J. Fractal Geom. 9 (2022), 1–23
DOI 10.4171/JFG/115

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Baker domains and non-convergent deformations

Rodrigo Robles and Guillermo Sienra

Abstract. For an entire transcendental function f with a non-completely invariant Baker domain U, we study the pinching process of paths in U with certain restrictions, that we call Baker laminations. We show that if some curve in the Baker lamination of f joins a point in the boundary of U with infinity, then the deformation does not converge. Thus, in this particular case, the boundary of the space of deformations of f is incomplete.

1. Introduction

This paper studies pinching deformations along certain paths contained in a Baker domain of an entire transcendental function.

Iterating an entire transcendental function f gives rise to a dynamical system which partitions the complex plane in two completely invariant sets: the *Fatou set* and the *Julia set*. The domain in $\mathbb{C}$ where the sequence of iterates $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family is called the Fatou set $F(f)$, and its complement is named the Julia set $J(f)$. The Fatou set is an open set, and the Julia set is a closed, perfect and uncountable set. If $F(f) \neq \emptyset$, the Julia set has no interior points, and both sets are unbounded in $\mathbb{C}$. The dynamics in the Julia set $J(f)$ is chaotic following Devaney [14]. See [6, 30] for a general explanation on the dynamics of these functions.

Let U be a connected component of $F(f)$, then $f^n(U) \subseteq F(f)$, and proper containment is possible. We say that U is *preperiodic* if there are $p > q \geq 0$ such that $f^p(U) \subseteq f^q(U)$; if $q = 0$, U is *p-periodic*. If the component U is not preperiodic, it is a *wandering domain.*

If U is a p-periodic Fatou component, we have the following classification for entire transcendental functions (see [6]):

(1) U is an *immediate attracting basin* of an attracting p-periodic point $z_0 \in U$ and $\lim_{n\to\infty} f^{np}(z) = z_0$ for every $z \in U$.
(2) U is a *parabolic basin* of a parabolic point $z_0 \in \partial U$ and $\lim_{n\to\infty} f^{np}(z) = z_0$ for every $z \in U$.

2020 Mathematics Subject Classification. Primary 37F10; Secondary 37F30, 37F45, 30D05.
Keywords. Entire functions, quasiconformal maps, holomorphic dynamics, Baker domains, laminations, pinching deformation.

(3) U is a *Siegel disk* where U is biholomorphic to $\mathbb{D}$, and $f^p|_U$ is analytically conjugated to an irrational rotation of the disk.

(4) U is a *Baker domain* where $\lim_{n\to\infty} f^{np}(z) = \infty$ for every $z \in U$.

A *completely invariant domain* U satisfies that $f^{-1}(U) = U$; particular things happen when this is the case (see [2,7,9,31]).

This article continues the research on deformations of holomorphic transcendental functions by pinching curves in Baker domains which started in [15]. The theory begins with deformations of rational maps by pinching attracting domains to parabolic domains (see [21,22,27,35]). In [15], it is proved that when we pinch certain entire transcendental function along some curves in a completely invariant Baker domain, the limit converges to another transcendental entire function. This can be interesting because the process of pinching defines a path in the deformation space of the function, so its limit, if it exists, describes a function at the boundary of such space. Sometimes such limit exists and sometimes it does not. For instance, if the paths where the pinching is taking place form a closed path which contains in its interior part of the Julia set, then the limit does not exist (see [27,35]); here, we obtain under the conditions stated in Theorem 2, that the limit of the pinching process does not exist, even if such closed path does not exist.

In this article, we show that if f is an entire transcendental map with a non-completely invariant Baker domain, there are deformations (by pinching) of f whose limit does not converge. In the next subsection, we define the curves where the pinching process is supported.

1.1. Baker laminations

In this context we introduce the following definition, which is a very natural setting for a pinching process.

Definition 1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function with a Baker domain U such that $f(U) = U$ and U is equipped with the hyperbolic metric, let Λ be a set of complete geodesics in U. We say that Λ is a *Baker lamination of* U, if the geodesics $\lambda \in \Lambda$, called *leaves* henceforth, satisfy:

(1) The leaves of the lamination do not accumulate in U.

(2) If $\lambda \in \Lambda$, then $f^n(\lambda) \in \Lambda$, with $n \in \mathbb{N}$. Also, $\lambda \subseteq U$ is in Λ, if $f^n(\lambda)$ is an element of Λ, for some $n > 0$.

(3) For any $\lambda, \lambda' \in \Lambda$, $\lambda \cap \lambda' = \emptyset$, when $\lambda \neq \lambda'$.

(4) For any $\lambda \in \Lambda$, there exists $\partial\lambda := \lim_{t\to\pm\infty} \lambda(t)$ and $\partial\lambda \subseteq \partial U \subseteq \widehat{\mathbb{C}}$.

The elements of the boundary $\partial\lambda$ are called *endpoints* and we write $\overline{\lambda} := \lambda \cup \partial\lambda$. Due to a theorem of Carathéodory, there is a dense subset in ∂U of points that are accessible from the interior of U. Also, ∞ is known to be always accessible from the interior of U. So, in every Baker domain, there is a geodesic that connects some point in ∂U with ∞ and there always exists some Baker lamination with a leaf connecting some point in ∂U with ∞. Figures 3.2–3.5 display some graphic examples of laminations in univalent Baker domains of Section 3.1.

Consider a cycle of Baker domains U with p-periodic components U_i, where $U = \bigcup_{i=0}^{p-1} U_i, f(U_i) = U_{i+1}$ if $i \neq p-1$ and $f(U_{p-1}) = U_0$, so $f^p(U_i) = U_i$. If Λ_0 is a Baker lamination in U_0 under f^p, we induce a Baker lamination in all the components U_i and in all U by defining $\Lambda_k = f^k(\Lambda_0)$, which is a Baker lamination in U_k and $\Lambda = \bigcup_{i=0}^{p-1} \Lambda_i$ is a Baker lamination in U.

Let us set $\mathcal{L} := \bigcup_{k\in\mathbb{N}} f^{-k}(\Lambda)$, the full orbit of Λ.

All these conditions were fulfilled in [15] to obtain the convergence of the maps in the limit of the pinching.

Geodesic laminations on surfaces where introduced by Thurston [36], precisely as a tool to deform Kleinian groups and in [34] to study dynamics of polynomials; since then their importance has been growing in geometry and dynamics.

1.2. Intuitive pinching process and results

In order to produce a deformation of a function in a complex variable, there is a well-known technique by means of quasiconformal maps, which for the sake of the reader, we explain in Section 2.2, however a nice geometric property of these kind of homeomorphisms $h : \mathbb{C} \to \mathbb{C}$ that preserve the orientation is the following (see, for instance, [20]): If Q is a quadrilateral on $\mathbb{C}$ and $M(Q)$ denotes its modulus, then

$$K(h) = \sup_{Q} \frac{M(h(Q))}{M(Q)} < \infty.$$

The value $K(h)$ is called the *quasiconformal dilatation* of h. This definition of a quasiconformal homeomorphism is equivalent to the one we give in Section 2.2, which is more useful to us. The equivalence is formally proven in [11].

