

A characterization of metric subspaces of full Assouad dimension

Yoshito Ishiki¹

Abstract. We introduce the notion of tiling spaces for metric spaces. The class of tiling spaces contains the Euclidean spaces, the middle-third Cantor set, and various self-similar spaces appearing in fractal geometry. For doubling tiling spaces, we characterize metric subspaces whose Assouad dimension coincides with that of the whole space.

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1. Introduction

Fraser and Yu [2] provided a characterization of subsets of a Euclidean space whose Assouad dimension coincides with that of the whole space. Namely, they proved in [2] that for every subset F of the N -dimensional Euclidean space \mathbb{R}^N , the following are equivalent:

- (1) F asymptotically contains arbitrary large arithmetic patches;
- (2) F satisfies the asymptotic Steinhaus property;
- (3) $\dim_A F = N$, where \dim_A stands for the Assouad dimension;
- (4) $\text{Cdim}_A F = N$, where Cdim_A stands for the conformal Assouad dimension;
- (5) F has a weak tangent with non-empty interior;
- (6) the closed unit ball $B(0, 1)$ in \mathbb{R}^N is a weak tangent to F .

They used this characterization to study the problem of whether specific subsets of Euclidean spaces related to number theory, such as the products of the set of

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all prime numbers, asymptotically contain higher dimensional arithmetic progression. Their results in [2] are related to the Erdős–Turán conjecture.

The purpose of this paper is to generalize the Fraser–Yu characterization to general metric spaces. In the proof of the Fraser–Yu characterization, they essentially used the fact that Euclidean spaces are tiled by congruent cubes. From this point of view, we introduce the notion of tiling spaces for metric spaces. The class of tiling spaces contains the Euclidean spaces, the middle-third Cantor set, and various self-similar spaces appearing in fractal geometry. For doubling tiling spaces, we characterize metric subspaces whose Assouad dimension coincides with that of the whole space.

We first define covering pairs as follows: For a set X , we denote by $\text{cov}(X)$ the set of all coverings of X . We call a map P from \mathbb{N} or \mathbb{Z} to $\text{cov}(X)$ a *covering structure on X* . We denote by P_n the value of P at n . For $T \in P_n$ and $k \in \mathbb{N}$, we put $[T]_k = \{A \in P_{n+k} \mid A \subset T\}$.

We call a pair (X, P) of a set X and a covering structure P on X a *covering pair*. We denote by $\text{dom}(P)$ the domain of the map P . Note that $\text{dom}(P)$ coincides with either \mathbb{N} or \mathbb{Z} .

Definition 1.1 (tiling set). Let (X, P) be a covering pair. For $N \in \mathbb{N}$, we say that (X, P) is an *N -tiling set* if it satisfies the following:

- (S1) for each pair $n, m \in \text{dom}(P)$ with $n < m$, and for each $A \in P_n$, we have $\text{card}([A]_{m-n}) = N^{m-n}$ and $A = \bigcup [A]_{m-n}$, where the symbol card stands for the cardinality;
- (S2) for each $n \in \text{dom}(P)$, and for each pair $A, B \in P_n$, there exist $m \in \text{dom}(P)$ and $C \in P_m$ such that $A \cup B \subset C$ and $m < n$.
- (S3) for each $n \in \text{dom}(P)$, for all $l, m \in \mathbb{N}$, and for each $A \in P_n$, we have

$$[A]_{m+l} = \bigcup_{T \in [A]_m} [T]_l.$$

We say that (X, P) is a *tiling set* if it is an N -tiling set for some N . For a tiling set (X, P) , we say that a subset T of X is a *tile* of (X, P) if there exists $n \in \text{dom}(P)$ such that $T \in P_n$.

We next specialize the notion of tiling sets for metric spaces. For a metric space X and for a subset A of X , we denote by $\delta(A)$ the diameter of A . For $p \in X$ and $r \in (0, \infty)$, we denote by $U(p, r)$ the open ball centered at p with radius r . For $h \in (0, \infty)$, and for a metric space X with metric d_X , we denote by hX the metric space (X, hd_X) .

Definition 1.2 (tiling space). Let $N \in \mathbb{N}$ and $s \in (0, \infty)$. Let X be a metric space. Let (X, P) be an N -tiling set. We say that (X, P) is an (N, s) -pre-tiling space if it satisfies the following:

- (T1) there exist $D_1 > 0$ and $D_2 > 0$ such that for each $n \in \text{dom}(P)$ and for each $A \in P_n$, we have $D_1 \leq \delta(A)/s^n \leq D_2$;
- (T2) there exists $E > 0$ such that for each $n \in \text{dom}(X)$ and for each $A \in P_n$, there exists a point $p_A \in A$ with $U(p_A, Es^m) \subset A$.

Furthermore, we say that an (N, s) -pre-tiling space (X, P) is an (N, s) -tiling space if it satisfies:

- (U) for each countable sequence $\{A_i\}_{i \in \mathbb{N}}$ of tiles of (X, P) , there exists a subsequence $\{A_{\phi(i)}\}_{i \in \mathbb{N}}$ such that $\{(\delta(A_{\phi(i)}))^{-1} A_{\phi(i)}\}_{i \in \mathbb{N}}$ converges to $(\delta(T))^{-1} T$ for some tile T of (X, P) in the sense of Gromov–Hausdorff.

We say that (X, P) is a tiling (resp. pre-tiling) space if it is an (N, s) -tiling (resp. (N, s) -pre-tiling) space for some N and s .

We say that two metric spaces X and Y are similar if there exists $h \in (0, \infty)$ with $d_{\text{GH}}(hX, Y) = 0$, where d_{GH} is the Gromov–Hausdorff distance. Similarity is an equivalence relation on metric spaces. Note that every metric space is similar to its completion.

Let X be a metric space. Let (X, P) be a tiling set. If the similarity classes of the tiles of (X, P) is finite, then the condition (U) is satisfied. Thus the condition (U) is considered as a generalization of the finiteness of the similarity classes of tiles. There exists a pre-tiling space failing the condition (U) whose tiles have infinite similarity classes (see Example 6.2). There exists a tiling space whose tiles have infinite similarity classes (see Example 6.3). The p -adic numbers (Example 5.4) are non-Euclidean examples of tiling spaces.

To state our main result, we use the notion of pseudo-cones introduced by the author [3]. Let X be a metric space. Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of subsets of X , and let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $(0, \infty)$. We say that a metric space P is a pseudo-cone of X approximated by $(\{A_i\}_{i \in \mathbb{N}}, \{u_i\}_{i \in \mathbb{N}})$ if $\lim_{i \rightarrow \infty} d_{\text{GH}}(u_i A_i, P) = 0$. We denote by $\text{PC}(X)$ the class of all pseudo-cones, and by $\text{KPC}(X)$ the class of all pseudo-cones approximated by a pair of a sequence $\{A_i\}_{i \in \mathbb{N}}$ of compact sets of X and a sequence $\{u_i\}_{i \in \mathbb{N}}$ in $(0, \infty)$. Let (X, P) be a pre-tiling space, and let F be a subset of X . We also denote by $\text{TPC}(F)$ the class of all pseudo-cones approximated by $(\{A_i \cap F\}_{i \in \mathbb{N}}, \{u_i\}_{i \in \mathbb{N}})$, where $\{A_i\}_{i \in \mathbb{N}}$ is a sequence of tiles of (X, P) and $\{u_i\}_{i \in \mathbb{N}}$ is a sequence in $(0, \infty)$. We emphasize that, in this paper, we use the Gromov–Hausdorff distance between not only compact metric spaces but also non-compact ones. Thus d_{GH} is not necessarily a metric.

Let (X, P) be a tiling set. Let A be a tile of (X, P) . We say that a subset F of X satisfies the *asymptotic Steinhaus property for A* if for each $\epsilon > 0$ and for each finite subset S of A , there exist a finite subset T of F , and $\delta \in (0, \infty)$ such that $d_{\text{GH}}(T, \delta \cdot S) \leq \delta \cdot \epsilon$.

Our main result of this paper is the following characterization:

Theorem 1.1. *Let (X, P) be a doubling tiling space. Then for every subset F of X the following are equivalent:*

- (1) $\dim_A F = \dim_A X$;
- (2) *there exists a tile A of (X, P) such that $A \in \text{PC}(F)$;*
- (3) *there exists a tile A of (X, P) such that $A \in \text{TPC}(F)$;*
- (4) *there exists a tile A of (X, P) such that $A \in \text{KPC}(F)$;*
- (5) *there exists a tile A of (X, P) such that F satisfies the asymptotic Steinhaus property for A .*

In Theorem 1.1, we need the assumption of the doubling property for X , which is equivalent to the finiteness of the Assouad dimension. The definition of the doubling property can be seen in Section 2. There exists a tiling space that is not doubling (see Example 6.1).

