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The formula for the quasicentral modulus in the case of spectral measures on fractals

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Abstract. We prove a general ampliation homogeneity result for the quasicentral modulus of an n-tuple of operators with respect to the $(p, 1)$ Lorentz normed ideal. We use this to prove a formula involving Hausdorff measure for the quasicentral modulus of n-tuples of commuting Hermitian operators whose spectrum is contained in certain Cantor-like self-similar fractals.

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1. Introduction

The quasicentral modulus $k_{\mathcal{J}}(\tau)$ is a number associated with an n-tuple τ of Hermitian operators relative to a normed ideal $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ of compact operators. It underlies many questions on normed ideal perturbations of n-tuples of operators (see the recent survey [11]), and it also had applications in non-commutative geometry in work on the spectral characterization of manifolds [1].

We proved in [7] that in the case of τ an n-tuple of commuting Hermitian operators and if the normed ideal is the $(n, 1)$-Lorentz ideal, which we denote by $\mathcal{C}_n^-$, the corresponding quasicentral modulus $k_n^-(\tau)$ has the property that $(k_n^-(\tau))^{1/n}$ is proportional to the integral with respect to n-dimensional Lebesgue measure of the multiplicity function of τ.

Here we prove a similar result in fractional dimension. More precisely, instead of a cube in $\mathbb{R}^n$ which contains the spectrum of τ, we assume there is a fixed self-similar fractal in $\mathbb{R}^n$ of Hausdorff dimension $p > 1$ containing the spectrum $\sigma(\tau)$.

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The analogous formula we prove has the exponent n replaced by p and the integral of the multiplicity function is with respect to p-Hausdorff measure. For technical reasons, the class of fractals is rather restricted: only certain totally disconnected sets, that is Cantor-like fractals are considered. One should certainly expect this can be extended to a larger class of fractals, the present paper being only a first step.

What made the extension of the formula to fractional dimension possible is a completely general ampliation homogeneity result $k_p^-(\tau \otimes I_m) = m^{1/p} k_p^-(\tau)$. Such a result was previously known only for $p = \infty$ (see [9]) and for $p = 1$ (see [7]). Note that in [7] we obtained easily an ampliation homogeneity result for $k_p(\tau)$, that is with $\mathcal{C}_p^-$ replaced by the Schatten–von Neumann class $\mathcal{C}_p$. For $p = 1$, we have $\mathcal{C}_1 = \mathcal{C}_1^-$, but for $p > 1$ the result for $k_p(\tau)$ turned out to be trivial after we showed in [8] that in this case $k_p(\tau) \in \{0, \infty\}$. The interesting quantity which replaces $k_p(\tau)$ is $k_p^-(\tau)$.

The paper has six sections including this introduction. Section 2 contains preliminaries about the quasicentral modulus. Section 3 is devoted to the ampliation homogeneity theorem. In Section 4 we collect preliminaries concerning the class of Cantor-type fractals we consider. The formula for $k_p^-(\tau)$ in the fractal setting is obtained in Section 5. Section 6 deals with concluding remarks.

2. Operator preliminaries

By $\mathcal{H}$ we denote a separable complex Hilbert space of infinite dimension and by $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, $\mathcal{R}(\mathcal{H})$ the bounded operators, the compact operators, and the finite rank operators, respectively. When no confusion can arise, we will simply write $\mathcal{K}$, $\mathcal{R}$ and we will denote by $\mathcal{R}_1^+(\mathcal{H})$ or $\mathcal{R}_1^+$ the finite rank positive contractions $0 \le A \le I$ on $\mathcal{H}$. The $(p, 1)$ Lorentz normed ideal of compact operators will be denoted by $(\mathcal{C}_p^-, |\cdot|_p^-)$. We recall that the norm is

$$|T|_p^- = \sum_{j \in \mathbb{N}} s_j \ j^{-1+1/p}$$

where $s_1 \ge s_2 \ge \cdots$ are the eigenvalues of $|T| = (T^*T)^{1/2}$ in decreasing order. If $(\mathcal{C}_p, |\cdot|_p)$ is the Schatten–von Neumann p-class, then $\mathcal{C}_1 = \mathcal{C}_1^-$. More on normed ideals can be found in [5] and [6].