As we mention above, in this article we consider paths of deformations of certain entire transcendental function f by contracting each leaf of a Baker lamination until it becomes a point in a pinching process, which we describe in detail in Section 4. The idea of the process is to endow each leaf $\lambda \in \Lambda$ of a lamination with a neighborhood $V_\delta(\lambda)$, so that all neighborhoods are disjoint. Let us define $V_\delta(\Lambda) = \bigcup_{\lambda\in\Lambda} V_\delta(\lambda)$ and consider $\mathcal{V} := \bigcup_{k\in\mathbb{N}} f^{-k}(V_\delta(\Lambda))$.

Then, we consider a family of quasiconformal maps h_t, $t \in [0, 1)$ supported in $\mathcal{V}$ with the property that h_0 is the identity and h_t deforms quadrilaterals in each $V_\delta(\lambda)$

by contracting along the leaf λ or equivalently, expanding along the transversals of λ. Then, for the entire maps $f_t = h_t \circ f \circ h_t^{-1}$, the result is that as $t \to 1$, each of the leaves of $\mathcal{L}$ get shorter and shorter until they collapse to a point (see Figure 4.2).

For an entire transcendental map f with a Baker domain U which is not completely invariant, our main theorem shows that if there is a curve λ_∞ in the Baker lamination of U that connects a point z_0 to ∞, then pinching along the lamination does not converge. More precisely, we prove the following result.

Theorem 2. *Let f be an entire transcendental map with a non-completely invariant Baker domain U. Consider a Baker lamination Λ, with a leaf λ_∞ having endpoints at z_0 and ∞, with z_0 a non-exceptional point in ∂U. Then, the pinching deformation along $\mathcal{L}$ does not converge.*

A class of functions satisfying the conditions of Theorem 2 are the known examples of Baker domains which are univalent and of hyperbolic type I, as defined in Section 2.1.

The next theorem states another situation where divergence of the pinching deformation occurs.

Theorem 3. *Let f be an entire transcendental map with a non-completely invariant Baker domain. Consider a Baker lamination $\mathcal{L}$, with a leaf λ_a having endpoints at non-exceptional points. If λ_a intersects the set of asymptotic values of f, then the pinching deformation along $\mathcal{L}$ does not converge.*

The structure of the paper is as follows. In Section 2, we review the properties of Baker domains and quasiconformal maps that we require. In Section 3, we define the collections of paths which are a very natural setting for the pinching process and that we call Baker laminations. These laminations are also going to be relevant in a forthcoming paper. This section also treats the case of Baker laminations in univalent Baker domains, which are the main examples for this situation. In Section 4, we review the technique of pinching that we require, and, in Section 5, we prove the main theorems. For completeness, in Section 6, we include a brief explanation on the Teichmüller space of f and its relation with the pinching process.

2. Preliminaries on Baker domains and quasiconformal theory

2.1. Baker domains

In the case that f has a Baker domain, Eremenko and Lyubich, in [17], prove that the closure of the set of critical and finite asymptotic values of f, $\text{sing}(f^{-1})$, is unbounded. Baker [2] shows that a Baker domain on f is simply connected. Bergweiler and

Eremenko [8] prove that the inverse image of a non-invariant Baker domain under the map f that omits a value is disconnected.

We describe the classification given by Cowen. Let U be a domain and let $f : U \to U$ be holomorphic. We say that a subdomain V of U is *absorbing for* f, if V is simply connected, $f(V) \subseteq V$ and for any compact subset K of U, there exists $n = n(K)$ such that $f^n(K) \subseteq V$. Let $H = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ be the right half-plane.

Definition 4. Let $f : U \to U$ be holomorphic. Then (V, φ, T, Ω) is called an *eventual conjugacy of* f *in* U, if the following statements hold:

(i) V is absorbing for f;

(ii) $\varphi : U \to \Omega \in \{H, \mathbb{C}\}$ is holomorphic and φ is univalent in V ;

(iii) T is a Möbius transformation mapping Ω onto itself and $\varphi(V)$ is absorbing for T;

(iv) $\varphi(f(z)) = T(\varphi(z))$ for $z \in U$.

The result of Cowen, when U is a Baker domain, can now be stated as follows.

Lemma 5. *Let $U \neq \mathbb{C}$ be a simply connected Baker domain of f. Then f has an eventual conjugacy (V, φ, T, Ω). Moreover, T and Ω may be chosen as exactly one of the following possibilities:*

(a) *$\Omega = H$ and $T(z) = \lambda z$, where $\lambda > 1$;*

(b) *$\Omega = H$ and $T(z) = z + i$ or $T(z) = z - i$;*

(c) *$\Omega = \mathbb{C}$ and $T(z) = z + 1$.*

For the existence of the different Baker domains that can appear in the dynamics of a map see, for instance, [10, 24, 33].

An important class of examples for this article are univalent Baker domains. Barański and Fagella [5] gave the classification of univalent Baker domains of entire transcendental functions. A point $\zeta \in \widehat{\mathbb{C}}$ in the boundary of a simply connected domain $U \subseteq \mathbb{C}$ is called *accessible from* U if there exists a curve $\gamma : [0, \infty) \to U$ which *lands* at ζ, i.e., $\gamma(t)$ tends to ζ as $t \to \infty$. In this context, an *access* is a homotopy class within the family of curves $\widehat{\gamma} : [0, 1] \to \widehat{\mathbb{C}}$, such that $\widehat{\gamma}((0, 1)) \subseteq U$ and $\widehat{\gamma}(1) = \zeta$, which is an equivalence relation.

Baker [3] showed that ∞ is accessible for every Baker domain U; Barański and Fagella [5] saw that the iterations of every point in U tend to ∞ through the same access. In the case that f is not univalent, Baker and Domínguez [4] proved that there exists infinitely many accesses to ∞ and, in particular, ∂U is disconnected.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental map and let $U \subseteq \mathbb{C}$ be an invariant univalent Baker domain. Then there exists a point $\zeta \in \widehat{\mathbb{C}}$, such that the backward iterates under $(f|_U)^{-1}$ of all points in U tend to ζ through the same access (which we called the *backward dynamical access*). Moreover, by [5], exactly one of the following occurs:

- $\zeta \neq \infty$ is an attracting fixed point (under $(f|_U)^{-1}$) in the boundary of U and U is of *hyperbolic type I.*
- $\zeta = \infty$ where the backward dynamical access is different from the forward one and U is of *hyperbolic type II.*
- $\zeta = \infty$ where the backward dynamical access is equal to the forward one and U is of *parabolic type.*

2.2. Quasiconformal theory

In this section, we give basic elements of quasiconformal theory that will be used in the construction of the pinching deformation. For further study, see Ahlfors [1], Lehto [25], Gardiner [20], Zakeri and Zeinalian [37] and, specifically for holomorphic dynamics and for this paper, Branner and Fagella [5].

We say that $f : I \to \mathbb{R}$ is *absolutely continuous on the interval I* if for every $\varepsilon > 0$, there exists $\delta > 0$ so that for finite intervals $(x_k, y_k) \subseteq I$ satisfying $\sum_k |x_k - y_k| < \delta$, this implies $\sum_k |f(x_k) - f(y_k)| < \varepsilon$. Now, a continuous real-valued function u is said to be *absolutely continuous on lines* (ACL) in a domain $U \subseteq \mathbb{C}$ if for each closed rectangle $\{x + iy \mid a \leq x \leq b,\ c \leq y \leq d\} \subseteq U$, the function $x \mapsto u(x + iy)$ is absolutely continuous in $[a, b]$ for almost all $y \in [c, d]$ and the function $y \mapsto u(x + iy)$ is absolutely continuous in $[c, d]$ for almost all $x \in [a, b]$. A complex function is *absolutely continuous in A* (ACL) if its real and imaginary parts are ACL in U.

