J. Fractal Geom. 9 (2022), 23–72
DOI 10.4171/JFG/111

Hidden positivity and a new approach to numerical computation of Hausdorff dimension: higher order methods

Richard S. Falk and Roger D. Nussbaum

Abstract. In 2018, the authors developed a new approach to the computation of the Hausdorff dimension of the invariant set of an iterated function system or IFS. In this paper, we extend this approach to incorporate high order approximation methods. We again rely on the fact that we can associate to the IFS a parametrized family of positive, linear, Perron–Frobenius operators L_s, an idea known in varying degrees of generality for many years. Although L_s is not compact in the setting we consider, it possesses a strictly positive C^m eigenfunction v_s with eigenvalue $R(L_s)$ for arbitrary m and all other points z in the spectrum of L_s satisfy $|z| \le b$ for some constant $b < R(L_s)$. Under appropriate assumptions on the IFS, the Hausdorff dimension of the invariant set of the IFS is the value $s = s_*$ for which $R(L_s) = 1$. This eigenvalue problem is then approximated by a collocation method at the extended Chebyshev points of each subinterval using continuous piecewise polynomials of arbitrary degree r. Using an extension of the Perron theory of positive matrices to matrices that map a cone K to its interior and explicit a priori bounds on the derivatives of the strictly positive eigenfunction v_s, we give rigorous upper and lower bounds for the Hausdorff dimension s_*, and these bounds converge rapidly to s_* as the mesh size decreases and/or the polynomial degree increases.

1. Introduction

In this paper, we continue previous work in finding rigorous estimates for the Hausdorff dimension of invariant sets for iterated function systems or IFS's. To describe the framework of the problem we are considering, we let $S \subset \mathbb{R}$ be a nonempty compact set, and for some positive integer m, let $\theta_p : S \to S$ and $g_p : S \to [0, \infty) \in C^m(S)$ for $1 \le p \le n < \infty$. If θ_p are contraction mappings, it is known that there exists a unique, compact, nonempty set $C \subset S$ such that $C = \bigcup_{p=1}^n \theta_p(C)$. The set C is called the *invariant set* for the IFS $\{\theta_p : 1 \le p \le n\}$.

2020 Mathematics Subject Classification. Primary 11K55, 37C30; Secondary 65J10.
Keywords. Hausdorff dimension, positive transfer operators, continued fractions.

For $s > 0$, define a bounded linear map $L_s : C(S) \to C(S)$ (often called a *Perron–Frobenius operator* or *linear transfer operator*) by

$$(L_s f)(t) = \sum_{p=1}^{n} [g_p(t)]^s f(\theta_p(t)), \quad \text{for } t \in S. \tag{1.1}$$

Under additional appropriate hypotheses (stated in the next section), L_s, considered as a map from $C^m(S)$ to itself, has a strictly positive eigenfunction $v_s \in C^m(S)$ with algebraically simple eigenvalue $\lambda_s = R(L_s)$, the spectral radius of L_s. In addition, all other points z in the spectrum of L_s satisfy $|z| \le b$ for some constant $b < R(L_s)$. A more precise statement of this result, along with other conclusions, is given in Theorem 2.1 in the next section. Note that in the C^m setting, L_s is, in general, not compact, has positive essential spectral radius and cannot be the limit in operator norm of a sequence of finite-dimensional linear operators. These difficulties do not usually arise if L_s can be studied in a Banach space of complex analytic functions; there is an extensive literature concerning the spectral theory of Perron–Frobenius operators which map a Banach space of analytic functions into itself. We prefer to work in the more general C^m setting so as to provide tools which also can be applied to some non-analytic examples, e.g., as in [14, Section 5].

The aim of this paper is to derive an approximation scheme that allows us to estimate $R(L_s)$ by the spectral radius of an associated matrix $\mathbf{L}_s$ which approximates the operator L_s in a weak sense and then to obtain rigorous bounds on the error $|R(L_s) - R(\mathbf{L}_s)|$. We then use this approximation scheme to estimate s_*, the unique number $s \ge 0$ such that $R(L_s) = 1$. Under appropriate assumptions, s_* equals the Hausdorff dimension of the invariant set associated to the IFS. This observation about Hausdorff dimension has been made, in varying degrees of generality by many authors. See, for example, [4,6–10,13,15,18–20,22–25,31,32,35,37–39,41]. There is also a large literature on the approximation of linear transform operators, not necessarily related to the computation of Hausdorff dimension, and often assuming the maps are analytic. We do not attempt to survey that literature, other than to cite one recent paper, [1], which has some connections to our work here, and contains many references to that literature.

In previous work, [14], the authors presented a new approach to the problem described in the preceding paragraph. We obtained rigorous upper and lower bounds for the Hausdorff dimension s_*, and these bounds exhibited second order convergence to s_* as the mesh size decreases. The approximate matrix was obtained by a collocation method using continuous piecewise linear functions, motivated by the fact that if such functions are nonnegative at the mesh points, they are nonnegative at all points of the interval in which they are defined. This property leads to nonnegative matrix approximations of the operator L_s. One would like these matrices to mimic the properties of the continuous operator L_s, which means they should satisfy the conclusions of the Perron theorem for positive matrices (matrices with strictly positive entries), i.e., they should have an eigenvalue of multiplicity one equal to the spectral radius of the matrix with corresponding positive eigenvector and all other eigenvalues of the matrix should have modulus less than the spectral radius. This is not true for nonnegative matrices, however, unless they have an additional property. One such property that would guarantee this is that the matrix $\mathbf{L}_s$ be *primitive*, i.e., there exists a positive integer p such that $\mathbf{L}_s^p$ is a positive matrix. Note that if $\mathbf{L}_s$ is *irreducible*, then the first two properties hold, but there can be other eigenvalues of the same modulus as the spectral radius. Unfortunately, the approximation scheme used led to matrices which are neither primitive nor irreducible. The remedy to obtain the desired properties was to note that the cone K of nonnegative vectors is not the natural cone in which such matrices should be studied. Using a more general notion of positivity of an operator L in which L maps a cone K into itself, one can still obtain the conclusions of the Perron theorem. This is important since we use the spectral radius of the approximate matrix $\mathbf{L}_s$ to approximate the spectral radius of L_s and the fact that there is a single dominant eigenvalue enables us to calculate it efficiently using some variant of the power method.

In this paper, we analyze a similar method obtained by approximation using higher order piecewise polynomials. As we shall see, the matrices resulting from the approximation scheme appear to be even more problematic, since they are not even nonnegative. Despite this fact, the use of an alternative cone, in place of the standard cone of nonnegative vectors, allows us to show that the conclusions of the classical Perron theorem also hold for the matrices of this paper. There is a substantial abstract theory which has been developed for finite-dimensional linear operators which are positive in the sense that they map a cone into itself. The survey paper [43], references in [43], and [28, Appendices A and B] provide a good starting point. However, the difficulty lies in finding such a cone that fits the application under study. We use the term *hidden positivity* to call attention to the fact that we are able to find such a cone for the approximate operators developed in this paper.

The cone we use is easiest to describe in the case of continuous, piecewise linear functions, and is defined as follows. On the interval $[0, 1]$, for a fixed integer N, let $h = 1/N$ and $x_i = ih$ for $i = 0, 1, \ldots, N$. The space of continuous, piecewise linear functions is just the finite-dimensional space of continuous functions that restricted to each subinterval $[x_i, x_{i+1}]$ are linear functions. Since a function w in this space is completely determined by its values $w_i = w(x_i)$, $i = 0, 1, \ldots, N$ (the *degrees of freedom* of w), we can also view w as the vector $[w_0, \ldots, w_N]$. For any integer $M > 0$, we then define the cone K_M by

$$K_M = \{w : w_i \le \exp(M|x_i - x_j|)w_j \text{ for } i, j = 0, 1, \ldots, N\}.$$

The cone for higher order piecewise polynomials is similar, but its description is more involved because of the more complicated nature of the degrees of freedom of such functions. The details are provided in Section 3.

One technical difference between piecewise linear functions and higher order piecewise polynomials is that in order to obtain the results described above, we must consider approximations $\mathbf{L}_{s,\nu}$ of the operator L_s^ν, where ν depends on the degree r of the piecewise polynomial approximation. As we observe in the next section, the operator L_s^ν has the same form as L_s, i.e.,

$$(L_s^\nu f)(t) = \sum_{\omega \in \Omega_\nu} [g_\omega(t)]^s f(\theta_\omega(t)),$$

where for $\nu \geq 1$, $\Omega_\nu = \{\omega = (p_1, p_2, \dots, p_\nu) : 1 \leq p_j \leq n \text{ for } 1 \leq j \leq \nu\}$, for $\omega = (p_1, p_2, \dots, p_\nu) \in \Omega_\nu$,

$$\theta_\omega(t) = (\theta_{p_1} \circ \theta_{p_2} \circ \cdots \circ \theta_{p_\nu})(t),$$

and $g_\omega(t)$ is defined in the next section. We note that under the weaker assumption that θ_ω is a contraction mapping for all $\omega \in \Omega_\nu$, there exists a unique compact set C such that $C = \bigcup_{\omega \in \Omega_\nu} \theta_\omega(C)$ and that necessarily $C = \bigcup_{p=1}^n \theta_p(C)$. By using the matrix $\mathbf{L}_{s,\nu}$, one reduces the domain of the operator to a finite set of subintervals whose total length is much less than the original length of the domain, resulting in many fewer mesh points. The downside, however, is that this advantage is completely offset by the increase in the number of terms in the operator L_s for each time the map is iterated, i.e., from n to n^ν. We note that since we have not found any case where the method fails if we do not iterate the matrix, we conjecture that this extra condition is an artifact of the method of proof, and not the method itself.

To obtain the conclusions of the Perron theorem, the key result is to show that for some $0 < M' < M$,

$$\mathbf{L}_{s,\nu}(K_M \setminus \{0\}) \subset K_{M'} \setminus \{0\}. \tag{1.2}$$

This enables us to apply results from the literature on mappings of a cone to itself to obtain the desired conclusions. Details of this connection, along with references to the relevant literature, are described in Section 4.

A main goal of our approach, in addition to proposing a new approximation scheme, is to provide rigorous upper and lower bounds for the Hausdorff dimension of the underlying IFS. This will follow directly if we are able to derive rigorous error bounds for $|[R(L_s)]^\nu - R(\mathbf{L}_{s,\nu})|$. In the case of piecewise linear functions, we obtained the bounds by using a simple and well-known result (cf. [14, Lemma 2.2]) that if A is a nonnegative matrix and w is a vector with strictly positive components, then (i) if $(Aw)_k \geq \lambda w_k$ for all components k, then $R(A) \geq \lambda$ and (ii) if $(Aw)_k \leq \lambda w_k$ for all components k, then $R(A) \leq \lambda$. Here, we use an analogous result for a matrix mapping a cone K to itself, in which we replace $\leq$ by $\leq_K$, i.e., $u \leq_K v$ if and only if $v - u \in K$.

Another key tool for obtaining rigorous upper and lower bounds for the Hausdorff dimension s_*, is to obtain and use explicit a priori bounds on the quantity $D^q v_s(x)/v_s(x)$ of the strictly positive eigenfunction v_s of L_s, where $D^q v_s$ denotes the qth derivative of v_s. Such estimates are derived in Section 7.

In order to improve the efficiency of our computation, in Section 7, we consider the possibility of replacing the original interval $S = [a, b]$ by a smaller interval $S_0 \subset S$ such that $\theta_\beta(S_0) \subset S_0$ for $\beta \in \mathcal{B}$. In particular, for the maps $\theta_\beta = 1/(x + \beta)$, setting $\gamma = \min\{\beta \in \mathcal{B}\}$ and $\Gamma = \max\{\beta \in \mathcal{B}\}$, we can reduce the interval S to $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, where

$$\mathfrak{a}_\infty = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{\Gamma}} \quad \text{and} \quad \mathfrak{b}_\infty = -\frac{\Gamma}{2} + \sqrt{\left(\frac{\Gamma}{2}\right)^2 + \frac{\Gamma}{\gamma}} = \frac{\Gamma}{\gamma}\mathfrak{a}_\infty.$$

For example, for the set $\{1, 2\}$, we reduce the interval $[0, 1]$ to $[(\sqrt{3} - 1)/2, \sqrt{3} - 1]$ of approximate length 0.366, while for the set $\{10, 11\}$, we reduce the interval $[0, 1]$ to $[0.0901, 0.0991]$ of length 0.009.

A main result of the paper, Theorem 8.8, says that under appropriate hypotheses, there is a computable constant H, such that

$$[R([1 + Hh^r]^{-1}\mathbf{L}_{s,\nu})]^{1/\nu} \leq \lambda_s \leq [R([1 - Hh^r]^{-1}\mathbf{L}_{s,\nu})]^{1/\nu},$$

where h denotes the maximum mesh size and r is the degree of the piecewise polynomial approximation. Using these inequalities, we can obtain rigorous upper and lower bounds on the Hausdorff dimension of the invariant set associated with the transfer operator L_s as follows. Let s_u and s_l denote values of s satisfying

$$[1 - Hh^r]^{-1} R(\mathbf{L}_{s_u,\nu}) < 1 \quad \text{and} \quad [1 + Hh^r]^{-1} R(\mathbf{L}_{s_l,\nu}) > 1.$$

It follows immediately from Theorem 8.8 that $\lambda_{s_u}^\nu < 1$ and $\lambda_{s_l}^\nu > 1$. Since the spectral radius λ_s of L_s is a strictly decreasing function of s, there will be a value s_* satisfying $s_l < s_* < s_u$ for which $\lambda_{s_*}^\nu = 1$ or, equivalently, $\lambda_{s_*} = 1$. The value s_* gives the Hausdorff dimension s_* of the invariant set associated with the transfer operator L_s. Since $s_u - s_l$ is of order h^r, by choosing h to be sufficiently small and/or r to be sufficiently large, we obtain a highly accurate estimate for s_*. As noted above, for a given s, $R([1 \pm Hh^r]^{-1}\mathbf{L}_{s,\nu})$ is easily computed by variants of the power method for eigenvalues, since the largest eigenvalue has multiplicity one and is the only eigenvalue of its modulus. Our theoretical results imply that $\mathbf{L}_{s,\nu}$ has an eigenvector w in $K := K_M$ with eigenvalue $R(L_{s,\nu})$ and that this eigenvector can be computed to high accuracy. Still, one might be concerned about possible errors in the computation of $R(\mathbf{L}_{s,\nu})$ and w. However, independently of how a purported eigenvector $w \in K$ for $\mathbf{L}_{s,\nu}$ is found, if $\alpha w \leq_K \mathbf{L}_{s,\nu} w \leq_K \beta w$, Lemma 4.1 in Section 4 implies that $\alpha \leq R(\mathbf{L}_{s,\nu}) \leq \beta$. This provides a means of giving rigorous bounds for $R(\mathbf{L}_{s,\nu})$.

In Section 9, we present some results of computations of the Hausdorff dimension s of invariant sets in $[0, 1]$ arising from continued fraction expansions. In this much studied case, one defines $\theta_p = 1/(x + p)$, for a positive integer p and $x \in [0, 1]$, and for a subset $\mathcal{B} \subset \mathbb{N}$, one considers the IFS $\{\theta_p : p \in \mathcal{B}\}$ and seeks estimates on the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for this IFS. This problem has previously been considered by many authors; see [3,6,7,15,17–20,23,24]. In this case, (1.1) becomes

$$(L_s f)(x) = \sum_{p \in \mathcal{B}} \Big(\frac{1}{x+p}\Big)^{2s} f\Big(\frac{1}{x+p}\Big), \quad \text{for } 0 \leq x \leq 1,$$

and one seeks a value $s \geq 0$ for which $\lambda_s := R(L_s) = 1$. Several of the papers listed above contain a large number of computations to various degrees of accuracy of the Hausdorff dimension of the IFS $\{\theta_p : p \in \mathcal{B}\}$ for various choices of the set $\mathcal{B}$. An early paper, [20], gives results for over 30 choices of $\mathcal{B}$, containing between two and five terms in the set $\mathcal{B}$, with results reported to an accuracy between 10^{-6} and 10^{-19}, depending on the problem studied. A *Mathematica* code implementing the algorithm is also provided. In [23], computations to four decimal places are given for over 35 choices of the set $\mathcal{B}$, ranging from two terms, to as many as 34 (this computation is to three decimal places), and also includes a computation of $E[1, 2]$ ($\mathcal{B} = \{1, 2\}$) accurate to 54 decimal places. In [24], eight examples of $\mathcal{B}$, consisting of two terms, are computed with accuracies ranging from 10^{-13} to 10^{-52}, depending on the choice of $\mathcal{B}$, although the authors note that for the sets $[10, 11]$ and $[10^2, 10^4]$, they were able to compute them to accuracies of 10^{-61} and 10^{-122}, respectively. This depends on the fact that the speed of convergence of their methods depends on the size of the smallest value of $p \in \mathcal{B}$. In [26], the Hausdorff dimension of $E[1, 2]$ is rigorously computed to 100 decimal places, although more digits are computed. It is less clear how well some of the approximation schemes employed in these papers work when $|\mathcal{B}|$ is moderately large or when different real analytic functions $\hat{\theta}_j : [0, 1] \to [0, 1]$ are used. Here and in [14], in the one-dimensional case, we present an alternative approach with much wider applicability that only requires the maps in the IFS to be C^m, for some finite value of m. As an illustration, we considered in [14], perturbations of the IFS for the middle thirds Cantor set for which the corresponding contraction maps are C^3, but not C^4.

The computations in Section 9 include choices of various sets of continued fractions, maximum mesh size h, piecewise polynomial degree r, and number of iterations ν (where $\nu = 1$ corresponds to the original map), including choices of ν for which the hypotheses of our theorems are satisfied, but also computations which obtain the same results when the mappings are not iterated. These results support our conjecture that our method also works in the non-iterated situation. To facilitate computation of further examples, a *Matlab* code is provided in the website `https://sites.math.rutgers.edu/~falk/hausdorff/codes.html`.

An outline of the paper is as follows. In the next section, we introduce further notation and state some preliminary results we will use in our analysis. Section 3 contains a description of the approximate problem and the cone we use to analyze it. Section 4 contains the theoretical results we will need to show that the matrices arising from the approximation scheme satisfy the conclusions of the Perron theorem. In Section 5, the main result is to determine conditions under which the matrix $\mathbf{L}_{s,\nu}$ satisfies (1.2). These conditions involve a number of constants, which we then estimate in Section 6, ultimately deriving bounds for $R(L_s)$ in terms of $[R(\mathbf{L}_{s,\nu})]^{1/\nu}$. In Section 7, we consider a method for reducing the size of the interval S on which the problem is defined, with the aim of reducing the number of mesh points that will be needed in the approximation scheme. In so doing, we are also able to improve the bound on two constants which are used in the error estimate for $R(L_s)$. Recall that condition (1.2) requires determining for each constant M, a constant $0 < M' < M$ such that (1.2) is satisfied. In Section 8, we provide a procedure for determining this constant. Finally, the numerical computations described above are given in Section 9.

It would be of considerable interest to extend the methods of this paper to the two-dimensional case, e.g., to the problem of obtaining rigorous estimates for the Hausdorff dimension of sets of complex continued fractions. We conjecture that such an extension can be done, but we leave it as an open problem for possible future work.

2. Notation and preliminaries

Let $C(S)$ denote the Banach space of continuous functions $f : S \to \mathbb{R}$, where S is a compact subset of $\mathbb{R}$. Assume the following:

(H0) For $1 \le p \le n < \infty$, $\theta_p : S \to S$ is a Lipschitz map.

(H1) For $1 \le p \le n < \infty$, $g_p : S \to [0, \infty)$ is a nonnegative continuous function which is not identically zero, and satisfies that for some constant $M_0 > 0$,

$$g_p(t_1) \le g_p(t_2) \exp(M_0 |t_1 - t_2|), \quad \text{for all } t_1, t_2 \in S, \ 1 \le p \le n.$$

We note that it is easy to show that (H1) is equivalent to assuming that $g_p(t) > 0$ for all $t \in S$, and

$$|\ln(g_p(t_1)) - \ln(g_p(t_2))| \le M_0 |t_1 - t_2|, \quad \text{for all } t_1, t_2 \in S, \ 1 \le p \le n.$$

For $s > 0$, define a bounded linear map $L_s : C(S) \to C(S)$ (often called a *Perron–Frobenius operator*) by (1.1), i.e.,

$$(L_s f)(t) = \sum_{p=1}^{n} [g_p(t)]^s f(\theta_p(t)), \quad \text{for } t \in S.$$

We shall need to consider the νth iterate of L_s, L_s^ν. For $\nu \geq 1$, let

$$\Omega_\nu = \{\omega = (p_1, p_2, \dots, p_\nu) : 1 \leq p_j \leq n \text{ for } 1 \leq j \leq \nu\},$$

and for $\omega = (p_1, p_2, \dots, p_\nu) \in \Omega_\nu$, define

$$\theta_\omega(t) = (\theta_{p_1} \circ \theta_{p_2} \circ \cdots \circ \theta_{p_\nu})(t)$$

and

$$g_\omega(t) = g_{p_1}(\theta_{p_2} \circ \cdots \circ \theta_{p_\nu}(t)) g_{p_2}(\theta_{p_3} \circ \cdots \circ \theta_{p_\nu}(t)) \cdots g_{p_{\nu-1}}(\theta_{p_\nu}(t)) g_{p_\nu}(t).$$

The reader can verify (e.g., see [35]) that for all $f \in C(S)$,

$$(L_s^\nu f)(t) = \sum_{\omega \in \Omega_\nu} [g_\omega(t)]^s f(\theta_\omega(t)).$$

Note that L_s^ν has the same form as L_s, except that L_s^ν has index set Ω_ν. To analyze the operator L_s^ν, we shall need stronger assumptions than (H0). We will thus assume the following:

(H2) Hypothesis (H0) is satisfied and there exist constants $C_0 \geq 1$ and $0 < \kappa < 1$ such that for all integers $\nu \geq 1$, all $\omega \in \Omega_\nu$, and all $t_1, t_2 \in S$,

$$|\theta_\omega(t_1) - \theta_\omega(t_2)| \leq C_0 \kappa^\nu |t_1 - t_2|.$$

Assuming (H1) and (H2), one can prove that for all $\omega \in \Omega_\nu$ and all $t_1, t_2 \in S$,

$$g_\omega(t_1) \leq \exp(M_0' |t_1 - t_2|) g_\omega(t_2),$$

where $M_0' = M_0 C_0 [(1 - \kappa^\nu)/(1 - \kappa)]$. The proof is left to the reader. The reader will notice that the above framework carries over to the more general case that S is a compact metric space with metric ρ. The hypotheses (H1) and (H2) take the same form except that $|t_1 - t_2|$ is replaced by $\rho(t_1, t_2)$.

