1, \delta_2, \dots) \in W_\theta^A \Big\},$$

where $\xi_1 = \alpha$ and

$$\xi_{j+1} = \begin{cases} \xi_j & \text{if } \delta_j = 0, \\ \overline{\xi_j} & \text{if } \delta_j \neq 0, \end{cases} \tag{3.2}$$

for $j > 0$.

Four examples of $X^3_{\alpha,\theta}$ are shown in Figure 5. Notice that $X^3_{\alpha,\theta}$ has a significant difference from $X_{\alpha,\theta}$ and $X^2_{\alpha,\theta}$: the $\prod_{j=1}^n \xi_j$ term found in $X^3_{\alpha,\theta}$ depends on the behavior of the sequence $(\delta_1, \delta_2, \cdots, \delta_n)$, while the products in $X_{\alpha,\theta}$ and $X^2_{\alpha,\theta}$ do not depend on that sequence.

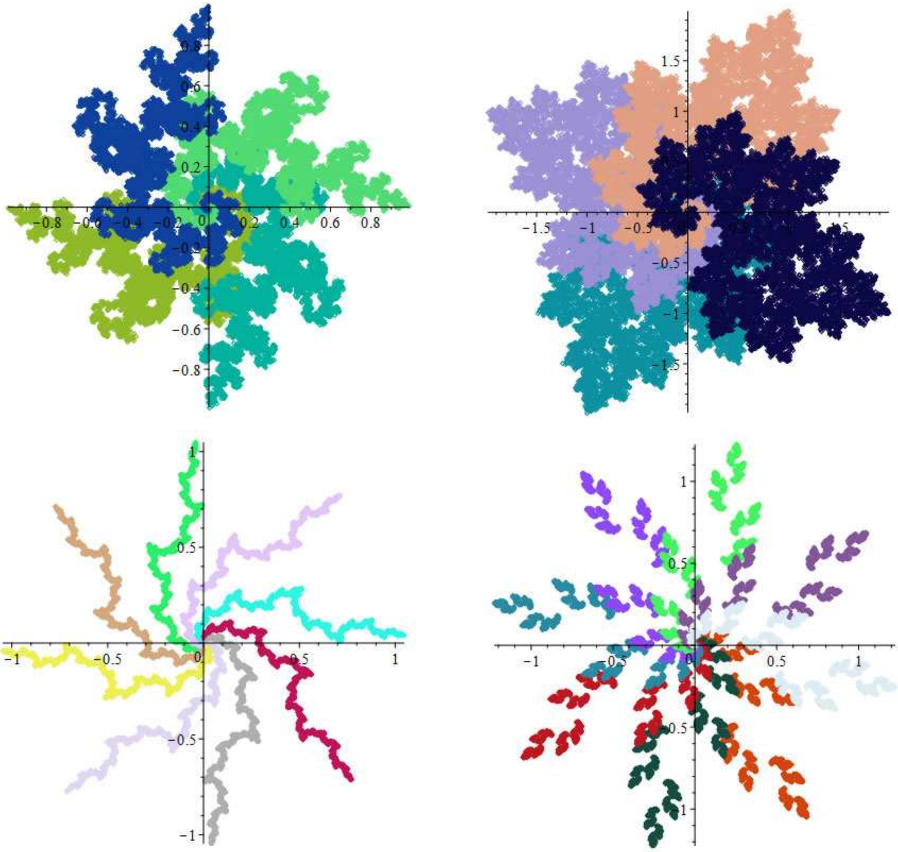

Figure 5. $X^3_{\alpha,\theta}$: $(\alpha,\theta) = \left(\frac{1+i}{2}, \frac{\pi}{2}\right)$ (top left), $(\alpha,\theta) = \left(\frac{1+i}{2}, -\frac{\pi}{2}\right)$ (top right), $(\alpha,\theta) = \left(\frac{2+i}{4}, \frac{\pi}{4}\right)$ (bottom left), $(\alpha,\theta) = \left(\frac{2+i}{4}, -\frac{\pi}{4}\right)$ (bottom right).

Let $j_1 := \min\{j : \delta_j \neq 0\}$ and define a subset of $X^3_{\alpha,\theta}$ as follows.

$$X^3_{1,\alpha,\theta} = \left\{ \sum_{n=1}^{\infty} \delta_n \prod_{j=1}^{n} \xi_j : \delta_{j_1} = 1, (\delta_1, \delta_2, \ldots) \in W^A_\theta \right\},$$

A straightforward modification of the proof of Lemma 1.5 gives the following Lemma 3.2 and Corollary 3.3.