A mapping $h : U \to V$ is *K-quasiconformal* if and only if h is a homeomorphism, h is ACL in U, and $|\partial_{\bar{z}} h| \leq k\, |\partial_z h|$ almost everywhere, with

$$k := \frac{K - 1}{K + 1} < 1.$$

In this context, the *complex dilatation* or *Beltrami coefficient* of h is defined as

$$\mu_h(z) := \frac{\partial_{\bar{z}} h(z)}{\partial_z h(z)}.$$

Conversely, let $\mu(z)$ be a measurable complex-valued function defined on U for which $\|\mu\|_\infty = k < 1$ almost everywhere, then we say that μ is a *k-Beltrami coefficient of U*. And then we ask if there is a quasiconformal map h satisfying the *Beltrami equation* $\partial_{\bar{z}} h(z) = \mu(z) \partial_z h(z)$. The answer is the next theorem, written as it is in the standard reference from Ahlfors and Bers [1]. Bojarski, in [12], explains that “the first sketch for a complete proof for the existence problem” is due to Morrey [32].

Theorem 6 (Measurable Riemann mapping theorem). *The Beltrami equation gives a one-to-one correspondence between the set of quasiconformal homeomorphisms of* $\widehat{\mathbb{C}}$ *that fix the points* $0, 1, \infty$*, and the set of measurable complex-valued functions* μ *on* $\widehat{\mathbb{C}}$ *for which* $\|\mu\|_\infty < 1$.

Extending the theory to Riemann surfaces S, S', if we have a homemorphism $h : S \to S'$ and there exists a $K \geq 1$ so that h is locally $K-$quasiconformal when it is expressed in all the charts, then h is *quasiconformal*. As explained in [5], $D_u f$ defines ellipses $E_u \subseteq T_u U$ via the inverse image of circles centered at the origin under $D_u f$ with Beltrami coefficient

$$\mu_h(z) := \frac{\partial_{\bar{z}} h(z)}{\partial_z h(z)}.$$

A *Beltrami form* or a *Beltrami differential* μ on a Riemann surface S is a $(-1, 1)$ differential on S, which is expressed as $\mu(z) d\bar{z}/dz$.

Let $h : S \to S'$ be a quasiconformal map between two Riemann surfaces S and S', with arbitrary charts $\varphi : U_S \to U$ and $\varphi' : U_{S'} \to U'$ on points $s \in S$ and $h(s) \in S'$ where $z = \varphi(s)$ and $z' = \varphi'(s)$, respectively. If μ' is a Beltrami form on S' then the pullback $h^*\mu'$ is defined as the Beltrami form on S, which in the chart φ has the Beltrami coefficient

$$(h^*\mu')_\varphi(z) = h^*(\varphi^*\mu_0) = (\varphi \circ h)^*\mu_0 = \mu_{\varphi\circ h},$$

where $\mu'_{\varphi'}(z')$ is the Beltrami coefficient of μ' in the chart φ' and $\mu_0 = 0$.

So, if we have a holomorphic map f, it is required that μ is invariant under pullback by f and we want to deform it via quasiconformal conjugation in such a way the deformations will be holomorphic; we can do it via a quasiconformal map such that $h^*\mu_0 = \mu$. The measurable Riemann mapping theorem guarantees its existence via integration (see [13]).

On the other hand, let $U \subseteq \mathbb{C}$, and let $TU = \bigcup_{u\in U} T_u U$ be the tangent bundle over U. An *almost complex structure* on U is a measurable field of ellipses $\sigma \subseteq TU$, i.e., we put an ellipse $E_u \subseteq T_u U$ defined up to scaling for almost every point $u \in U$, with semi-major axis M, semi-minor axis m, and $\theta \in [0, \pi)$ the chosen argument of the direction of the minor axis, such that the map $u \mapsto \mu(u)$ from U to $\mathbb{D}$ is Lebesgue measurable, where $\mu(u) = \frac{M-m}{M+m} e^{i2\theta}$ and is denoted as the Beltrami coefficient of E_u.

Also, σ_0 is defined as the *standard complex structure* made with a field of circles, i.e., $M = m$.

Actually, the Beltrami coefficient of a quasiconformal map h has the same information of the Beltrami coefficient of an ellipse E_u. This fact gives an equivalence between quasiconformal maps and almost complex structures.

3. Baker laminations and univalent Baker domains

The pinching deformations is done along differentiable curves and a specific neighborhood of each curve. Since a Baker domain U of an entire transcendental function is simply connected then there exists a uniformization $\psi : \mathbb{H} \to U$ of U, hence inherits a hyperbolic metric from ψ. Here $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. We choose the pinching curves to be sets of geodesics in U and the neighborhoods to be hyperbolic neighborhoods as follows:

Let $\alpha := t + i(\pi/2)$ with $t \in \mathbb{R}$ and $B_\delta := \mathbb{R} \times (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta) \subseteq \mathbb{C}$ with $\delta < \frac{\pi}{2}$. Applying the exponential map, we define $\beta := \exp(\alpha) = \{it \mid t \in \mathbb{R}^+\}$ and

$$V'_\delta(\beta) := \exp(B_\delta) \subseteq \mathbb{H},$$

where $V'_\delta(\beta)$ is called a *good neighborhood of thickness* δ of the complete geodesic β. If γ is any other complete geodesic in $\mathbb{H}$, there is a unique oriented isometry $M \in \mathrm{PSL}(2, \mathbb{R})$ of $\mathbb{H}$ such that $M(\beta) = \gamma$ and we say that

$$V'_\delta(\gamma) := M(V'_\delta(\beta))$$

is a *good neighborhood* (of thickness δ) for γ (see Figure 3.1).

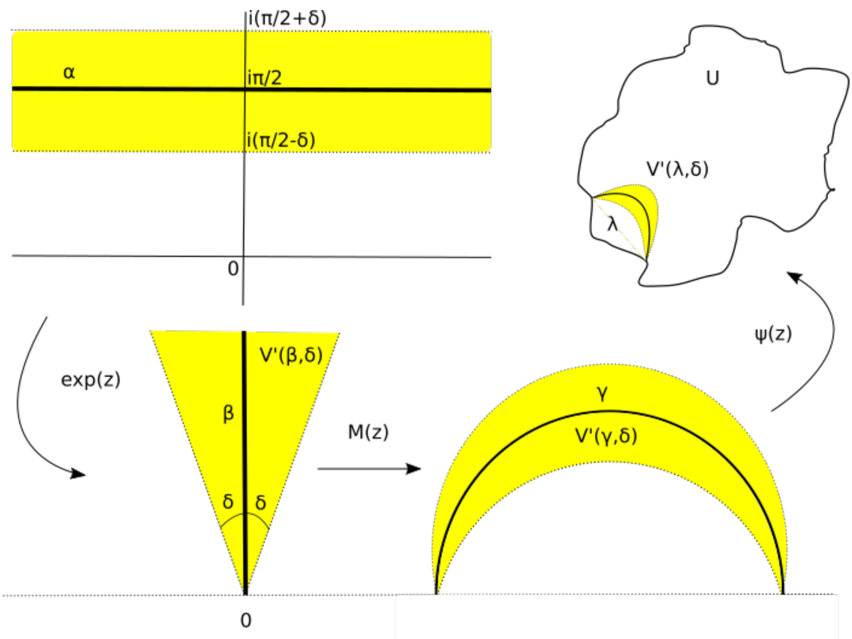

Figure 3.1. Definition of a good neighborhood of a geodesic.