The following result provides some theoretical background which will be essential for our later work concerning the operator L_s. This theorem is a special case of [34, Corollary 6.6]. We refer to [33, Section 3] for a brief discussion of the essential spectrum, which is mentioned in Theorem 2.1 below.

Theorem 2.1. *Assume that the hypotheses* (H1) *and* (H2) *are satisfied, S is a finite union of compact intervals, and L_s is given by* (1.1)*, where $s > 0$. Assume also that $\theta_i \in C^m(S)$ and $g_i \in C^m(S)$ for some positive integer m. Let $\Lambda_s : Y := C^m(S) \to Y$ be the bounded linear operator given by* (1.1)*, but considered as a map from Y to itself, so $L_s(f) = \Lambda_s(f)$ for $f \in Y$. If $R(L_s)$ (respectively, $R(\Lambda_s)$) denotes the spectral radius of L_s (respectively, of Λ_s) and $\rho(\Lambda_s)$ denotes the essential spectral radius of Λ_s and κ is as in* (H2)*, then*

$$\rho(\Lambda_s) \leq \kappa^m R(\Lambda_s) \quad \text{and} \quad R(\Lambda_s) = R(L_s) =: \lambda_s > 0.$$

Let $\hat{\Lambda}_s$ denote the complexification of Λ_s. If $\sigma(\hat{\Lambda}_s)$ denotes the spectrum of $\hat{\Lambda}_s$ and we define $\sigma(\Lambda_s) := \sigma(\hat{\Lambda}_s)$, then if $z \in \sigma(\Lambda_s)$ and $\rho(\Lambda_s) < |z|$, z is an isolated point of $\sigma(\Lambda_s)$ and is an eigenvalue of Λ_s of finite algebraic multiplicity. Moreover, there exists $b_s < \lambda_s$ such that

$$\sigma(\Lambda_s) \setminus \{\lambda_s\} \subset \{z \in \mathbb{C} : |z| \leq |b_s|\}.$$

There exists a strictly positive eigenfunction $v_s \in C^m(S)$ with eigenvalue $\lambda_s > 0$, and λ_s is an algebraically simple eigenvalue of Λ_s. If $u \in Y$ and $u(t) > 0$ for all $t \in S$, there exists a positive real number α (dependent on u) such that

$$\lim_{k\to\infty} \left(\frac{1}{\lambda_s}\right)^k \Lambda_s^k(u) = \alpha v_s, \tag{2.1}$$

where the convergence is in the norm topology on Y.

Remark 2.2. In our work here, it will be important to have estimates on

$$\sup\left\{\frac{\left|\frac{d^j v_s}{dt^j}(t)\right|}{v_s(t)} : t \in S\right\},$$

where $1 \leq j \leq m$. Note that if we take $u := 1$ in (2.1), we find that for $t \in S$ and $1 \leq j \leq m$,

$$\frac{\left|\frac{d^j v_s}{dt^j}(t)\right|}{v_s(t)} = \lim_{k\to\infty} \frac{\left|\sum_{\omega\in\Omega_k} \frac{d^j g_\omega^s}{dt^j}(t)\right|}{\sum_{\omega\in\Omega_k} g_\omega(t)^s}, \tag{2.2}$$

and the convergence in (2.2) is uniform in $t \in S$. If u is as in (2.1), we also obtain from (2.1) that

$$\lim_{k\to\infty} \frac{\Lambda_s^{k+1} u}{\Lambda_s^k u} = \lambda_s,$$

where the convergence to the constant function λ_s is in the norm topology on Y.

3. Approximation of the spectral radius of L_s

Returning to the notation of (1.1), we want to approximate $R(L_s)$ by the spectral radius of an appropriate finite-dimensional linear map $\mathbf{L}_s$. To do so, we assume that $S = [a, b]$ in (H1) and (H2), with $a < b$, and let $\hat{S}$ denote a union of disjoint subintervals $[a_i, b_i] \subset [a, b]$, $i = 1, \dots, I$. We also assume throughout this section that $\theta_p(\hat{S}) \subset \hat{S}$ for $1 \le p \le n$. Further subdivide each interval $[a_i, b_i]$ into N_i equally spaced subintervals $[t^i_{j-1}, t^i_j]$, $j = 1, \dots, N_i$, of width

$$h_i = \frac{b_i - a_i}{N_i}, \quad \text{for } 1 \le i \le I. \tag{3.1}$$

Set $h = \max_{1 \le i \le I} h_i$.

Next, for $0 \le k \le r$, let $c^i_{j,k}$ denote an element of $[t^i_{j-1}, t^i_j]$, with $c^i_{j,0} = t^i_{j-1}$, $c^i_{j,r} = t^i_j$, and $c^i_{j,k} < c^i_{j,k+1}$ for $0 \le k < r$. Given values $F^i_{j,k} = F(c^i_{j,k})$, we then define a piecewise polynomial $\mathcal{F}$ on $\hat{S}$ as follows: For $t^i_{j-1} \le x \le t^i_j$, $1 \le j \le N_i$, and $1 \le i \le I$,

$$\mathcal{F}|_{[t^i_{j-1}, t^i_j]}(x) = \mathcal{F}^i_j(x) = \sum_{k=0}^{r} l^i_{j,k}(x) F^i_{j,k}, \tag{3.2}$$

where

$$l^i_{j,k}(x) = \frac{\prod_{\substack{l=0 \\ l \ne k}}^{r} (x - c^i_{j,l})}{\prod_{\substack{l=0 \\ l \ne k}}^{r} (c^i_{j,k} - c^i_{j,l})}.$$

Since $c^i_{j,r} = c^i_{j+1,0}$, $\mathcal{F} \in \mathbf{V}^r_h$, the space of continuous piecewise polynomials of degree less than or equal to r, whose degrees of freedom are the $Nr + I =: Q$ values $F^i_{j,k}$, where $N = \sum_{i=1}^{I} N_i$.

Note that we can simplify our expressions by choosing points $\hat{c}_k \in [-1, 1]$ for $0 \le k \le r$, with $\hat{c}_k < \hat{c}_{k+1}$ for $0 \le k < r$, $\hat{c}_0 = -1$ and $\hat{c}_r = 1$. If we then define

$$c^i_{j,k} = t^i_{j-1} + \frac{h_i(1 + \hat{c}_k)}{2},$$

and write $x \in [t^i_{j-1}, t^i_j] \subset [a_i, b_i]$ in the form $x = t^i_{j-1} + h_i(1 + \hat{x})/2$, where $\hat{x} \in [-1, 1]$, we obtain

$$l^i_{j,k}(x) = \hat{l}_k(\hat{x}) = \frac{\prod_{\substack{l=0 \\ l \ne k}}^{r} (\hat{x} - \hat{c}_l)}{\prod_{\substack{l=0 \\ l \ne k}}^{r} (\hat{c}_k - \hat{c}_l)}. \tag{3.3}$$

Because we seek to make use of high order piecewise polynomials, it is important to choose the points $\hat{c}_k$ to avoid the large errors that can occur in polynomial interpolation due to Runge's phenomenon (e.g., when equally spaced interpolation points are used). Since, for our analysis, we shall need the function $\mathcal{F}(x)$ in (3.2) to be continuous, we choose the points $\hat{c}_k$ to be the extended Chebyshev points in $[-1, 1]$ given by

$$\hat{c}_k = \frac{-\cos\left(\frac{2k+1}{2r+2}\pi\right)}{\cos\left(\frac{\pi}{2r+2}\right)}, \quad \text{for } k = 0, \dots, r, \tag{3.4}$$

obtained by rescaling the usual Chebyshev nodes. Then,

$$c^i_{j,k} = t^i_{j-1} + \frac{h_i}{2}\left(1 - \frac{\cos\left(\frac{2k+1}{2r+2}\pi\right)}{\cos\left(\frac{\pi}{2r+2}\right)}\right), \quad \text{for } k = 0, \dots, r. \tag{3.5}$$

We note that another possible choice is to use the augmented Chebyshev points, consisting of the roots of the Chebyshev polynomial of degree $r-1$ shifted to the interval $[t^i_{j-1}, t^i_j]$ plus the endpoints t^i_{j-1} and t^i_j.

With this notation, we can now define, for $s > 0$, the linear map $\mathbf{L}_s : \mathbb{R}^Q \to \mathbb{R}^Q$. If $\mathbf{F} = \{F^i_{j,k}\} \in \mathbb{R}^Q$, we define

$$\mathbf{L}_s(\mathbf{F})(c^i_{j,k}) = \sum_{p=1}^{n} g_p(c^i_{j,k})^s \mathcal{F}(\theta_p(c^i_{j,k})),$$

where $\mathcal{F}$ is defined above. Equivalently, we can also think of the operator $\mathbf{L}_s$ as a map from the space $\mathbf{V}^r_h$ to itself, if we replace $\mathbf{F}$ by the function $\mathcal{F}$, and given the values $\{G^i_{j,k}\} = \sum_{p=1}^n g_p(c^i_{j,k})^s \mathcal{F}(\theta_p(c^i_{j,k}))$, we define $\mathcal{G}(x)$ as follows: For $t^i_{j-1} \le x \le t^i_j$, $1 \le j \le N_i$, and $1 \le i \le I$,

$$\mathcal{G}|_{[t^i_{j-1}, t^i_j]}(x) = \mathcal{G}^i_j(x) = \sum_{k=0}^{r} l^i_{j,k}(x) G^i_{j,k}.$$

Given a positive real number M, we next define $K_M \subset \mathbb{R}^Q$ as the set of all $\mathbf{F} \in \mathbb{R}^Q$ such that for all $\xi = c^i_{j,k}$ and $\eta = c^{i'}_{j',k'}$, with $1 \le i, i' \le I$, $1 \le j \le N_i$, $1 \le j' \le N_{i'}$ and $0 \le k, k' \le r$,

$$F(\xi) \le \exp(M|\xi - \eta|)F(\eta). \tag{3.6}$$

Note that to verify (3.6), it suffices to verify it whenever ξ and η are two consecutive points in the linear ordering inherited from $\mathbb{R}$ of the points $\{c^i_{j,k}\}$.

One can easily verify that if $\mathbf{F} \in K_M$, then either (a) $F^i_{j,k} = 0$ for all $1 \le i \le I$, $1 \le j \le N_i$, and $0 \le k \le r$, or (b) $F^i_{j,k} > 0$ for all i, j, k in this range. In case (b), one has for all $1 \le i, i' \le I$, $1 \le j \le N_i$, $1 \le j' \le N_{i'}$, $0 \le k, k' \le r$,

$$|\ln(F^i_{j,k}) - \ln(F^{i'}_{j',k'})| \le M|c^i_{j,k} - c^{i'}_{j',k'}|, \tag{3.7}$$

and (3.7) implies that (3.6) holds.

One might hope to prove that the spectral radius $R(\mathbf{L}_s)$ of $\mathbf{L}_s$ closely approximates the spectral radius $R(L_s)$, and we shall see that this is true if the Lipschitz constant $C_0\kappa$ in (H2) (corresponding to the case $\nu = 1$ and the operator $R(L_s)$) and the constant h in (3.1) are sufficiently small. However, if $C_0\kappa$ is not sufficiently small, we can instead work with the operator L_s^ν, where ν is a positive integer, and the corresponding Lipschitz constant in (H2) is then $C_0\kappa^\nu$. This, in turn, means that we will have to replace $\mathbf{L}_s$ by $\mathbf{L}_{s,\nu} : \mathbb{R}^Q \to \mathbb{R}^Q$, where, ν is a positive integer and in our earlier notation,

$$(\mathbf{L}_{s,\nu}(\mathbf{F}))(c_{j,k}^i) = \sum_{\omega \in \Omega_\nu} g_\omega(c_{j,k}^i)^s \mathcal{F}(\theta_\omega(c_{j,k}^i)). \tag{3.8}$$

4. Cones, positive eigenvectors and Birkhoff's contraction constant

As noted in Section 1, we would like to have the approximating matrices defined in the previous section mimic the properties of the infinite-dimensional, bounded linear operator L_s, which means they should satisfy the conclusions of the Perron theorem for positive matrices, i.e., they should have an eigenvalue of multiplicity one equal to the spectral radius of the matrix with corresponding positive eigenvector and that all other eigenvalues of the matrix should have modulus less than the spectral radius. However, the matrix $\mathbf{L}_s$ defined in the previous section is not even a nonnegative matrix once the degree r of the piecewise polynomial satisfies $r > 1$. The reason for this, seen by constructing the matrix, is that the Lagrange basis functions for a polynomial of degree strictly larger than 1 are not always positive.

The remedy, also used in the case $r = 1$, when the resulting matrix was nonnegative, but not *primitive* or *irreducible*, is to base the analysis on a cone different from the usual cone of nonnegative functions. More precisely, by using the cone K_M defined in the previous section, we shall show that the conclusions of the classical Perron theorem also hold for the matrices of this paper.

To outline our method of proof, it is convenient to describe, at least in the finite-dimensional case, some basic definitions and classical theorems concerning linear maps $L : \mathbb{R}^N \to \mathbb{R}^N$ which leave a cone $K \subset \mathbb{R}^N$ invariant. In doing so, we shall closely follow the analogous description in [14]. Recall that a closed subset K of $\mathbb{R}^N$ is called a *closed cone* if (i) $ax + by \in K$ whenever $a \geq 0$, $b \geq 0$, $x \in K$ and $y \in K$, and (ii) if $x \in K \setminus \{0\}$, then $-x \notin K$. If K is a closed cone, K induces a partial ordering on $\mathbb{R}^N$ denoted by $\leq_K$ (or simply $\leq$ if K is obvious) by $u \leq_K v$ if and only if $v - u \in K$. If $u, v \in K$, we shall say that u and v are *comparable* (with respect to K) and we shall write $u \sim_K v$ if there exist positive scalars a and b such that $v \leq_K au$ and $u \leq_K bv$. The equivalence relation given by being *comparable with respect to* K partitions K into equivalence classes of comparable elements. We shall henceforth assume that $\mathrm{int}(K)$, the interior of K, is nonempty. Then, an easy argument shows that all elements of $\mathrm{int}(K)$ are comparable. Generally, if $x_0 \in K$ and $K_{x_0} := \{x \in K : x \sim_K x_0\}$, all elements of K_{x_0} are comparable.

Following standard notation, if $u, v \in K$ are comparable elements, we define

$$M\Big(\frac{u}{v}; K\Big) = \inf\big\{\beta > 0 : u \le \beta v\big\},$$
$$m\Big(\frac{u}{v}; K\Big) = M\Big(\frac{v}{u}; K\Big)^{-1} = \sup\big\{\alpha > 0 : \alpha v \le u\big\}.$$

If u and v are comparable elements of $K \setminus \{0\}$, we define *Hilbert's projective metric* $d(u, v; K)$ by

$$d(u, v; K) = \ln\Big(M\Big(\frac{u}{v}; K\Big)\Big) + \ln\Big(M\Big(\frac{v}{u}; K\Big)\Big).$$

We make the convention that $d(0, 0; K) = 0$. If $x_0 \in K \setminus \{0\}$, then for all $u, v, w \in K_{x_0}$, one can prove that (i) $d(u, v; K) \ge 0$, (ii) $d(u, v; K) = d(v, u; K)$, and (iii) $d(u, v; K) + d(v, w; K) \ge d(u, w; K)$. Thus d restricted to K_{x_0} is almost a metric, but $d(u, v; K) = 0$ if and only if $v = tu$ for some $t > 0$ and generally, $d(su, tv; K) = d(u, v; K)$ for all $u, v \in K_{x_0}$ and all $s > 0$ and $t > 0$. If $\|\cdot\|$ is any norm on $\mathbb{R}^N$ and $S := \{u \in \mathrm{int}(K) : \|u\| = 1\}$ (or, more generally, if $x_0 \in K \setminus \{0\}$ and $S = \{x \in K_{x_0} : \|x\| = 1\}$), then $d(\cdot, \cdot; K)$, restricted to $S \times S$, gives a metric on S; it is known that S is a complete metric space with this metric.

With these preliminaries, we can describe a special case of the Birkhoff–Hopf theorem. We refer to [2,21,40] for the original papers and to [11,12] for an exposition of a general version of this theorem and further references to the literature. We remark that P. P. Zabreiko, M. A. Krasnosel'skij, Y. V. Pokornyi, and A. V. Sobolev independently obtained closely related theorems; we refer to [27] for details. If K is a closed cone as above, $S = \{x \in \mathrm{int}(K) : \|x\| = 1\}$, and $L : \mathbb{R}^N \to \mathbb{R}^N$ is a linear map such that $L(\mathrm{int}(K)) \subset \mathrm{int}(K)$, we define $\Delta(L; K)$, *the projective diameter* of L by

$$\begin{aligned}\Delta(L; K) &= \sup\big\{d(Lx, Ly; K) : x, y \in K \text{ and } Lx \sim_K Ly\big\}\\ &= \sup\big\{d(Lx, Ly; K) : x, y \in \mathrm{int}(K)\big\}.\end{aligned}$$

The Birkhoff–Hopf theorem implies that if $\Delta := \Delta(L; K) < \infty$, then L is a contraction mapping with respect to Hilbert's projective metric. More precisely, if we define $\lambda = \tanh(\frac{1}{4}\Delta) < 1$, then for all $x, y \in K \setminus \{0\}$ such that $x \sim_K y$, we have

$$d(Lx, Ly; K) \le \lambda d(x, y; K),$$

and the constant λ is optimal.

If we define $\Phi : S \to S$ by $\Phi(x) = L(x)/\|L(x)\|$, it follows that Φ is a contraction mapping with a unique fixed point $v \in S$, and v is necessarily an eigenvector of L with eigenvalue $r(L) := r$, where r is the spectral radius of L. Furthermore, given any

$x \in \mathrm{int}(K)$, there are explicitly computable constants M and $c < 1$ (see [11, Theorem 2.1]) such that for all $k \geq 1$,

$$\left\| \frac{L^k(x)}{\|L^k(x)\|} - v \right\| \leq Mc^k;$$

the latter inequality is exactly the sort of result we need. Furthermore, it is proved in [11, Theorem 2.3] that $r = r(L)$ is an algebraically simple eigenvalue of L and that if $\sigma(L)$ denotes the spectrum of L and $q(L)$ denotes the *spectral clearance of* L,

$$q(L) := \sup \left\{ \frac{|z|}{r(L)} : z \in \sigma(L) \text{ and } z \neq r(L) \right\},$$

then $q(L) < 1$ and $q(L)$ can be explicitly estimated.

The main issue, then, is to find a suitable cone satisfying the hypotheses outlined above. We shall show in the sections that follow that the cone K_M, defined in the previous section, is such a cone. To do so, we shall show that there exists M', $0 < M' < M$, such that $L(K_M \setminus \{0\}) \subset K_{M'} \setminus \{0\}$. After correcting the typo in the formula for $d_2(f, g)$ on [29, p. 286], it follows from [29, Lemma 2.12] that

$$\sup \{d(f, g; K_M) : f, g \in K_{M'} \setminus \{0\}\} \leq 2 \ln \left(\frac{M + M'}{M - M'} \right) + 2 \exp(M'(b - a)) < \infty,$$

where now $S := [a, b]$ in (H1) and (H2) (cf. Section 3). This implies that $\Delta(L; K_M) < \infty$, which in turn implies that L has a normalized eigenvector $v \in K_{M'}$ with positive eigenvalue $r = r(L) =$ the spectral radius of L. Furthermore, r has algebraic multiplicity 1, $q(L) < 1$, and $\lim_{k \to \infty} \|L^k(x)/\|L^k(x)\| - v\| = 0$ for all $x \in K_M \setminus \{0\}$. Thus, it suffices to prove, for an appropriate map L, that for some $M' < M$,

$$L(K_M \setminus \{0\}) \subset K_{M'} \setminus \{0\}. \tag{4.1}$$

Note that if the map L satisfies (4.1), then it is not difficult to show that $L(K_M \setminus \{0\}) \subset \mathrm{int}(K_M)$. An alternative approach is then to apply [44, Theorem 4.4], which concludes that $R(L)$ is a simple eigenvalue, greater than the magnitude of any other eigenvalue, and that an eigenvector corresponding to $R(L)$ lies in $\mathrm{int}(K)$. In any case, the key step is to show for an appropriate matrix L and cone K_M, that (4.1) is satisfied.