We shall also use the following notation for operations on n-tuples of operators, in line with [7]. If $\tau = (T_i)_{1\le i\le n} \in (\mathcal{B}(\mathcal{H}))^n$ and $X, Y \in \mathcal{B}(\mathcal{H})$, then we use

$$\begin{aligned} X\tau Y &= (XT_iY)_{1\le i\le n}, \\ [X,\tau] &= ([X,T_i])_{1\le i\le n}, \\ \tau^* &= (T_i^*)_{1\le i\le n}. \end{aligned}$$

If also $\sigma = (S_i)_{1\le i\le n} \in (\mathcal{B}(\mathcal{H}))^n$, then we write

$$\begin{aligned} \sigma + \tau &= (S_i + T_i)_{1\le i\le n}, \\ \sigma \oplus \tau &= (S_i \oplus T_i)_{1\le i\le n}, \\ \tau \otimes I_m &= (T_i \otimes I_m)_{1\le i\le n}, \end{aligned}$$

where I_m is the identity operator on $\mathbb{C}^m$. When we identify $\mathcal{H} \otimes \mathbb{C}^m$ and $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, then we also have

$$\tau \otimes I_m \cong \underbrace{\tau \oplus \cdots \oplus \tau}_{m\text{-times}}.$$

Further, we consider norms

$$\|\tau\| = \max_{1\le i\le n} \|T_i\|, \quad |\tau|_{\mathcal{J}} = \max_{1\le i\le n} |T_i|_{\mathcal{J}}.$$

The quasicentral modulus of an n-tuple $\tau = (T_i)_{1\le i\le n}$ with respect to a normed ideal $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ (see [7] and [9]) is the number

$$k_{\mathcal{J}}(\tau) = \liminf_{A\in\mathcal{R}_1^+} |[\tau, A]|_{\mathcal{J}},$$

where the lim inf is with respect to the natural order on $\mathcal{R}_1^+$. This definition is also equivalent to $k_{\mathcal{J}}(\tau)$ being

$$\inf\left\{\alpha \in [0,\infty] \mid \alpha = \lim_{k\to\infty} |[A_k, \tau]|_{\mathcal{J}},\ A_k \uparrow I,\ A_k \in \mathcal{R}_1^+\right\}$$

or the same with $w - \lim_{k\to\infty} A_k = I$ instead of $A_k \uparrow I$. If $\mathcal{J} = \mathcal{C}_p^-$ we denote $k_{\mathcal{J}}(\tau)$ by $k_p^-(\tau)$.

We should also record as the next proposition the results in [7, Proposition 1.4 and Proposition 1.6].

Proposition 2.1. *If* $\tau^{(j)} \in \mathcal{B}(\mathcal{H})^n$, $j \in \mathbb{N}$, *then*

$$\max_{j=1,2} k_{\mathcal{J}}(\tau^{(j)}) \leq k_{\mathcal{J}}(\tau^{(1)} \oplus \tau^{(2)}) \leq k_{\mathcal{J}}(\tau^{(1)}) + k_{\mathcal{J}}(\tau^{(2)})$$

and

$$k_{\mathcal{J}}\Big(\bigoplus_{j\in\mathbb{N}}\tau^{(j)}\Big) = \lim_{m_i\to\infty} k_{\mathcal{J}}\Big(\bigoplus_{1\leq j\leq m}\tau^{(j)}\Big).$$

If $\lambda^{(j)} \in \mathbb{C}^n$ *and* $\lambda^{(j)} \otimes I_{\mathcal{H}} \in \mathcal{B}(\mathcal{H})^n$, *then*

$$k_{\mathcal{J}}(\tau^{(1)} \oplus \cdots \oplus \tau^{(m)}) = k_{\mathcal{J}}\big((\tau^{(1)} - \lambda^{(1)} \otimes I_{\mathcal{H}}) \oplus \cdots \oplus (\tau^{(m)} - \lambda^{(m)} \otimes I_{\mathcal{H}})\big).$$

Finally, if τ is an n-tuple of commuting Hermitian operators, then we denote by $\sigma(\tau) \subset \mathbb{R}^n$ the joint spectrum and by $E(\tau;\omega)$ the spectral projection of τ for the Borel set $\omega \subset \mathbb{R}^n$.

3. Ampliation homogeneity

Theorem 3.1. *If* τ *is an* n*-tuple of bounded operators and* $1 \leq p \leq \infty$, *then*

$$k_p^-(\tau \otimes I_m) = m^{1/p} k_p^-(\tau).$$

The cases $p = 1$ and $p = \infty$ have already been proved ([7, Proposition 1.5] and [9, Proposition 3.9]).

We begin the proof with a couple of lemmas.