Now, consider that $\lambda \in U$ is a complete geodesic and $\gamma \in \mathbb{H}$ any geodesic such that $\psi(\gamma) = \lambda$, to which we put a good neighborhood $V'_\delta(\gamma)$, and we define $V_\delta(\lambda) := \psi(V'_\delta(\gamma))$.

3.1. Univalent Baker domains

The simplest situation for constructing Baker laminations is when the domain is an univalent Baker domain of hyperbolic type I.

In this case we extend the uniformization $\psi : \mathbb{H} \to U$ to the boundary and we name it $\overline{\psi} : \overline{\mathbb{H}} \to \overline{U}$, and we do it such that $\overline{\psi}(\infty) = (\infty)$ and $\overline{\psi}(0) = \zeta$, where ζ is the repelling fixed point from section 2.1, in fact all points of ∂U different to ζ tend to ∞ under iteration.

Baker laminations in U can be classified in three categories:

(a) Consider the geodesic λ_∞ which goes from ζ to ∞, which is invariant under f. It is the only leaf in Λ. In this case, $\Lambda = \{\lambda_\infty\}$ is a Baker lamination and $\mathcal{L} = \bigcup_n f^{-n}(\lambda_\infty)$ (see Figure 3.2).

(b) Geodesic leaves that none of their endpoints are attached to ∞ (see Figure 3.3).

(c) The geodesic lamination that contains a leaf of type (a) and leaf of type (b) (see Figure 3.4).

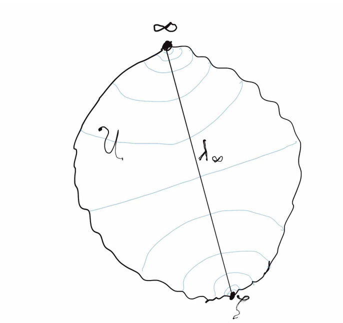

Figure 3.2. Case (a). In light blue, the action of f in U and the geodesic λ_∞.

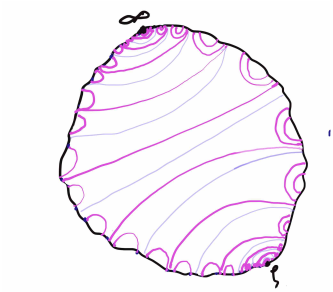

Figure 3.3. Case (b). A Baker lamination with no endpoints attached to ∞.

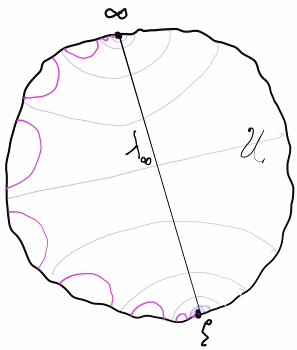

Figure 3.4. Case (c). An example of a mixture of case (a) and case (b).

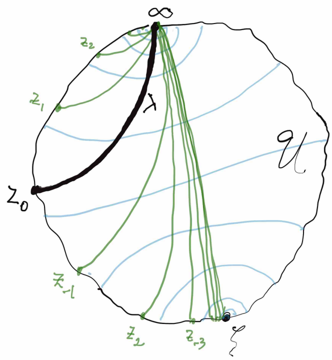

Figure 3.5. The geodesic λ and all of its forward and backward iterates in U. This is not a Baker lamination.

Theorem 2 states that the pinching along laminations of case (a) or (c) does not converge. The case (b) will be studied in a forthcoming paper.

Note that in cases (a) and (c), there cannot exist a leaf $\lambda \in \Lambda$ connecting $z_0 \in \partial U$ to ∞ with $z_0 \neq \zeta$. Because then, by condition (2) in the definition of Baker laminations, there would be preimages of λ accumulating on λ_∞ in contradiction with condition (1). The situation is illustrated in Figure 3.5.

Let us interpret geometrically the above situation. If U is a Riemann surface and f is an endomorphism of U, the *grand orbit* of $z \in U$ is defined as the set $\{z' \in U \mid f^n(z') = f^m(z) \text{ for some } n, m \geq 0\}$. We say that $z \sim z'$ if their grand orbits are the same, which is an equivalence relation and we call it the *grand orbit relation*. Under this relation, we build the quotient space U/f. So, if U is of hyperbolic type I, $f|_U$ is conjugated to $g : \mathbb{H} \to \mathbb{H}$ with $g(w) = aw$, $a \in \mathbb{R}$. Let $\beta = \{it \mid t > 0\}$, then $\mathbb{H}/g$ is an annulus $\mathcal{A}$ with core geodesic $\tilde{\beta} := \beta/g$, i.e., the unique closed geodesic. Since $g(1) = a$, length$(\tilde{\beta}) = \log(a) = \pi/\text{mod}(\mathcal{A})$ (see [28]).

An example by Bergweiler (see, for instance, [5]), shows that each of the family of functions $f_n(z) = n - (n-1)\log(n) + nz - e^z$, $n > 1$, has a Baker domain U of hyperbolic type I, which is not completely invariant and contains a left half-plane $\{\mathrm{Re}(z) < n\}$. Its boundary is locally connected, and $\mathrm{Postcrit}(f) \cap U = \emptyset$, ζ is real, with multiplier n. In this case we set λ_∞ to be the interval $(-\infty, \zeta)$.

4. Pinching deformation on Baker laminations

It is pretty common to study the theory of holomorphic dynamics introducing deformations of a map via conjugation classes, i.e., analyzing certain space of functions. One of these tools is the pinching deformations introduced by [27] to prove that the component of J-stability is unbounded in $\mathbb{CP}^{2d+1}$ for rational maps with disconnected Julia sets and with connected Julia sets with some restrictions on accesses. Tan Lei, in [35], generalized this concept and gave it a different approach. In general terms, it consists on taking a function f and a curve γ with an attracting fixed point and a repelling fixed point as boundary points and deform f, via quasiconformal conjugations, shrinking γ to a point, i.e., to fuse the attracting point with the repelling fixed point creating a parabolic fixed point. It is this approach we want to adapt to transcendental entire maps with Baker domains, where the final points of the pinching curves are both in $J(f)$.

We will follow [22] closely to build the deformations in this particular case.

Let f be an entire transcendental mapping with at least one periodic cycle of Baker domains of period p, $U = \{U_0, U_1, \ldots, U_{p-1}\}$, with a Baker lamination Λ in U.

Take $L_b, L_y, L_r \in \mathbb{R}$ such that $0 < L_b < L_y < L_r$ and a function $\tau : [0,1) \to [L_r, \infty)$ such that $\tau \in C^1[0,1)$ and it is an increasing function. With τ we build the closed set $\mathbf{M} \subseteq \mathbb{R}^2$ bounded by

$$\big([0,1] \times \{L_b\}\big) \cup \big(\{0\} \times [L_b, L_r]\big) \cup \big(\{1\} \times [L_b, \infty)\big) \cup \big\{(t, \tau(t)) \bigm| t \in [0,1)\big\}.$$

Now choose $v_t(y)$ such that $v_t(y) = y$ for $y \in \big[L_b, L_y\big]$ and $(t, y) \mapsto (t, v_t(y))$ is a C^1-diffeomorphism from $[0,1] \times [L_b, L_r] \setminus \{(1, L_r)\}$ onto $\mathbf{M}$ (see Figure 4.1).