Remark 1.1. Let (X, P) be a tiling space. If X is doubling, then for every tile $T \in P$ we have $\dim_A T = \dim_A X$ (see Corollary 4.3). If X is not doubling, then the equality does not necessarily hold. For example, the tiling space constructed in Example 6.1 has infinite Assouad dimension, and possesses a tile of finite Assouad dimension.

If a tiling space (X, P) satisfies the assumption that the conformal dimensions of all the tiles of (X, P) and X are equal to $\dim_A X$, then the condition that $\text{Cdim}_A F = \dim_A X$ is equivalent to the conditions (1)–(5) stated in Theorem 1.1. The assumption mentioned above seems to be quite strong. Indeed, the author does not know an example satisfying the assumption except the Euclidean spaces. We do not deal with the conformal dimensions of tiling spaces.

Attractors of iterated function systems on metric spaces are studied as canonical examples of fractals, and their Hausdorff dimensions are investigated (see e.g., [1, Chapter 9], [4], and [5]).

Definition 1.3. Let X be a complete metric space. For $L \in (0, \infty)$, a map $f: X \rightarrow X$ is said to be an L -similar transformation on X if for all $x, y \in X$ we have $d_X(f(x), f(y)) = Ld_X(x, y)$. Let $N \geq 2$ and let $s \in (0, 1)$. We say that \mathcal{S} is an (N, s) -similar iterated function system on X if \mathcal{S} consists of N many s -similar transformations on X , say $\mathcal{S} = \{S_i\}_{i=0}^{N-1}$. A non-empty subset F of X is said to be an attractor of the iterated function system \mathcal{S} if F is compact and $F = \bigcup_{i=0}^{N-1} S_i(F)$. Since X is complete, an attractor of \mathcal{S} always uniquely exists (see [1, Chapter 9] for the Euclidean setting). We write $A_{\mathcal{S}}$ as the attractor of \mathcal{S} . We say that the system \mathcal{S} satisfies the strong open set condition if there exists an open set V of X such that

- (O1) $\bigcup_{i=0}^{N-1} S_i(V) \subset V$;
- (O2) $\{S_i(V)\}_{i=0}^{N-1}$ are pairwise disjoint;
- (O3) $V \cap A_{\mathcal{S}} \neq \emptyset$.

Let W be the set of all words generated by $\{0, \dots, N - 1\}$. For each word $w = w_0 \cdots w_l$, we write $S_w = S_{w_l} \circ \cdots \circ S_{w_0}$, where each w_i belongs to $\{0, \dots, N - 1\}$. We define a map $P_{\mathcal{S}}: \mathbb{N} \rightarrow \text{cov}(A_{\mathcal{S}})$ by

$$(P_{\mathcal{S}})_n = \{S_w(A_{\mathcal{S}}) \mid w \in W \text{ and } |w| = n\}, \tag{1}$$

where $|w|$ stands for the length of the word w .

Similar iterated function systems provide us a plenty of tiling spaces.

Theorem 1.2. For $N \geq 2$ and $s \in (0, 1)$, let \mathcal{S} be an (N, s) -similar iterated function system on a complete metric space with the strong open set condition. Let $A_{\mathcal{S}}$ be the attractor of \mathcal{S} , and $P_{\mathcal{S}}$ the map defined by (1). Then the pair $(A_{\mathcal{S}}, P_{\mathcal{S}})$ is an (N, s) -tiling space.

By Theorem 1.2, for instance, the middle-third Cantor set and the Sierpiński gasket are tiling spaces for some suitable covering structures (see Subsection 5.3), and we can apply Theorem 1.1 to them.

The organization of this paper is as follows. In Section 2, we review the definitions and basic properties of the Assouad dimension and the Gromov–Hausdorff distance. In Section 3, we discuss basic properties of tiling spaces. In Section 4, we prove Theorem 1.1. In Section 5, we show Theorem 1.2, and provide tiling spaces induced from similar iterated function systems. In Section 6, we exhibit counterexamples related to our characterization of tiling spaces.

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2. Preliminaries

In this paper, we denote by \mathbb{N} the set of all non-negative integers.

2.1. Metric spaces. Let X be a metric space. The symbol d_X stands for the metric of X . We denote by $B(x, r)$ (resp. $U(x, r)$) the closed (resp. open) ball centered at x with radius r . For a subset A of X , and for $r \in (0, \infty)$, we denote by $B(A, r)$ the closed ball centered at A with radius r defined by

$$B(A, r) = \{x \in X \mid \inf_{a \in A} d(x, a) \leq r\}.$$

For a subset A of X , we set

$$\alpha(A) = \inf\{d_X(x, y) \mid x, y \in A \text{ and } x \neq y\}.$$

We say that A is a *separated set* if there exists $r \in (0, \infty)$ with $\alpha(A) \geq r$.

Let $p \in [1, \infty]$. For two metric spaces X and Y , we denote by $X \times_p Y$ the product metric space of X and Y with the ℓ^p -product metric $d_{X \times_p Y}$ defined by

$$d_{X \times_p Y}((a, b), (c, d)) = \begin{cases} (d_X(a, c)^p + d_Y(b, d)^p)^{1/p} & \text{if } p \in [1, \infty), \\ \max\{d_X(a, c), d_Y(b, d)\} & \text{if } p = \infty. \end{cases}$$

2.2. Assouad dimension. For $N \in \mathbb{N}_{\geq 1}$, we say that a metric space X is *N -doubling* if for every bounded set S of X , there exists a subset F of X such that $S \subset B(F, \delta(S)/2)$ and $\text{card}(F) \leq N$, where $B(F, \delta(S)/2)$ is the closed ball centered at F with radius $\delta(S)/2$. A metric space is said to be *doubling* if it is N -doubling for some N .

Let X be a metric space. For a bounded set S of X and $r \in (0, \infty)$, we denote by $\mathcal{B}_X(S, r)$ the minimum integer N such that S can be covered by at most N bounded sets with diameter at most r . We denote by $\mathcal{A}(X)$ the set of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every bounded set S of X and for every $r \in (0, \infty)$ we have $\mathcal{B}_X(S, r) \leq C(\delta(S)/r)^\beta$. We also denote by $\mathcal{C}(X)$ the set of all $\gamma \in (0, \infty)$ such that there exists $C \in (0, \infty)$ such that

for every bounded set S of X and for every separated subset M of S , we have $\text{card}(M) \leq C(\delta(S)/\alpha(M))^\nu$.

The Assouad dimension $\text{dim}_A X$ of a metric space X is defined as $\inf(\mathcal{A}(X))$ if $\mathcal{A}(X)$ is non-empty; otherwise, $\text{dim}_A(X) = \infty$.

By the definitions, we have the next two propositions.

Proposition 2.1. *For every metric space X , the following are equivalent:*

- (1) X is doubling;
- (2) $\mathcal{A}(X)$ is non-empty;
- (3) $\mathcal{C}(X)$ is non-empty;
- (4) $\text{dim}_A X < \infty$.

Proposition 2.2. *For every metric space X , we have*

$$\text{dim}_A X = \inf(\mathcal{C}(X)).$$

2.3. Gromov–Hausdorff distance. For a metric space Z , and for subsets A, B of Z , we denote by $d_H(A, B; Z)$ the Hausdorff distance between A and B in Z . For two metric spaces X and Y , the Gromov–Hausdorff distance $d_{GH}(X, Y)$ between X and Y is defined as the infimum of all values $d_H(i(X), j(Y); Z)$, where Z is a metric space and $i: X \rightarrow Z$ and $j: Y \rightarrow Z$ are isometric embeddings.

By the definition of the Gromov–Hausdorff distance, we have:

Proposition 2.3. *Let $h \in (0, \infty)$. Let X and Y be metric spaces. Then*

$$d_{GH}(hX, hY) = h d_{GH}(X, Y).$$

Let X be a metric space. For $\epsilon \in (0, \infty)$, we define a function

$$d_X^\epsilon: X \times X \longrightarrow [0, \infty)$$

by

$$d_X^\epsilon(x, y) = (d_X(x, y))^\epsilon.$$

If d_X^ϵ is a metric, then we denote by X^ϵ the metric space (X, d_X^ϵ) . The metric space X^ϵ is called a *snowflake of X* . Note that if $\epsilon \in (0, 1)$, then d_X^ϵ is a metric.

Proposition 2.4. *Let $\epsilon \in (0, 1)$. Let X and Y be metric spaces. Then*

$$d_{GH}(X^\epsilon, Y^\epsilon) = d_{GH}(X, Y)^\epsilon.$$

Proof. Let Z be a metric space. For all subsets $A, B \subset Z$, we have

$$d_H(A, B; Z^\epsilon) = d_H(A, B; Z)^\epsilon.$$

This leads to the proposition. □

Let X be a metric space. Let $\epsilon \in (0, \infty)$. We say that a subset S of X is an ϵ -net if S is finite and $B(S, \epsilon) = X$, where $B(S, \epsilon)$ is the closed ball centered at S with radius r .