Lemma 3.1. *Let* $X_j \in \mathcal{C}_p^-$, $j \in \mathbb{N}$, $p \in [1,\infty]$ *be so that* $|X_j|_p^- \leq C$. *If* $\lim_{j\to\infty} \|X_j\| = 0$, *then*

$$\lim_{j\to\infty} (|X_j \otimes I_m|_p^- - m^{1/p}|X_j|_p^-) = 0.$$

Proof. Let $s_1^{(j)} \geq s_2^{(j)} \geq \cdots$ be the eigenvalues of $(X_j^* X_j)^{1/2}$. Then,

$$\begin{aligned}|X_j \otimes I_m|_p^- &= \sum_{k\in\mathbb{N}} s_k^{(j)}\big((m(k-1)+1)^{-1+1/p} + \cdots + (mk)^{-1+1/p}\big)\\ &\geq m^{1/p} \sum_{k\in\mathbb{N}} s_k^{(j)} k^{-1+1/p} = m^{1/p}|X_j|_p^-.\end{aligned}$$

On the other hand, given $\epsilon > 0$ there is N so that

$$k \geq N \implies (m(k-1)+1)^{-1+1/p} + \cdots + (mk)^{-1+1/p} \leq (1+\epsilon)m^{1/p}k^{-1+1/p}.$$

This gives

$$|X_j \otimes I_m|_p^- \le (1+\epsilon)m^{1/p}\sum_{k\ge N} s_k^{(j)} k^{-1+1/p} + N\|X_j\|$$
$$\le (1+\epsilon)m^{1/p}|X_j|_p^- + N\|X_j\|.$$

Thus,

$$0 \le |X_j \otimes I_m|_p^- - m^{1/p}|X_j|_p^- \le \epsilon m^{1/p}|X_j|_p^- + N\|X_j\|.$$

Since $\epsilon > 0$ is arbitrary and $\|X_j\| \to 0$, we get the desired result when $j \to \infty$. □

Corollary 3.1. *Let $X_j = (X_{ji})_{1\le i\le n}$ be n-tuples of operators so that $|X_j|_p^- \le C$ where $p \in [1,\infty]$ and $\lim_{j\to\infty}\|X_j\| = 0$. Then,*

$$\lim_{j\in\infty}(m^{1/p}|X_j|_p^- - |X_j \otimes I_m|_p^-) = 0.$$

Lemma 3.2. *If $\tau \in (\mathcal{B}(\mathcal{H}))^n$ and $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ is a normed ideal so that $k_{\mathcal{J}}(\tau) < \infty$, then there are $B_j \in \mathcal{R}_1^+$ so that $B_j \uparrow I$ and*

$$\lim_{j\to\infty}|[\tau, B_j]|_{\mathcal{J}} = k_J(\tau), \quad \lim_{j\to\infty}\|[\tau, B_j]\| = 0.$$

Proof. It suffices to show that given $\epsilon > 0$ and P a finite rank Hermitian projector we can find $B \in \mathcal{R}_1^+$ so that $B \ge P$ and

$$|[B,\tau]|_{\mathcal{J}} \le k_{\mathcal{J}}(\tau) + \epsilon, \quad \|[B,\tau]\| \quad \le \epsilon.$$

Such B can be constructed as follows. We find recursively

$$P = P_1 \le P_2 \le P_3 \le \cdots$$

finite rank Hermitian projectors and $A_j \in \mathcal{R}_1^+$ so that

$$A_j \ge P_j, \quad |[A_j,\tau]|_{\mathcal{J}} \le k_{\mathcal{J}}(\tau) + \epsilon,$$
$$\tau P_j = P_{j+1}\tau P_j, \quad \tau^* P_j = P_{j+1}\tau^* P_j, \quad P_{j+1} \ge A_j.$$

If we put $Q_j = P_{j+1} - P_j$ if $j \ge 1$ and $Q_0 = P_1 = P$, then

$$Q_r \tau Q_s \ne 0 \implies |r-s| \le 1$$

and

$$A_j = (Q_0 + \cdots + Q_{j-1}) + Q_j A_j Q_j.$$

This gives

$$Q_r[\tau, A_j]Q_s \neq 0 \implies |r-s| \leq 1 \text{ and } j-1 \leq r, s \leq j+1.$$

It follows that if $|k-j| \geq 4$ $([\tau, A_j])^*[\tau, A_k] = 0$, then

$$\|[\tau, A_4 + A_8 + \cdots + A_{4N}]\| \leq 2\|\tau\|.$$

Thus, if $B_N = N^{-1}(A_4 + A_8 + \cdots + A_{4N})$, then $B_N \geq P$, $B_N \in \mathcal{R}_1^+$, $|[B_N, \tau]|_{\mathcal{J}} < k_{\mathcal{J}}(\tau) + \epsilon$, and

$$\|[B_N, \tau]\| \leq 2N^{-1}\|\tau\|.$$

Thus, if $2N^{-1}\|\tau\| < \epsilon$, we may take $B = B_N$. □

Lemma 3.3. *If $m \in \mathbb{N}$ and $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ is a normed ideal and $\tau \in (B(\mathcal{H}))^n$ is so that $k_{\mathcal{J}}(\tau) < \infty$, then there are $A_j \in \mathcal{R}_1^+$, $A_j \uparrow I$ so that*

$$k_{\mathcal{J}}(\tau \otimes I_m) = \lim_{j\to\infty} |[\tau \otimes I_m, A_j \otimes I_m]|_{\mathcal{J}}$$

and $\lim_{j\to\infty} \|[\tau, A_j]\| = 0$.