For $t = 1$, $y \neq L_r$ consider this technical assumption: For any $L' < L_r$, there is $t(L') \in (0,1)$ with $t(L') \to 1$ as $L' \to L_r$ such that for any $(s, y) \in (t(L'), 1] \times [L_b, L']$, we take $v_s(y) = v_{t(L')}(y)$.

With v_t, we build a map $\widetilde{P}_t$ defined on the strip $\{x + iy \mid x \in \mathbb{R}, y \in [L_b, L_r]\}$ with $t \in [0,1]$, where

$$\widetilde{P}_t(x + iy) = x + i v_t(y)$$

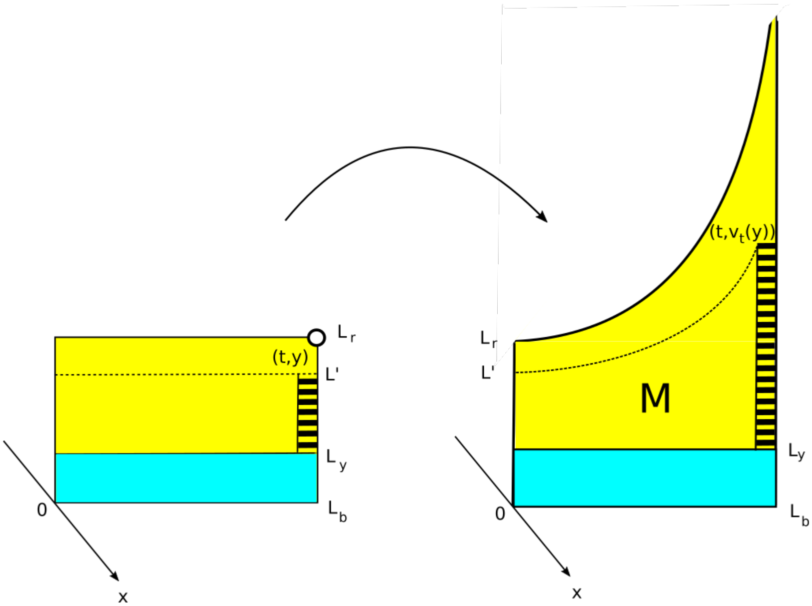

Figure 4.1. The diffeomorphism $(t, y) \mapsto (t, v_t(y))$.

and it has the next properties:

(1) It commutes with any real translation.

(2) It is the identity map on $\mathbb{R} \times [L_b, L_y]$.

(3) Its coefficient of the Beltrami form is

$$\frac{\partial_{\bar{z}} \widetilde{P}_t}{\partial_z \widetilde{P}_t}(x + iy) = \frac{1 - \partial_y v_t(y)}{1 + \partial_y v_t(y)},$$

which is continuous on

$$(t, x + iy) \in \big([0, 1] \times \mathbb{R} \times [L_b, L_r]\big) \setminus \big\{(1, x, L_r) : x \in \mathbb{R}\big\},$$

its norm for every $t \in [0, 1)$ is locally uniformly bounded away from 1 and tends to 1 as $(t, x, y) \to (1, x, L_r)$.

Now we have to connect these bands with those in the Baker lamination. If $B_\delta^+ := \mathbb{R} \times [\frac{\pi}{2}, \frac{\pi}{2} + \delta]$, i.e., the upper part of B_δ, we define the map $S_+ : \mathbb{R} \times [L_b, L_r] \to B_\delta^+$ as $S_+(z) = \frac{\delta}{L_r - L_b}(\bar{z} + iL_r) + i(\frac{\pi}{2})$. Also for $\Psi : \mathbb{H} \to U$ a Riemann mapping and $M \in \mathrm{PSL}(2, \mathbb{R})$, we define the map $\phi_+ := \Psi \circ A \circ \exp \circ S_+ : \mathbb{R} \times [L_b, L_r] \to V^+ \subseteq U_i$ where $V^+ \subseteq V$, with V a good neighborhood of $\lambda \in \Lambda$, and with a well defined inverse branch $\psi_+ : V^+ \to \mathbb{R} \times [L_b, L_r]$.

For $t \in [0,1)$, let $(\sigma_t')_+ := (\tilde{P}_t \circ \psi_+)^*(\sigma_0)$ be the pullback of the standard almost complex structure on B_δ^+.

Similarly, for $B_\delta^- := \mathbb{R} \times [\frac{\pi}{2} - \delta, \frac{\pi}{2}]$ we define the map $S_- : \mathbb{R} \times [L_b, L_r] \to B_\delta^-$ as $S_-(z) = \frac{\delta}{L_r - L_b}(z - iL_r) + i(\frac{\pi}{2})$ for $V^- \subseteq V$, making the deformation is symmetric. We define the map $\phi_- := \Psi \circ M \circ \exp \circ S_- : \mathbb{R} \times [L_b, L_r] \to V^- \subseteq U_i$, with inverse branch $\psi_- : V^- \to \mathbb{R} \times [L_b, L_r]$. For $t \in [0,1)$, let $(\sigma_t')_- := (\tilde{P}_t \circ \psi_-)^*(\sigma_0)$ be the pullback of the standard almost complex structure on B_δ^-.

Then, we spread $(\sigma_t')_+$ and $(\sigma_t')_-$ to the grand orbit $\mathcal{V}$ by defining

$$\sigma_t := \bigcup_n \left((f^n)^*(\sigma_t')_+ \cup (f^n)^*(\sigma_t')_-\right),$$

and we define σ_t outside of $\mathcal{V}$ on the Riemann sphere by setting $\sigma_t := \sigma_0$.

4.1. Almost complex structures associated to a Baker lamination

Let f be an entire transcendental mapping with at least one periodic cycle of Baker domains of period p, $U = \{U_0, U_1, \ldots, U_{p-1}\}$, with a Baker lamination Λ in U. Consider an almost complex structure $(\sigma_t)_{t\in[0,1)}$ as defined above in each of the leaves of the Baker lamination in U_0 in such a way that if $l_1, l_2 \in \Lambda \cap U_0$ and $l_1 = f^m(l_2)$, for some $m \in \mathbb{N}$, then the almost complex structure on the neighborhood of l_1 is the pushforward of the almost complex structure on the neighborhood of l_2. Now define an almost complex structure in $\Lambda \cap U_k$ by pushforward the almost complex structures in $\Lambda \cap U_0$, for $k = 1, \ldots, p-1$. Extend the almost complex structure to the grand orbit $\mathcal{V}$ by pulling back the almost complex structures of each $U_k \in U$.

Definition 7. Let f be an entire transcendental map with at least one periodic cycle of Baker domains $U = \{U_0, U_1, \ldots, U_{p-1}\}$ with a Baker lamination Λ in U. The family of almost complex structures $(\sigma_t)_{t\in[0,1)}$ defined above, defines a *pinching deformation* of f with support in $\mathcal{V}$. These structures come with quasiconformal maps $h_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ via integration by the measurable Riemann mapping theorem, that we can normalize assuming h_t fixes ∞ and two points $p, q \in J(f)$. Then, $f_t := h_t \circ f \circ h_t^{-1}$ is an holomorphic map for $t \in [0,1)$.