A metric space X is said to be *totally bounded* if for each $\epsilon \in (0, \infty)$ the space X contains an ϵ -net. A metric space X is totally bounded if and only if X is approximated by its finite subset in the sense of Gromov–Hausdorff.

By the definitions of the total boundedness and d_{GH} , we have:

Proposition 2.5. *Let X be a totally bounded metric space, and Y a metric space. If $d_{GH}(X, Y) \leq \epsilon$, then there exists a finite subset E of Y such that $d_{GH}(X, E) \leq 2\epsilon$.*

3. Properties of spaces with tiling structures

We discuss basic properties of tiling sets, and (pre-)tiling spaces.

Proposition 3.1. *Let (X, P) be a tiling set. Then for every pair $n, m \in \text{dom}(P)$ with $n < m$, and for every $A \in P_m$, there exists a unique $B \in P_n$ with $A \subset B$.*

Proof. Let (X, P) be an N -tiling set for some N . Suppose that there exist $B, C \in P_n$ with $B \neq C$ and $A \subset B \cap C$. By the condition (S2), there exist $k \in \text{dom}(P)$ and $D \in P_k$ such that $B \cup C \subset D$ and $k < n$. By the condition (S1), we have $\text{card}([D]_{m-k}) = N^{m-k}$.

Put $[D]_{m-k} = \{T_i\}_{i=1}^{N^{m-k}}$. We may assume that $T_1 = B$ and $T_2 = C$. Then the condition (S3) yields $[D]_{m-k} = \bigcup_{i=1}^{N^{m-k}} [T_i]_{m-n}$. By $A \subset B \cap C$, we have $A \in [T_1]_{m-n} \cap [T_2]_{m-n}$, and hence

$$[D]_{m-k} = ([T_1]_{m-n} \setminus \{A\}) \cup \bigcup_{i=2}^{N^{m-k}} [T_i]_{m-n}.$$

Since the condition (S1) implies that for every $i \in \{1, \dots, N^{m-k}\}$ we have $\text{card}([T_i]_{m-n}) = N^{m-n}$, we obtain

$$\begin{aligned} \text{card}([D]_{m-k}) &\leq (\text{card}([T_1]_{m-n}) - 1) + \sum_{i=2}^{N^{m-k}} \text{card}([T_i]_{m-n}) \\ &= N^{m-k} - 1 < N^{m-k}. \end{aligned}$$

This is a contradiction. □

By the conditions (S2) and (T1), we have the following propositions:

Proposition 3.2. *Let (X, P) be a pre-tiling space. If X is bounded, then one has $\text{dom}(P) = \mathbb{N}$ and $P_0 = \{X\}$.*

Proof. By the condition (T1) and the boundedness of X , the tiling index $\text{dom}(P)$ must be \mathbb{N} . If P_0 would have two elements, then this contradicts the condition (S2) and $\text{dom}(P) = \mathbb{N}$. Thus $P_0 = \{X\}$. \square

Proposition 3.3. *Let (X, P) be a pre-tiling space. The space X is bounded if and only if $\text{dom}(P) = \mathbb{N}$. Equivalently, the space X is unbounded if and only if $\text{dom}(P) = \mathbb{Z}$.*

Proof. By Proposition 3.2, it suffices to show that if $\text{dom}(P) = \mathbb{N}$, then X is bounded. This holds true by the condition (T1). \square

For a subset A of a metric space X , we denote by $\text{INT}(A)$ the interior of A in X .

Lemma 3.4. *Let (X, P) be a pre-tiling space. For every distinct pair $A, B \in P_n$, we have $\text{INT}(A) \cap \text{INT}(B) = \emptyset$.*

Proof. If for some distinct $A, B \in P_n$ we have $\text{INT}(A) \cap \text{INT}(B) \neq \emptyset$, then, by the condition (T1), there exist $k \in \text{dom}(P)$ and $C \in P_k$ such that $C \subset \text{INT}(A) \cap \text{INT}(B)$. This contradicts Proposition 3.1. \square

Lemma 3.5. *Every bounded pre-tiling space is totally bounded.*

Proof. Let (X, P) be a bounded (N, s) -pre-tiling space, and let D_2 be a constant appearing in the condition (T1). Proposition 3.3 implies $\text{dom}(P) = \mathbb{N}$. For each $n \in \mathbb{N}$, and for each $A \in [X]_n$, take a point $q_A \in A$. By the conditions (S1) and (T1), the set $\{q_A \in X \mid A \in [X]_n\}$ is a $(D_2 s^n)$ -net of X . \square

Since a totally bounded complete metric space is compact, we have:

Corollary 3.6. *Every bounded complete pre-tiling space is compact.*

Since a totally bounded metric space is separable, we obtain:

Corollary 3.7. *Every bounded pre-tiling space is separable.*

We next show the countability of tiling structures.

Proposition 3.8. *Let (X, P) be a pre-tiling space. Then each P_n is a countable family.*

Proof. By Propositions 3.2 and 3.3, we may assume that X is unbounded. Take a sequence $\{T_i\}_{i \in \mathbb{N}}$ of tiles of (X, P) such that for each $i \in \mathbb{N}$ we have $T_i \in P_{-i}$ and $T_i \subset T_{i+1}$. By the condition (S2) and Proposition 3.1, we have $X = \bigcup_{i \in \mathbb{N}} T_i$. Then we obtain $P_n = \bigcup_{-i \leq n} [T_i]_{n+i}$. This shows the proposition. \square

By Propositions 3.7 and 3.8, we obtain:

Corollary 3.9. *Every pre-tiling space is separable.*

Let $f: X \rightarrow Y$ be a map between metric spaces. Let $L \in [1, \infty)$ and $\gamma \in (0, \infty)$. We say that f is (L, γ) -homogeneously bi-Hölder if for all $x, y \in X$ we have

$$L^{-1}d_X(x, y)^\gamma \leq d_Y(f(x), f(y)) \leq Ld_X(x, y)^\gamma.$$

A homogeneously bi-Hölder map means an (L, γ) -homogeneously bi-Hölder map for some L and γ . A map $f: X \rightarrow Y$ is L -bi-Lipschitz if it is $(L, 1)$ -homogeneously bi-Hölder.

Remark 3.1. Let X be an ultrametric space. For every $\gamma \in (0, \infty)$, the function d_X^γ is also an ultrametric. Therefore the identity map $\text{id}: X \rightarrow X^\gamma$ is $(1, \gamma)$ -homogeneously bi-Hölder for any $\gamma \in (0, \infty)$.

Lemma 3.10. *Let $f: X \rightarrow Y$ be a surjective (L, γ) -homogeneously bi-Hölder map between metric spaces. Then for every $x \in X$ and for every $r \in (0, \infty)$ we have*

$$f(B(x, r)) \subset B(f(x), Lr^\gamma) \subset f(B(x, L^{2/\gamma}r)).$$

By Lemma 3.10, we find that being a pre-tiling space is invariant under homogeneously bi-Hölder maps.

Proposition 3.11. *Every homogeneously bi-Hölder image of an arbitrary pre-tiling space is a pre-tiling space. More precisely, the image of an arbitrary (N, s) -pre-tiling space under an (L, γ) -homogeneously bi-Hölder map is an (N, s^γ) -pre-tiling space.*

Since bi-Lipschitz maps are homogeneously bi-Hölder, we have:

Corollary 3.12. *Every bi-Lipschitz image of an arbitrary (N, s) -pre-tiling space is an (N, s) -pre-tiling space.*

In spite of the virtue of Proposition 3.11, a homogeneously bi-Hölder image of a tiling space is not always a tiling space (see Example 6.2).

By Proposition 2.4, we find that a specific bi-Hölder image of a tiling space is a tiling space.

Proposition 3.13. *Let (X, P) be an (N, s) -tiling space and let $\epsilon \in (0, 1)$. Then (X^ϵ, P) is an (N, s^ϵ) -tiling space.*

Let X be a metric space and $P: \text{dom}(P) \rightarrow \text{cov}(X)$. Define a map

$$P^C: \text{dom}(P) \longrightarrow \text{cov}(X)$$

by

$$P_n^C = \{\text{CL}(A) \mid A \in P_n\},$$

where CL is the closure operator in X . The following proposition allows us to assume that tiles of pre-tiling spaces are closed sets.