A key part of the paper is to obtain upper and lower bounds on $R(L_s)$ using the approximations developed in this paper. To do so, we will use an extension to cones of a well known result for positive matrices.

Lemma 4.1. *Suppose $L(K_M \setminus \{0\}) \subset K_{M'}$ and $\mathcal{V}_s \in K_M \setminus \{0\}$. Then, if there exist positive constants α and β such that*

$$\alpha \mathcal{V}_s \leq_{K_M} L\mathcal{V}_s \leq_{K_M} \beta \mathcal{V}_s,$$

then $\alpha \leq R(L) \leq \beta$.

5. Theory for the discrete approximation

The main result of this section, Theorem 5.8, is to show that under appropriate hypotheses, the matrix operator $\mathbf{L}_{s,\nu}$ defined in Section 3, satisfies

$$\mathbf{L}_{s,\nu}(K_M(T) \setminus \{0\}) \subset K_{M'}(T) \setminus \{0\},$$

where T will be as in (5.1) below.

Throughout this section, we shall assume that (H1) and (H2) are satisfied and that $S = [a, b]$ with $a < b$, where S is as in (H1) and (H2). We shall also assume that (H3), given below, is satisfied, and we shall use the notation of (H1), (H2) and (H3).

(H3) For a given positive integer ν, there exist pairwise disjoint, nonempty compact intervals $[a_i, b_i] \subset S$, $1 \le i \le I$ (where $S := [a,b]$ is as in (H0)–(H2)) with the following property: For every $\omega \in \Omega_\nu$, there exists $i = i(\omega)$, $1 \le i \le I$, such that $\theta_\omega(S) \subset [a_i, b_i]$.

Remark 5.1. Assume that the hypotheses (H0)–(H2) are satisfied and that for some positive integer ν', $\theta_{\omega_1}(S)$ and $\theta_{\omega_2}(S)$ are disjoint whenever ω_1 and ω_2 are unequal elements of $\Omega_{\nu'}$. Label the elements of $\Omega_{\nu'}$ as ω_i, $1 \le i \le I$, and define $[a_i, b_i] = \theta_{\omega_i}(S)$. Then, for all positive integers $\nu \ge \nu'$, (H3) is satisfied. More generally, for $1 \le i \le I$, one could take $[a_i, b_i]$ to be any interval contained in $[a, b]$ such that $\theta_{\omega_i}(S) \subset [a_i, b_i]$ as long as $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ for $1 \le i, j \le I$. Note that (H3) is also trivially satisfied if we take $I = 1$ and $[a_1, b_1] = [a, b]$.

Remark 5.2. In the context of (H3), it is possible to assume that $b_i < a_{i+1}$ for $1 \le i < I$ by relabeling, so the intervals are linearly ordered as subsets of $\mathbb{R}$. Thus, we shall make this assumption if convenient.

We now follow the notation of Section 3. If we define

$$T := \left\{c^i_{j,k} : 1 \le i \le I, 1 \le j \le N_i \text{ and } 0 \le k \le r\right\}, \tag{5.1}$$

then T is a finite subset of $\mathbb{R}$ and a compact metric space. Recall that we consider the finite-dimensional vector space $V = V(T)$ of dimension $Q = Nr + I$ of all maps $F : T \to \mathbb{R}$ and $K_M(T)$ is then defined as in Section 4 or (3.6), i.e., $F \in K_M(T) \setminus \{0\}$ if and only if

$$|\ln(F(\xi)) - \ln(F(\eta))| \le M|\xi - \eta|, \quad \text{for all } \xi, \eta \in T.$$

Note that V is linearly isomorphic to $\mathbb{R}^Q$.

A central question for our approach is to determine under what conditions on $\mathbf{L}_{s,\nu}$ (see (3.8)) $\mathbf{L}_{s,\nu}(K_M(T)) \subset K_{M'}(T)$ for some M, M' with $0 < M' < M$. To do so, we begin by recalling some classical results.

Lemma 5.3 (See [30], [36, pp. 121–123] or [42]). *Let $p(t)$ be a real-valued polynomial of degree less than or equal to r. Then,*

$$\max_{-1\le t\le 1} |p'(t)| \le r^2 \max_{-1\le t\le 1} |p(t)|.$$

A proof of the following estimate is given in [5] and refinements for $r \ge 5$ can be found in [16].

Lemma 5.4. *If $\hat{l}_k(\hat{x})$, $0 \le k \le r$, are defined by (3.3), then*

$$\max_{-1\le \hat{x}\le 1} \sum_{k=0}^{r} |\hat{l}_k(\hat{x})| \le \frac{2}{\pi}\ln(r+1) + \frac{3}{4} =: \psi(r),$$

where ln *denotes the natural logarithm.*

It will also be convenient to have some elementary estimates concerning the numbers $c^i_{j,k}$, $1 \le j \le N_i$, $0 \le k \le r$, in (3.5). If x is a real number, $[x]$ denotes the greatest integer $m \le x$. If r is an integer, it follows that $[r/2] = r/2$ if r is even and $[r/2] = (r-1)/2$ is r is odd. The next lemma is a straightforward exercise and is left to the reader.

Lemma 5.5. *Let the numbers $c^i_{j,k}$ be defined by (3.5). Then, for $0 \le k \le r$ and $1 \le i \le I$,*

$$|c^i_{j,k} - c^i_{j,[r/2]}| \le \begin{cases} \frac{h_i}{2}, & \text{if } r \text{ is even},\\ \frac{h_i}{2}\big[1 + \tan\big(\frac{\pi}{[2r+2]}\big)\big], & \text{if } r \text{ is odd}. \end{cases}$$

For $1 \le j \le N_i$ and $0 \le k < r$,

$$\min_{1\le j\le N_i, 0\le k<r} |c^i_{j,k} - c^i_{j,k+1}| = 2h_i\Big[\sin\Big(\frac{\pi}{[2r+2]}\Big)\Big]^2,$$

$$\max_{1\le j\le N_i, 0\le k<r} |c^i_{j,k} - c^i_{j,k+1}| = \begin{cases} h_i \tan\big(\frac{\pi}{[2r+2]}\big), & \text{if } r \text{ is odd},\\ h_i \sin\big(\frac{\pi}{[2r+2]}\big), & \text{if } r \text{ is even}. \end{cases}$$

Since the first result in Lemma 5.5 will be used later, we define the constant $\eta(r)$ for a positive integer r by:

$$\eta(r) = \begin{cases} \frac{1}{2}, & \text{if } r \text{ even},\\ \frac{1}{2}\big[1 + \tan\big(\frac{\pi}{[2r+2]}\big)\big], & \text{if } r \text{ odd}. \end{cases} \tag{5.2}$$

In addition, for a positive integer ν and $\omega \in \Omega_\nu$, we define constants $M_0(\nu)$ and $c(\nu)$ such that for all $\omega \in \Omega_\nu$,

$$g_\omega \in K_{M_0(\nu)}(S) \quad \text{and} \quad \mathrm{lip}(\theta_\omega|_S) \le c(\nu). \tag{5.3}$$

We already know (see Section 2) that $M_0(\nu) \le M_0 C_0(1-\kappa^\nu)/(1-\kappa)$, where M_0 is as in (H1) and C_0 and κ are as in (H2), and (H2) implies that $c(\nu) \le C_0\kappa^\nu$. However, in specific examples which we shall study later, these estimates can be significantly improved.

Lemma 5.6. *Suppose that $\tau \in \mathbb{R}$, $\epsilon > 0$ and $\hat{c}_k$, $0 \le k \le r$ is a normalized extended Chebyshev point as in (3.4), and $c_k = \tau + \frac{\epsilon}{2}(1+\hat{c}_k)$. If $x \in [\tau, \tau+\epsilon]$, let $\hat{x} \in [-1,1]$ denote the unique point such that $x = \tau + \frac{\epsilon}{2}[1+\hat{x}]$. Assume that $M > 0$, $\Gamma = \{c_k : 0 \le k \le r\}$ and $F \in K_M(\Gamma) \setminus \{0\}$, so $|\ln(F(\xi)) - \ln(F(\eta))| \le M|\xi - \eta|$ for all $\xi, \eta \in \Gamma$. Let $\mathcal{F} : [\tau, \tau+\epsilon] \to \mathbb{R}$ denote the unique polynomial of degree less than or equal to r such that $\mathcal{F}(c_k) = F(c_k)$ for $0 \le k \le r$. Let $\eta(r)$ be as in (5.2) and $\psi(r)$ as in Lemma 5.4, and define $u = M\epsilon\eta(r)$. If*

$$\psi(r)u\exp(u) < 1,$$

then $\mathcal{F}(x) > 0$ for all $x \in [\tau, \tau+\epsilon]$. If $\psi(r)u\exp(u) < 1$ and

$$C := \frac{[2\eta(r)r^2\psi(r)]\exp(u)M}{1-\psi(r)u\exp(u)},$$

then for all $x, y \in [\tau, \tau+\epsilon]$,

$$\left|\ln(\mathcal{F}(x)) - \ln(\mathcal{F}(y))\right| \le C|x-y|.$$

Proof. Recall that for $0 \le k \le r$,

$$l_k(x) = \frac{\prod_{\substack{l=0\\l\neq k}}^r (x - c_l)}{\prod_{\substack{l=0\\l\neq k}}^r (c_k - c_l)} \quad \text{and} \quad \hat{l}_k(\hat{x}) = \frac{\prod_{\substack{l=0\\l\neq k}}^r (\hat{x} - \hat{c}_l)}{\prod_{\substack{l=0\\l\neq k}}^r (\hat{c}_k - \hat{c}_l)},$$

and $l_k(x) = \hat{l}_k(\hat{x})$ for $x = \tau + \frac{\epsilon}{2}[1+\hat{x}]$ and, writing $F_k = F(c_k)$,

$$\mathcal{F}(x) = \sum_{k=0}^r l_k(x)F(c_k) = \sum_{k=0}^r l_k(x)F_k.$$

Recalling that $\sum_{k=0}^r l_k(x) = 1$ for all $x \in [\tau, \tau+\epsilon]$, we obtain

$$\mathcal{F}(x) = \sum_{k=0}^r l_k(x)F_k = F_{[r/2]}\Bigg(1 + \sum_{\substack{k=0\\k\neq[r/2]}}^r l_k(x)\Big[\frac{F_k}{F_{[r/2]}} - 1\Big]\Bigg) = F_{[r/2]}[1+\phi(x)]$$

$$= F_{[r/2]}\Bigg(1 + \sum_{\substack{k=0\\k\neq[r/2]}}^r \hat{l}_k(\hat{x})\Big[\frac{F_k}{F_{[r/2]}} - 1\Big]\Bigg) =: F_{[r/2]}[1+\hat{\phi}(\hat{x})],$$

where as usual, $x = \tau + \frac{\epsilon}{2}[1+\hat{x}]$ and $\hat{x} \in [-1,1]$.

Since $F \in K_M(\Gamma) \setminus \{0\}$, we have for $0 \le k \le r$, $k \ne [r/2]$,

$$\exp\big(-M|c_k - c_{[r/2]}|\big) \le \frac{F_k}{F_{[r/2]}} \le \exp\big(M|c_k - c_{[r/2]}|\big).$$

Because Lemma 5.5 (with $h_i = \epsilon$) implies that $|c_k - c_{[r/2]}| \le \eta(r)\epsilon$,

$$\exp(-M\eta(r)\epsilon) - 1 \le \frac{F_k}{F_{[r/2]}} - 1 \le \exp(M\eta(r)\epsilon) - 1.$$

Using the mean value theorem and writing $u = M\eta(r)\epsilon$, it follows that

$$-u \le \frac{F_k}{F_{[r/2]}} - 1 \le u\exp(u),$$

so

$$\Big|\frac{F_k}{F_{[r/2]}} - 1\Big| \le u\exp(u).$$

Using Lemma 5.4, it follows that

$$|\hat{\phi}(\hat{x})| \le \sum_{\substack{k=0 \\ k\ne[r/2]}}^{r} |\hat{l}_k(\hat{\xi})|\Big|\frac{F_k}{F_{[r/2]}} - 1\Big| \le \psi(r)u\exp(u), \tag{5.4}$$

so if $\psi(r)u\exp(u) < 1$, $1 + \hat{\phi}(\hat{x}) > 0$, and $\mathcal{F}(x) > 0$ for all $x \in [\tau, \tau+\epsilon]$. For the remainder of the proof, we assume that $\psi(r)u\exp(u) < 1$.

If $x, y \in [\tau, \tau+\epsilon]$, our previous calculations show that

$$\begin{aligned}|\ln\mathcal{F}(x) - \ln\mathcal{F}(y)| &= \big|\ln[1+\phi(x)] - \ln[1+\phi(y)]\big| = \Big|\int_{1+\phi(y)}^{1+\phi(x)} \frac{1}{s}\,ds\Big| \\ &\le \Big|\frac{\phi(x)-\phi(y)}{2}\Big|\Big[\frac{1}{1+\phi(x)} + \frac{1}{1+\phi(y)}\Big],\end{aligned}$$

where we have used the fact that $(1/s)$ is a convex function and hence the integral is bounded by the trapezoidal rule approximation.

Now, by the mean value theorem, for some $\hat{\xi}$ lying between $\hat{x}$ and $\hat{y}$ and hence $\hat{\xi} \in [-1, 1]$,

$$\begin{aligned}|\phi(x) - \phi(y)| &= |\hat{\phi}(\hat{x}) - \hat{\phi}(\hat{y})| = |\hat{\phi}'(\hat{\xi})||\hat{x} - \hat{y}| \\ &\le \frac{2}{\epsilon}|x-y| \max_{-1\le\hat{\xi}\le1} |\hat{\phi}'(\hat{\xi})| \le \frac{2}{\epsilon}|x-y|r^2 \max_{-1\le\hat{\xi}\le1} |\hat{\phi}(\hat{\xi})|,\end{aligned}$$

where in the last step we have used Markov's polynomial inequality (Lemma 5.3).

Recalling our earlier estimate for $|\hat{\phi}(\hat{\xi})|$ in (5.4), we obtain

$$|\phi(x) - \phi(y)| \le 2r^2\psi(r)\eta(r)M\exp(u)|x-y|$$

and

$$\frac{1}{2}\left[\frac{1}{1+\phi(x)} + \frac{1}{1+\phi(y)}\right] \le \frac{1}{1-\psi(r)u\exp(u)},$$

which implies that

$$|\ln(\mathcal{F}(x)) - \ln(\mathcal{F}(y))| \le C|x-y|,$$

with the constant C defined in the statement of the lemma. This concludes the proof of Lemma 5.6. ■

Lemma 5.7. *Let the notation be as in Section 3 and T be as defined by (5.1). Suppose that $F: T \to \mathbb{R}$ is an element of $K_M(T)\setminus\{0\}$ and let $\mathcal{F}: \bigcup_{i=1}^{I}[a_i, b_i] \to \mathbb{R}$ be defined by (3.2). For $1 \le i \le I$, define $u_i = Mh_i\eta(r)$ and assume that*

$$\psi(r)u_i\exp(u_i) < 1.$$

Then, $\mathcal{F}(x) > 0$ for all $x \in [a_i, b_i]$. If we define C_i by

$$C_i := \frac{[2\eta(r)r^2\psi(r)]\exp(u_i)M}{1-\psi(r)u_i\exp(u_i)}, \tag{5.5}$$

then for all $x, y \in [a_i, b_i]$,

$$|\ln(\mathcal{F}(x)) - \ln(\mathcal{F}(y))| \le C_i|x-y|. \tag{5.6}$$

Proof. Recall that $[a_i, b_i] = \bigcup_{j=1}^{N_i}[t^i_{j-1}, t^i_j]$, where $t^i_j - t^i_{j-1} = h_i = (b_i - a_i)/N_i$. If we write $h_i = \epsilon$, note that whenever $x, y \in [t^i_{j-1}, t^i_j]$ for some j, Lemma 5.7 implies that (5.6) is satisfied. Thus we can assume that $x, y \in [a_i, b_i]$ and that there does not exist j, $1 \le j \le N_i$, such that x and y are both elements of $[t^i_{j-1}, t^i_j]$. We can also assume that $x < y$ and select j_1, $1 \le j_1 \le N_i$, such that $x \in [t^i_{j_1-1}, t^i_{j_1}]$ and j_2, $1 \le j_2 \le N_i$, such that $y \in [t^i_{j_2-1}, t^i_{j_2}]$. By our assumptions, it must be true that $j_1 < j_2$. If we apply Lemma 5.7 to $\mathcal{F}(x)$ and $\mathcal{F}(t^i_{j_1})$, we obtain

$$\left|\ln(\mathcal{F}(t^i_{j_1})) - \ln(\mathcal{F}(x))\right| \le C_i(t^i_{j_1} - x).$$

Similarly, if we apply Lemma 5.7 to $\mathcal{F}(t^i_{j_2-1})$ and $\mathcal{F}(y)$, we obtain

$$\left|\ln(\mathcal{F}(y)) - \ln(\mathcal{F}(t^i_{j_2-1}))\right| \le C_i(y - t^i_{j_2-1}).$$

Since $\mathcal{F}(t^i_{j_1}) = F(t^i_{j_1})$ and $\mathcal{F}(t^i_{j_2-1}) = F(t^i_{j_2-1})$ and $F \in K_M(T)$, we obtain

$$\left|\ln(\mathcal{F}(t^i_{j_2-1})) - \ln(\mathcal{F}(t^i_{j_1}))\right| \le M(t^i_{j_2-1} - t^i_{j_1}) \le C_i(t^i_{j_2-1} - t^i_{j_1}),$$

where we have used the fact that $C_i > M$. Combining these inequalities, we find that

$$
\begin{aligned}
&\big|\ln(\mathcal{F}(y)) - \ln(\mathcal{F}(x))\big| \\
&\quad \le \big|\ln(\mathcal{F}(y)) - \ln(\mathcal{F}(t^i_{j_2-1}))\big| + \big|\ln(\mathcal{F}(t^i_{j_2-1})) - \ln(\mathcal{F}(t^i_{j_1}))\big| \\
&\qquad + \big|\ln(\mathcal{F}(t^i_{j_1})) - \ln(\mathcal{F}(x))\big| \\
&\quad \le C_i(y-x),
\end{aligned}
$$

which proves Lemma 5.7. ∎

Up to this point, we have only used the fact that $F \in K_M(T)$, where T is defined in (5.1) and the notation is as in Section 3. We now exploit the fact that $\operatorname{lip}(\theta_\omega|_S) \le c(\nu)$ for all $\omega \in \Omega_\nu$.

Theorem 5.8. *Let notation be as in Section 3 and for positive real numbers $M' < M$, let $K_M(T)$ and $K_{M'}(T)$ be as defined earlier. Recall that $h_i = (b_i - a_i)/N_i$, $1 \le i \le I$, and $h = \max\{h_i : 1 \le i \le I\}$. Assume that the hypotheses* (H1), (H2) *and* (H3) *are satisfied, and that*

$$\psi(r)u \exp u < 1, \tag{5.7}$$

where we now set

$$u = Mh\eta(r).$$

If $F \in K_M(T) \setminus \{0\}$ and $\mathcal{F}$ is the piecewise polynomial approximation of F of degree less than or equal to r on $\hat{S} = \bigcup_{i=1}^{I} [a_i, b_i]$, then $\mathcal{F}(x) > 0$ for all $x \in \hat{S}$.

Define $C := \max\{C_i : 1 \le i \le I\}$, where C_i is defined by (5.5) and let $V(T) := \mathbb{R}^Q$ and $\mathbf{L}_{s,\nu} : V(T) \to V(T)$ be defined by (3.8). Assume the above hypotheses are satisfied and also assume that

$$c(\nu)C := \frac{c(\nu)[2\eta(r)r^2\psi(r)]\exp(u)M}{1 - \psi(r)u\exp(u)} < M' - sM_0(\nu). \tag{5.8}$$

Then, it follows that $\mathbf{L}_{s,\nu}(K_M(T) \setminus \{0\}) \subset K_{M'}(T) \setminus \{0\}$.