Proof. Let G be the group $\mathcal{S}_m \rtimes \mathbb{Z}_2^m$ of permutation matrices with ± 1 entries and $g \to U_g$ its representation on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ which is $\otimes I_{\mathcal{H}}$ the representation on $\mathbb{C}^m$. Then, the commutant $\{U_g : g \in G\}'$ is $B(\mathcal{H}) \otimes I_m$ and the map $\Phi : B(\mathcal{H}^m) \to \mathcal{B}(\mathcal{H}) \otimes I_m$ given by $\Phi(X) = |G|^{-1} \sum_{g\in G} U_g X U_g^*$ is the projection of norm one which preserves the trace. If $B \in R_1^+(\mathcal{H}^m)$, then

$$|[B, \tau \otimes I_m]|_{\mathcal{J}} = |U_g[B, \tau \otimes I_m]U_g^*|_{\mathcal{J}} = |[U_g B U_g^*, \tau \otimes I_m]|_{\mathcal{J}},$$

which gives by taking the mean over G

$$|[\Phi(B), \tau \otimes I_m]|_{\mathcal{J}} \leq |[B, \tau \otimes I_m]|_{\mathcal{J}}$$

and clearly $B_j \uparrow I \otimes I_m$ implies $\Phi(B_j) \uparrow I \otimes I_m$. Thus, if $A_j \otimes I_m = \Phi(B_j)$ and $B_j \uparrow I \otimes I_m$ are so that

$$\lim_{j\to\infty} |[B_j, \tau \otimes I_m]|_{\mathcal{J}} = k_{\mathcal{J}}(\tau \otimes I_m)$$

and

$$\lim_{j\to\infty} |[B_j, \tau \otimes I_m]\| = 0,$$

then

$$\limsup_{j\to\infty} |[A_j \otimes I_m, \tau \otimes I_m]|_{\mathcal{J}} \leq k_{\mathcal{J}}(\tau \otimes I_m)$$

and

$$\lim_{j\to\infty} \|[A_j \otimes I, \tau \otimes I_m]\| = 0.$$

Since, on the other hand,

$$\liminf_{j\to\infty} |[A_j \otimes I_m, \tau \otimes I_m]|_{\mathcal{J}} \geq k_{\mathcal{J}}(\tau \otimes I_m),$$

we conclude that

$$\lim_{j\to\infty} |[A_j \otimes I_m, \tau \otimes I_m]|_{\mathcal{J}} = k_{\mathcal{J}}(\tau).$$ □

Proof of Theorem 3.1. Using Lemma 3.3, we can find $A_j \in \mathcal{R}_1^+$, $A_j \uparrow I$ when $j \to \infty$, so that

$$\lim_{j\to\infty} |[A_j, \tau] \otimes I_m]|_p^- = k_p^-(\tau \otimes I_m)$$

and

$$\lim_{j\to\infty} \|[A_j, \tau]\| = 0.$$

Using Corollary 3.1, we infer that

$$\lim_{j\to\infty} m^{1/p} |[A_j; \tau]|_p^- = k_p^-(\tau \otimes I_m)$$

which implies that

$$m^{1/p} k_p^-(\tau) \leq k_p^-(\tau \otimes I_m).$$

On the other hand, Lemma 3.2 shows that there are $A_j \uparrow I$, $A_j \in \mathcal{R}_1^+$ so that

$$\lim_{j\to\infty} |[\tau, A_j]|_p^- = k_p^-(\tau)$$

and

$$\lim_{j\to\infty} \|[\tau, A_j]\| = 0.$$

Then, by Corollary 3.1 we get that

$$\begin{aligned} m^{1/p} k_p^-(\tau) &= \lim_{j\to\infty} |[\tau, A_j] \otimes I_m|_p^- \\ &= \lim_{j\to\infty} |[\tau \otimes I_m, A_j \otimes I_m]|_p^- \\ &\geq k_p^-(\tau \otimes I_m) \end{aligned}$$

which concludes the proof. □

4. Fractal preliminaries

For simplicity, the fractal context will be certain totally disconnected Cantor-like self-similar sets on which a certain Hausdorff measure is a Radon measure on Borel sets.