Proposition 3.14. *Let (X, P) be an (N, s) -pre-tiling space. Then (X, P^C) is also an (N, s) -pre-tiling space. Moreover, if (X, P) satisfies the condition (U), then so does (X, P^C) .*

Proof. By the definition of P^C , the condition (S3) is satisfied. From Lemma 3.4 and the condition (T2), it follows that for each pair $n, m \in \text{dom}(P)$ with $n < m$ and for each $A \in P_n$, if $S, T \in [A]_{m-n}$ satisfy $S \neq T$, then $\text{CL}(S) \neq \text{CL}(T)$. Hence the condition (S1) is satisfied. By $\text{CL}(A \cup B) = \text{CL}(A) \cup \text{CL}(B)$, the conditions (S2) is satisfied. Then the pair (X, P^C) is an N -tiling set. By the facts that $\delta(A) = \delta(\text{CL}(A))$ and that if $A \subset B$, then $\text{CL}(A) \subset \text{CL}(B)$, we conclude that (X, P^C) is a pre-tiling space. Since for every subset A of X we have $d_{\text{GH}}(A, \text{CL}(A)) = 0$, we obtain the latter part of the proposition. \square

Let X be a metric space. We say that a covering pair (X, P) is *self-similar* if there exists $s \in (0, 1)$ such that for each $n \in \text{dom}(P)$, for each $A \in P_n$ and for each $B \in P_{n+1}$, we have $d_{\text{GH}}(sA, B) = 0$. By the definition of the self-similarity, we have:

Lemma 3.15. *Let X be a metric space. If a covering pair (X, P) is self-similar, then (X, P) satisfies the conditions (T1) and (U).*

For a product of pre-tiling spaces, we obtain:

Proposition 3.16. *Let $p \in [1, \infty]$. Let (X, P) and (Y, Q) be (N, s) -pre-tiling spaces with $\text{dom}(P) = \text{dom}(Q)$. Define a covering structure $R: \text{dom}(P) \rightarrow \text{cov}(X)$ by $R_n = \{A \times B \mid A \in P_n, B \in Q_n\}$. Then the covering pair $(X \times_p Y, R)$ is a (N^2, s) -pre-tiling space.*

Proof. Since (X, P) and (Y, Q) satisfy the conditions (S1), (S2) and (S3), so does $(X \times_p Y, R)$. By the definition of the ℓ^p -product metric, we conclude that the conditions (T1) and (T2) are satisfied. Hence the covering pair $(X \times_p Y, R)$ is a (N^2, s) -pre-tiling space. \square

Remark 3.2. The author does not know whether it is true that if X and Y satisfy (U), then so does $X \times_p Y$ for any $p \in [1, \infty]$.

4. Tiling spaces and the Assouad dimensions

In this section, we prove Theorem 1.1.

Proposition 4.1. *Let (X, P) be a doubling pre-tiling space. Then for every $W \in (0, \infty)$ there exists $M_W \in \mathbb{N}_{\geq 1}$ such that for each $m \in \text{dom}(P)$ and for each subset S of X with $\delta(S) \leq Ws^m$, we have*

$$\text{card}(\{A \in P_m \mid A \cap S \neq \emptyset\}) \leq M_W.$$

Proof. Let D_2 and E be constants appearing in the conditions (T1) and (T2). Let $W \in (0, \infty)$, and take $m \in \text{dom}(P)$ and a subset S of X satisfying $\delta(S) \leq Ws^m$. For each $A \in P_m$ with $A \cap S \neq \emptyset$, let $p_A \in A$ be a point appearing in the condition (T2). Set

$$Z = \{p_A \in X \mid A \in P_m, A \cap S \neq \emptyset\}.$$

By Lemma 3.4 and the condition (T2), we have

$$\text{card}(Z) = \text{card}(\{A \in P_m \mid A \cap S \neq \emptyset\})$$

and $\alpha(Z) \geq Es^m$. By the condition (T2), we have $\delta(A) \leq D_2s^m$ for every $A \in P_m$. For every point $x \in S$, we have $Z \subset B(x, (D_2 + W)s^m)$. Thus we have $\delta(Z) \leq 2(D_2 + W)s^m$. By Proposition 2.1, we can take $\gamma \in \mathcal{C}(X)$. Then we have

$$\text{card}(\{A \in P_m \mid A \cap S \neq \emptyset\}) \leq C \left(\frac{\delta(Z)}{\alpha(Z)} \right)^\gamma \leq C \left(\frac{2(D_2 + W)}{E} \right)^\gamma$$

for some $C \in (0, \infty)$. This completes the proof. \square

Definition 4.1. Let (X, P) be a tiling space, and let F be a subset of X . For each pair $n, m \in \text{dom}(P)$ with $n < m$ and for each $B \in P_n$, we put

$$\mathcal{Q}_{n,m}^F(B) = \{A \in P_m \mid A \cap F \neq \emptyset, A \subset B\}. \tag{2}$$

We show the following:

Lemma 4.2. *Let (X, P) be a doubling pre-tiling space. Let F be a subset of X . Let Δ be the infimum of $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for each pair $n, m \in \text{dim}(P)$ with $n < m$ and for each $B \in P_n$ we have*

$$\mathcal{Q}_{n,m}^F(B) \leq C(s^{n-m})^\beta. \tag{3}$$

Then we have

$$\dim_A F = \Delta.$$

Proof. Let (X, P) be a doubling (N, s) -pre-tiling space, and let D_1 and D_2 be constants appearing in the condition (T1). Take $\beta \in (0, \infty)$ satisfying (3) with $\Delta < \beta$. Let S be a bounded subset of F . Take $n \in \text{dom}(P)$ with $D_1 s^{n-1} \leq \delta(S) < D_1 s^n$. Let $r \in (0, \infty)$, and take $m \in \text{dom}(P)$ with $s^{m-1} \leq r < s^m$. Applying Proposition 4.1 to D_1 , we obtain a constant M_{D_1} stated in the proposition. Then S can be covered by at most M_{D_1} members in P_n , and hence by (3) the set S can be covered by at most $M_{D_1} C(s^{n-m})^\beta$ members in P_m . In particular, we have

$$\mathcal{B}_X(S, r) \leq M_{D_1} C(s^{n-m})^\beta \leq M_{D_1} D_1^{-\beta} C s^\beta (\delta(S)/r)^\beta.$$

This implies $\beta \in \mathcal{A}(X)$. Hence, $\dim_A F \leq \Delta$.

We next prove the opposite inequality. Take $\beta \in \mathcal{A}(X)$ and $B \in P_n$. The set $B \cap F$ is a bounded set of X with $\delta(B \cap F) \leq D_2 s^n$. Thus $B \cap F$ can be covered by at most $C(s^{n-m})^\beta$ bounded sets with diameter at most $D_2 s^m$. Write these bounded sets as A_1, A_2, \dots, A_N , where $N \leq C(s^{n-m})^\beta$. Applying Proposition 4.1 to D_2 , we obtain a constant M_{D_2} stated in the proposition. Then each A_i can be covered by at most M_{D_2} members in P_m . Hence we have $\mathcal{Q}_{n,m}^F(B) \leq M_{D_2} C(s^{n-m})^\beta$. This implies $\Delta \leq \dim_A(F)$. □

Applying Lemma 4.2 to a whole pre-tiling space or to a tile of it, we obtain the following:

Corollary 4.3. *Let (X, P) be a doubling (N, s) -pre-tiling space. Then for every tile T of (X, P) we have*

$$\dim_A T = \dim_A X = \log(N)/\log(s^{-1}).$$

By the virtue of the condition (U), we obtain the following lemma:

Lemma 4.4. *Let (X, P) be a doubling tiling space. Let F be a subset of X . Let D_2 be a constant appearing in the condition (T1). If $\text{TPC}(F)$ contains no tiles of (X, P) , then there exists $k \in \mathbb{N}$ such that for each $n \in \text{dom}(P)$ and for each $B \in P_n$ we have $d_{\text{GH}}(B, B \cap F) > D_2 s^{n+k}$.*

Proof. Suppose that for each $k \in \mathbb{N}$ there exist $n_k \in \text{dom}(P)$ and $B_k \in P_{n_k}$ such that $d_{\text{GH}}(B_k, B_k \cap F) \leq D_2 s^{n_k+k}$. By the condition (T1), we have

$$d_{\text{GH}}(B_k, B_k \cap F) \leq (D_2/D_1)s^k \cdot \delta(B_k), \tag{4}$$

where D_1 is a constant appearing in the condition (T1). By the condition (U), there exists a subsequence $\{B_{\phi(k)}\}_{k \in \mathbb{N}}$ of $\{B_k\}_{k \in \mathbb{N}}$ such that $\delta(B_{\phi(k)})^{-1} B_{\phi(k)}$ converges to $(\delta(T))^{-1} T$ for some tile T of (X, P) . From (4) it follows that

$$d_{\text{GH}}((\delta(B_{\phi(k)}))^{-1} B_{\phi(k)}, (\delta(B_{\phi(k)}))^{-1} (B_{\phi(k)} \cap F)) \leq (D_2/D_1)s^{\phi(k)}.$$

By $s^{\phi(k)} \rightarrow 0$ as $k \rightarrow \infty$, we conclude that $T \in \text{TPC}(X)$. This is a contradiction. □