Proof. Suppose that $F \in K_M(T) \setminus \{0\}$, which implies that $F(\xi) > 0$ for all $\xi \in T$. Since $u \ge u_i$ for $1 \le i \le I$, (5.7) implies that $\psi(r)u_i \exp u_i < 1$ for $1 \le i \le I$. It follows from Lemma 5.7 that $\mathcal{F}(x) > 0$ for all $x \in [a_i, b_i]$, $1 \le i \le I$, so $\mathcal{F}(x) > 0$ for all $x \in \hat{S}$. Since (H1) implies that $g_\omega(x)^s > 0$ for all $x \in S = [a, b]$ and for all $\omega \in \Omega_\nu$,

$$\mathbf{L}_{s,\nu}(F)(\xi) = \sum_{\omega \in \Omega_\nu} [g_\omega(\xi)]^s \mathcal{F}(\theta_\omega(\xi)),$$

and $g_\omega(x)^s \mathcal{F}(\theta_\omega(x)) > 0$ for all $x \in \hat{S}$ and certainly for all $\xi \in T$, it suffices to prove that the map $\xi \mapsto g_\omega(\xi)^s \mathcal{F}(\theta_\omega(\xi))$ belongs to $K_{M'}(T) \setminus \{0\}$ for every $\omega \in \Omega_\nu$. We know that for all $x, y \in S$,

$$|\ln(g_\omega(x)^s) - \ln(g_\omega(y)^s)| \le sM_0(\nu)|x-y|,$$

so it suffices to prove that for all $x, y \in \hat{S}$,

$$|\ln(\mathcal{F}(\theta_\omega(x)) - \ln(\mathcal{F}(\theta_\omega(y))| \le [M' - sM_0(\nu)]\,|x-y|.$$

For each fixed $\omega \in \Omega_\nu$, (H3) implies that there exists $i = i(\omega)$ such that $\theta_\omega(\hat{S}) \subset [a_i, b_i]$. Writing $x' = \theta_\omega(x) \in [a_i, b_i]$ and $y' = \theta_\omega(y) \in [a_i, b_i]$, Lemma 5.7 implies that

$$|\ln(\mathcal{F}(x')) - \ln(\mathcal{F}(y'))| \le C|x'-y'| \le c(\nu)C|x-y|,$$

so (5.8) completes the proof. ∎

Remark 5.9. Assume that $\psi(r)u\exp(u) < 1$. Notice that for a given positive integer r, a necessary condition that (5.8) be satisfied is that

$$c(\nu)2r^2\psi(r)\eta(r) < \frac{M' - sM_0(\nu)}{M}. \tag{5.9}$$

For a given $M' < M$, if (5.9) is satisfied, then (5.8) will be satisfied if h is sufficiently small.

Remark 5.10. The reader will note that in our definition of $\mathbf{L}_{s,\nu}(F)$ for $F \in V(T)$, we arrange that $\mathcal{F}|_{[a_i,b_i]}$ is a piecewise polynomial map of degree less than or equal to r. In some applications, it is desirable to make $\mathcal{F}|_{[a_i,b_i]}$ a piecewise polynomial map of degree $\le r_i$, which leads to a generalization of the definition of $\mathbf{L}_{s,\nu}$. An analogue of Theorem 5.8 which handles this more general case can be proved by an argument similar to the proof of Theorem 5.8. Because of considerations of length, we omit the proof.

6. Estimating $R(L_s)$ by the spectral radius of $\mathbf{L}_{s,\nu}$

In the previous section (cf. Theorem 5.8), we determined conditions under which

$$\mathbf{L}_{s,\nu}(K_M(T) \setminus \{0\}) \subset K_{M'}(T) \setminus \{0\},$$

for $M' < M$. The main result of this section is to show that under this condition, $R(\mathbf{L}_{s,\nu})$, the spectral radius of $\mathbf{L}_{s,\nu}$, satisfies

$$\lambda_s^\nu(1 - Hh^r) \le R(\mathbf{L}_{s,\nu}) \le \lambda_s^\nu(1 + Hh^r),$$

for some constant H to be specified below. Using this result, we obtain the following explicit bounds on the spectral radius λ_s of L_s.

$$[R([1 + Hh^r]^{-1}\mathbf{L}_{s,\nu})]^{1/\nu} \le \lambda_s \le [R([1 - Hh^r]^{-1}\mathbf{L}_{s,\nu})]^{1/\nu},$$

where the entries of the matrices $[1 + Hh^r]^{-1}\mathbf{L}_{s,\nu}$ and $[1 - Hh^r]^{-1}\mathbf{L}_{s,\nu}$ differ by $O(h^r)$.

Throughout this section, we shall assume the hypotheses and notation of (H1), (H2) and (H3); $v_s(\cdot)$ will always denote the positive eigenvector v_s of L_s guaranteed by Theorem 2.1. In particular, S and $[a_i, b_i]$, $1 \le i \le I$, will be as in (H3) and (as can be guaranteed by relabeling) we shall assume that $b_i < a_{i+1}$ for $1 \le i < I$.

We shall further denote by E and χ, constants for which the following two inequalities are satisfied:

$$\sup_{a \le x \le b} \frac{\left|\frac{d^p v_s(x)}{dx^p}\right|}{v_s(x)} \le E(s, p) =: E, \tag{6.1}$$

where p is a positive integer, and

$$v_s(x_1) \le v_s(x_2) \exp(2s|x_1 - x_2|/\chi), \quad \text{for all } x_1, x_2 \in S, \tag{6.2}$$

where $\chi := \chi(s, \{\theta_i, g_i : 1 \le i \le n\})$. Using Theorem 2.1 and Remark 2.2, in the next section, we shall see that for some interesting examples, it is possible to obtain sharp estimates on the constants E and χ such that (6.1) and (6.2) below are satisfied. These estimates will refine earlier results in [14].

Using the notation of Section 3, if $\mathcal{V}_s(x)$ is the piecewise polynomial interpolant of $v_s(x)$ of degree less than or equal to r at the extended Chebyshev points in $[a_i, b_i]$, $1 \le i \le I$, then on each subinterval $[t^i_{j-1}, t^i_j]$, $j = 1, \ldots, N_i$, we have, using standard estimates for polynomial interpolation,

$$v_s(x) - \mathcal{V}_s(x) = \frac{v_s^{(r+1)}(\xi_x)}{(r+1)!} \prod_{k=0}^{r} (x - c^i_{j,k}),$$

for some $\xi_x \in [t^i_{j-1}, t^i_j]$. If we write, as done previously,

$$c^i_{j,k} = t^i_{j-1} + h_i(1 + \hat{c}_k)/2, \quad x = t^i_{j-1} + h_i(1 + \hat{x})/2 \quad \text{and} \quad \hat{x} \in [-1, 1],$$

then for $x \in [a_i, b_i]$,

$$|v_s(x) - \mathcal{V}_s(x)| \le |v_s^{(r+1)}(\xi_x)| h_i^{r+1} m_{r+1},$$

where

$$m_{r+1} = \frac{1}{2^{r+1}(r+1)!} \max_{\hat{x} \in [-1,1]} \left| \prod_{k=0}^{r} (\hat{x} - \hat{c}_k) \right|. \tag{6.3}$$

Using (6.1) and (6.2), we see that

$$|v_s^{(r+1)}(\xi_x)| \le E \exp\Big(\frac{2sh_i}{\chi}\Big) v_s(x),$$

so

$$|v_s(x) - \mathcal{V}_s(x)| \le E h_i^{r+1} m_{r+1} v_s(x) \exp\Big(\frac{2sh_i}{\chi}\Big).$$

Defining, for $1 \le i \le I$,

$$G_{r,i} := E m_r \exp\Big(\frac{2sh_i}{\chi}\Big), \tag{6.4}$$

(6.3) implies that for $1 \le i \le I$ and $x \in [a_i, b_i]$

$$(1 - G_{r+1,i} h_i^{r+1}) v_s(x) \le \mathcal{V}_s(x) \le (1 + G_{r+1,i} h_i^{r+1}) v_s(x). \tag{6.5}$$

In order to make (6.3) explicit, we need a formula for $\max_{\hat{x} \in [-1,1]} |\prod_{k=0}^{r} (\hat{x} - \hat{c}_k)|$. The result and proof, which we provide below, are slight modifications of the well-known corresponding bound and proof when $\hat{c}_k$ are taken to be the zeros of the standard Chebyshev polynomial.

Lemma 6.1. *If $r \ge 2$ is a positive integer and $\hat{c}_k$ is defined by (3.4), then*

$$\max_{\hat{x} \in [-1,1]} \Big| \prod_{k=0}^{r} (\hat{x} - \hat{c}_k) \Big| = \frac{1}{2^r} \Bigg[\frac{1}{\cos\big(\frac{\pi}{[2r+2]}\big)} \Bigg]^{r+1}.$$

Proof. If we set $w = \hat{x} \cos(\pi/[2r+2])$, where $|\hat{x}| \le 1$, then $|w| \le \cos(\pi/[2r+2])$. For notational convenience, we write $\alpha = 1/\cos(\pi/[2r+2])$, and we obtain

$$\begin{aligned}
&\max_{\hat{x} \in [-1,1]} \Big| \prod_{k=0}^{r} (\hat{x} - \hat{c}_k) \Big| \\
&\quad = \alpha^{r+1} \max \Bigg\{ \Bigg| \prod_{k=0}^{r} \Big(w + \cos\Big(\frac{[2k+1]\pi}{[2r+2]} \Big) \Big) \Bigg| : |w| \le \cos\Big(\frac{\pi}{[2r+2]} \Big) \Bigg\} \\
&\quad = \alpha^{r+1} \max \Bigg\{ \Bigg| \prod_{k=0}^{r} \Big(w - \cos\Big(\frac{[2k+1]\pi}{[2r+2]} \Big) \Big) \Bigg| : |w| \le \cos\Big(\frac{\pi}{[2r+2]} \Big) \Bigg\}.
\end{aligned}$$

If we define $q_{r+1}(w) = \prod_{k=0}^{r} (w - \cos([2k+1]\pi/[2r+2]))$, $q_{r+1}(w)$ is a polynomial of degree $r+1$ which vanishes at the points $\cos([2k+1]\pi/[2r+2])$, $0 \le k \le r$, and has leading term w^{r+1}; and these properties uniquely determine $q_{r+1}(w)$.

Recall that for integers $r \ge 0$, $\cos([r+1]\theta) = p_{r+1}(\cos\theta)$ for $0 \le \theta \le \pi$, where $p_{r+1}(w)$ is the *Chebyshev* polynomial of degree $r+1$. These polynomials satisfy $p_1(w) = w$, $p_2(w) = 2w^2 - 1$, and for $r \ge 2$, the recurrence relation $p_{r+1}(w) = 2wp_r(w) - p_{r-1}(w)$. Using the recursion relation for $p_{r+1}(w)$ and induction, it also follows that the coefficient of w^{r+1} in $p_{r+1}(w)$ is 2^r. Since $\cos([r+1]\theta) = 0$ when $\theta = [2k+1]\pi/[2r+2]$ for $0 \le k \le r$, we see that $p_{r+1}(w) = 0$ when $w = \cos([2k+1]\pi/[2r+2])$, for $0 \le k \le r$. It follows that for $w = \cos(\theta)$ and $0 \le \theta \le \pi$,

$$q_{r+1}(w) = \frac{1}{2^r} p_{r+1}(w) = \frac{1}{2^r}\cos([r+1]\theta),$$

so

$$\max_{|w|\le 1} |q_{r+1}(w)| = \frac{1}{2^r}.$$

However,

$$\max\left\{|q_{r+1}(w)| : |w| \le \cos\left(\frac{\pi}{[2r+2]}\right)\right\}$$
$$= \frac{1}{2^r}\max\left\{|\cos([r+1]\theta)| : \frac{\pi}{[2(r+1)^2]} \le \theta \le \frac{(2r+1)\pi}{2r+2}\right\}.$$

Since $\pi/[2(r+1)^2] < \pi/(r+1) < (2r+1)\pi/(2r+2)$ and also because we have that $|\cos([r+1]\pi/(r+1))| = 1$,

$$\frac{1}{2^r}\max\left\{|q_{r+1}(w)| : |w| \le \cos\left(\frac{\pi}{[2r+2]}\right)\right\} = \frac{1}{2^r},$$

which completes the proof. ∎

If E and χ are defined by (6.1) and (6.2), we can use Lemma 6.1 to estimate the constant $G_{r+1,i}$ in (6.4) more precisely:

$$G_{r+1,i} = E\exp\left(\frac{2sh_i}{\chi}\right)\left[\frac{1}{(r+1)!}\right]\left[\frac{1}{2\cos\left(\frac{\pi}{[2r+2]}\right)}\right]^{r+1}\frac{1}{2^r}, \tag{6.6}$$

and with this estimate of $G_{r+1,i}$, (6.5) is satisfied for all $x \in [a_i, b_i]$, $1 \le i \le I$. For notational convenience, we define h and G_{r+1} by $h = \max_{1\le i\le I} h_i$ and

$$G_{r+1} = \max_{1\le i\le I} G_{r+1,i} = E\exp\left(\frac{2sh}{\chi}\right)\left[\frac{1}{(r+1)!}\right]\left[\frac{1}{2\cos\left(\frac{\pi}{[2r+2]}\right)}\right]^{r+1}\frac{1}{2^r}. \tag{6.7}$$

Lemma 6.2. *Define $\lambda_s = R(\Lambda_s) = R(L_s)$, where Λ_s and L_s are as in Theorem 2.1. Let $[a_i, b_i]$, $1 \le i \le I$, be as in* (H3). *For $1 \le i \le I$, let N_i, h_i and $\mathcal{V}_s$ be as defined in the fourth paragraph of this section. Assume that, for $1 \le i \le I$,*

$$\left[\sin\left(\frac{\pi}{2r+2}\right)\right]^2 h_i \le a_{i+1} - b_i.$$

Define $h_{min} = \min_{1\le i\le I} h_i$ and $\mu = h/h_{min}$. Then, for all $x \in [a_i, b_i]$, $1 \le i \le I$, we have

$$(1 - G_{r+1,i}h_i^{r+1})v_s(x) \le \mathcal{V}_s(x) \le (1 + G_{r+1,i}h_i^{r+1})v_s(x), \tag{6.8}$$

and

$$(1 - G_{r+1}h^{r+1})\lambda_s^\nu v_s(x) \leq (L_s^\nu \mathcal{V}_s)(x) \leq (1 + G_{r+1}h^{r+1})\lambda_s^\nu v_s(x). \tag{6.9}$$

If we define M_1 by

$$M_1 = \Big[\mu \frac{G_{r+1}h^r}{1 - G_{r+1}^2 h^{2r+2}}\Big]\Bigg[\frac{1}{\sin\left(\frac{\pi}{[2r+2]}\right)}\Bigg]^2 + \frac{2s}{\chi} \tag{6.10}$$

and

$$T = \left\{c_{j,k}^i : 1 \leq i \leq I, 1 \leq j \leq N_i \text{ and } 0 \leq k \leq r\right\} \subset S,$$

then $\mathcal{V}_s|_T \in K(2s/\chi; T)$ and $L_s^\nu \mathcal{V}_s|_T = \mathbf{L}_{s,\nu}(\mathcal{V}_s|_T) \in K(M_1; T)$.

Proof. To simplify the exposition, we shall denote G_{r+1} as G. Equation (6.5) gives (6.8) and (6.2) implies that $v_s \in K(2s/\chi; S)$. Since $\mathcal{V}_s|_T = v_s|_T$, it follows that $\mathcal{V}_s|_T \in K(2s/\chi; S)$. If we observe that we have $1 - Gh^{r+1} \leq 1 - G_{r+1,i}h_i^{r+1}$ and $1 + G_{r+1,i}h_i^{r+1} \leq 1 + Gh^{r+1}$ for $1 \leq i \leq I$, we derive from (6.5) that for $1 \leq i \leq I$ and $x \in [a_i, b_i]$,

$$(1 - Gh^{r+1})v_s(x) \leq \mathcal{V}_s(x) \leq (1 + Gh^{r+1})v_s(x).$$

Applying L_s^ν to this inequality, we obtain (6.9) and in particular, (6.9) holds for all $x \in T$. A little thought shows that for all $x \in T$,

$$[L_s^\nu \mathcal{V}_s](x) = \mathbf{L}_{s,\nu}(\mathcal{V}_s|_T)(x).$$

If $x, y \in T \cap [a_i, a_{i+1}]$, $1 \leq i \leq I$, and $x \neq y$, we obtain from (6.9) that

$$\begin{aligned}(L_s^\nu \mathcal{V}_s)(x) \leq (1 + Gh^{r+1})\lambda_s^\nu v_s(x) &\leq (1 + Gh^{r+1}) \exp\Big(\frac{2s|x-y|}{\chi}\Big)v_s(y) \\ &\leq \frac{1 + Gh^{r+1}}{1 - Gh^{r+1}} \exp\Big(\frac{2s|x-y|}{\chi}\Big)(L_s^\nu \mathcal{V}_s)(y).\end{aligned}$$

Taking logarithms on both sides of the above inequality, and noting that x and y are interchangeable in the inequality, we find that

$$\big|\ln[(L_s^\nu \mathcal{V}_s)(x)] - \ln[(L_s^\nu \mathcal{V}_s)(y)]\big| \leq \frac{2s}{\chi}|x-y| + \big[\ln(1 + Gh^{r+1}) - \ln(1 - Gh^{r+1})\big].$$

Using the trapezoidal rule and the convexity of the map $u \mapsto 1/u$,

$$\begin{aligned}\ln(1 + Gh^{r+1}) - \ln(1 - Gh^{r+1}) &= \int_{1-Gh^{r+1}}^{1+Gh^{r+1}} \frac{1}{u}\,du \\ \leq \frac{1}{2}\Big[\frac{1}{1 - Gh^{r+1}} + \frac{1}{1 + Gh^{r+1}}\Big]\Big[2Gh^{r+1}\Big] &= \frac{2Gh^{r+1}}{1 - G^2h^{2r+2}}.\end{aligned}$$

To prove that $L_s^\nu \mathcal{V}_s|_T \in K(M_1; T)$, it will suffice to prove that

$$\frac{2s}{\chi}|x-y| + \frac{2Gh^{r+1}}{1-G^2h^{2r+2}} \leq \frac{2s}{\chi}|x-y| + \Big[\mu\frac{Gh^r}{1-G^2h^{2r+2}}\Big]\Big[\frac{1}{\sin\big(\frac{\pi}{[2r+2]}\big)}\Big]^2|x-y|, \tag{6.11}$$

whenever $x, y \in ([a_i, b_i] \cap T) \cup \{a_{i+1}\}$ for $1 \leq i \leq I$ or $x, y \in ([a_I, b_I] \cap T)$. (Of course, we assume, as we can, that $x \neq y$.) A calculation shows that this is true if

$$2h \leq \mu\Big[\frac{1}{\sin\big(\frac{\pi}{[2r+2]}\big)}\Big]^2|x-y|.$$

If $x, y \in [a_i, b_i]$, we know that $|x-y| \geq 2h_i[\sin(\pi/(2r+2))]^2$, so it suffices to prove that $h \leq \mu h_i$, which follows from the definition of μ. We can assume that $x < y$, so if $x, y \in [a_i, b_i]$, the same argument applies. If $y = a_{i+1}$, $|x-y| \geq |a_{i+1} - b_i|$, and we assume that $|a_{i+1} - b_i| \geq 2h_i[\sin(\pi/(2r+2))]^2$, so again the same argument applies and gives (6.11). ∎

Remark 6.3. If $I = 1$, the condition on $a_{i+1} - b_i$ is vacuous and $\mu = 1$.

Our next lemma will play a crucial role in relating $R(\mathbf{L}_{s,\nu})$ to $R(L_s^\nu)$.

Lemma 6.4. *Let notation and assumptions be as in Lemma 6.2. Let $G := G_{r+1}$ be as in (6.7) and M_1 as in (6.10). Assume that $H := H_{r+1}$ is a constant with $H > G$ and assume that $h < 1$. Define M_2 by*

$$M_2 = M_1 + \frac{G}{H}\Big[\frac{\mu}{1-[\frac{G}{H}h]^2}\Big]\frac{1}{\big[\sin\big(\frac{\pi}{[2r+2]}\big)\big]^2}.$$

If $K = K(M_2; T)$, we have

$$\lambda_s^\nu \mathcal{V}_s(1-Hh^r) \leq_K \mathbf{L}_{s,\nu}\mathcal{V}_s|_T \leq_K \lambda_s^\nu \mathcal{V}_s(1+Hh^r). \tag{6.12}$$

Proof. Our previous results show that $(L_s^\nu \mathcal{V}_s)(x) = (\mathbf{L}_{s,\nu}\mathcal{V}_s)(x)$ for all $x \in T$, and, for all $x \in S = [a, b]$,

$$\lambda_s^\nu \mathcal{V}_s(x)\frac{1-Gh^{r+1}}{1+Gh^{r+1}} \leq (L_s^\nu \mathcal{V}_s)(x) \leq \lambda_s^\nu \mathcal{V}_s(x)\frac{1+Gh^{r+1}}{1-Gh^{r+1}}.$$

Recalling that $\mathcal{V}_s(x) = v_s(x)$ for $x \in T$, we have for $x \in T$,

$$\begin{aligned}\lambda_s^\nu \mathcal{V}_s(x)(1+Hh^r) - (L_s^\nu \mathcal{V}_s)(x) &\leq \lambda_s^\nu \mathcal{V}_s(x)(1+Hh^r) - \lambda_s^\nu(1-Gh^{r+1})\mathcal{V}_s(x) \\ &= \lambda_s^\nu(Hh^r + Gh^{r+1})\mathcal{V}_s(x) \\ &= \lambda_s^\nu h^r \mathcal{V}_s(x)(1+[\tfrac{G}{H}]h)H.\end{aligned}$$

If $y \in T$, a similar argument shows that

$$\begin{aligned}\lambda_s^\nu \mathcal{V}_s(y)(1+Hh^r) - (L_s^\nu \mathcal{V}_s)(y) &\geq \lambda_s^\nu \mathcal{V}_s(y)(1+Hh^r) - \lambda_s^\nu(1+Gh^{r+1})\mathcal{V}_s(y) \\ &= \lambda_s^\nu h^r \mathcal{V}_s(y)(1-[\tfrac{G}{H}]h)H.\end{aligned}$$

Using Lemma 6.2 and the above estimates, we find that

$$\begin{aligned}\frac{\lambda_s^\nu \mathcal{V}_s(x)(1+Hh^r) - (L_s^\nu \mathcal{V}_s)(x)}{\lambda_s^\nu \mathcal{V}_s(y)(1+Hh^r) - (L_s^\nu \mathcal{V}_s)(y)} &\leq \frac{\mathcal{V}_s(x)(1+[\frac{G}{H}]h)}{\mathcal{V}_s(y)(1-[\frac{G}{H}]h)} \\ &\leq \exp(M_1|x-y|)\frac{1+[\frac{G}{H}]h}{1-[\frac{G}{H}]h}. \qquad (6.13)\end{aligned}$$

The right hand side of (6.12) will follow from (6.13) if we prove that, for all $x, y \in T$ with $x \neq y$,

$$\exp(M_1|x-y|)\frac{1+[\frac{G}{H}]h}{1-[\frac{G}{H}]h} \leq \exp(M_2|x-y|). \qquad (6.14)$$

As in Lemma 6.2, it suffices to verify (6.14) for all points $x \neq y$, $x, y \in [a_i, a_{i+1}] \cap T$, $1 \leq i \leq I$, where $a_{I+1} = b_I$.