We consider a non-empty compact set $K \subset \mathbb{R}^n$ and a N-tuple of maps

$$F_i(x) = \lambda(x - b(i)) + b(i), \quad 1 \le i \le N,$$

where $0 < \lambda < 1$ and $b(i) \in \mathbb{R}^n$, so that

$$K = \bigcup_{1 \le i \le N} F_i K,$$

and we assume that

$$i_1 \ne i_2 \implies F_{i_1} K \cap F_{i_2} K = \emptyset.$$

Note that the open set condition for the contractions F_i (see [4, p. 121]) in this case can be satisfied with an open neighborhood of K. Both the Hausdorff measure and the box dimension of K are equal to

$$p = \frac{\log N}{\log(1/\lambda)}$$

by [4, Theorem 8.6], and the p-Hausdorff measure of K is finite and non-zero. This Hausdorff measure is often referred to as the *Hutchinson measure of K*. Note also the uniqueness of K given the maps F_i, $1 \le i \le N$ ([4, Theorem 8.3]).

If $w \in \{1, \ldots, N\}^m$, then we put $|w| = m$ and define $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and $K_w = F_w K$. In particular, if $|w| = |w'|$, then K_w and K_{w_1} are congruent and have the same p-Hausdorff measure and diameter. Moreover,

$$K = \bigcup_{|w| = L} K_w$$

and, for any $L \in \mathbb{N}$, the union is disjoint. On K there is a unique Radon measure μ so that

$$\mu(K_w) = N^{-|w|} = \lambda^{|w|p}$$

and on Borel sets $\mu = cH_p$ (where H_p denotes the p-Hausdorff measure and c is a constant). We use the l^∞-norm $|(x_1, \ldots, x_n)| = \max_{1 \le i \le n} |x_i|$ on $\mathbb{R}^n$. Note that

$$\operatorname{diam}(K_w) = c \cdot \lambda^{|w|}$$

for some constant c. We shall also assume that the Hausdorff dimension $p \ge 1$.

If τ is a n-tuple of commuting Hermitian operators on $\mathcal{H}$ with spectrum $\sigma(\tau) \subset K$, like with Lebesgue measure, here on K with μ, that is with H_p, then the Hilbert space splits as

$$\mathcal{H} = \mathcal{H}_{\text{psing}} \oplus \mathcal{H}_{\text{pac}},$$

where $\mathcal{H}_{\text{psing}}$ and $\mathcal{H}_{\text{pac}}$ are reducing subspaces for τ and consist of vectors ξ so that

$$\langle E(\tau; \cdot)\xi, \xi\rangle$$

is singular with respect to μ and, respectively, absolutely continuous with respect to μ, that is with respect to H_p. (Use for instance in [3, Section 1.6.3].) For more on the Hausdorff measure and on fractals, see [3] and [4].

5. k_p^- in the fractal setting

In this section we study $k_p^-(\tau)$, where τ is a n-tuple of commuting Hermitian operators with $\sigma(\tau) \subset K$, in the context of Section 4. We assume $p \geq 1$ (and for certain results we will require $p > 1$).

Lemma 5.1. *Assume τ is a n-tuple of commuting Hermitian operators with $\sigma(\tau) \subset K$ and with a cyclic vector ξ. Then, for some constant C depending only on K,*

$$k_p^-(\tau) \leq C(H_p(\sigma(\tau)))^{1/p}.$$

Proof. Let $\Omega(L) = \{w \mid |w| = L,\ K_w \cap \sigma(\tau) \neq \emptyset\}$ and $G(L) = \bigcup_{w \in \Omega(L)} K_w$. Then, $G(L)$ is open in K and there is L_0 so that, for a given $\epsilon > 0$, we have $L \geq L_0 \implies H_p(G(L)) \leq H_p(\sigma(\tau)) + \epsilon$. Let further $E_w = E(\tau; K_w)$ and observe that, since $G(L) \supset \sigma(\tau)$,

$$\sum_{w \in \Omega(L)} E_w = I.$$

We also have

$$H_p(G_L) = |\Omega(L)| H_p(K) \lambda^{L_p}.$$

Let P_L be the orthogonal projection onto $\sum_{w \in \Omega(L)} \mathbb{C} E_w \xi$ and P_w the orthogonal projection onto $\mathbb{C} E_w \xi$. Remark that the non-zero $E_w \xi$ are an orthogonal basis on $P_L \mathcal{H}$ and rank $P_L \leq |\Omega_L|$. We have $[P_L, E_w] = 0$ if $|w| = L$ and