We next prove the following:

Lemma 4.5. *Let (X, P) be a doubling (N, s) -tiling space. Let F be a subset of X . If $\text{TPC}(F)$ contains no tiles of (X, P) , then we have $\dim_A F < \log(N)/\log(s^{-1})$.*

Proof. Let D_1 and D_2 be constants appearing in the condition (T1). Set $d = \log(N)/\log(s^{-1})$. Take $k \in \mathbb{N}$ stated in Lemma 4.4. Put $L = s^k$. By Lemma 4.4, for each $n \in \text{dom}(P)$ and for each $B \in P_n$, we have $d_H(B, B \cap F; X) > D_2 s^{n+k}$. Thus we can take a point $x \in B$ such that for every $y \in F$ we have $d_X(x, y) > D_2 s^{n+k}$. Take $C \in [B]_k$ with $x \in C$, then by the condition (T1) we have $C \cap F = \emptyset$. Therefore we obtain the following:

Sublemma 4.6. *For each $n \in \text{dom}(P)$ and for each $B \in P_n$, there exists $C \in P_{n+k}$ with $C \subset B$ and $C \cap F = \emptyset$.*

Fix $a, b \in \text{dom}(P)$ with $a > b$ and $B \in P_b$. Take $v \in \mathbb{N}$ such that

$$D_1 s^{b+k(v+1)} \leq D_2 s^a < D_1 s^{b+kv}. \tag{5}$$

Since $D_2 s^a < D_1 s^{b+kv}$, we have $b + kv < a$. Hence for each $A \in P_{b+kv}$ the set $[A]_{a-(b+kv)}$ is non-empty. Let W be the set of all words generated by

$\{0, \dots, N^k - 1\}$ whose length is at most v . Note that W contains the empty word. For $w \in W$, we denote by $|w|$ the length of w . For $u, v \in W$, we denote by uv the word product of u and v .

Let the set $\bigcup_{i=0}^v [B]_{ki}$ be indexed by W , say $\{T_w\}_{w \in W}$ such that for each $w \in W$ we have $T_w \in [B]_{k|w|}$, and such that if $|w| < v - 1$, then $T_{w0} \cap F = \emptyset$. This is possible by Sublemma 4.6.

For each $i \in \{1, \dots, v\}$, define a set H_i by

$$H_i = \{w0 \mid w \in W, |w| = i - 1 \text{ and all entries of } w \text{ are not } 0\}.$$

Put $R_w = [T_w]_{a-(b+k|w|)}$. Remark that $R_w = \{A \in P_a \mid A \subset T_w\}$. Put $H = \bigcup_{i=1}^v H_i$. Note that for all distinct $v, w \in H$, the sets R_v and R_w are disjoint.

Let $G = \bigcup_{w \in H} R_w$. We find that $G = \prod_{i=1}^v \prod_{w \in H_i} R_w$. Let $\Omega_{b,a}^F(B)$ be the quantity defined in Definition 4.1. Then we have

$$\Omega_{b,a}^F(B) \leq \text{card}([B]_{a-b}) - \text{card}(G).$$

Since $d = \log(N)/\log(s^{-1})$, we have

$$\begin{aligned} \text{card}(R_w) &= N^{a-b-k|w|} = s^{-d(a-b-k|w|)}, \\ \text{card}(H_i) &= (N^k - 1)^{i-1} = (L^{-d} - 1)^{i-1}. \end{aligned}$$

By these equalities, we obtain

$$\begin{aligned} \text{card}(G) &= \text{card}\left(\prod_{i=1}^v \prod_{w \in H_i} R_w\right) = \sum_{i=1}^v \text{card}\left(\prod_{w \in H_i} R_w\right) \\ &= \sum_{i=1}^v \sum_{w \in H_i} s^{-d(a-b-k|w|)} = \sum_{i=1}^v \sum_{w \in H_i} s^{-d(a-b-ki)} \\ &= \sum_{i=1}^v s^{d(b-a)} s^{kdi} (L^{-d} - 1)^{i-1} = s^{d(b-a)} \sum_{i=1}^v L^{di} (L^{-d} - 1)^{i-1}. \end{aligned}$$

Since for each $w \in H$, we have $T_w \cap F = \emptyset$, by the definition (2) of $\Omega_{b,a}^F(B)$, we also obtain

$$\Omega_{b,a}^F(B) \leq N^{a-b} - \text{card}(G) = (s^{b-a})^d \left(1 - \sum_{i=1}^v L^{di} (L^{-d} - 1)^{i-1}\right).$$

Note that we have

$$\begin{aligned} \sum_{i=1}^v L^{di} (L^{-d} - 1)^{i-1} &= (L^{-d} - 1)^{-1} \sum_{i=1}^v (1 - L^d)^i \\ &= (L^d - 1)^{-1} (1 - L^d) (1 - (1 - L^d)^v) L^{-d} = 1 - (1 - L^d)^v. \end{aligned}$$

By (5), we have $L^{v+1} \leq (D_2/D_1)s^{a-b}$, then

$$\log((D_2/D_1)s^{a-b})/\log L - 1 \leq v,$$

and hence

$$\begin{aligned} Q_{b,a}^F(B) &\leq (s^{b-a})^d \left(1 - \sum_{i=1}^v L^{di} (L^{-d} - 1)^{i-1}\right) = (s^{b-a})^d (1 - L^d)^v \\ &\leq (s^{b-a})^d (1 - L^d)^{\log((D_2/D_1)s^{a-b})/\log L - 1} \\ &= (s^{b-a})^d \frac{1}{1 - L^d} ((D_1/D_2)s^{b-a})^{-\log(1-L^d)/\log L} \\ &= \frac{1}{1 - L^d} \left(\frac{D_1}{D_2}\right)^{-\log(1-L^d)/\log L} (s^{b-a})^{d - \log(1-L^d)/\log L}. \end{aligned}$$

By Lemma 4.2, we obtain

$$\dim_A(F) \leq d - \log(1 - L^d)/\log L < d.$$

This completes the proof. \square

Lemma 4.7. *Let (X, P) be a pre-tiling space. Then the following are equivalent:*

- (1) *there exists a tile A of (X, P) such that $A \in \text{PC}(X)$;*
- (2) *there exists a tile A of (X, P) such that F satisfies the asymptotic Steinhaus property for A .*

Proof. We first show that (2) \implies (1). Take a tile A of (X, P) stated in (2). For each $n \in \mathbb{N}_{\geq 1}$, take a $(1/n)$ -net S_n of A . By the condition (2), we can take a finite subset T_n of F and $\delta_n \in (0, \infty)$ such that $d_{\text{GH}}(T_n, \delta_n S_n) < \delta_n/n$. Set $u_n = \delta_n^{-1}$, then $d_{\text{GH}}(u_n T_n, S_n) \leq 1/n$. Hence we have $\lim_{n \rightarrow \infty} d_{\text{GH}}(u_n T_n, A) = 0$. This implies $A \in \text{PC}(X)$.

We next show (1) \implies (2). Take a tile A of (X, P) stated in (1). Take a finite subset S of A . Since $A \in \text{PC}(F)$, there exist a sequence $\{T_n\}_{n \in \mathbb{N}}$ of subsets of F and a sequence $\{u_n\}_{n \in \mathbb{N}}$ in $(0, \infty)$ with $\lim_{n \rightarrow \infty} d_{\text{GH}}(u_n T_n, A) = 0$. By Lemma 3.5, the tile A is totally bounded. Then, by Proposition 2.5, for each $\epsilon \in (0, \infty)$, we can take a finite subset Y_N of T_N such that $d_{\text{GH}}(u_n Y_N, A) < \epsilon$. Since S is finite, we can take a subset U_N of Y_N such that $d_{\text{GH}}(U_N, u_n^{-1} S) < u_n^{-1} \cdot \epsilon$. Thus F satisfies the asymptotic Steinhaus property for A . \square

In order to complete the proof of Theorem 1.1, we recall the following theorem proved by the author [3].

Theorem 4.8. *Let X be a metric space. If $P \in \text{PC}(X)$, then we have*

$$\dim_A P \leq \dim_A X.$$

Proof of Theorem 1.1. By the definitions, both the implications (3) \implies (2) and (4) \implies (2) are true. Theorem 4.8 implies that (2) \implies (1). Lemma 4.5 is equivalent to (1) \implies (3). Lemma 4.7 states that (2) \iff (5) is true. Therefore it suffices to show that (2) \implies (4).