Arguing as in Lemma 6.2, we see that

$$\ln\left(1+\left[\frac{G}{H}\right]h\right) - \ln\left(1-\left[\frac{G}{H}\right]h\right) \leq \frac{G}{H}\frac{2h}{1-([\frac{G}{H}]h)^2}.$$

If we take the log of both sides of (6.14), it suffices to prove that

$$\begin{aligned}&M_1|x-y| + \frac{G}{H}\frac{2h}{1-([\frac{G}{H}]h)^2} \\ &\quad \leq M_1|x-y| + \frac{G}{H}\left[\frac{\mu}{1-[(\frac{G}{H})h]^2}\right]\frac{1}{\left[\sin\left(\frac{\pi}{[2r+2]}\right)\right]^2}|x-y|.\end{aligned}$$

As we proved in Lemma 6.2, all $x, y \in [a_i, a_{i+1}] \cap T$ with $x \neq y$ satisfy $|x-y| \geq 2h_i[\sin(\pi/(2r+2))]^2$. Since $2|x-y| \geq 2h_i/[\sin(\pi/[2r+2])]^2$, we see after simplification, the above inequality will be satisfied if $h \leq \mu h_i$, which holds by the definition of μ. This proves the right hand side of (6.12). The proof of the left hand side of inequality (6.12) follows by an exactly analogous argument and is left to the reader. ∎

Our next theorem connects $R(\mathbf{L}_{s,\nu})$ and $R(L_s^\nu)$. To use the theorem, we shall need to estimate various constants, and we shall carry this out in the next section for an important class of examples.

Theorem 6.5. *Let notation and assumptions be as in Lemma 6.2 and let H and M_2 be as in Lemma 6.4. Assume that ν, h, s and r have been selected so that $\mathbf{L}_{s,\nu}(K(M;T)) \subset K(M';T)$ where $0 < M' < M$ and $M \geq M_2$ (see Theorem 5.8). Then, we have that $R(\mathbf{L}_{s,\nu})$, the spectral radius of $\mathbf{L}_{s,\nu}$, satisfies*

$$\lambda_s^\nu(1 - Hh^r) \leq R(\mathbf{L}_{s,\nu}) \leq \lambda_s^\nu(1 + Hh^r).$$

Proof. Our previous results show that $\mathbf{L}_{s,\nu}$ has a unique, strictly positive eigenvector $w_{s,\nu} \in K(M;T)$ with $\|w_{s,\nu}\| = 1$. The eigenvalue corresponding to $w_{s,\nu}$ is $R(\mathbf{L}_{s,\nu})$. Furthermore, for every $u \in K(M;T) \setminus \{0\}$, $\lim_{m\to\infty}(\mathbf{L}_{s,\nu}^m u/\|\mathbf{L}_{s,\nu}^m u\|) = w_{s,\nu}$, with convergence in the sup norm topology on $\mathbb{R}^Q$.

If we use (6.12), but define $K = K(M;T)$, then because $M \geq M_2$ and $K(M;T) \supset K(M_2;T)$, we obtain

$$\lambda_s^\nu \mathcal{V}_s(1 - Hh^r) \leq_K \mathbf{L}_{s,\nu}\mathcal{V}_s \leq_K \lambda_s^\nu \mathcal{V}_s(1 + Hh^r).$$

The theorem now follows directly from Lemma 4.1. ∎

Our ultimate goal has been to provide rigorous upper and lower bounds on $\lambda_s = R(L_s)$ in terms of the eigenvalues of computable matrices, as was done in [14]. This follows immediately from Theorem 6.5.

Theorem 6.6. *Under the hypotheses of Theorem 6.5, we have*

$$[(1 + Hh^r)^{-1}R(\mathbf{L}_{s,\nu})]^{1/\nu} \leq \lambda_s \leq [(1 - Hh^r)^{-1}R(\mathbf{L}_{s,\nu})]^{1/\nu},$$

where the entries of the matrices $[1 + Hh^r]^{-1}\mathbf{L}_{s,\nu}$ and $[1 - Hh^r]^{-1}\mathbf{L}_{s,\nu}$ differ by $O(h^r)$.

7. Calculating the optimal interval $[a, b]$ and estimating E and χ

Throughout this section, we shall assume at least the hypotheses of Theorem 2.1, so L_s has a strictly positive C^m eigenfunction v_s. We shall take $S = [a, b], a < b$ in (H2). If S_0 is a closed, nonempty subset of S and $\theta_i(S_0) \subset S_0$ for $1 \leq i \leq n$, then $L_s : C(S) \to C(S)$ induces a bounded linear operator $L_{s,S_0} : C(S_0) \to C(S_0)$. It is often desirable to replace the original interval $[a, b]$ by a smaller interval (or union of intervals) $S_0 \subset [a, b]$ such that $\theta_i(S_0) \subset S_0$ for $1 \leq i \leq n$, and we first describe a class of examples for which this can be easily done. Note that $v_s|_{S_0}$ is strictly positive; and since $L_{s,S_0}(v_s|_{S_0}) = R(L_s)(v_s|_{S_0})$, the following lemma implies that $R(L_{s,S_0}) = R(L_s)$. Although this is a special case of another well-known result, a proof is provided for the readers' convenience.

Lemma 7.1. *Let S_0 be a compact metric space, $W := C(S_0)$ and $P = \{f \in W : f(t) \geq 0 \text{ for all } t \in S_0\}$. Assume that $L : W \to W$ is a bounded linear operator such that $L(P) \subset P$. If $w \in W$ and $w(t) > 0$ for all $t \in S_0$, then*

$$R(L) = \lim_{k\to\infty} \|L^k\|^{1/k} = \lim_{k\to\infty} \|L^k w\|^{1/k}.$$

If, in addition, $Lw = \lambda w$, then $\lambda = R(L)$, and there exists a constant $C \geq 1$ such that $\|L^k\| \leq C\lambda^k$ for all positive integers k.

Proof. Since S_0 is compact, there exists $\alpha > 0$ such that $w(t) \geq \alpha$ for all $t \in S_0$. If $f \in W$ and $\|f\| \leq 1$, it follows that for all $t \in S_0$,

$$-\frac{1}{\alpha}w(t) \leq f(t) \leq \frac{1}{\alpha}w(t).$$

Because L is order-preserving in the partial ordering from P,

$$-\frac{1}{\alpha}(L^k w)(t) \leq (L^k f)(t) \leq \frac{1}{\alpha}(L^k w)(t)$$

for all positive integers k, which implies that

$$\|L^k\|^{1/k} = \big(\sup\big\{\|L^k f\| : f \in W \text{ and } \|f\| \leq 1\big\}\big)^{1/k} \leq \Big(\frac{1}{\alpha}\Big)^{1/k}\|L^k w\|^{1/k}.$$

We also have that

$$\Big(\frac{1}{\alpha}\Big)^{1/k}\|L^k w\|^{1/k} \leq \Big(\frac{1}{\alpha}\Big)^{1/k}\|w\|^{1/k}\|L^k\|^{1/k}.$$

Since $\lim_{k\to\infty}\|L^k\|^{1/k} = R(L)$ and $\lim_{k\to\infty}(1/\alpha)^{1/k} = \lim_{k\to\infty}\|w\|^{1/k} = 1$, we conclude that $\lim_{k\to\infty}\|L^k w\|^{1/k} = R(L)$.

If $Lw = \lambda w$, the above argument shows that

$$\frac{1}{\alpha}\lambda^k w(t) \leq (L^k f)(t) \leq \frac{1}{\alpha}\lambda^k w(t),$$

which implies $\|L^k\| \leq \frac{1}{\alpha}\|w\|\lambda^k$, so the lemma is satisfied with $C = \frac{1}{\alpha}\|w\| \geq 1$. ■

Let $S_0 = [\mathfrak{a}_0, \mathfrak{b}_0]$ be a compact interval of real numbers, $\mathfrak{a}_0 < \mathfrak{b}_0$, and let $\mathcal{B}$ be a finite set of real numbers. For each $\beta \in \mathcal{B}$, $\theta_\beta : S_0 \to \mathbb{R}$. We make the following hypothesis:

(H4) (i) For each $\beta \in \mathcal{B}$, $\theta_\beta : S_0 \to S_0$ and θ_β is a continuous map.

(ii) There exist $\gamma \in \mathcal{B}$ and $\Gamma \in \mathcal{B}$ such that for all $x \in S_0$ and all $\beta \in \mathcal{B}$, $\theta_\Gamma(x) \leq \theta_\beta(x) \leq \theta_\gamma(x)$.

(iii) $x \mapsto \theta_\gamma(x)$ and $x \mapsto \theta_\Gamma(x)$ are strictly decreasing functions on S_0.

The example we have in mind is that $\mathcal{B}$ is a finite set of distinct real numbers with $\beta \geq \gamma > 0$ for all $\beta \in \mathcal{B}$ and $\theta_\beta(x) = 1/(x+\beta)$ and $S_0 = [0, 1/\gamma]$, but there seems no gain in specializing at this point.

Lemma 7.2. *Assume* (H4) *is satisfied. Define* $\mathfrak{a}_1 = \theta_\Gamma(\mathfrak{b}_0)$ *and* $\mathfrak{b}_1 = \theta_\gamma(\mathfrak{a}_0)$*. Then,* $\mathfrak{a}_0 \leq \mathfrak{a}_1 < \mathfrak{b}_1 \leq \mathfrak{b}_0$*, and* $\theta_\beta(x) \in [\mathfrak{a}_1, \mathfrak{b}_1]$ *for all* $x \in S_0$ *and all* $\beta \in \mathcal{B}$*.*

Proof. Property (i) in (H4) implies that $\mathfrak{a}_0 \leq \mathfrak{a}_1 \leq \mathfrak{b}_0$ and $\mathfrak{a}_0 \leq \mathfrak{b}_1 \leq \mathfrak{b}_0$. Property (ii) implies that $\theta_\Gamma(\mathfrak{b}_0) = \mathfrak{a}_1 \leq \theta_\gamma(\mathfrak{b}_0)$ and Property (iii) implies that $\theta_\gamma(\mathfrak{b}_0) < \theta_\gamma(\mathfrak{a}_0) = \mathfrak{b}_1$, so $\mathfrak{a}_1 < \mathfrak{b}_1$. For all $x \in [\mathfrak{a}_0, \mathfrak{b}_0]$ and all $\beta \in \mathcal{B}$, $\theta_\Gamma(x) \leq \theta_\beta(x)$ (Property (ii)) and $\theta_\Gamma(\mathfrak{b}_0) \leq \theta_\Gamma(x)$ (Property (iii)), so $\mathfrak{a}_1 \leq \theta_\beta(x)$. Similarly, $\theta_\beta(x) \leq \theta_\gamma(x)$ and $\theta_\gamma(x) \leq \theta_\gamma(\mathfrak{a}_0) = \mathfrak{b}_1$, so $\theta_\beta(x) \leq \mathfrak{b}_1$. ∎

Lemma 7.3. *Assume* (H4) *is satisfied. Also assume that for* $1 \leq j \leq k$*, we have found an increasing sequence of real numbers* $\mathfrak{a}_0 \leq \mathfrak{a}_1 \leq \cdots \leq \mathfrak{a}_k$ *and a decreasing sequence of real numbers* $\mathfrak{b}_0 \geq \mathfrak{b}_1 \geq \ldots \geq \mathfrak{b}_k$ *such that* $\mathfrak{b}_j - \mathfrak{a}_j > 0$ *for* $1 \leq j \leq k$ *and* $\theta_\beta([\mathfrak{a}_j, \mathfrak{b}_j]) \subset [\mathfrak{a}_{j+1}, \mathfrak{b}_{j+1}]$ *for* $0 \leq j \leq k-1$ *and all* $\beta \in \mathcal{B}$*. Define* $\mathfrak{a}_{k+1} = \theta_\Gamma(\mathfrak{b}_k)$ *and* $\mathfrak{b}_{k+1} = \theta_\gamma(\mathfrak{a}_k)$*. Then, we have* $\mathfrak{a}_k \leq \mathfrak{a}_{k+1}$*,* $\mathfrak{b}_{k+1} \leq \mathfrak{b}_k$*,* $\mathfrak{b}_{k+1} - \mathfrak{a}_{k+1} > 0$ *and* $\theta_\beta([\mathfrak{a}_k, \mathfrak{b}_k]) \subset [\mathfrak{a}_{k+1}, \mathfrak{b}_{k+1}]$ *for all* $\beta \in B$*.*

Proof. Apply Lemma 7.2 with $[\mathfrak{a}_k, \mathfrak{b}_k]$ taking the place of $[\mathfrak{a}_0, \mathfrak{b}_0]$ and $\theta_\Gamma(\mathfrak{b}_k) = \mathfrak{a}_{k+1}$ taking the place of $\mathfrak{a}_1$ and $\theta_\gamma(\mathfrak{a}_k) = \mathfrak{b}_{k+1}$ taking the place of $\mathfrak{b}_1$. ∎

It follows from Lemma 7.3 that if (H4) holds and if we inductively define sequences $\mathfrak{a}_k$ and $\mathfrak{b}_k$ by $\mathfrak{a}_{k+1} = \theta_\Gamma(\mathfrak{b}_k)$ and $\mathfrak{b}_{k+1} = \theta_\gamma(\mathfrak{a}_k)$ for $k \geq 0$, then for all integers $k \geq 0$, $\mathfrak{a}_k < \mathfrak{b}_k$, $\mathfrak{a}_{k+1} \geq \mathfrak{a}_k$, $\mathfrak{b}_{k+1} \leq \mathfrak{b}_k$, and $\theta_\beta([\mathfrak{a}_k, \mathfrak{b}_k]) \subset [\mathfrak{a}_{k+1}, \mathfrak{b}_{k+1}]$ for all $\beta \in \mathcal{B}$. It follows that $\lim_{k\to\infty} \mathfrak{a}_k =: \mathfrak{a}_\infty$ and $\lim_{k\to\infty} \mathfrak{b}_k =: \mathfrak{b}_\infty$ both exist.

Lemma 7.4. *Assume* (H4) *is satisfied and let the notation be as above. Then, we have* $\theta_\beta([\mathfrak{a}_\infty, \mathfrak{b}_\infty]) \subset [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$ *for all* $\beta \in \mathcal{B}$ *and* $\theta_\gamma(\mathfrak{a}_\infty) = \mathfrak{b}_\infty$ *and* $\theta_\Gamma(\mathfrak{b}_\infty) = \mathfrak{a}_\infty$*, so* $\theta_\Gamma \circ \theta_\gamma(\mathfrak{a}_\infty) = \mathfrak{a}_\infty$ *and* $\theta_\gamma \circ \theta_\Gamma(\mathfrak{b}_\infty) = \mathfrak{b}_\infty$*. If* $\beta_1, \beta_2, \ldots, \beta_k$ *are elements of* $\mathcal{B}$ *and* $x \in [\mathfrak{a}_0, \mathfrak{b}_0]$*, then* $(\theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_k})(x) \in [\mathfrak{a}_k, \mathfrak{b}_k]$*. If* $(\theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_k})(x) = x$ *for some* $x \in [\mathfrak{a}_0, \mathfrak{b}_0]$ *and some elements* $\beta_1, \beta_2, \ldots, \beta_k$ *of* $\mathcal{B}$*, then* $x \in [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$*.*

Proof. Since $\lim_{k\to\infty} \mathfrak{a}_k =: \mathfrak{a}_\infty$, $\lim_{k\to\infty} \mathfrak{b}_k =: \mathfrak{b}_\infty$, $\theta_\gamma(\mathfrak{a}_k) = \mathfrak{b}_{k+1}$ and $\theta_\Gamma(\mathfrak{b}_k) = \mathfrak{a}_{k+1}$, it follows from the continuity of θ_γ and θ_Γ that $\theta_\gamma(\mathfrak{a}_\infty) = \mathfrak{b}_\infty$ and $\theta_\Gamma(\mathfrak{b}_\infty) = \mathfrak{a}_\infty$.

If $x \in [\mathfrak{a}_0, \mathfrak{b}_0]$ and $\beta_1, \beta_2, \ldots, \beta_k$ are elements of $\mathcal{B}$, repeated applications of Lemma 7.2 show that $\theta_{\beta_k}(x) \in [\mathfrak{a}_1, \mathfrak{b}_1]$, $(\theta_{\beta_{k-1}} \circ \theta_{\beta_k})(x) \in [\mathfrak{a}_2, \mathfrak{b}_2]$, and generally that $(\theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_k})(x) \in [\mathfrak{a}_k, \mathfrak{b}_k]$. If $x \in [\mathfrak{a}_0, \mathfrak{b}_0]$ and $\beta_1, \beta_2, \ldots, \beta_k$ are elements of $\mathcal{B}$ are such that $(\theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_k})(x) = x$, it follows that $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$. Now the same argument can be repeated to show that $x \in [\mathfrak{a}_{2k}, \mathfrak{b}_{2k}]$ and generally that

$x \in [\mathfrak{a}_{mk}, \mathfrak{b}_{mk}]$ for every positive integer m. Since $\bigcap_{m\geq 1}[\mathfrak{a}_{mk}, \mathfrak{b}_{mk}] = [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, we conclude that $x \in [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$. ∎

Remark 7.5. Under the hypotheses of Lemma 7.4, if $(\theta_\Gamma \circ \theta_\gamma)(x) = x$ or $(\theta_\gamma \circ \theta_\Gamma)(x) = x$ for some $x \in [\mathfrak{a}_0, \mathfrak{b}_0]$, then $\mathfrak{a}_\infty \leq x \leq \mathfrak{b}_\infty$, so $\mathfrak{a}_\infty$ is the least fixed point of $\theta_\Gamma \circ \theta_\gamma$ in $[\mathfrak{a}_0, \mathfrak{b}_0]$ and $\mathfrak{b}_\infty$ is the greatest fixed point of $\theta_\gamma \circ \theta_\Gamma$ in $[\mathfrak{a}_0, \mathfrak{b}_0]$.

Lemma 7.4 provides a way of obtaining invariant intervals J such that $\theta_\beta(J) \subset J$ for all $\beta \in \mathcal{B}$. However, it is frequently the case that we have more information than given in (H4), and then one can give more flexible methods to find invariant intervals. The following lemma, whose proof we omit, describes a commonly occurring class of examples where such methods are available.

Lemma 7.6. *Let the hypotheses and notation be as in Lemma 7.4. Suppose also that there exist intervals $J_1 = [x_1, \mathfrak{a}_\infty]$ and $J_2 = [\mathfrak{b}_\infty, x_2]$ with $\mathfrak{a}_0 \leq x_1 < \mathfrak{a}_\infty$ and $\mathfrak{b}_\infty < x_2 < \mathfrak{b}_0$, such that $\theta_\gamma(J_2) \subset [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, $\theta_\Gamma(J_1) \subset [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, $\mathrm{lip}(\theta_\gamma|_{J_1}) \leq c_1$, $\mathrm{lip}(\theta_\Gamma|_{J_2}) \leq c_2$, and $c_1 c_2 < 1$. If $\xi_1 \in J_1$ is chosen so that $\xi_2 = \theta_\gamma(\xi_1) \in J_2$, then $\theta_\beta([\xi_1, \xi_2]) \subset [\xi_1, \xi_2]$ for all $\beta \in \mathcal{B}$. Similarly, if $\eta_2 \in J_2$ is chosen so that $\eta_1 = \theta_\Gamma(\eta_2) \in J_1$, then $\theta_\beta([\eta_1, \eta_2]) \subset [\eta_1, \eta_2]$ for all $\beta \in \mathcal{B}$.*

We shall use $\mathcal{B}$ as an index set, so the operator L_s can be written

$$(L_s f)(x) = \sum_{\beta \in \mathcal{B}} g_\beta(x)^s f(\theta_\beta(x)),$$

where $\theta_\beta(S_0) \subset S_0$ for all $\beta \in \mathcal{B}$ and $S_0 = [\mathfrak{a}_0, \mathfrak{b}_0]$. If the conditions of Theorem 2.1 are satisfied, L_s has a strictly positive, C^m eigenfunction. Assuming (H4) and the hypotheses of Theorem 2.1, the observation in the first paragraph of this section implies that to compute $R(L_s)$, we can, in the notation of Lemma 7.4, replace $[\mathfrak{a}_0, \mathfrak{b}_0]$ by $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$ or by $[\mathfrak{a}_k, \mathfrak{b}_k]$ for any integer $k \geq 1$. In fact, we could use any interval $J \subset [\mathfrak{a}_0, \mathfrak{b}_0]$ with $\theta_\beta(J) \subset J$ for all $\beta \in \mathcal{B}$ (compare Lemma 7.6).