$L_1 \geq L_2 \implies P_{L_1} \geq P_{L_2}$, since each E_w with $|w| = L_2$ is the sum of $E_{w'}$ with $|w'| = L_1$. Remark also that

$$\begin{aligned}\|[P_L,\tau]\| &= \max_{|w|=L} \|[P_L,\tau]E_w\| \\ &= \max_{|w|=L} \|[P_w, E_w\tau]\| \\ &\leq \max_{|w|=L} 2\mathrm{diam}(\sigma(E_w\tau \mid E_w\mathcal{H})) \\ &\leq \max_{|w|=L} 2\mathrm{diam}(K_w) \leq 2\lambda^L \mathrm{diam}(K).\end{aligned}$$

If $P_L \uparrow P$, then $P\xi = \xi$ and $[P,\tau] = 0$ since $\|[P_L,\tau]\| \to 0$. The vector ξ being cyclic, this gives $P = I$, so $P_L \uparrow I$ as $L \to \infty$. Hence,

$$\begin{aligned}|[P_L,\tau]|_p^- &\leq p(\mathrm{rank}[P_L,\tau])^{1/p} \cdot \|[P_L,\tau]\| \\ &\leq p(2|\Omega(L)|)^{1/p} \cdot 2\lambda^L \mathrm{diam}(K) \\ &= p \cdot 2^{1+1/p} \cdot \Big(\frac{H_p(G_L)}{H_p(K)}\lambda^{-Lp}\Big)^{1/p} \cdot \lambda^L \mathrm{diam}\, K \\ &\leq p \cdot 2^{1+1/p}(H_p(K))^{-1/p} \cdot \mathrm{diam}\, K(H_p(\sigma(\tau)) + \epsilon)^{1/p}.\end{aligned}$$

Since $\epsilon > 0$,

$$|[P_L,\tau]|_p^- \leq c(H_p(\sigma(\tau)))^{1/p}$$

where $c = p \cdot 2^{1+1/p} \cdot \mathrm{diam}\, K(H_p(K))^{-1/p}$. Letting $L \to \infty$, we get

$$k_p^-(\sigma(\tau)) \leq c(H_p(\sigma(\tau)))^{1/p}. \qquad \square$$

Lemma 5.2. *Assume $\sigma(\tau) \subset K$ and that the spectral measure $E(\tau;\cdot)$ is singular with respect to H_p. Then,*

$$k_p^-(\tau) = 0.$$

Proof. Since τ is the orthogonal sum of n-tuples with cyclic vector, it suffices to prove the lemma when τ has a cyclic unit vector ξ. The absolute continuity class on K of $E(\tau;\cdot)$ is then the same as the absolute continuity class of the scalar measure $\nu = \langle E(\tau;\cdot)\xi,\xi\rangle$.

Given $\epsilon > 0$, we can find a compact set C_m which is a finite union of K_w so that $H_p(C_m) \leq \epsilon$ and $\nu(K\backslash C_m) < 2^{-m}$. Since $\sigma(\tau \mid E(\tau;C_m)\mathcal{H}) \subset C_m$, the preceding lemma gives that

$$k_p^-(\tau \mid E(\tau;C_m)\mathcal{H}) \leq c \cdot H_p(C_m) \leq c \cdot \epsilon.$$

On the other hand, $\nu(K\backslash C_m) \leq 2^{-m}$ gives

$$\|\xi - E(\tau;C_m)\xi\|^2 \leq 2^{-m}.$$

Since $(\tau)''$ is a maximal abelian von Neumann algebra in $\mathcal{B}(\mathcal{H})$, ξ being cyclic is also separating, and we infer $E(\tau; C_m) \xrightarrow{w} I$ and hence $E(\tau; C_m) \xrightarrow{s} I$ as $m \to \infty$. Therefore, we can find $A_m \in \mathcal{R}_1^+$ such that one has $A_m \leq E(\tau; C_m)$, $|[A_m, \tau]|_p^- \leq k_p^-(\tau \mid E(\tau; C_m))$, and $A_m \uparrow I$. It follows that

$$k_p^-(\tau) \leq c \cdot \epsilon$$

and, ϵ being arbitrary, $k_p^-(\tau) = 0$. □

Note that on K the p-Hausdorff measure satisfies an Ahlfors regularity condition

$$C^{-1} r^p \leq H_p(B(x,r)) \leq C r^p$$

if $r \leq 1$ for some $C > 0$. The right half of this $H_p(B(x,r)) \leq C r^p$ if $r \leq 1$ is the sub-regularity condition where $p > 1$, required in [2, Corollary 4.7], to show that $k_p^-(\tau_K) > 0$, where τ_K is the n-tuple of multiplication operators by the coordinate functions in $L^2(K, H_p \mid K)$. Thus we have the following result.