Let (X, P) be a doubling tiling space. By Lemma 3.5, every tile of a tiling space is totally bounded. From this property and Proposition 2.5, it follows that a tile of (X, P) in $\text{PC}(F)$ is approximated by a sequence of scalings of finite sets of F in the sense of Gromov–Hausdorff. This completes the proof of Theorem 1.1. \square

5. Tiling spaces induced from iterated function systems

5.1. Attractors. We first prove Theorem 1.2.

Let X be a complete metric space. Let $N \in \mathbb{N}_{\geq 2}$ and $s \in (0, 1)$. Let $\mathcal{S} = \{S_i\}_{i=0}^{N-1}$ be an (N, s) -similar iterated function system on X . Assume that the attractor $A_{\mathcal{S}}$ of \mathcal{S} satisfies the strong open set condition. Let V be an open set appearing in the strong open set condition. Let W be the set of all words generated by $\{0, \dots, N - 1\}$. For every $q \in A_{\mathcal{S}}$, and for each $w \in W$, put $q_w = S_w(q)$.

We first prove that $(A_{\mathcal{S}}, P_{\mathcal{S}})$ is an N -tiling set, where $P_{\mathcal{S}}$ is the map defined in Definition 1.3.

Lemma 5.1. *The covering pair $(A_{\mathcal{S}}, P_{\mathcal{S}})$ is an N -tiling set.*

Proof. By the definition of $P_{\mathcal{S}}$, we see that the condition (S3) is satisfied. We next verify that the condition (S1) is satisfied. Take $q \in A_{\mathcal{S}} \cap V$. For each pair $n, m \in \mathbb{N}$ with $n < m$, and for each $B \in P_n$, by the definitions of the attractor and P , we have $B = \bigcup [B]_{m-n}$.

We show $\text{card}([B]_{m-n}) = N^{m-n}$. By the definition of $P_{\mathcal{S}}$, we have

$$[B]_{m-n} = \{S_w(B) \mid w \in W \text{ and } |w| = m - n\}.$$

Write $B = S_v(A_{\mathcal{S}})$, where $v \in W$. By the condition (O2) in the strong open set condition, the family $\{S_{vw}(V) \mid w \in W \text{ and } |w| = m - n\}$ is pairwise disjoint. This implies that if $w \neq w'$ with $|w| = |w'| = m - n$, then $q_{vw} \neq q_{vw'}$. Hence the set $\{q_{vw} \in A_{\mathcal{S}} \mid w \in W \text{ and } |w| = m - n\}$ consists of N^{m-n} elements. Since for each $w \in W$ with $|w| = m - n$ we have $q_{vw} \in S_{vw}(V) \cap S_{vw}(A_{\mathcal{S}})$, we obtain $\text{card}([B]_{m-n}) = N^{m-n}$.

By the boundedness of A_S , the pair (A_S, P_S) satisfies the condition (S2). Therefore the pair (A_S, P_S) is an N -tiling set. \square

We next prove that (A_S, P_S) is an (N, s) -tiling space.

Lemma 5.2. *The attractor A_S of \mathcal{S} is contained in $\text{CL}(V)$. Moreover, for every $w \in W$, we have $S_w(A_S) \subset \text{CL}(S_w(V))$.*

Proof. Take $q \in A_S \cap V$. Put $M_0 = \{q\}$, and for each $n \in \mathbb{N}_{\geq 1}$ put $M_n = \bigcup_{i=0}^{N-1} S_i(M_{n-1})$. Then M_n converges to A_S in the Hausdorff topology, in particular, $\text{CL}(\bigcup_{n \in \mathbb{N}} M_n) = A_S$. By the definition, for each $n \in \mathbb{N}$ we have $M_n \subset V$. Thus $A_S \subset \text{CL}(V)$. Since S_w is a topological embedding for any $w \in W$, the latter part follows from the former one. \square

Proof of Theorem 1.2. Since (A_S, P_S) is self-similar, by Lemma 3.15 the covering pair (A_S, P_S) satisfies the conditions (T1) and (U). It suffices to show that (A_S, P_S) satisfies the condition (T2). By $A_S \cap V \neq \emptyset$, we can take $q \in A_S \cap V$. Then there exists $E \in (0, \infty)$ such that $U(q, E) \subset V$. Since \mathcal{S} consists of s -similar transformations, for each $w \in W$ we have $B(q_w, Es^{|w|}) \subset S_w(V)$. By Lemma 5.2, for each $w \in W$ we have $S_w(A_S) \subset \text{CL}(S_w(V))$, thus the ball $B(q_w, Es^{|w|})$ in X meets only $S_w(A_S)$. Hence the ball $B(q_w, Es^{|w|})$ in A_S is a subset of $S_w(A_S)$. Therefore we conclude that (A_S, P_S) satisfies the condition (T2). This completes the proof of Theorem 1.2. \square

Remark 5.1. An iterated function system \mathcal{S} on a metric space is said to satisfy the *open set condition* if \mathcal{S} satisfies the conditions (O1) and (O2) in the strong open set condition. Schief [4] proved that the open set condition implies the strong open set condition in the Euclidean setting. Schief [5] also proved that the implication mentioned above does not hold in a general setting (see [5, Example 3.1]).

5.2. extended attractors. We can construct an unbounded tiling space induced from a similar iterated function system.

Definition 5.1 (extended attractor). Let $N \in \mathbb{N}_{\geq 2}$ and $s \in (0, 1)$. Let \mathcal{S} be an (N, s) -similar iterated function system on a complete metric space with the strong open set condition, say $\mathcal{S} = \{S_i\}_{i=0}^{N-1}$. Define a sequence $\{F_k\}_{k \in \mathbb{N}}$ of metric spaces by $F_k = s^{-k}A_S$, where A_S is the attractor of \mathcal{S} . Note that for each $k \in \mathbb{N}$, each S_i is an s -similar transformation on F_k . By the definition of A_S , we find that $S_0(F_{k+1})$ is isometric to F_k . Thus we can identify F_k with $S_0(F_{k+1})$, and we may consider that $F_k \subset F_{k+1}$ for each $k \in \mathbb{N}$. Put $E_S = \bigcup_{k \in \mathbb{N}} F_k$. Note that E_S

is unbounded. Let W be the set of all words generated by $\{0, \dots, N - 1\}$. Define a map $Q_S: \mathbb{Z} \rightarrow \text{cov}(E_S)$ by

$$(Q_S)_n = \{S_w(F_k) \mid w \in W \text{ and } |w| - k = n\}. \tag{6}$$

We call E_S an *extended attractor of S* .

Similarly to Theorem 1.2, we obtain the following:

Theorem 5.3. *For $N \in \mathbb{N}_{\geq 2}$ and $s \in (0, \infty)$, let S be an (N, s) -similar iterated function system on a complete metric space with the strong open set condition. Let E_S be the extended attractor of S , and Q_S the map defined by (6). Then (E_S, Q_S) is an unbounded (N, s) -tiling space.*

Proof. Since all the tiles of (E_S, Q_S) are similar to A_S , by a similar argument to Lemma 5.1, we see that the condition (S1) is satisfied. Lemma 3.15 implies that (E_S, Q_S) satisfies the conditions (T1) and (U). By $E_S = \bigcup_{k \in \mathbb{N}} F_k$, and by the definition of Q_S , the conditions (S2) and (S3) are satisfied. Thus (E_S, Q_S) is an N -tiling set. Similarly to the proof of Theorem 1.2, we see that the condition (T2) is satisfied. Therefore the pair (E_S, Q_S) is an (N, s) -tiling space. \square

5.3. Examples of attractors

Example 5.1 (The middle-third Cantor set). Let C be the middle-third Cantor set. For each $i \in \{0, 1\}$, define a map $f_i: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_i(x) = \frac{1}{3}x + \frac{2}{3}i.$$

Put $S = \{f_0, f_1\}$. Then S is a $(2, 3^{-1})$ -similar iterated function system on \mathbb{R} , and C is the attractor of S . The open set $(0, 1)$ satisfies the conditions (O1), (O2) and (O3), and hence S satisfies the strong open set condition. Let $P_S: \mathbb{N} \rightarrow C$ be the map defined in Definition 1.3. Theorem 1.2 implies that (C, P_S) is a $(2, 3^{-1})$ -tiling space.

Example 5.2 (the Sierpiński gasket). Referring to the cubic roots of unity, put $w_0 = (1, 0)$, $w_1 = 2^{-1}(-1, \sqrt{3})$ and $w_2 = 2^{-1}(-1, -\sqrt{3})$. For each $i \in \mathbb{N}$, we define a map $f_i: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$f_i(x) = \frac{1}{2}x + \frac{1}{2}w_i.$$

Put $\mathcal{S} = \{f_0, f_1, f_2\}$. Then \mathcal{S} is a $(3, 2^{-1})$ -similar iterated function system on \mathbb{R}^2 . The attractor $A_{\mathcal{S}}$ of \mathcal{S} is called the Sierpiński gasket. The interior V of the triangle with vertices $\{w_0, w_1, w_2\}$ satisfies the conditions (O1), (O2) and (O3). Thus \mathcal{S} satisfies the strong open set condition. Let $P_{\mathcal{S}}: \mathbb{N} \rightarrow \text{cov}(A_{\mathcal{S}})$ be the map defined in Definition 1.3. Then Theorem 1.2 implies that $(A_{\mathcal{S}}, P_{\mathcal{S}})$ is a $(3, 2^{-1})$ -tiling space.