Lemma 3.2. *$X^3_{1,\alpha,\theta}$ is a closed set.*

Corollary 3.3. *$X^3_{\alpha,\theta}$ is a closed set.*

Proposition 3.4. *$X^3_{1,\alpha,\theta}$ satisfies the set equation*

$$X^3_{1,\alpha,\theta} = \psi_1(X^3_{1,\alpha,\theta}) \cup \psi_2(X^3_{1,\alpha,\theta}),$$

where $\{\psi_1, \psi_2\}$ is the IFS from (3.1).

Proof. Let $x = \sum_{n=1}^{\infty} \delta_n \prod_{l=1}^{n} \xi_l \in X^3_{1,\alpha,\theta}$, where (ξ_l) depends on (δ_n) as in (3.2). If $\delta_1 = 0$, set $\delta'_j := \delta_{j+1}$ and $\xi'_j := \xi_{j+1}$ for $j = 1, 2, \ldots$. Then (δ'_j) satisfies the alternating condition with its first nonzero digit equal to 1, and (ξ'_j) depends on (δ'_j) as in (3.2), with first term α. So

$$x = \alpha \sum_{j=1}^{\infty} \delta'_j \prod_{l=1}^{j} \xi'_l \in \psi_1(X^3_{1,\alpha,\theta}).$$

If $\delta_1 = 1$, set $\delta'_j := e^{i\theta}\overline{\delta_{j+1}}$ and $\xi'_j := \overline{\xi_{j+1}}$ for $j = 1, 2, \ldots$. Since the second nonzero digit of (δ_n) is $e^{i\theta}$, the sequence (δ'_j) satisfies the alternating condition with its first nonzero digit equal to 1, so

$$x = \alpha + \alpha e^{i\theta} \sum_{j=1}^{\infty} \overline{\delta'_j} \prod_{l=1}^{j} \overline{\xi'_l} \in \psi_2(X^3_{1,\alpha,\theta}).$$

Thus $X^3_{1,\alpha,\theta} \subset \psi_1(X^3_{1,\alpha,\theta}) \cup \psi_2(X^3_{1,\alpha,\theta})$. The reverse inclusion follows analogously. □

Since the set equation $X = \psi_1(X) \cup \psi_2(X)$ has a unique nonempty compact solution, Theorem 3.5 follows immediately from Lemma 3.2 and Proposition 3.4.

Theorem 3.5. *Let $K^3_{\alpha,\theta}$ be the self-similar set generated by the IFS from* (3.1). *Then*

$$\bigcup_{k=0}^{p-1} (e^{i\theta})^k K^3_{\alpha,\theta} = X^3_{\alpha,\theta}.$$

Remark 3.6. We originally found the results of Theorems 1.2, 2.5, and 3.5 using a technique similar to the one in [4] (considering the unique solution of a functional equation analogous to (0.2)). However, to avoid unnecessary technicalities, we have chosen to include only the more direct proofs using Propositions 1.7, 2.4, and 3.4. For the details of the original proofs, see [5].

Acknowledgments. This research was done mainly during a visit to Utrecht University, Netherlands. The first author is grateful to Prof. K. Dajani for her warm hospitality and encouragement. Also, we appreciate Prof. A. Vince for pointing out the simpler proof of Theorem 1.2 and a referee for providing us a proof of Lemma 1.5. Lastly, we greatly appreciate Prof. P. Allaart for his helpful comments and suggestions in preparing this paper.

References

[1] C. Davis and D. E. Knuth, Number representations and dragon curves I. *J. Recreational Math.* **3** (1970), no. 2, 66–81. MR 3287868

[2] K. J. Falconer, *Fractal geometry.* Mathematical foundations and applications. Third edition. John Wiley & Sons, hichester, 2014. MR 3236784 Zbl 1285.28011

[3] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747. MR 0625600 Zbl 0598.28011

[4] K. Kawamura, On the classification of self-similar sets determined by two contractions on the plane. *J. Math. Kyoto Univ.* **42** (2002), no. 2, 255–286. MR 1966837 Zbl 1048.28004

[5] K. Kawamura and A. Allen, Revolving fractals. Preprin, 2019. arXiv:1905.05924v1 [math.DS]

[6] P. Lévy, Les courbes planes ou gauches et les surfaces composées de parties semblables au tout, *J. Ecole Polytechn.* (3) **144** (1938), 227–247. JFM 64.0706.03 Zbl 0019.37201

[7] M. Mizutani and S. Ito, Dynamical systems on dragon domains. *Japan J. Appl. Math.* **4** (1987), no. 1, 23–46. MR 0899202 Zbl 0647.58029

[8] A. Young, unpublished manuscript, 2015.

Received January 7, 2020; revised July 10, 2020

Kiko Kawamura, Department of Mathematics, University of North Texas,
1155 Union Circle #311430, Denton, TX 76203-5017, USA

e-mail: kiko.kawamura@unt.edu

Andrew Allen, Department of Mathematics, University of North Texas,
1155 Union Circle #311430, Denton, TX 76203-5017, USA

e-mail: andrew.allen@unt.edu