Lemma 5.3 ([2]). *Assume $p > 1$. Then $k_p^-(\tau_K) > 0$.*

More generally, if $\omega \subset K$ is a Borel set, let τ_ω be the n-tuple of multiplication operators by the coordinate functions in $L^2(\omega, H_p \mid \omega)$ (this is the same as $\tau_K \mid L^2(\omega, H_p \mid \omega)$ since $L^2(\omega, H_p \mid \omega) \subset L^2(K, H_p \mid K)$). A key part of the proof of the main theorem will be to evaluate $k_p^-(\tau_\omega)$ for increasingly general ω, along lines similar of Lebesgue measure on $\mathbb{R}^n$ considered in [7].

We also define a constant

$$\gamma_K = \frac{(k_p^-(\tau_K))^p}{H_p(K)},$$

where p is the Hausdorff dimension of K. Lemma 5.1 and Lemma 5.3 imply that $0 < k_p^-(\tau_K) < \infty$, so that $0 < \gamma_K < \infty$.

Theorem 5.1. *Let τ be a n-tuple of commuting Hermitian operators with $\sigma(\tau) \subset K$ and assume $p > 1$. Then,*

$$(k_p^-(\tau))^p = \gamma_K \int_K m(x) dH_p(x),$$

where m is the multiplicity function of τ.

Proof. Using Lemma 5.2 and the decomposition $\mathcal{H} = \mathcal{H}_{\text{psing}} \oplus \mathcal{H}_{\text{pac}}$, the proof reduces to the case when the spectral measure of τ is absolutely continuous with respect to H_p, that is when $\mathcal{H} = \mathcal{H}_{\text{pac}}$. In view of Proposition 2.1, a further reduction is possible to the case when τ has finite cyclicity, that is when the multiplicity function is bounded. Since, when τ has a cyclic vector and H_p-absolutely continuous spectral measure, τ is unitarily equivalent to a τ_ω, the proof reduces to the case when $\tau = \tau_{\omega_1} \oplus \cdots \oplus \tau_{\omega_m}$ for some Borel sets $\omega_j \subset K$, $1 \leq j \leq m$. In view of the last assertion in Proposition 2.1, the theorem holds for $\tau_{\omega_1} \oplus \cdots \oplus \tau_{\omega_m}$ if and only if it holds for

$$\tau_{F_{w_1}(\omega_1)} \oplus \cdots \oplus \tau_{F_{w_m}(\omega_m)},$$

where $|w_1| = \cdots = |w_m|$, because

$$\tau_{F_{w_j}(\omega_j)} \simeq F_{w_j}(\tau_{\omega_j}).$$

We may then choose $|w_j|$ sufficiently large and so that the $F_{w_j}(\omega_j)$ $(1 \leq j \leq m)$ are disjoint, which implies that

$$\tau_{F_{w_1}(\omega_1)} \oplus \cdots \oplus \tau_{F_{w_m}(\omega_m)} \simeq \tau_\omega,$$

where

$$\omega = F_{w_1}(\omega_1) \cup \cdots \cup F_{w_m(\omega_m)}.$$

Thus, the proof has been reduced to showing that

$$(k_p^-(\tau_\omega))^p = \gamma_K H_p(\omega).$$

First, assume that ω is a finite union of K_w. Since K_w is a disjoint union of $K_{w'}$ with $|w'| \geq |w|$, we may assume that

$$\omega = K_{w_1} \cup \cdots \cup K_{w_m},$$

where $|w_1| = \cdots = |w_m|$ and $w_1, \ldots, w_m$ are distinct. These K_{w_j} are congruent and, using again the last assertion in Proposition 2.1, the proof of this case reduces to proving the theorem for $\tau = \tau_{K_w} \otimes I_m$. The multiplicity function is m times the indicator function of K_w so that the right-hand side in the formula we want to prove is

$$\begin{aligned}\gamma_K m H_p(K_w) &= \frac{(k_p^-(\tau_K))^p}{H_p(K)} \cdot m \cdot \lambda^{p|w|} \cdot H_p(K) \\ &= m \cdot (\lambda^{|w|} k_p^-(\tau_K))^p = m(k_p^-(\tau_{K_w}))^p.\end{aligned}$$

By Theorem 3.1, the left-hand side equals

$$(k_p^-(\tau_{K_w} \otimes I_m))^p = (m^{1/p} k_p^-(\tau_{K_w}))^p.$$

Next, we prove the theorem for τ_ω when $\omega \subset K$ is a general open subset. Let $\omega^{(L)}$ be the union of the $K_w \subset \omega$ with $|w| \le L$. The $\omega^{(L)}$ are clopen subsets of K and are finite unions of K_w, so that the theorem holds for $\tau_{\omega^{(L)}}$ and for τ_ω is obtained using Proposition 2.1, which gives $k_p^-(\tau_{\omega^{(L)}}) \uparrow k_p^-(\tau_\omega)$.