Example 5.3 (Euclidean spaces). Consider the N -dimensional normed vector space \mathbb{R}^N with ℓ^p -metric, where $p \in [1, \infty]$.

Let $A = \{v \in \mathbb{R}^N \mid \text{the entries of } v \text{ are } 0, 1 \text{ or } -1\}$. Since A has cardinality 3^N , it is indexed by $\{1, \dots, 3^N\}$, say $A = \{v(i)\}_{i=1}^{3^N}$. For each $i \in \{1, \dots, 3^N\}$, define a $(1/3)$ -similar transformation $f_i: \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$f_i(x) = \frac{1}{3}x + \frac{1}{3}v(i).$$

Put $\mathcal{S} = \{f_i\}_{i=1}^{3^N}$. Then \mathcal{S} is a $(3^N, 3^{-1})$ -similar iterated function system on \mathbb{R}^N , and $[-2^{-1}, 2^{-1}]^N$ is the attractor of \mathcal{S} . The open set $(-2^{-1}, 2^{-1})^N$ satisfies the conditions (O1), (O2) and (O3). Hence \mathcal{S} satisfies the strong open set condition. Let $P_{\mathcal{S}}: \mathbb{N} \rightarrow \text{cov}([-2^{-1}, 2^{-1}]^N)$ be the map defined in Definition 1.3, then this map is described as

$$(P_{\mathcal{S}})_n = \{3^{-n}v + 3^{-n}[-2^{-1}, 2^{-1}]^N \mid v \in \mathbb{Z}^N\}.$$

Theorem 1.2 implies that $([-2^{-1}, 2^{-1}]^N, P_{\mathcal{S}})$ is a $(3^N, 3^{-1})$ -tiling space.

We next consider the extended attractor $E_{\mathcal{S}}$ of \mathcal{S} . Since

$$\mathbb{R}^N = \bigcup_{i \in \mathbb{N}} [-2^{-1} \cdot 3^i, 2^{-1} \cdot 3^i]^N,$$

the space $E_{\mathcal{S}}$ is isomeric to \mathbb{R}^N with ℓ^p -metric. Under a natural identification between $E_{\mathcal{S}}$ and \mathbb{R}^N , the map $Q_{\mathcal{S}}: \mathbb{Z} \rightarrow \text{cov}(\mathbb{R}^N)$ defined in Definition 5.1 is described as

$$(Q_{\mathcal{S}})_n = \{3^{-n}v + 3^{-n}[-2^{-1}, 2^{-1}]^N \mid v \in \mathbb{Z}^N\}.$$

Theorem 5.3 implies that $(\mathbb{R}^N, Q_{\mathcal{S}})$ is a $(3^N, 3^{-1})$ -tiling space.

Applying Theorem 1.1 to the tiling space $(\mathbb{R}^N, P_{\mathcal{S}})$ discussed in Example 5.3, we obtain the Fraser–Yu characterization in [2] in a slightly different formulation:

Corollary 5.4. *For every subset F of \mathbb{R}^N , the following are equivalent:*

- (1) $\dim_A F = \dim_A \mathbb{R}^N$;
- (2) $[0, 1]^N \in \text{PC}(F)$;
- (3) $[0, 1]^N \in \text{TPC}(F)$;
- (4) $[0, 1]^N \in \text{KPC}(F)$;
- (5) F satisfies the asymptotic Steinhaus property for $[0, 1]^N$.

Proof. Let P_S be the map described in Example 5.3. Since all the tiles of (\mathbb{R}^N, P_S) are similar to $[0, 1]^N$, Theorem 1.1 leads to the claim. \square

Example 5.4 (p -adic numbers). Let p be a prime number and let v_p be the p -adic valuation. Let \mathbb{Q}_p be the set of all p -adic numbers. Let $s \in (0, 1)$, and define $d_{\mathbb{Q}_p}(x, y) = s^{v_p(x-y)}$, then $d_{\mathbb{Q}_p}$ is an ultrametric on \mathbb{Q}_p . For each $k \in \{0, \dots, p-1\}$, define an s -similar transformation $f_k: \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ by

$$f_k(x) = xp + k.$$

Put $\mathcal{S} = \{f_k\}_{k=0}^{p-1}$. Then \mathcal{S} is a (p, s) -similar iterated function system on \mathbb{Q}_p . The ball $B(0, 1)$ centered at 0 with radius 1 in \mathbb{Q}_p is the attractor of \mathcal{S} . Since for each $k \in \{0, \dots, p-1\}$ we have $f_k(B(0, 1)) = B(k, s)$, and since $d_{\mathbb{Q}_p}$ is an ultrametric, the open set $B(0, 1)$ satisfies the conditions (O1), (O2) and (O3). Thus \mathcal{S} satisfies the strong open set condition. Let $P_S: \mathbb{N} \rightarrow \text{cov}(B(0, 1))$ be the map defined in Definition 1.3, then this map is described as

$$(P_S)_n = \{B(a, s^{-n}) \mid a \in \mathbb{Q}_p\}.$$

By Theorem 1.2, we conclude that $(B(0, 1), P_S)$ is a (p, s) -tiling space.

We next consider the extended attractor of \mathcal{S} . Since

$$\mathbb{Q}_p = \bigcup_{i \in \mathbb{N}} B(0, s^{-i}),$$

the space E_S is isometric to \mathbb{Q}_p . Under a natural identification between E_S and \mathbb{Q}_p , the map $Q_S: \mathbb{Z} \rightarrow \text{cov}(\mathbb{Q}_p)$ defined in Definition 5.1 is described as

$$(Q_S)_n = \{B(a, s^{-n}) \mid a \in \mathbb{Q}_p\}.$$

Theorem 5.3 implies that (\mathbb{Q}_p, Q_S) is a (p, s) -tiling space. By Corollary 4.3, we obtain $\dim_A \mathbb{Q}_p = \log(p) / \log(s^{-1})$.

6. Counterexamples

We first provide a tiling space that is not doubling.

Example 6.1. Let $N \in \mathbb{N}_{\geq 2}$ and $s \in (0, 1)$. Let T be the set of all sequences $x: \mathbb{N} \rightarrow \{0, \dots, N - 1\}$ satisfying that $x_0 \in \{0, \dots, N - 2\}$. The set T can be described as

$$T = \{0, \dots, N - 2\} \times \prod_{n=1}^{\infty} \{0, \dots, N - 1\}.$$

For $x, y \in T$, define a valuation $v: T \times T \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$v(x, y) = \begin{cases} \min\{n \in \mathbb{N} \mid x_n \neq y_n\} & \text{if } x \neq y, \\ \infty & \text{if } x = y. \end{cases}$$

For each $i \in \mathbb{N}$, let T_i be the metric space (T, d_i) , where the metric d_i is defined by $d_i(x, y) = s^{v(x,y)+i}$.

For each $i \in \mathbb{N}$, the symbol o_i stands for the sequence whose all entries are 0 in T_i . For each $k \in \mathbb{N}$, put $O_k = \prod_{i=k}^{\infty} \{o_i\}$, and

$$X(k) = \left(\prod_{i \in \mathbb{N}_{\geq k}} T_i \right) / O(k).$$

Namely, $X(k)$ is the set constructed by identifying the zero sequences in the set $\prod_{i \in \mathbb{N}_{\geq k}} T_i$. Put $X = X(0)$. We may consider that for each $k \in \mathbb{N}$ we have $X(k + 1) \subset X(k)$ and $T_k \subset X$. The symbol o stands for the zero sequence in X . The point o is the identified point in X .

We define a function $d_X: X \times X \rightarrow \mathbb{R}_{\geq 0}$ by

$$d_X(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in T_i \text{ for some } i, \\ d_i(x, o) + d_j(o, y) & \text{if } x \in T_i \text{ and } y \in T_j \text{ for some } i \neq j. \end{cases}$$

The function d_X is a metric on X .

We next define a tiling structure on X . Let W be the set of all words whose 0-th entries are in $\{0, \dots, N - 2\}$ and other entries are in $\{0, \dots, N - 1\}$. Remark that the set W does not contain the empty word. For each word $w = w_0 \cdots w_l \in W$, we define

$$(T_i)_w = \{x \in T_i \mid x_0 = w_0, \dots, x_l = w_l\},$$

where $w_0 \in \{0, \dots, N - 2\}$ and $w_i \in \{0, \dots, N - 1\}$ ($i \geq 1$). For each $k \in \mathbb{N}$ and for each $l \in \mathbb{N}_{\geq 1}$, we define

$$S_{k,l} = \{(T_k)_w \mid w \in W \text{ and } |w| = l\}.$$

We define a map $P: \mathbb{N} \rightarrow \text{cov}(X)$ by

$$P_n = \{X(n)\} \cup \bigcup_{k+l=n} \mathcal{S}_{k,l}.$$

We first show that (X, P) is an N -tiling set. By the definition of P , the condition (S3) is satisfied. For each $w \in W$, we have $(T_i)_w = \bigcup_{v=0}^{N-1} (T_i)_{wv}$. For each $n \in \mathbb{N}$, we have

$$X(n) = X(n+1) \cup \bigcup_{v=0}^{N-2} (T_n)_v.$$

Thus, the conditions (S1) is satisfied. By the boundedness of X , the condition (S2) is satisfied. Thus, the pair (X, P) is an N -tiling set.