For the remainder of this section, we shall assume the following:

(H5) $\mathcal{B}$ is a finite set of distinct real numbers and $\gamma = \min\{\beta : \beta \in \mathcal{B}\} \geq 1$. For every $\beta \in \mathcal{B}$, we define $\theta_\beta : [0, 1/\gamma] =: [\mathfrak{a}_0, \mathfrak{b}_0] \to [0, 1/\gamma]$ by $\theta_\beta(x) = (x + \beta)^{-1}$.

We shall write $\Gamma = \max\{\beta : \beta \in \mathcal{B}\}$ and $\gamma = \min\{\beta : \beta \in \mathcal{B}\}$ and always assume that $\gamma < \Gamma$. The reader can check that $\{\theta_\beta : \beta \in \mathcal{B}\}$ satisfies the conditions of (H4) with $\theta_\gamma(x) = (x + \gamma)^{-1}$ and $\theta_\Gamma(x) = (x + \Gamma)^{-1}$. Using the calculations in the following paragraph, the reader can check that the conditions of Lemma 7.6 are also satisfied.

We assume that the sequences $\{\mathfrak{a}_k : k \geq 1\}$ and $\{\mathfrak{b}_k : k \geq 1\}$ are defined as in Lemmas 7.3 and 7.4, with $\mathfrak{a}_0 = 0$ and $\mathfrak{b}_0 = 1/\gamma$, and $\mathfrak{a}_\infty$ and $\mathfrak{b}_\infty$ defined as in Lemma 7.4. Since $\mathfrak{a}_\infty$ is a fixed point of $\theta_\Gamma \circ \theta_\gamma$ in $[0, 1/\gamma]$ and $\mathfrak{b}_\infty$ is a fixed point of $\theta_\gamma \circ \theta_\Gamma$ in $[0, 1/\gamma]$, one can easily solve the equations

$$x = (\theta_\Gamma \circ \theta_\gamma)(x) = \frac{x+\gamma}{\Gamma x + 1 + \Gamma\gamma} \quad \text{and} \quad x = (\theta_\gamma \circ \theta_\Gamma)(x) = \frac{x+\Gamma}{\gamma x + 1 + \Gamma\gamma}$$

to obtain

$$\mathfrak{a}_\infty = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{\Gamma}} \quad \text{and} \quad \mathfrak{b}_\infty = -\frac{\Gamma}{2} + \sqrt{\left(\frac{\Gamma}{2}\right)^2 + \frac{\Gamma}{\gamma}}. \tag{7.1}$$

One can verify that $\mathfrak{b}_\infty = (\Gamma/\gamma)\mathfrak{a}_\infty$, so $0 < \mathfrak{a}_\infty < \mathfrak{b}_\infty < 1/\gamma$.

Since our index set is $\mathcal{B}$, we slightly abuse the previous notation and, for a positive integer ν, we define the set of ordered ν-tuples of elements of $\mathcal{B}$ by

$$\Omega_\nu = \{(\beta_1, \beta_2, \dots, \beta_\nu) : \beta_j \in \mathcal{B} \text{ for } 1 \le j \le \nu\}.$$

For each $\omega = (\beta_1, \beta_2, \dots, \beta_\nu) \in \Omega_\nu$, we define $\theta_\omega = \theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_\nu}$. Our first task is to estimate $c(\nu)$ (see (5.3)), which gives an upper bound for $\mathrm{lip}(\theta_\omega)$, $\omega \in \Omega_\nu$.

If $\omega = (\beta_1, \beta_2, \dots, \beta_\nu) \in \Omega_\nu$ and $\beta \in \mathcal{B}$, define a matrix

$$M_\beta = \begin{pmatrix} 0 & 1 \\ 1 & \beta \end{pmatrix}.$$

It is proved in [14, Section 6] that

$$M = M_{\beta_1} M_{\beta_2} \cdots M_{\beta_\nu} = \begin{pmatrix} A_{\nu-1} & A_\nu \\ B_{\nu-1} & B_\nu \end{pmatrix},$$

where A_j and B_j are defined inductively by $A_0 = 0$, $A_1 = 1$, $B_0 = 1$, $B_1 = \beta_1$ and, generally, for $1 \le j \le \nu$, by

$$A_{j+1} = A_{j-1} + \beta_{j+1} A_j \quad \text{and} \quad B_{j+1} = B_{j-1} + \beta_{j+1} B_j. \tag{7.2}$$

Note that $\det(M_\beta) = -1$, so $\det(M) = (-1)^\nu$. Standard results for Möbius transforms now imply that for $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$, $0 \le k \le \infty$,

$$\begin{aligned} (\theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_\nu})(x) &= \frac{A_{\nu-1}x + A_\nu}{B_{\nu-1}x + B_\nu}, \\ \frac{d}{dx}(\theta_{\beta_1} \circ \theta_{\beta_2} \circ \cdots \circ \theta_{\beta_\nu})(x) &= \frac{(-1)^\nu}{(B_{\nu-1}x + B_\nu)^2}. \end{aligned} \tag{7.3}$$

If we define $\tilde{B}_0 = 1$, $\tilde{B}_1 = \gamma$ and $\tilde{B}_{j+1} = \tilde{B}_{j-1} + \gamma \tilde{B}_j$ for $j \ge 1$, then because $\gamma \le \beta$ for all $\beta \in \mathcal{B}$, it is straightforward to prove that $\tilde{B}_j \le B_j$ for $0 \le j \le \nu$, where B_j is defined by (7.2). It follows that for all $\omega \in \Omega_\nu$ and $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$,

$$|\theta_\omega'(x)| \le [\tilde{B}_{\nu-1}\mathfrak{a}_k + \tilde{B}_\nu]^{-2},$$

which implies that, for $\theta_\omega : [\mathfrak{a}_k, \mathfrak{b}_k] \to \mathbb{R}$,

$$\max\left\{\mathrm{lip}(\theta_\omega) : \omega \in \Omega_\nu\right\} = [\tilde{B}_{\nu-1}\mathfrak{a}_k + \tilde{B}_\nu]^{-2} =: c(\nu). \tag{7.4}$$

It remains to give an exact formula for the right hand side of (7.4). The linear difference equation $\tilde{B}_{j+1} = \tilde{B}_{j-1} + \gamma\tilde{B}_j$ has solutions of the form λ^j for $j \geq 0$, which leads to the formula $\lambda^{n+1} = \lambda^{n-1} + \gamma\lambda^n$, or for $\lambda \neq 0$, $\lambda^2 = 1 + \lambda\gamma$. Hence,

$$\lambda = \lambda_+ = \frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 + 4} \quad \text{and} \quad \lambda = \lambda_- = \frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 + 4}. \tag{7.5}$$

The general solution of the difference equation is then

$$c_1\lambda_+^j + c_2\lambda_-^j = \tilde{B}_j, \quad \text{for } j \geq 0, \tag{7.6}$$

where c_1 and c_2 must be chosen so that $\tilde{B}_0 = 1$ and $\tilde{B}_1 = \gamma$. A calculation gives

$$c_1 = \frac{\sqrt{\gamma^2 + 4} + \gamma}{2\sqrt{\gamma^2 + 4}} \quad \text{and} \quad c_2 = \frac{\sqrt{\gamma^2 + 4} - \gamma}{2\sqrt{\gamma^2 + 4}}. \tag{7.7}$$

Summarizing the above discussion, we obtain the following lemma.

Lemma 7.7. *Assume* (H4) *holds and consider* θ_β, $\beta \in \mathcal{B}$, *as a map of* $[\mathfrak{a}_k, \mathfrak{b}_k]$ *to itself, for* $0 \leq k \leq \infty$, *where* $\mathfrak{a}_0 = 0$, $\mathfrak{b}_0 = 1$, $\mathfrak{a}_k = \theta_\Gamma(\mathfrak{b}_{k-1})$ *and* $\mathfrak{b}_k = \theta_\gamma(\mathfrak{a}_{k-1})$ *for* $k \geq 1$ *and* $\mathfrak{a}_\infty$ *and* $\mathfrak{b}_\infty$ *are given by* (7.1). *Then, for* $j \geq 1$, $\tilde{B}_j$ *is given by* (7.6), *where* λ_+ *and* λ_- *are given by* (7.5) *and* c_1 *and* c_2 *by* (7.7).

Remark 7.8. Because $\lambda_+ > 1$ and $-1 < \lambda_- = -1/\lambda_+ < 0$ for all $\gamma > 0$, $c_1\lambda_+^j$ is the dominant term in (7.6) as j increases; and one can check that $|c_2\lambda_-^j| < 1/2$ for all $j \geq 0$. Of course, for moderate values of j, one can easily compute $\tilde{B}_j$ from its recurrence formula. It is clear that the constant $c(\nu)$ in (7.4) is minimized by working on the interval $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$.

Assuming (H5) holds, we now define for $s > 0$, $L_s : C([0, 1]) \to C([0, 1])$ by

$$(L_s f)(x) = \sum_{\beta \in \mathcal{B}} |\theta_\beta'(x)|^s f(\theta_\beta(x)) = \sum_{\beta \in \mathcal{B}} g_\beta(x)^s f(\theta_\beta(x)).$$

It is well known that for ν a positive integer,

$$(L_s^\nu f)(x) = \sum_{\beta \in \mathcal{B}} |\theta_\omega'(x)|^s f(\theta_\omega(x)) = \sum_{\omega \in \Omega_\nu} g_\omega(x)^s f(\theta_\omega(x)).$$

Because $\theta_\omega([\mathfrak{a}_k, \mathfrak{b}_k]) \subset [\mathfrak{a}_k, \mathfrak{b}_k]$ for $0 \leq k \leq \infty$ and $\omega \in \Omega_\nu$, we can also consider L_s^ν as a map from $C([\mathfrak{a}_k, \mathfrak{b}_k])$ to itself and as noted earlier, this does not change the spectral radius of L_s^ν. Thus, we shall consider L_s^ν as a map from $C([\mathfrak{a}_k, \mathfrak{b}_k])$ into itself, with optimal results obtained by taking $k = \infty$.

We need to find a constant $M_0(\nu)$ (compare (5.3)) such that for all $\omega \in \Omega_\nu$, $g_\omega(x) := |\theta'_\omega(x)| \in K(M_0(\nu); [\mathfrak{a}_k, \mathfrak{b}_k])$. In this case, this is equivalent to proving that for all $\omega \in \Omega_\nu$, $x \mapsto \ln(|\theta'_\omega(x)|)$ is a Lipschitz map on $[\mathfrak{a}_k, \mathfrak{b}_k]$ with Lipschitz constant less than or equal to $M_0(\nu)$. If $\omega = (\beta_1, \beta_2, \dots, \beta_\nu)$, we have, by (7.3), that

$$|\theta'_\omega(x)| = \frac{1}{(B_{\nu-1}x + B_\nu)^2},$$

so

$$\ln(|\theta'_\omega(x)|) = -2\ln(B_{\nu-1}x + B_\nu).$$

Thus, it suffices to choose $M_0(\nu)$ so that for all $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$ and all $\omega \in \Omega_\nu$,

$$\begin{aligned}\left|\frac{d}{dx}\ln(|\theta'_\omega(x)|)\right| &= 2\frac{B_{\nu-1}}{B_{\nu-1}x + B_\nu} \\ &= 2\left[\frac{1}{x + (\frac{B_\nu}{B_{\nu-1}})}\right] \le 2\left[\frac{1}{\mathfrak{a}_k + (\frac{B_\nu}{B_{\nu-1}})}\right] \le M_0(\nu).\end{aligned}$$

If we define $x_j = B_{j-1}/B_j$ for $1 \le j < \nu$, then since $B_{j+1} = B_{j-1} + \beta_{j+1}B_j$ for $1 \le j \le \nu$, we get $B_{j+1}/B_j = B_{j-1}/B_j + \beta_{j+1}$ or $x_{j+1} = 1/(x_j + \beta_{j+1}) = \theta_{\beta_{j+1}}(x_j)$ for $1 \le j \le \nu$. Since $x_1 = 1/\beta_1 \in [\mathfrak{a}_1, \mathfrak{b}_1] = [1/(\Gamma + 1/\gamma), 1/\gamma]$, it follows from Lemma 7.3 that $x_{j+1} \in [\mathfrak{a}_{j+1}, \mathfrak{b}_{j+1}]$ for $1 \le j < \nu$, so $1/x_{j+1} = B_{j+1}/B_j \in [\mathfrak{b}_{j+1}^{-1}, \mathfrak{a}_{j+1}^{-1}]$ and $\mathcal{B}_\nu/B_{\nu-1} \ge \mathfrak{b}_\nu^{-1}$. It follows that for $\omega \in \Omega_\nu$ and $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$, we can take

$$M_0(\nu) = \frac{2}{\mathfrak{a}_k + \mathfrak{b}_\nu^{-1}}. \tag{7.8}$$

By adding the exponent s, one easily derives that for all $\omega \in \Omega_\nu$, $\nu \ge 1$ and $0 \le k \le \infty$,

$$g_\omega(\cdot)^s = |\theta'_\omega(\cdot)|^s \in K\Big(\frac{2s}{\mathfrak{a}_k + \mathfrak{b}_\nu^{-1}};\ [\mathfrak{a}_k, \mathfrak{b}_k]\Big). \tag{7.9}$$

Note that we could replace $\mathfrak{b}_\nu^{-1}$ by $\mathfrak{a}_{\nu-1} + \gamma$.

We summarize the above discussion in the following lemma.

Lemma 7.9. *Assume* (H5) *holds and let* $\mathfrak{a}_k$ *and* $\mathfrak{b}_k$, $0 \le k \le \infty$, *be as described in Lemma* 7.7. *If* $\omega \in \Omega_\nu$, $\nu \ge 1$, *the map* $x \mapsto \ln(|\theta'_\omega(x)|^s)$, $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$ *is Lipschitz with Lipschitz constant* $\le 2s/(\mathfrak{a}_k + \mathfrak{b}_\nu^{-1})$, *so* (7.9) *is satisfied.*

It remains to estimate, for $0 \le k \le \infty$, $\max_{x \in [\mathfrak{a}_k, \mathfrak{b}_k]} |(d^j v_s/dx^j)(x)|/v_s(x)$, where v_s denotes the unique (to within normalization) strictly positive eigenfunction of L_s. The basic idea is to exploit (2.2), as was done in [14, Section 6], but our results will refine those in [14].

Our previous calculations show that for $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$, $0 \le k \le \infty$, we have

$$g_\omega(x)^s = |\theta'_\omega(x)|^s = \frac{1}{B_{\nu-1}^{2s}\big(x + \frac{B_\nu}{B_{\nu-1}}\big)^{2s}}.$$

It follows that for $j \ge 1$, and letting D denote d/dx, we have

$$\frac{(-1)^j (D^j[(g_\omega)^s])(x)}{g_\omega(x)} = \frac{2s(2s+1)\cdots(2s+j-1)}{\big(x + \frac{B_\nu}{B_{\nu-1}}\big)^j}. \tag{7.10}$$

The same argument used in Lemma 7.9 shows that

$$\mathfrak{b}_\nu^{-1} \le \frac{B_\nu}{B_{\nu-1}} \le \mathfrak{a}_\nu^{-1}, \tag{7.11}$$

so if $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$, we derive from (7.10) and (7.11) that

$$\begin{aligned} \frac{2s(2s+1)\cdots(2s+j-1)}{[\mathfrak{b}_k + \mathfrak{a}_\nu^{-1}]^j} &\le \frac{(-1)^j (D^j[(g_\omega)^s])(x)}{g_\omega(x)^s} \\ &\le \frac{2s(2s+1)\cdots(2s+j-1)}{(\mathfrak{a}_k + \mathfrak{b}_\nu^{-1})^j}. \end{aligned} \tag{7.12}$$

It follows from (7.12) that for $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$,

$$\begin{aligned} \frac{2s(2s+1)\cdots(2s+j-1)}{[\mathfrak{b}_k + \mathfrak{a}_\nu^{-1}]^j} &\le \frac{(-1)^j \sum_{\omega \in \Omega_\nu} (D^j[(g_\omega)^s])(x)}{\sum_{\omega \in \Omega_\nu} g_\omega(x)^s} \\ &\le \frac{2s(2s+1)\cdots(2s+j-1)}{(\mathfrak{a}_k + \mathfrak{b}_\nu^{-1})^j}. \end{aligned} \tag{7.13}$$

Taking limits as $\nu \to \infty$ in (7.13) and using (2.2), we find that for $x \in [\mathfrak{a}_k, \mathfrak{b}_k]$,

$$\begin{aligned} \frac{2s(2s+1)\cdots(2s+j-1)}{[\mathfrak{b}_k + \mathfrak{a}_\infty^{-1}]^j} &\le \frac{(-1)^j \big(\frac{d^j v_s}{dx^j}\big)(x)}{v_s(x)} \\ &\le \frac{2s(2s+1)\cdots(2s+j-1)}{(\mathfrak{a}_k + \mathfrak{b}_\infty^{-1})^j}. \end{aligned} \tag{7.14}$$

Notice that we can replace $\mathfrak{a}_\infty^{-1}$ by $\mathfrak{b}_\infty + \Gamma$ and $\mathfrak{b}_\infty^{-1}$ by $\mathfrak{a}_\infty + \gamma$ in (7.14).

As one can easily see, the lower bound in (7.14) increases as k increases and the upper bound decreases as k increases, so the optimal bounds are obtained when $k = \infty$ and apply to the interval $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$.

We summarize the above results in the following lemma.

Lemma 7.10. *Let v_s denote the unique (to with normalization) strictly positive eigenfunction of L_s. Assume* (H4) *holds and let $\mathfrak{a}_k$ and $\mathfrak{b}_k$, $k \ge 0$ be as in Lemma* 7.4. *Then, v_s satisfies* (7.14).

Remark 7.11. Since, in Lemma 7.10, we have specified the coefficient g_β and the maps θ_β for $\beta \in \mathcal{B}$, now Lemma 7.10 gives us a simple formula for the constant $E(s,p) = E$ in (6.1),

$$\max_{x\in[\mathfrak{a}_k,\mathfrak{b}_k]} \frac{\left|\left(\frac{d^p v_s}{dx^p}\right)(x)\right|}{v_s(x)} \leq \frac{(2s)(2s+1)\cdots(2s+p-1)}{(\mathfrak{a}_k+\mathfrak{b}_\infty^{-1})^p} =: E,$$

where p and k are positive integers and $1 \leq k \leq \infty$. Here we have allowed the interval to vary with k, but we may eventually restrict to $k = \infty$.

It remains to find a constant χ (compare (6.2)) such that for all $x_1, x_2 \in [\mathfrak{a}_k, \mathfrak{b}_k]$,

$$v_s(x_1) \leq \exp\left(\frac{2s|x_1 - x_2|}{\chi}\right) v_s(x_2).$$

It follows from (7.14) that if $\mathfrak{a}_k \leq x_1 \leq x_2 \leq \mathfrak{b}_k$, then

$$-\int_{x_1}^{x_2} \frac{v_s'(x)}{v_s(x)}\, dx = \ln(v_s(x_1)) - \ln(v_s(x_2)) \leq \frac{2s}{\mathfrak{a}_k + \mathfrak{b}_\infty^{-1}} |x_2 - x_1|,$$

which implies that

$$v_s(x_1) \leq \exp\left(\frac{2s|x_1 - x_2|}{[\mathfrak{a}_k + \mathfrak{b}_\infty^{-1}]}\right) v_s(x_2). \tag{7.15}$$

If $x_2 \leq x_1$, we know that $v_s(x_2) \geq v_s(x_1)$, so (7.15) is satisfied for all $x_1, x_2 \in [\mathfrak{a}_k, \mathfrak{b}_k]$. In particular, (7.15) is satisfied if the roles of x_1 and x_2 are reversed, which implies that the map $x \mapsto \ln(v_s(x))$ is Lipschitz on $[\mathfrak{a}_k, \mathfrak{b}_k]$ with Lipschitz constant $2s/(\mathfrak{a}_k + \mathfrak{b}_\infty^{-1})$. Summarizing, we have the following lemma.

Lemma 7.12. *If* $\chi = \mathfrak{a}_k + \mathfrak{b}_\infty^{-1}$, *then* (6.2) *is satisfied on* $[\mathfrak{a}_k, \mathfrak{b}_k]$.

8. Computation of $R(L_s)$

In this section we shall describe how to use the results of Sections 5–7 to obtain rigorous, high order estimates for $R(L_s) = \lambda_s$. As a subcase, we shall obtain rigorous estimates for the Hausdorff dimension of certain fractal objects described by iterated function systems.