Furthermore, we say that a pinching deformation *converges uniformly* if $h_t \rightrightarrows H$ (double arrow means uniform convergence, with respect to the spherical metric) and the non-trivial fibers of H are the leaves in the lamination $\mathcal{L}$, in the sense that $\operatorname{diam}_s(h_t(\bar{\gamma})) \to 0$, as $t \to 1$, for each $\gamma \in \mathcal{L}$. Here, $\operatorname{diam}_s(A)$ denotes the spherical diameter of a set $A \subseteq \widehat{\mathbb{C}}$.

See Figure 4.2 for a visualization of the pinching process at the time t. Consider that we have put two bands in the domain and contradomain of $\tilde{P}_t$ that are not well drawn mathematically but help to visualize the situation.

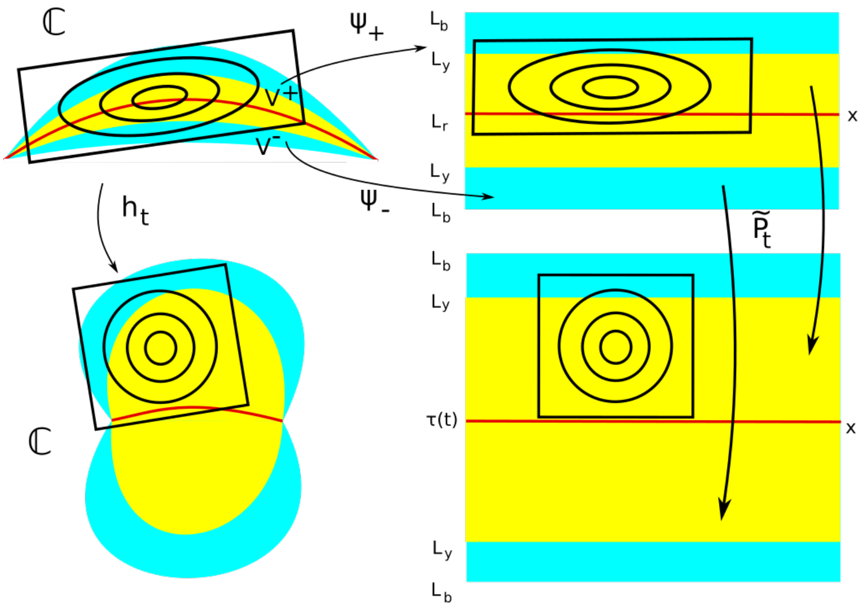

Figure 4.2. The map $h_t(z)$ for the pinching process.

We are interested in showing the circumstances when a pinching deformation converges uniformly or does not.

In this context, we have the next lemma.

Lemma 8 ([22, Lemma A]). *Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous surjective map. For $t \in [0, 1)$, let $F_t, G_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ two families of homeomorphisms. Assume that F_t, G_t converge uniformly with respect to the spherical metric to continuous maps F_1, G_1 respectively, and g maps each fiber of F_1 into a fiber of G_1. Then*

$$g_t := G_t \circ g \circ F_t^{-1} : \mathbb{C} \to \mathbb{C}$$

converges uniformly with respect to the spherical metric to a continuous map g_1, and

$$g_1 \circ F_1 = G_1 \circ g.$$

Taking $F_t = G_t := h_t$ in the lemma above and $g := f$, it implies that if $h_t \rightrightarrows H$, then $g_t := f_t \rightrightarrows F$ in the spherical metric.

On the other hand, let us observe that if a sequence of functions $f_t : \mathbb{C} \to \mathbb{C}$ does not converge in the spherical metric (restricted to $\mathbb{C}$), then it does not converge in the Euclidean metric of $\mathbb{C}$.

We make some remarks related to the convergence of a pinching deformation.

Remark 9. From condition (3) above, the Beltrami coefficient of $\widetilde{P}_t(x+iy)$ is

$$\frac{1-\partial_y v_t(y)}{1+\partial_y v_t(y)}.$$

By the formula in Section 2.2, we have that the quasiconformal dilation of $\widetilde{P}_t$ is $K(\widetilde{P}_t) = \frac{1}{\partial_y v_t(y)}$. Therefore $K(\widetilde{P}_t \circ \psi_\pm) = \frac{1}{\partial_y v_t(y)}$, which has support in the grand orbit of $\bigcup_{\lambda\in\Lambda} Y(\lambda) = \mathcal{Y}$. Since $\partial_y v_t(y) \to 0$ when $y \to L_r$ and $t \to 1$, then $K(\widetilde{P}_t \circ \psi_\pm) \to \infty$. From the definition of quasiconformal maps h_t by the structures σ_t, we obtain that $K(h_t(Q)) \to \infty$, for a quadrilateral Q that intersects $\lambda \subseteq \mathcal{L}$. Therefore, for z_1 and z_2 two points in a leaf $\lambda \subseteq \mathcal{L}$, by taking a quadrilateral with two opposite sides intersecting the segment λ in z_1 and z_2, we have that $d_e(h_t(z_1), h_t(z_2)) \to 0$ as $t \to 1$. See the left-hand side of Figure 4.2.

Remark 10. Also, from condition (3) on $\widetilde{P}_t$, we have that the coefficient of Beltrami of $\widetilde{P}_t$ is locally uniformly bounded away from 1 at any point $(t, x, y) \neq (1, x, L_r)$, therefore following the argument in Remark 9, we have that $K(h_t(Q))$ is bounded away from ∞ and so no quadrilateral in the complement of $\mathcal{L}$ shrinks to a point in the process of pinching. This implies that if D is any disc in $\mathbb{C} \setminus \mathcal{L}$, then $h_t(D)$ is homeomorphic to D for $t \in [0, 1)$. When $t = 1$, some problems may appear (see Remark 11 in Section 5).

5. Main theorem

As mentioned in the introduction, the deformations of some completely invariant Baker domains were studied in [15]. In particular, in that article it is proven that the Fatou function $f(z) = e^{-z} + z + 1$ can be pinched to obtain the Baker–Domínguez function $f(z) = e^{-z} + z$ (see Figure 5.1). The Fatou function has a completely invariant doubly parabolic Baker domain that contains the right half-plane and it is pinched, so the result is an infinite union of invariant doubly parabolic Baker domains.

In Theorem 2, we consider the case when the Baker domain is not completely invariant, we prove that for certain curves, the pinching process along such curves is divergent.

Proof of Theorem 2. As z_0 is a non-exceptional point, $\overline{\bigcup_{n=1}^{\infty} f^{-n}(z_0)} = J(f)$ and so there is a subsequence $\{z_{n_k}\} := \{f^{-n_k}(z_0)\} \to z_0$, as $n_k \to \infty$. Then, we have a family of curves $\{\gamma_{n_k}\} := \{f^{-n_k}(\lambda_\infty)\} \subseteq \mathcal{L} \subseteq \{f^{-n_k}(U)\}$ with $\{z_{n_k}\}$ and ∞ as endpoints of each one.