We next show that (X, P) is an (N, s) -tiling space. By the definition of the metric d_X , for each $n \in \mathbb{N}$, and for all $k \in \mathbb{N}$ and $l \in \mathbb{N}_{\geq 1}$ with $k + l = n$, for each $(T_i)_w \in \mathcal{S}_{k,l}$, we have $\delta((T_i)_w) = s^{-n}$ and $\delta(X(n)) = s^{-n} + s^{-n-1}$. By $s^{-n} \leq s^{-n} + s^{-n-1} \leq 2s^{-n}$, the condition (T1) is satisfied. By the definition of $\{T_i\}_{i \in \mathbb{N}}$, for every $a \in T_i$ we have $(T_i)_w = B(a, s^{-n})$. For every $a \in (T_n)_1$ we also have

$$B(a, s^{-n-1}) \subset T_n \subset X(n).$$

Then the condition (T2) is satisfied. For each $n \in \mathbb{N}$, the spaces $sX(n)$ and $X(n+1)$ are isometric to each other. For all $i, j \in \mathbb{N}$ and for all $u, v \in W$, the spaces $(T_i)_u$ and $(T_j)_v$ are similar. Thus the tiles of (X, P) have two similarity classes, and hence the condition (U) is satisfied. Therefore (X, P) is an (N, s) -tiling space.

For each $n \in \mathbb{N}$, we have $\text{card}(\{A \in P_n \mid o \in A\}) = (n - 1)(N - 1) + N$. By Proposition 4.1, and by $(n - 1)(N - 1) + N \rightarrow \infty$ as $n \rightarrow \infty$, we conclude that X is not doubling.

Remark 6.1. Due to the Brouwer characterization of the middle-third Cantor set, the space X constructed in Example 6.1 is homeomorphic to the middle-third Cantor set. Indeed, the space X is topologically 0-dimensional and compact, and it has no isolated points.

Remark 6.2. In Example 6.1, by replacing the role of \mathbb{N} with that of \mathbb{Z} , we also obtain an unbounded non-doubling (N, s) -tiling space that is not locally compact. Therefore being a tiling space does not imply the local compactness.

We next construct a pre-tiling space that is not a tiling space. The space constructed below is also a bi-Lipschitz image of a tiling space.

Example 6.2. Let $2^{\mathbb{N}}$ be the set of all sequences valued in $\{0, 1\}$. For $x, y \in 2^{\mathbb{N}}$, define a valuation $v: 2^{\mathbb{N}} \times 2^{\mathbb{N}} \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$v(x, y) = \begin{cases} \min\{n \in \mathbb{N} \mid x_n \neq y_n\} & \text{if } x \neq y, \\ \infty & \text{if } x = y. \end{cases}$$

For $n \in \mathbb{N}$, set $a_n = (1 - 1/(n + 3))2^{-(n+3)}$. Let X be a metric space $(2^{\mathbb{N}}, d_X)$ with metric d_X defined by $d_X(x, y) = a_{v(x,y)}$. Let Y be a metric space $(2^{\mathbb{N}}, d_Y)$ with metric d_Y defined by $d_Y(x, y) = 2^{-v(x,y)}$. The spaces X and Y are ultrametric spaces. Define two maps $P, Q: \mathbb{N} \rightarrow \text{cov}(2^{\mathbb{N}})$ by $P_n = \{B(x, a_n) \mid x \in X\}$ and $Q_n = \{B(x, 2^{-n}) \mid x \in Y\}$. Then (X, P) and (Y, Q) are 2-tiling sets.

We now prove that (Y, Q) is a $(2, 2^{-1})$ -tiling space. For each $i \in \{0, 1\}$, define a map $f_i: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by

$$(f_i(x))_j = \begin{cases} i & \text{if } j = 0, \\ x_{j-1} & \text{if } j \geq 1, \end{cases}$$

where $(f_i(x))_j$ is the j -th entry of $f_i(x)$. Then $\{f_0, f_1\}$ is a $(2, 2^{-1})$ -similar iterated function system on Y , and Y is the attractor of $\{f_0, f_1\}$. The map $P_{\{f_0, f_1\}}: \mathbb{N} \rightarrow \text{cov}(2^{\mathbb{N}})$ coincides with the map Q . Therefore Theorem 1.2 implies that the pair (Y, Q) is a $(2, 2^{-1})$ -tiling space.

Note that the tiles of (X, P) have infinitely many similarity classes. The similarity classes of the tiles of (X, P) do not contain that of (Y, Q) .

The identity map $id: X \rightarrow Y$ is bi-Lipschitz, in particular, the metric space X is a bi-Lipschitz image of Y . Since (Y, Q) is a $(2, 2^{-1})$ -pre-tiling space, by Corollary 3.12, so is (X, P) .

Take a sequence $\{A_i\}_{i \in \mathbb{N}}$ of tiles of (X, P) with $A_i \in P_i$. For each $N \in \mathbb{N}$ and for each $n \in \mathbb{N}$, we have $|a_{n+N}/a_N - 2^{-n}| < 1/(N + 2)$. Then the sequence $(\delta(A_i))^{-1}A_i$ converges to Y in the sense of Gromov–Hausdorff. Thus (X, P) is a pre-tiling space which does not satisfy the condition (U).

In summary, the pre-tiling space (X, P) is a non-tiling space which is a bi-Lipschitz image of the tiling space (Y, Q) .

Combining the metric spaces provided in Example 6.2, we construct a tiling space whose tiles have infinitely many similarity classes.

Example 6.3. Let (X, P) and (Y, Q) be the pre-tiling space and the tiling space constructed in Example 6.2, respectively. Put $Z = X \sqcup Y$ and define a metric d_Z

on Z by

$$d_Z(x, y) = \begin{cases} d_X(x, y) & \text{if } x, y \in X, \\ d_Y(x, y) & \text{if } x, y \in Y, \\ 2 & \text{if } x \text{ and } y \text{ lie in distinct components.} \end{cases}$$

We now define a map $R: \mathbb{N} \rightarrow \text{cov}(X \sqcup Y)$ by $R_0 = \{Z\}$, and by $R_n = P_{n-1} \cup Q_{n-1}$ for $n \in \mathbb{N}_{\geq 1}$. Since (X, P) and (Y, Q) are $(2, 2^{-1})$ -pre-tiling spaces, the pair (Z, R) is a $(2, 2^{-1})$ -pre-tiling space.

We now prove that (Z, R) satisfies the condition (U). Take a sequence $\{A_n\}_{n \in \mathbb{N}}$ of tiles of (Z, R) . Then there exists a subsequence $\{A_{n_i}\}_{i \in \mathbb{N}}$ of $\{A_n\}_{n \in \mathbb{N}}$ consisting of either tiles of (X, P) or that of (Y, Q) . If $\{A_{n_i}\}_{i \in \mathbb{N}}$ consists of tiles of (X, P) , then by the argument in Example 6.2, there exists a subsequence $\{A_{m_i}\}_{i \in \mathbb{N}}$ of $\{A_{n_i}\}_{i \in \mathbb{N}}$ such that $\delta(A_{m_i})^{-1}A_{m_i}$ converges to either Y or $\delta(T)^{-1}T$ for some tile T of (X, P) . In the case where $\{A_{n_i}\}_{i \in \mathbb{N}}$ consists of tiles of (Y, Q) , since (Y, Q) satisfies the condition (U), there exists a subsequence $\{A_{m_i}\}_{i \in \mathbb{N}}$ of $\{A_{n_i}\}_{i \in \mathbb{N}}$ such that $\delta(A_{m_i})^{-1}A_{m_i}$ converges to $\delta(T)^{-1}T$ for some tile T of (Y, Q) . Therefore (Z, R) satisfies the condition (U).

In this way, we obtain a $(2, 2^{-1})$ -tiling space (Z, R) whose tiles have infinitely many similarity classes.

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Yoshito Ishiki, Graduate School of Pure and Applied Sciences, University of Tsukuba,
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571, Japan

Present Affiliation: Photonics Control Technology Team,
RIKEN Center for Advanced Photonics, 2-1 Hirasawa, Wako, Saitama 351-0198, Japan

e-mail: yoshito.ishiki@riken.jp