Finally, let $\omega \subset K$ be a Borel set and let C be compact and G in K be open so that $C \subset \omega \subset G$ and $H_p(G \backslash C) < \epsilon$ for a given $\epsilon > 0$. We have $|k_p^-(\tau_\omega) - k_p^-(\tau_G)| \le |k_p^-(\tau_{G\backslash C})| = (\gamma_K \epsilon)^{1/p}$, using the fact that $G\backslash C$ is open in K and Proposition 2.1. Thus,

$$
\begin{aligned}
&|k_p^-(\tau_\omega) - (\gamma_K H_p(\omega))^{1/p}| \\
&\quad \le |k_p^-(\tau_\omega) - k_p^-(\tau_G)| + |k_p^-(\tau_G) - (\gamma_K H_p(\omega))^{1/\omega}| \\
&\quad \le (\gamma_K \epsilon)^{1/p} + |(\gamma_K H_p(G))^{1/p} - (\gamma_K H_p(\omega))^{1/p}| \\
&\quad \le (\gamma_K \epsilon)^{1/p} + |(\gamma_K (H_p(\omega) + \epsilon))^{1/p} - (\gamma_K H_p(\omega))^{1/p}|.
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, we get $k_p^-(\tau_\omega) = (\gamma_K H_p(\omega))^{1/p}$. □

Corollary 5.1. *Assume $\sigma(\tau) \subset K$ and $p > 1$. Then, $k_p^-(\tau) = 0$ if and only if the spectral measure of τ is singular with respect to H_p.*

Remark 5.1. In [8] we showed for a n-tuple τ and a normed ideal $\mathcal{J}$ that there is a largest reducing subspace for τ on which $k_{\mathcal{J}}$ vanishes. In the case of commuting n-tuples of Hermitian operators and $\mathcal{J} = \mathcal{C}_n^-$ this subspace is the subspace where the spectral measure is singular with respect to the Lebesgue measure. The theorem we proved in this section shows that if $\sigma(\tau) \subset K$ and $p > 1$, then the largest reducing subspace on which k_p^- vanishes for the restriction of τ is precisely $\mathcal{H}_{\text{psing}}$.

6. Concluding remarks

Remark 6.1. It is natural to wonder whether in general

$$(k_p^-(\tau_1 \oplus \tau_2))^p = (k_p^-(\tau_1))^p + (k_p^-(\tau_2))^p,$$

which would be much more than the ampliation homogeneity we proved. If $p = 1$, this is known to be true [7]. For $1 < p \le \infty$, this is an open problem. While a negative answer would not be surprising, it is certainly desirable to clarify this issue.

Remark 6.2. To get results for more general self-similar fractals than the Cantor-like K, we believe that it may be useful to replace k_p^- by $\tilde{k}_p^-$, the variant of k_p^- considered in [7], pp. 13–16. This amounts to extending the norms of normed ideals to n-tuples, not by the max of norm on the components, but by the norm of $(T_1^*T_1 + \cdots + T_n^*T_n)^{1/2}$, that is the modulus in the polar decomposition of the column $\begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix}$. This $|\tau|_{\tilde{\mathcal{J}}}$ has the advantage over $|\tau|_{\mathcal{J}}$ of being invariant under rotations, that is if $(u_{ij})_{1\le i,j\le n}$ is a unitary matrix and the n-tuple $\left(\sum_j u_{ij}T_j\right)_{1\le i\le n}$ has the same $\sim \mathcal{J}$-norm as $\tau = (T_i)_{1\le i\le n}$. In particular $\tilde{k}_p^-$ may be better suited to handle self-similar sets K when we use more general $F_i(x) = \lambda U_i(X - b(i)) + b(i)$, where $U_i \in O(n)$. In particular, it is quite straightforward to use $\sim$-norms in §3 and to see that ampliation homogeneity still holds for $\tilde{k}_p^-$, which we record as the next theorem.

Theorem 6.1. *If τ is a n-tuple of bounded operators and $1 \le p \le \infty$, then*

$$\tilde{k}_p^-(\tau \otimes I_m) = m^{1/p}\tilde{k}_p^-(\tau).$$

Remark 6.3. In [10] we give an extension in another direction to the formula for $k_n^-(\tau)$ in [7] to hybrid perturbations.

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