Since U is not completely invariant we have that the curves $\{\gamma_{n_k}\}$ are not in U for $n_k \neq 0$ and they are disjoint in $\mathbb{C}$ (see [8]). Thus, for $\varepsilon_0 > 0$, there is a natural number

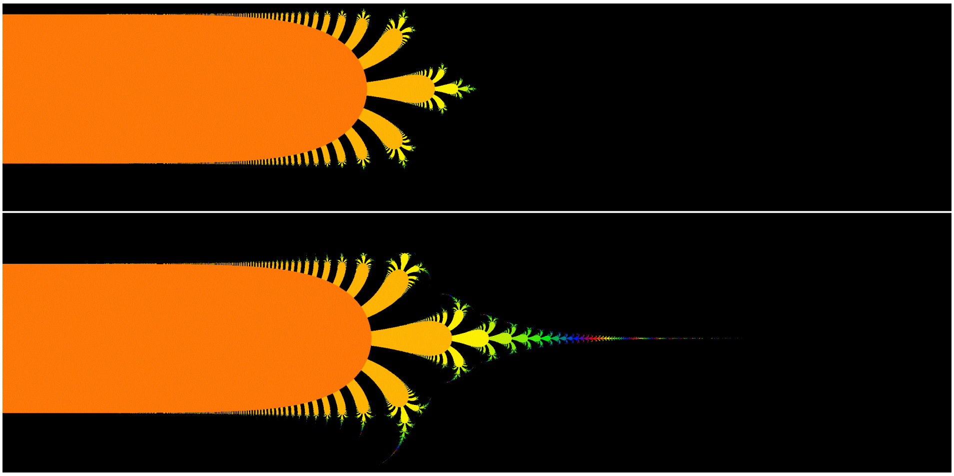

Figure 5.1. Pinching the Fatou function (above) to the Baker–Domínguez function (below).

N_{ε_0} such that for every $n_k > N_{\varepsilon_0}$, the subsequence $\{z_{n_k}\}$ is contained in $D_{\varepsilon_0}(z_0) \cap (\widehat{\mathbb{C}} \setminus U)$. Notice that ∞ is accessible from $\widehat{\mathbb{C}} \setminus h_t(U)$ by the curves $h_t(\{\gamma_{n_k}\})$ (all these curves are attached to ∞) (see Figure 5.1).

The duality between λ_∞ being in U and $\{\gamma_{n_k}\}$ not being in U is the heart of the problem of convergence, as we will see now.

By the classification of the Baker laminations in Section 3.1, $f(\lambda_\infty) = \lambda_\infty$. Then

$$f_t\left(h_t\left(\lambda_\infty\right)\right) = \left(h_t \circ f \circ h_t^{-1}\right)\left(h_t\left(\lambda_\infty\right)\right) = h_t(f(\lambda_\infty)) = h_t\left(\lambda_\infty\right).$$

Let us assume that the pinching deformation along $\mathcal{L}$, converges uniformly via the quasiconformal maps h_t, so $h_t \rightrightarrows H$ as in Definition 7. Then, for $\gamma \in \mathcal{L}$, $\operatorname{diam}_s(h_t(\bar{\gamma})) \to 0$ as $t \to 1$. Since h_t fixes infinity for all t, then $h_t(\gamma)$ tends to infinity, if $\gamma \in \mathcal{L}$. In particular, $h_t(\gamma_{n_k}) \to \infty$ as $t \to 1$.

Notice that the set

$$C_t = \bigcup_{k=0}^{\infty} h_t(\gamma_{n_k}) \cup h_t(\lambda_\infty),$$

disconnects the complex plane in two regions Ω_i, such that $\Omega_i \cap U \neq \emptyset$, $i = 1, 2$. We have two cases: either some region $h_t(\Omega_i)$ collapse to ∞ as $t \to 1$, i.e., $h_1(\Omega_i) = \infty$ or none of the regions collapse. By hypothesis, we have that f_t converges uniformly to a entire transcendental function g and the non-trivial fibers of the pinching are the leaves of the lamination $\mathcal{L}$, so the family $h_t(\Omega_i)$ can not collapses to a point when $t \to 1$, $i = 1, 2$ (see Figure 5.2). In Remark 11 below, we explain why we imposed this requirement on the fibers, in our situation.

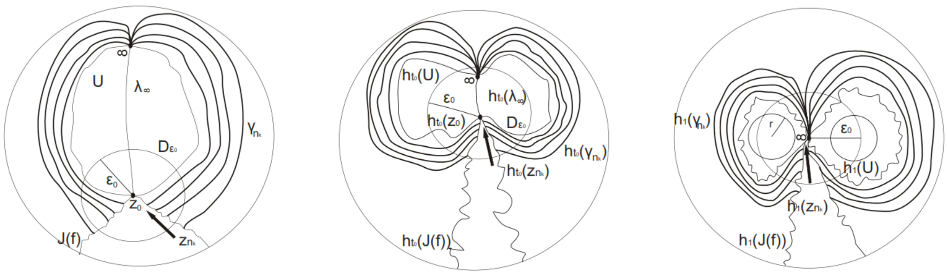

Figure 5.2. The pinching process in the proof of Theorem 2. In this figure, there are two different regions in the complement of C_t, that intersects $h_t(U)$.

By continuity of h_t, $\lim_{n_k\to\infty} h_t(z_{n_k}) = h_t(z_0)$ for every $t \in [0,1]$, and it follows that $\{h_t(z_{n_k})\} \subseteq D_{\varepsilon_0}(h_t(z_0)) \cap (\widehat{\mathbb{C}} \setminus h_t(U))$, for some $\varepsilon_0 > 0$. Since the pinching deformation is convergent, then by the discussion above,

$$\lim_{t\to 1} h_t(z_0) = h_1(z_0) = \infty.$$

Since, by hypothesis, the regions $h_t(\Omega_i)$ do not collapse, they contain an open set for $t \in [0,1]$. Then there are two open discs, one at each side of $h_t(\lambda_\infty)$ and contained in $h_t(U \setminus \Lambda)$ of radius $r_t > 0$, such that $0 < r_t < \mathrm{diam}_s(h_t(\gamma_{n_k}))$, for $n_k > N_{\varepsilon_0}$, for all $0 \le t \le 1$.

Observe that there is t_0 such that $\mathrm{diam}_s(\infty, h_{t_0}(z_0)) < \varepsilon_0$, therefore $D_{\varepsilon_0}(h_t(z_0)) \cap (\widehat{\mathbb{C}} \setminus h_t(U))$ has two components. One component contains the endpoints $\{h_t(z_{n_k})\}$, the other component contains the access to ∞ from $\widehat{\mathbb{C}} \setminus h_t(U)$.

This implies that for every curve in $\{h_t(\gamma_{n_k})\}$, its intersection with $D_{\varepsilon_0}(h_t(z_0))$ has two components. But this is a contradiction, because the convergence of the pinching implies that $\mathrm{diam}_s(h_t(\gamma_{n_k})) \to 0$ when $t \to 1$. Thus the pinching along $\mathcal{L}$ does not converge uniformly. ■

Remark 11. Observe that for every $t \in [0,1]$, we have $h_t(\Omega) \cap J(f_t) \neq \emptyset$. If $p \in J(f)$, we have that $p_t := h_t(p) \in J(f_t)$ and, by Montel's theorem, there is an $m \ge 0$ such that for V_{p_t} any neighborhood of p_t, $f_t^m(V_{p_t}) \cap h_t(\Omega) \neq \emptyset$. The integer m depends on V_{p_t}. Therefore $V_{p_t} \cap f_t^{-m}(h_t(\Omega)) \neq \emptyset$.