For simplicity, we shall restrict attention to the class of maps $\theta_\beta : [0,1] \to [0,1]$, where $\theta_\beta(x) = 1/(x+\beta)$ for $\beta \in \mathcal{B}$ and $\mathcal{B}$ as in (H5) of Section 7. For $s \geq 0$, we define $L_s : C[0,1] \to C[0,1]$ by

$$(L_s f)(x) = \sum_{\beta\in\mathcal{B}} |\theta_\beta'(x)|^s f(\theta_\beta(x)).$$

Recall (see Lemmas 7.3 and 7.4) that we define $\mathfrak{a}_0 = 0$, $\mathfrak{b}_0 = 1/\gamma$, $\mathfrak{a}_{k+1} = \theta_\Gamma(\mathfrak{b}_k)$, $\mathfrak{b}_{k+1} = \theta_\gamma(\mathfrak{a}_k)$, $\mathfrak{a}_\infty = \lim_{k\to\infty} \mathfrak{a}_k$, and $\mathfrak{b}_\infty = \lim_{k\to\infty} \mathfrak{b}_k$. Since $\theta_\beta([\mathfrak{a}_k, \mathfrak{b}_k]) \subset [\mathfrak{a}_k, \mathfrak{b}_k]$ for $0 \le k \le \infty$ and for all $\beta \in \mathcal{B}$, and since L_s has a strictly positive eigenvector on $[0, 1/\gamma]$, L_s induces a bounded linear operator

$$L_{s,[\mathfrak{a}_k,\mathfrak{b}_k]} : C([\mathfrak{a}_k, \mathfrak{b}_k]) \to C([\mathfrak{a}_k, \mathfrak{b}_k])$$

and $R(L_s) = R(L_{s,[\mathfrak{a}_k,\mathfrak{b}_k]})$, Various constants are optimized by working on $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, so we shall abuse notation and also use L_s to denote L_s as an operator on $C([\mathfrak{a}_\infty, \mathfrak{b}_\infty])$ (or, sometimes, $C([\mathfrak{a}_k, \mathfrak{b}_k])$). For a given positive integer r, we assume that (H3) is satisfied, but with $S := [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$.

Thus, $[a_i, b_i] \subset S$, $1 \le i \le I$, denote pairwise disjoint intervals that satisfy the conditions of (H3). Given positive integers N_i, $1 \le i \le I$, we write $h_i = (b_i - a_i)/N_i$. As in Section 3 (see (3.5)), we define mesh points $c^i_{j,k} \in [a_i, b_i]$, $1 \le i \le I$, $1 \le j \le N_i$, $0 \le k \le r$ and $T := \{c^i_{j,k}\}$ for i, j, k in the ranges given above. As in Section 3, if v_s is the positive eigenvector of L_s on S, $\mathcal{V}_s : \hat{S} := \bigcup_{i=1}^{I} [a_i, b_i] \to \mathbb{R}$ is the polynomial interpolant of v_s of degree $\le r$ on $[t^i_{j-1}, t^i_j]$ for $1 \le i \le I$, $1 \le j \le N_i$, so we have $\mathcal{V}_s(x) = v_s(x)$ for all $x \in T$.

Our general approach will be as follows: Given $s > 0$, we must find r, ν, M and h such that the conditions of Theorem 5.8 are satisfied. First, we choose a positive integer $r \ge 2$, where r is the piecewise polynomial degree in (3.6). Once r has been chosen, we select a positive integer ν such that (compare Remark 5.9)

$$c(\nu)[2\eta(r) r^2 \psi(r)] =: \kappa_1 < 1. \tag{8.1}$$

Here, $c(\nu)$ is as in (5.3), and for our case an exact formula for $c(\nu)$ is provided by (7.4), where we shall take $\mathfrak{a}_k = \mathfrak{a}_\infty$ in (7.4). Also, $\psi(r)$ is as in Lemma 5.4 and $\eta(r)$ as in (5.2). As a practical matter, we demand that κ_1 not be too close to 1, say $\kappa_1 \le 4/5$. Note that for fixed r, this means that ν must be sufficiently large and hence $c(\nu)$ sufficiently small, so that (8.1) is satisfied. We next choose κ_2 with $\kappa_1 < \kappa_2 < 1$ and κ_2 not too close to 1. A simple choice is $\kappa_2 = (1 + \kappa_1)/2$. We define (see Theorem 5.8), $M' = \kappa_2 M$. If we write $u := M\eta(r)h$, the conditions of Theorem 5.8 take the form

$$\psi(r) u \exp(u) < 1, \tag{8.2}$$

$$\frac{\kappa_1 \exp(u)}{1 - \psi(r) u \exp(u)} < \kappa_2 - \frac{s M_0(\nu)}{M}. \tag{8.3}$$

Here, $M_0(\nu)$ is as in (5.3), and in our case Lemma 7.9 ensures that

$$M_0(\nu) \le \frac{2}{\mathfrak{a}_\infty + \mathfrak{a}_{\nu-1} + \gamma}.$$

Since $\exp(u)/(1-\psi(r)u\exp(u)) > 1$, (8.3) implies that

$$\kappa_1 < \kappa_2 - \frac{sM_0(\nu)}{M}.$$

We choose $M > 0$ such that

$$M = \frac{4s}{\mathfrak{a}_\infty + \mathfrak{a}_{\nu-1} + \gamma}\frac{1}{\kappa_2 - \kappa_1} \geq \frac{2sM_0(\nu)}{\kappa_2 - \kappa_1}, \tag{8.4}$$

which implies that

$$\kappa_2 - \frac{sM_0(\nu)}{M} \geq \kappa_2 - \frac{\kappa_2 - \kappa_1}{2} > \kappa_1.$$

Also note that since $\mathfrak{a}_\infty + \mathfrak{a}_{\nu-1} + \gamma < \chi$, we have that

$$M \geq \frac{4s}{\chi}\frac{1}{\kappa_2 - \kappa_1}. \tag{8.5}$$

Given an M that satisfies (8.4), we can choose $h = \max_i h_i$ sufficiently small, i.e., $h \leq \mathfrak{h}_0$, that (8.2) and (8.3) are satisfied. Recall, however, that we also have to insure that the constant M, defined by (8.4) also satisfies $M \geq M_2$, where M_2 is as in Lemma 6.4 and M_1 is given by (6.10). As we shall see, this may require a further restriction on the size of h.

The constants M_1 and M_2 are defined in terms of $G_{r+1} := G$ (compare (6.7)), $\chi := 2\mathfrak{a}_\infty + \gamma$, s, and H (compare Lemma 6.4), and it is desirable to choose H to be small. The constant E in G_{r+1} is given by Remark 7.11 with $p = r + 1$. By using (6.6) in Section 7 and the estimates for E and χ in Lemmas 7.10 and 7.12, we find that we can write

$$G = E\exp\left(\frac{2sh}{\mathfrak{a}_\infty + \mathfrak{b}_\infty^{-1}}\right)\frac{1}{(r+1)!}\left[\frac{1}{2\cos\left(\frac{\pi}{2r+2}\right)}\right]^{r+1}\frac{1}{2^r},$$

where

$$E := \frac{(2s)(2s+1)\cdots(2s+r)}{(\mathfrak{a}_\infty + \mathfrak{b}_\infty^{-1})^{r+1}} = \frac{(2s)(2s+1)\cdots(2s+r)}{(2\mathfrak{a}_\infty + \gamma)^{r+1}}, \tag{8.6}$$

and we have used the fact that $\mathfrak{b}_\infty = 1/(\mathfrak{a}_\infty + \gamma)$. Note that in the application of (7.14) for E, one must take $j = r + 1$. A calculation gives

$$G := G_{r+1} = 2\exp\left(\frac{2sh}{2\mathfrak{a}_\infty + \gamma}\right)\left[\frac{(2s)(2s+1)\cdots(2s+r)}{2\cdot 4\cdot 6\cdots(2r+2)}\right] \cdot \left[\frac{1}{2\mathfrak{a}_\infty + \gamma}\right]^{r+1}\left[\frac{1}{2\cos\left(\frac{\pi}{2r+2}\right)}\right]^{r+1}. \tag{8.7}$$

Finally, we define

$$D := D_{r+1} = \left[\frac{1}{\sin\left(\frac{\pi}{2r+2}\right)}\right]^2 G_{r+1}, \tag{8.8}$$

and

$$H_{r+1} = \mu D_{r+1}\frac{\chi}{2s} = \mu\left[\frac{1}{\sin\left(\frac{\pi}{2r+2}\right)}\right]^2 G_{r+1}\frac{(2\alpha_\infty+\gamma)}{2s}. \tag{8.9}$$

To estimate M_1 and M_2, we shall need estimates on these quantities.

Lemma 8.1. *Assume that $r \geq 2$, $0 < s < 2$ and $2\alpha_\infty + \gamma \geq 1$. Then, G_r is a decreasing function of r.*

Proof. For $r \geq 2$ and $0 < s \leq 2$, since $2j + 2 \geq 2s + j$ for $j \geq 2$, it follows that

$$\prod_{j=0}^{r}\frac{2s+j}{2j+2} \leq \left(\frac{2s}{2}\right)\left(\frac{2s+1}{4}\right) = s\frac{2s+1}{4}.$$

By using the Taylor series for $\cos(\theta)$, we see that

$$\frac{1}{2\cos\left(\frac{\pi}{2r+2}\right)} \leq \frac{1}{2-\left[\frac{\pi}{2r+2}\right]^2}.$$

Using these estimates in (8.7) gives

$$G_{r+1} \leq 2\exp\left(\frac{2sh}{2\alpha_\infty+\gamma}\right)\left[s\frac{2s+1}{4}\right]\left[\frac{1}{2\alpha_\infty+\gamma}\right]^{r+1}\left[\frac{1}{2-\left[\frac{\pi}{2r+2}\right]^2}\right]^{r+1},$$

which implies that $\lim_{r\to\infty} G_{r+1} = 0$. Furthermore, for $r \geq 2$ and $2\alpha_\infty + \gamma \geq 1$, another calculation shows that

$$\frac{G_{r+1}}{G_r} = \left[\frac{2s+r}{2r+2}\right]\left[\frac{1}{2\alpha_\infty+\gamma}\right]\left[\frac{1}{2\cos\left(\frac{\pi}{[2r+2]}\right)}\right]\left[\frac{\cos\left(\frac{\pi}{2r}\right)}{\cos\left(\frac{\pi}{2r+2}\right)}\right]^r < 1,$$

so G_r is a decreasing function for integer $r \geq 2$. ∎

If we set

$$u_{r+1} = \frac{1}{\chi}\left[\frac{1}{2\cos\left(\frac{\pi}{2r+2}\right)}\right],$$

a calculation gives

$$u_3 = \frac{1}{\chi\sqrt{3}} \quad\text{and}\quad u_4 = \frac{1}{\chi\sqrt{2+\sqrt{2}}},$$

which gives

$$G_3 = 2s\Big[\frac{1}{\chi\sqrt{3}}\Big]^3\Big[\frac{2s+1}{4}\Big]\Big[\frac{s+1}{3}\Big]\exp\Big(\frac{2sh}{\chi}\Big), \tag{8.10}$$

$$G_4 = 2s\Big[\frac{1}{2+\sqrt{2}}\Big]^2\Big[\frac{1}{\chi}\Big]^4\Big[\frac{2s+1}{4}\Big]\Big[\frac{s+1}{3}\Big]\Big[\frac{2s+3}{8}\Big]\exp\Big(\frac{2sh}{\chi}\Big),$$

$D_3 = 4G_3$ and $D_4 = [4/(2-\sqrt{2})]G_4$. Then, we have the following.

Lemma 8.2. *If* $0 \le s \le 3/2$, $r \ge 2$ *and* $\chi := 2\alpha_\infty + \gamma \ge 1$, *then*

$$\frac{G_{r+2}}{G_{r+1}} \le \Big[\frac{3}{4\sqrt{2+\sqrt{2}}}\Big]\frac{1}{\chi}, \quad \textit{so} \quad G_{r+2} \le \Big[\frac{3}{4\sqrt{2+\sqrt{2}}}\Big(\frac{1}{\chi}\Big)\Big]^{r-1} G_3.$$

Proof. A calculation gives

$$\frac{G_{r+2}}{G_{r+1}} = \Big[\frac{2s+r+1}{2r+4}\Big]u_{r+2}\Big[\frac{u_{r+2}}{u_{r+1}}\Big]^{r+1}.$$

Since $0 < u_{r+2} < u_{r+1}$, it follows that

$$\frac{G_{r+2}}{G_{r+1}} \le \Big[\frac{2s+r+1}{2r+4}\Big]\frac{1}{\chi}\Big[\frac{1}{2\cos\big(\frac{\pi}{2r+4}\big)}\Big],$$

and if $r \ge 2$ and $s \le 3/2$,

$$\frac{G_{r+2}}{G_{r+1}} \le \Big[\frac{3}{4\chi}\Big]\Big[\frac{1}{2\cos\big(\frac{\pi}{8}\big)}\Big] \le \Big[\frac{3}{4[\sqrt{2+\sqrt{2}}]}\Big]\frac{1}{\chi}$$

and so

$$G_{r+2} \le \Big[\frac{3}{4[\sqrt{2+\sqrt{2}}]}\Big(\frac{1}{\chi}\Big)\Big]^{r-1} G_3.$$

Furthermore, since we assume that $\chi \ge 1$,

$$G_3 = \Big[\frac{2s}{\chi}\Big]\Big[\frac{1}{\chi^2 3\sqrt{3}}\Big]\Big[\frac{2s+1}{4}\Big]\Big[\frac{s+1}{3}\Big]\exp\Big(\frac{2sh}{\chi}\Big) \tag{8.11}$$

$$\le \Big[\frac{2s}{\chi^3}\Big]\Big[\frac{5}{18\sqrt{3}}\Big]\exp(3h) \le \sqrt{3}/2, \tag{8.12}$$

for $h \le \ln(9/5)/3 \le 0.2$. ∎

Lemma 8.3. *Assume* $0 \le s \le 3/2$, $r \ge 2$, $\chi \ge 1$ *and* D_{r+1} *is as in* (8.8). *Then,*

$$D_{r+2} \le \Big[\frac{3}{4\chi}\Big]^{r-1} D_3.$$

Proof. A calculation gives

$$\frac{D_{r+2}}{D_{r+1}} = \left[\frac{u_{r+2}}{u_{r+1}}\right]^{r+1} u_{r+2} \left[\frac{\sin\left(\frac{\pi}{[2r+2]}\right)}{\sin\left(\frac{\pi}{[2r+4]}\right)}\right]^2 \left[\frac{2s+r+1}{2r+4}\right]$$
$$\leq \frac{1}{\chi}\left[\frac{1}{2\cos\left(\frac{\pi}{2r+4}\right)}\right]\left[\frac{\sin\left(\frac{\pi}{[2r+2]}\right)}{\sin\left(\frac{\pi}{[2r+4]}\right)}\right]^2 \left[\frac{r+4}{2r+4}\right].$$

Using the Taylor series expansions for $\sin(u)$, $u \geq 0$, we have $\sin(u) \leq u$ and $\sin(u) \geq u - u^3/6$. Hence,

$$\frac{\sin\left(\frac{\pi}{[2r+2]}\right)}{\sin\left(\frac{\pi}{[2r+4]}\right)} \leq \left[\frac{2r+4}{2r+2}\right]\left[1 - \frac{1}{6}\left(\frac{\pi}{2r+4}\right)^2\right]^{-1}.$$

Noting that for $r \geq 2$, the expression on the right hand side of the above is a decreasing function of r, as are the other two functions of r in the bound for D_{r+2}/D_{r+1}, we see that an upper bound for D_{r+2}/D_{r+1} is obtained by setting $r = 2$ in each of the expressions above. This gives

$$\frac{D_{r+2}}{D_{r+1}} \leq \left[\frac{1}{2\chi\cos\left(\frac{\pi}{8}\right)}\right]\left[\frac{4}{3}\right]^2\left[1 - \frac{1}{6}\left(\frac{\pi}{8}\right)^2\right]^{-2}\left[\frac{3}{4}\right] \leq \frac{3}{4\chi},$$

and so

$$D_{r+2} \leq \left[\frac{3}{4\chi}\right] D_{r+1} \leq \left[\frac{3}{4\chi}\right]^{r-1} D_3. \qquad \blacksquare$$

The following bound on H_r is a direct consequence of the above estimates.

Lemma 8.4. *Assume* $0 \leq s \leq 3/2$, $r \geq 2$. *Then,*

$$H_{r+2} \leq \mu\left[\frac{\chi}{2s}\right]\left[\frac{3}{4\chi}\right]^{r-1} 4G_3.$$

Lemma 8.5. *Suppose* $0 \leq s \leq 3/2$, $r \geq 2$, M_1 *is as in* (6.10), M_2 *is as in Lemma* 6.4, *and* $H_{r+1} = \mu D_{r+1}(\chi/2s)$. *Then,*

$$M_1 = \frac{\mu D_{r+1} h^r}{1 - G_{r+1}^2 h^{2r+2}} + \frac{2s}{\chi},$$

and

$$M_2 = M_1 + \left[\frac{2s}{\chi}\right]\frac{1}{1 - \left[\frac{G_{r+1}}{H_{r+1}} h\right]^2} \tag{8.13}$$
$$= \frac{2s}{\chi}\left[1 + \frac{H_{r+1} h^r}{1 - G_{r+1}^2 h^{2r+2}} + \left(1 - h^2 \frac{2s}{\mu\chi}\sin^2\left(\frac{\pi}{[2r+2]}\right)\right)^{-1}\right]. \tag{8.14}$$

Proof. Applying the definitions of M_1, M_2, D_{r+1}, G_{r+1} and H_{r+1}, we get

$$\begin{aligned} M_2 &= M_1 + \Big[\frac{\mu G_{r+1}}{H_{r+1}}\Big]\Bigg[\frac{1}{1-[\frac{G_{r+1}}{H_{r+1}}h]^2}\Bigg]\Bigg[\frac{1}{[\sin(\frac{\pi}{[2r+2]})]^2}\Bigg] \\ &= M_1 + \Big[\frac{2s}{\chi}\Big]\frac{1}{1-[\frac{G_{r+1}}{H_{r+1}}h]^2} \\ &= \frac{2s}{\chi} + \frac{\mu D_{r+1}h^r}{1-G_{r+1}^2 h^{2r+2}} + \Big[\frac{2s}{\chi}\Big]\frac{1}{1-[\frac{G_{r+1}}{H_{r+1}}h]^2} \\ &= \frac{2s}{\chi} + \Big[\frac{2s}{\chi}\Big]\frac{H_{r+1}h^r}{1-G_{r+1}^2 h^{2r+2}} + \Big[\frac{2s}{\chi}\Big]\frac{1}{1-(\frac{2s}{[\mu\chi]})[\sin(\frac{\pi}{[2r+2]})h]^2} \\ &= \frac{2s}{\chi}\Big[1 + \frac{H_{r+1}h^r}{1-G_{r+1}^2 h^{2r+2}} + \Big(1 - h^2\frac{2s}{\mu\chi}\sin^2\Big(\frac{\pi}{[2r+2]}\Big)\Big)^{-1}\Big]. \qquad \blacksquare \end{aligned}$$

Remark 8.6. Note that D_{r+1} has the factor $2s/\chi$, so the identity (8.13) for M_2 does not blow up as $s \to 0$.

Remark 8.7. If we replace in G_3 and D_3, the quantity $\exp(2sh/\chi)$ by $\exp(2s\mathfrak{h}_0/\chi)$, where $\mathfrak{h}_0$ is chosen so that (8.2) and (8.3) are satisfied for $0 < h \le \mathfrak{h}_0$, then we easily obtain a bound for M_2 in the form

$$M_2 \le \frac{2s}{\chi}\Big[2 + H_{r+1}h^r(1+\tilde{c}h^{2r+2}) + ch^2\Big], \tag{8.15}$$

where $\tilde{c}$ and c are easily computable from (8.13). Note that if $0 \le s \le 3/2$, $\chi = 2\alpha_\infty + \gamma \ge 1$ and $r \ge 2$, then $\chi \ge 1$ and $\sin^2(\pi/[2r+2]) \le 1/4$. So, for $h \le 1/\sqrt{3}$,

$$\Big(1 - h^2\frac{2s}{\mu\chi}\sin^2\Big(\frac{\pi}{[2r+2]}\Big)\Big)^{-1} \le \Big(1 - \frac{3h^2}{4}\Big)^{-1} \le 1 + h^2,$$

i.e., we can take $c = 1$. Similarly, for $h \le 0.2$, $G_{r+1}^2 \le 3/4$, so we can also take $\tilde{c} = 1$.

Recall that we have to insure that $M > M_2$. From (8.5), we have that

$$M \ge \Big[\frac{4s}{\chi}\Big]\frac{1}{\kappa_2 - \kappa_1}.$$

Comparing this expression to the bound for M_2 given in (8.15) and noting that $\kappa_2 - \kappa_1 < 1$, M will be $> M_2$ if we choose $h \le \mathfrak{h}_1$ sufficiently small so that it also satisfies

$$2 + H_{r+1}h^r(1 + h^{2r+2}) + h^2 \le \frac{2}{\kappa_2 - \kappa_1}. \tag{8.16}$$

We can now state versions of Theorem 6.5 and Theorem 6.6 in the context of this section.