Assuming that the functions f_t converge uniformly to an entire function g and $h_t(\Omega)$ collapses to ∞ as $t \to 1$ then there exists $p \in J(f)$ but $p \notin \Omega$, such that $p_t \in J(f_t)$ but $p_t \notin h_t(\Omega)$ for $t \in [0,1]$; otherwise, $J(g) = \infty$. Therefore, $p_1 \in J(g)$ and for any neighborhood V_{p_1} of p_1, there is an inverse branch of ∞ in V_{p_1}. This implies that p_1 is either a prepole or the accumulation point of different preimages of ∞ and p_1 is an essential singularity, so g is not an entire function.

Example 12 (An example of Theorem 2). Consider the example at the end of Section 3.1, where $f(z) = 2 - \log(2) + 2z - e^z$ has a Baker domain U of hyperbolic type I, which is not completely invariant and contains the left half-plane $\{\operatorname{Re} z < 2\}$. The lamination on U, λ_∞ consists of one leaf which is the interval $(-\infty, \zeta)$, where the point $\zeta = z_0$ is a fixed point of f in $J(f)$ and $\mathcal{L} = \bigcup_n f^{-n}(\Lambda)$. This is case (a). Hence, by the theorem above, pinching along $\mathcal{L}$ does not converge uniformly.

Observe that in this example, the core curve of the cylinder U/f is pinched and the limit surface exists, but the limit function does not.

On the other hand, there is a possibility that a Baker lamination intersects the set of asymptotic values of a map f, then we have Theorem 3.

Proof of Theorem 3. If a leaf $\lambda_a \in \mathcal{L}$ intersects the set of asymptotic values of f, then there is a leaf $\sigma \in \mathcal{L}$ with $f(\sigma) = \lambda_a$ such that σ is in some component of the inverse image of U. Moreover, σ has one of its extreme points at ∞. Then, we follow the same argument as in the proof of Theorem 2, to show that the pinching deformation does not converge. ■

Remark 13. The set of quasiconformal deformations of a map is really a class of quasiconformal maps, as we will see in Section 6. This is in order to avoid trivial situations. For instance, in case that $h_t(z)$ converges, the quasiconformal maps $\tilde{h}_t(z) := h_t(z)/(1-t)$ does not converge, even though it integrates the same structure. However, in our theorems above, we show that the pinching deformation does not converge, no matter which integrating map h_t is chosen for each t.

6. Teichmüller space and pinching

Extending the work of Sullivan and McMullen on rational maps [29], the global study of deformations of transcendental entire functions is first carried out by Harada and Taniguchi [23], in the case that the singular values are a discrete set of $\mathbb{C}$. Then Fagella and Henriksen in [18, 19] generalize the results of Harada and Taniguchi without restrictions. The aim of this section is to relate briefly pinching deformations and Teichmüller space.

We say that two quasiconformal automorphisms φ_1, φ_2 of V are *equivalent* if there exists a conformal automorphism ψ of V such that $\varphi_1 = \psi \circ \varphi_2$. $\operatorname{Def}(f, V)$, the *deformation space* of f, is the set of equivalence classes of quasiconformal automorphisms φ of V, which satisfies $\varphi \circ f = g \circ \varphi$ for some holomorphic map g on V. We say that such a g is *quasiconformally conjugated* to f. The measurable Riemann mapping theorem implies that $\operatorname{Def}(f, V)$ is identified with the unit ball of the space of all invariant Beltrami differentials for f (see [29]). Then, we identify the set of all quasiconformal automorphisms h of V admitting a quasiconformal isotopy to the identity compatible with f, and we denote it by $QC_0(V)$.

The group $QC_0(V)$ acts on $\mathrm{Def}(f, V)$ by $\omega_*\phi = \phi \circ \omega^{-1}$.

Definition 14. The Teichmüller space of f in V, denoted by $T(f, V)$, is the space $\mathrm{Def}(f, V)/QC_0(V)$.

If $f : V \to V$ is a covering map, then V/f denotes the set of grand orbits of f, with the quotient topology. We have the following theorem [19,23,29]: if $f : V \to V$ is a holomorphic covering map and the grand orbit relation of f is discrete and V/f is connected, then V/f is a Riemann surface and $T(f, V) \simeq T(V/f)$.

When $V = U \setminus S$, where U is a Baker domain and S is the grand orbit of the singular values of f that are in U, then $f : V \to V$ is a covering map. When U is a univalent Baker domain, we have that S is empty and $f : U \to U$ is a covering map. Since U is attached always to infinity, and f is an entire transcendental map, the conformal maps ψ in the definition of $\mathrm{Def}(f, V)$ belong to the Affine conformal group of the complex plane.

The class of quasiconformal pinching maps h_t defined in the previous sections, restrict to U and are elements in $\mathrm{Def}(f, U)$ so their class $[h_t]$ is in $T(f, U)$. The question is what kind of path is the map $\Gamma : [0, 1) \to T(f, V)$, defined as $\Gamma(t) = [h_t]$.

Let us consider the Finsler metric in $T(f, V)$ given by the Teichmüller metric in $T(V/f)$, this makes $T(f, V)$ a complete metric space. Then, the so-called Teichmüller rays, which are associated to quadratic differentials, are geodesic rays that move to infinity from their origin by a theorem of Marden (see, for instance, [26, Section 2]). Such geodesics are described by scaling horizontal and vertical foliations of a quadratic differential.

To answer the question above, let us consider from here on, the simplest case of an entire map f with U an invariant Baker domain of hyperbolic type I, with Baker lamination Λ which consists of only one leaf which is the geodesic λ_∞, as discussed in Section 3.1. In this case, U is uniformized by the upper half plane $\mathbb{H}$ by a biholomorphism and the function f is conjugate to the transformation $T(z) = az$, for some $a \in \mathbb{R}$, as explained in Section 2.1. The curve λ_∞ in the uniformization corresponds to the imaginary axis in $\mathbb{H}$ and $T(U/f)$ isometric to $T(\mathbb{H}/T)$. Pinching $\mathbb{H}$ along the imaginary axis is equivalent, in the Fuchsian context, to the family of transformations $T_t(z) = h_t \circ T \circ h_t^{-1}(z)$, where h_t are the quasiconformal maps $h_t(z) = z|z|^{-t}$, $t \in [0, 1)$. So if φ is the uniformization of U, pinching along λ_∞ is equivalent to considering the one parameter family f_t, where $\varphi(f_t(z)) = T_t(\varphi(z))$.

Now, observe that the Beltrami coefficient

$$\mu_t(z) = \frac{\partial h_t/\partial \bar{z}}{\partial h_t/\partial z} = K_t \frac{z}{\bar{z}},$$

where $K_t = \frac{-t/2}{1-t/2}$ and for the quadratic differential $\psi_t(z) = K_t \frac{1}{z^2} dz^2$, we have that $\mu_t = \frac{|\psi_t|}{\psi_t}$. This implies that the Beltrami coefficient μ_t is extremal, so the map $t \mapsto [\mu_t] \subseteq T(\mathbb{H}/T)$ is an isometry with respect to the Euclidean metric in $[0, 1)$ and the Teichmüller metric in $T(\mathbb{H}/T)$, according to [16, Theorem 6]. So the pinching process along λ_∞ is a geodesic in $T(U/f)$.

Acknowledgements. We would like to thank the referee for his suggestions and very interesting improvements for the realization of the paper.

Funding. The authors were supported by CONACYT project number CB15/255633 and PAPIIT IN102515.

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Received 15 September 2019; revised 30 October 2021.

Rodrigo Robles
Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México (UNAM), Mexico; montero@ciencias.unam.mx

Guillermo Sienra
Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México (UNAM), Mexico; guillermo.sienra@gmail.com