Theorem 8.8. *Assume that $r \geq 2$, $\chi \geq 1$ and $0 \leq s \leq 3/2$, and let ν, κ_1 and κ_2 be as described at the beginning of this section. Let M_2 and H_{r+1} be as described in Lemma 8.5 and select M such that (8.4) is satisfied. Finally, assume that $h = \max_{i \in I} h_i \leq \min(\mathfrak{h}_0, \mathfrak{h}_1, 0.2)$. Then, with $u = M\eta(r)h$, (8.2) and (8.3) are satisfied and $M > M_2$. Furthermore, $\mathbf{L}_{s,\nu}(K(M;T)) \subset K(M';T)$, where $M' = \kappa_2 M < M$ (compare Theorem 6.5). In addition, we have that for $H = H_{r+1}$,*

$$\lambda_s^{\nu}(1 - Hh^r) \leq R(\mathbf{L}_{s,\nu}) \leq \lambda_s^{\nu}(1 + Hh^r).$$

Proof. The fact that $M_2 > M$ follows directly from the computations above. Our selection of r, ν, h, M and $M' = \kappa_2 M$ shows that the inequality in Theorem 5.2 is satisfied, so $\mathbf{L}_{s,\nu}(K(M;T)) \subset K(M';T)$. The inequality for $R(\mathbf{L}_{s,\nu})$ in Theorem 8.8 follows directly from Theorem 6.5. ∎

Theorem 8.9. *Under the hypotheses of Theorem 8.8, we have*

$$[(1 + Hh^r)^{-1} R(\mathbf{L}_{s,\nu})]^{1/\nu} \leq \lambda_s \leq [(1 - Hh^r)^{-1} R(\mathbf{L}_{s,\nu})]^{1/\nu},$$

where the entries of the matrices $[1 + Hh^r]^{-1}\mathbf{L}_{s,\nu}$ and $[1 - Hh^r]^{-1}\mathbf{L}_{s,\nu}$ differ by $O(h^r)$.

Using the inequalities of Theorem 8.8, we can obtain rigorous upper and lower bounds on the Hausdorff dimension s_* of the invariant set associated with the transfer operator L_s as follows. Let s_l and s_u denote values of s satisfying

$$(1 - Hh^r)^{-1} R(\mathbf{L}_{s_u,\nu}) < 1 \quad \text{and} \quad (1 + Hh^r)^{-1} R(\mathbf{L}_{s_l,\nu}) > 1,$$

respectively. It follows immediately from Theorem 8.8 that $\lambda_{s_u}^{\nu} < 1$ and $\lambda_{s_l}^{\nu} > 1$. Since the spectral radius λ_s of L_s is a decreasing function of s, there will be a value s_* satisfying $s_l < s_* < s_u$ for which $\lambda_{s_*}^{\nu} = 1$, or equivalently $\lambda_{s_*} = 1$. The value s_* gives the Hausdorff dimension s_* of the invariant set associated with the transfer operator L_s.

9. Numerical computations

In this final section, we present results of computations of the Hausdorff dimension s for various choices of sets of continued fractions, maximum mesh size h, piecewise polynomial degree r, and number of iterations ν of the map, where $\nu = 1$ corresponds to the original map. These computations include choices of the above parameters (especially the number of iterations ν), for which the hypotheses of our theorems are satisfied, but also computations which obtain the same results when the mappings are not iterated (denoted by $\nu = 1^*$ in Table 1 below). We note the obvious fact that as the number of iterations increase, the complexity of the operator increases, so the time to compute the maximum eigenvalue of the resulting matrix operator increases as well. To keep the time involved in the various computations to be reasonable, we have elected to compute to fewer digits those sets with a large value of ν, especially if the set E contains many digits. We have also done additional computations, not reported in Table 1, which indicate that the method is robust with respect to the choices of h, r and ν.

Set E	r	h	ν
	s		
$E[1,2]$	14	0.0002	7
	0.531 280 506 277 205 141 624 468 647 368 471 785 493 059 109 018 398		
$E[1,3]$	8	5.0e-05	6
	0.454 489 077 661 828 743 845 777 611 651		
$E[1,4]$	8	5.0e-05	6
	0.411 182 724 774 791 776 844 805 904 696		
$E[2,3]$	8	5.0e-05	3
	0.337 436 780 806 063 636 304 494 910 387		
$E[2,4]$	8	5.0e-05	3
	0.306 312 768 052 784 030 277 908 307 445		
$E[3,4]$	8	5.0e-05	3
	0.263 737 482 897 426 558 759 863 384 275		
$E[3,7]$	18	0.01	3
	0.224 923 947 191 778 989 184 480 593 490		
$E[10,11]$	20	0.002	2
	0.146 921 235 390 783 463 311 108 628 515 904 073 067 083 129 676 755		
$E[10^2,10^4]$	20	0.002	1
	0.052 246 592 638 658 878 652 588 416 300 508 181 012 676 284 431 681		
$E[1,2,3]$	5	0.0001	5
	0.705 660 908 028 738		
$E[1,3,4]$	5	0.0001	5
	0.604 242 257 756 515		
$E[1,3,5]$	8	0.001	6
	0.581 366 821 182 975		
$E[1,4,7]$	6	.001	6
	0.517 883 757 006 911		

continued on next page

$E[2,3,4]$	16	0.005	4	
	0.480 696 222 317 573 041 322 515 564 711			
$E[1,2,3,4]$	8	0.005	6	
	0.788 945 557 483 153			
$E[2,3,4,5]$	16	0.005	4	
	0.559 636 450 164 776 713 312 144 913 530			
$E[1,2,3,4,5]$	5	0.0005	5	
	0.836 829 443 681 209			
$E[2,4,6,8,10]$	7	0.005	3	
	0.517 357 030 937 017			
$E[1,\dots,10]$	10	0.01	1*	
	0.925 737 591 146 765			
$E[1,\dots,34]$	10	0.01	1*	
	0.980 419 625 226 980			
$E[1,3,5,\dots,33]$	10	0.01	1*	
	0.770 516 008 717 163			
$E[2,4,6,\dots,34]$	10	0.01	1*	
	0.633 471 970 241 089			

Table 1. Computation of Hausdorff dimension s for various choices of sets of continued fractions, maximum mesh size h, piecewise polynomial degree r, and number of iterations ν.

In addition to the results presented in Table 1, we have also used our method to compute the Hausdorff dimension of the set $E[1,2]$ with degree $r = 36$, $h = 0.001$ and $\nu = 1$ using multiple precision with 108 digits. Although this choice does not satisfy the hypotheses of our theorem, the result agrees to 100 decimal places with the result in [26]. While we do not have a proof that our method also works in the non-iterated situation, we do not have any examples where it fails. We conjecture that the limitation is the method of proof and not the underlying method.

We next discuss how to control the size of some of the constants that appear in the error estimates. We begin with the constant $\mu = \max_i h_i / \min_i h_i$. Recall that to satisfy hypothesis (H3), for a given positive integer ν, we need to determine pairwise disjoint, nonempty compact intervals $[a_i, b_i] \subset S$, $1 \le i \le I$, such that for every $\omega \in \Omega_\nu$, there exists $i = i(\omega)$, $1 \le i \le I$, such that $\theta_\omega(S) \subset [a_i, b_i]$. By the results in the previous section, we can take $S = [\mathfrak{a}_\infty, \mathfrak{b}_\infty]$ and we then order the $\omega \in \Omega_\nu$, such that the sets $\theta_\omega(S)$ are ordered with $a_i < a_{i+1}$. Although we could use the domain consisting of the union of the sets $\theta_\omega(S)$, this can lead to very small subinterval sizes. Instead, we determine a new domain by iterating only ν' times, while still using the mappings obtained by ν iterations to calculate the mapping L. The constant ν' is determined so that the length of the smallest interval will be greater than or equal to h_{max}/μ.

Although we have not done the computations using interval arithmetic, we have only included the number of digits in each computation that we expect to be correct, which is always less than the number of digits provided by *Matlab* for the precision we have specified. For computations that use more digits than provided by standard *Matlab* computations, we have used the *Advanpix* multiprecision toolbox.

Because the theory developed in the previous sections involves the computation and estimates for many constants and parameters, we next provide, for the benefit of the reader, details for the specific example of $E[1,4,7]$, corresponding to the computation for $r = 6$, $h = 0.001$, $\nu = 6$, and $s = 0.518$, shown in Table 1. The computational domain is determined by only iterating $\nu' = 2$ times and the total number of subintervals used is 191. With these choices, $\gamma = 1$ and $\Gamma = 7$. Then, by (7.1),

$$\mathfrak{a}_\infty = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 + \frac{\gamma}{\Gamma}} \approx 0.127 \quad \text{and} \quad \mathfrak{b}_\infty = \frac{\Gamma}{\gamma}\mathfrak{a}_\infty \approx 0.8875.$$

Working on the interval $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, we have $\chi = \mathfrak{a}_\infty + \mathfrak{b}_\infty^{-1} = 2\mathfrak{a}_\infty + \gamma \approx 1.25$. From (7.5)–(7.7), we have that for $\nu = 6$,

$$c(\nu) = [\tilde{B}_{\nu-1}\mathfrak{a}_\infty + \tilde{B}_\nu]^{-2} \approx 0.0051.$$

From (5.2), we have that $\eta(r) = 1/2$ and from Lemma 5.4 that

$$\psi(r) = \frac{2}{\pi}\ln(r+1) + \frac{3}{4} \approx 2.0.$$

From (7.8), since we are working on the interval $[\mathfrak{a}_\infty, \mathfrak{b}_\infty]$, we take

$$M_0(\nu) = \frac{2}{\mathfrak{a}_\infty + \mathfrak{b}_\nu^{-1}} = \frac{2}{\mathfrak{a}_\infty + \mathfrak{a}_{\nu-1} + \gamma} \approx 1.5954.$$

Setting

$$\kappa_1 = c(\nu)2\eta(r)r^2\psi(r) \approx 0.364 < 1,$$

we choose $\kappa_2 = (1+\kappa_1)/2 \approx 0.6823$, so that $\kappa_1 < \kappa_2 < 1$. Next, we choose

$$M = \frac{4s}{\mathfrak{a}_\infty + \mathfrak{a}_{\nu-1} + \gamma}\frac{1}{\kappa_2 - \kappa_1} \approx 5.2022,$$

and $M' = \kappa_2 M$. One of the conditions on the mesh size $h = \max_{i\in I} h_i$ is that it is sufficiently small so that (8.2) and (8.3) are satisfied. For our example, setting

$u = M\eta(r)h$, we have

$$\psi(r)u\exp(u) \approx 0.0052 < 1,$$
$$\frac{\kappa_1 \exp(u)}{1-\psi(r)u\exp(u)} \approx 0.3674 < 0.5234 \approx \kappa_2 - \frac{sM_0(\nu)}{M}.$$

The second condition, coming from (8.16), is that

$$2 + H_{r+1}(h^r + h^{2r+2}) + h^2 \le \frac{2}{\kappa_2 - \kappa_1}, \tag{9.1}$$

where, combining (8.8), (8.9) and Lemma 8.3, we have

$$H_{r+1} \le \mu\frac{\chi}{2s}D_{r+1} \le \mu\frac{\chi}{2s}\Big[\frac{3}{4\chi}\Big]^{r-2}D_3 \le \mu\frac{\chi}{2s}\Big[\frac{3}{4\chi}\Big]^{r-2}4G_3,$$

where

$$G_3 = 2s\Big[\frac{1}{\chi\sqrt{3}}\Big]^3\Big[\frac{2s+1}{4}\Big]\Big[\frac{s+1}{3}\Big]\exp\Big(\frac{2sh}{\chi}\Big).$$

Combining these results and setting μ, the ratio of the maximum subinterval size to the minimum subinterval size, to be equal to 3.76, the value calculated by the computer code, we get

$$H_{r+1} \le 0.0608.$$

Since $2/(\kappa_2 - \kappa_1) \ge 6$, it is clear that (9.1) is satisfied.

We thus have satisfied the conditions of Theorems 8.8 and 8.9, which guarantee the rigorous bounds we use to obtain rigorous upper and lower bounds on the Hausdorff dimension of the set $E[1,4,7]$.

References

[1] O. F. Bandtlow and J. Slipantschuk, *Lagrange approximation of transfer operators associated with holomorphic data*. 2020, arXiv:2004.03534

[2] G. Birkhoff, Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.* **85** (1957), 219–227 Zbl 0079.13502 MR 87058

[3] J. Bourgain and A. Kontorovich, On Zaremba's conjecture. *Ann. of Math. (2)* **180** (2014), no. 1, 137–196 Zbl 1428.11128 MR 3194813

[4] R. Bowen, Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* (1979), no. 50, 11–25 Zbl 0439.30032 MR 556580

[5] L. Brutman, On the Lebesgue function for polynomial interpolation. *SIAM J. Numer. Anal.* **15** (1978), no. 4, 694–704 Zbl 0391.41002 MR 510554

[6] R. T. Bumby, Hausdorff dimensions of Cantor sets. *J. Reine Angew. Math.* **331** (1982), 192–206 Zbl 0468.10032 MR 647383

[7] R. T. Bumby, Hausdorff dimension of sets arising in number theory. In *Number theory (New York, 1983–84)*, pp. 1–8, Lecture Notes in Math. 1135, Springer, Berlin, 1985 Zbl 0575.28004 MR 803348

[8] V. Chousionis, D. Leykekhman, and M. Urbański, On the dimension spectrum of infinite subsystems of continued fractions. *Trans. Amer. Math. Soc.* **373** (2020), no. 2, 1009–1042 Zbl 1437.37032 MR 4068257

[9] T. W. Cusick, Continuants with bounded digits. *Mathematika* **24** (1977), no. 2, 166–172 Zbl 0371.10023 MR 472721

[10] T. W. Cusick, Continuants with bounded digits. II. *Mathematika* **25** (1978), no. 1, 107–109 Zbl 0388.10021 MR 498413

[11] S. P. Eveson and R. D. Nussbaum, Applications of the Birkhoff-Hopf theorem to the spectral theory of positive linear operators. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 3, 491–512 Zbl 0834.47029 MR 1317492

[12] S. P. Eveson and R. D. Nussbaum, An elementary proof of the Birkhoff-Hopf theorem. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 1, 31–55 Zbl 0834.47028 MR 1297895

[13] K. Falconer, *Techniques in fractal geometry*. John Wiley & Sons, Ltd., Chichester, 1997 Zbl 0869.28003 MR 1449135

[14] R. S. Falk and R. D. Nussbaum, C^m eigenfunctions of Perron-Frobenius operators and a new approach to numerical computation of Hausdorff dimension: applications in $\mathbb{R}^1$. *J. Fractal Geom.* **5** (2018), no. 3, 279–337 Zbl 1436.37031 MR 3827801

[15] I. J. Good, The fractional dimensional theory of continued fractions. *Proc. Cambridge Philos. Soc.* **37** (1941), 199–228 Zbl 67.0988.03 MR 4878

[16] R. Günttner, Evaluation of Lebesgue constants. *SIAM J. Numer. Anal.* **17** (1980), no. 4, 512–520 Zbl 0439.41004 MR 584726

[17] S.-M. Heinemann and M. Urbański, Hausdorff dimension estimates for infinite conformal IFSs. *Nonlinearity* **15** (2002), no. 3, 727–734 Zbl 1015.37019 MR 1901102

[18] D. Hensley, The Hausdorff dimensions of some continued fraction Cantor sets. *J. Number Theory* **33** (1989), no. 2, 182–198 Zbl 0689.10060 MR 1034198

[19] D. Hensley, Continued fraction Cantor sets, Hausdorff dimension, and functional analysis. *J. Number Theory* **40** (1992), no. 3, 336–358 Zbl 0745.28005 MR 1154044

[20] D. Hensley, A polynomial time algorithm for the Hausdorff dimension of continued fraction Cantor sets. *J. Number Theory* **58** (1996), no. 1, 9–45 Zbl 0858.11039 MR 1387719

[21] E. Hopf, An inequality for positive linear integral operators. *J. Math. Mech.* **12** (1963), 683–692 Zbl 0115.32501 MR 0165325

[22] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747 Zbl 0598.28011 MR 625600

[23] O. Jenkinson, On the density of Hausdorff dimensions of bounded type continued fraction sets: the Texan conjecture. *Stoch. Dyn.* **4** (2004), no. 1, 63–76 Zbl 1089.28006 MR 2069367

[24] O. Jenkinson and M. Pollicott, Computing the dimension of dynamically defined sets: E_2 and bounded continued fractions. *Ergodic Theory Dynam. Systems* **21** (2001), no. 5, 1429–1445 Zbl 0991.28009 MR 1855840

[25] O. Jenkinson and M. Pollicott, Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets. *Amer. J. Math.* **124** (2002), no. 3, 495–545 Zbl 1002.37023 MR 1902887

[26] O. Jenkinson and M. Pollicott, Rigorous effective bounds on the Hausdorff dimension of continued fraction Cantor sets: a hundred decimal digits for the dimension of E_2. *Adv. Math.* **325** (2018), 87–115 Zbl 1386.30039 MR 3742587

[27] M. A. Krasnosel'skij, J. A. Lifshits, and A. V. Sobolev, *Positive linear systems*. Sigma Series in Applied Mathematics 5, Heldermann, Berlin, 1989 Zbl 0674.47036 MR 1038527

[28] B. Lemmens and R. Nussbaum, *Nonlinear Perron-Frobenius theory*. Cambridge Tracts in Mathematics 189, Cambridge University Press, Cambridge, 2012 Zbl 1246.47001 MR 2953648

[29] B. Lemmens and R. Nussbaum, Birkhoff's version of Hilbert's metric and its applications in analysis. In *Handbook of Hilbert geometry*, pp. 275–303, IRMA Lect. Math. Theor. Phys. 22, Eur. Math. Soc., Zürich, 2014 Zbl 1310.51001 MR 3329884

[30] A. A. Markov, *On a question by D. I. Mendeleev*. Zap. Imp. Akad. Nauk SPb **62** (1890), 1–24

[31] R. D. Mauldin and M. Urbański, Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc. (3)* **73** (1996), no. 1, 105–154 Zbl 0852.28005 MR 1387085

[32] R. D. Mauldin and M. Urbański, *Graph directed Markov systems*. Cambridge Tracts in Mathematics 148, Cambridge University Press, Cambridge, 2003 Zbl 1033.37025 MR 2003772

[33] R. D. Nussbaum, The radius of the essential spectrum. *Duke Math. J.* **37** (1970), 473–478 Zbl 0216.41602 MR 264434

[34] R. D. Nussbaum, C^m positive eigenvectors for linear operators arising in the computation of Hausdorff dimension. *Integral Equations Operator Theory* **84** (2016), no. 3, 357–393 Zbl 1341.47013 MR 3463454

[35] R. D. Nussbaum, A. Priyadarshi, and S. Verduyn Lunel, Positive operators and Hausdorff dimension of invariant sets. *Trans. Amer. Math. Soc.* **364** (2012), no. 2, 1029–1066 Zbl 1280.37029 MR 2846362

[36] T. J. Rivlin, *Chebyshev polynomials. From approximation theory to algebra and number theory. Corrected reprint of the 1990 2nd edition.* Dover Publications, Mineola, New York, 2020 Zbl 1458.41001 MR 1060735

[37] D. Ruelle, *Thermodynamic formalism*. Encyclopedia of Mathematics and its Applications 5, Addison-Wesley Publishing Co., Reading, Mass., 1978 Zbl 0401.28016 MR 511655

[38] D. Ruelle, Bowen's formula for the Hausdorff dimension of self-similar sets. In *Scaling and self-similarity in physics (Bures-sur-Yvette, 1981/1982)*, pp. 351–358, Progr. Phys. 7, Birkhäuser Boston, Boston, MA, 1983 MR 733478

[39] H. H. Rugh, On the dimensions of conformal repellers. Randomness and parameter dependency. *Ann. of Math. (2)* **168** (2008), no. 3, 695–748 Zbl 1181.37066 MR 2456882

[40] H. Samelson, On the Perron-Frobenius theorem. *Michigan Math. J.* **4** (1957), 57–59 Zbl 0077.02303 MR 86041

[41] A. Schief, Self-similar sets in complete metric spaces. *Proc. Amer. Math. Soc.* **124** (1996), no. 2, 481–490 Zbl 0844.28004 MR 1301047

[42] A. Shadrin, Twelve proofs of the Markov inequality. In *Approximation theory: a volume dedicated to Borislav Bojanov*, pp. 233–298, Prof. M. Drinov Acad. Publ. House, Sofia, 2004 MR 2117091

[43] B.-S. Tam, A cone-theoretic approach to the spectral theory of positive linear operators: the finite-dimensional case. *Taiwanese J. Math.* **5** (2001), no. 2, 207–277 Zbl 0990.15009 MR 1832167

[44] J. S. Vandergraft, Spectral properties of matrices which have invariant cones. *SIAM J. Appl. Math.* **16** (1968), 1208–1222 Zbl 0186.05701 MR 244284

Received 25 August 2020; revised 27 February 2021.

Richard S. Falk
Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA;
falk@math.rutgers.edu

Roger D. Nussbaum
Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA;
nussbaum@math.rutgers.edu