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Ahlfors regular conformal dimension of metrics on infinite graphs and spectral dimension of the associated random walks

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Abstract. Quasisymmetry is a well-studied property of homeomorphisms between metric spaces, and the Ahlfors regular conformal dimension is a quasisymmetric invariant. In the present paper, we consider the Ahlfors regular conformal dimension of metrics on infinite graphs, and show that this notion coincides with the critical exponent of p-energies. Moreover, we give a relation between the Ahlfors regular conformal dimension and the spectral dimension of a graph.

1. Introduction

Quasisymmetry is a well-studied property of homeomorphisms between metric spaces and, roughly speaking, means that the homeomorphism in question preserves ratios of distances. The Ahlfors regular conformal dimension is a quasisymmetric invariant of metric spaces, which gives a measure of the simplest (in a certain sense) quasisymmetrically equivalent space. The purpose of this paper is to study the Ahlfors regular conformal dimension of discrete unbounded metric spaces, and show relations between the Ahlfors regular conformal dimensions and spectral dimensions of such spaces.

Quasisymmetry was introduced by Tukia and Väisälä in [19] to generalize the notion of quasiconformal mappings on the complex plane. In [19], quasisymmetry was given as a property of a homeomorphism between two metric spaces. A specialization of this was given by Kigami [10], for the comparison of metrics on the same underlying space. This is the definition we will use.

Definition 1.1 (Kigami's quasisymmetry). Let X be a set and d, ρ be metrics on X, and let

$$\theta : [0, \infty) \to [0, \infty)$$

be a homeomorphism. Then we say d is θ-*quasisymmetric* to ρ if for any $x, y, z \in X$

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with $x \neq z$,

$$\frac{\rho(x,y)}{\rho(x,z)} \leq \theta\Big(\frac{d(x,y)}{d(x,z)}\Big).$$

Moreover, if d is θ-quasisymmetric to ρ for some θ, then we say that d is quasisymmetric to ρ and write $d \sim_{\mathrm{QS}} \rho$.

For example, $d \sim_{\mathrm{QS}} d^\alpha$ for any $\alpha \in (0,1)$ and any metric space (X,d). This notion was also called "quasisymmetrically related by the identity map" in [6], and "quasisymmetrically equivalent" in [4]. Note that if there exists a quasisymmetric map $f : (X,d) \to (Y,\rho)$, then d is quasisymmetric to the pull-back metric ρ^* in the sense of Kigami's definition and we can identify (Y,ρ) with (X,ρ^*).

Quasisymmetry has been studied in various fields. For example, a quasi-isometric map (the definition is in [13, Definition 3.2.11], for example) between Gromov hyperbolic spaces induces a quasisymmetric map; see [13, Theorem 3.2.13 and Section 3.6], or [16]. There is also much research about quasisymmetry and Gromov hyperbolic spaces; see [13], for example. Quasisymmetry is a weaker notion of bi-Lipschitz equivalence, which has been studied extensively for decades; see [6] or [18], for example. From the viewpoint of global analysis, it is notable that a quasisymmetric modification preserves the volume doubling property, which plays an important role in heat kernel estimates. This idea is used in [10,11], and there is a recent application to circle packing graphs in [14].

The Ahlfors regular conformal dimension is a relatively new quasisymmetric invariant. It was introduced by Bourdon and Pajot [3] (see also Bonk and Kleiner [2]), and is defined as follows.

Definition 1.2 (Ahlfors regularity). Let (X,d) be a metric space, μ be a Borel measure on (X,d) and $\alpha > 0$. We say μ is α-*Ahlfors regular* with respect to (X,d) if there exists $C > 0$ such that

$$C^{-1}r^\alpha \leq \mu(B_d(x,r)) \leq Cr^\alpha \quad \text{for any } x \in X \text{ and } r_x \leq r \leq \operatorname{diam}(X,d),$$

where $r_x = r_{x,d} = \inf_{y \in X\setminus\{x\}} d(x,y)$ and $B_d(x,r) = \{y \in X \mid d(x,y) < r\}$. The space (X,d) is called α-*Ahlfors regular* if there exists a Borel measure μ such that μ is α-Ahlfors regular with respect to (X,d).

Definition 1.3 (Ahlfors regular conformal dimension). Let (X,d) be a metric space. The *Ahlfors regular conformal dimension* (or *ARC dimension* in short) of (X,d) is defined by

$$\begin{aligned}\dim_{\mathrm{AR}}(X,d) = \inf\big\{\alpha \;\big|\; &\text{there exists a metric } \rho \text{ on } X \text{ such that} \\ &\rho \text{ is } \alpha\text{-Ahlfors regular and } d \sim_{\mathrm{QS}} \rho\big\},\end{aligned}$$

where $\inf\emptyset = \infty$.

The ARC dimension is related to the conformal dimension, another well-known quasisymmetric invariant introduced by Pansu [15] in 1989. In this paper, we will extend the notion of ARC dimension to discrete metric spaces. Note that the ARC dimension has mainly been studied on bounded metric spaces without isolated points, in which case $r_x = 0$ and $\mathrm{diam}(X, d) < \infty$.

The ARC dimension is related to the well-known Cannon's conjecture, which claims that for any hyperbolic group G whose boundary is homeomorphic to the 2-dimensional sphere, there exists a discrete, cocompact and isometric action of G on the hyperbolic space $\mathbb{H}^3$. Bonk and Kleiner [2] proved that Cannon's conjecture is equivalent to the following: If G is a hyperbolic group whose boundary is homeomorphic to the 2-dimensional sphere, then there exists a metric that attains the value of the ARC dimension of the boundary.

It is not easy to calculate the ARC dimension in general. Motivated by [4, 5], Kigami [12] gave a method to calculate the ARC dimension as a critical exponent of a p-energy, which is defined by successive division of the original metric space. Furthermore, [12] gives inequalities between the ARC dimension and the p-spectral dimensions.

In this paper, we extend the results of [12] to infinite graphs and give a relation between the spectral dimensions and the ARC dimension. Our main results need a lot of notions, so we postpone the detailed definitions to Sections 2 and 4, and explain the main results through examples.

In our study, it will be useful to consider partitions of graphs that arise as the edges are successively unified. One of the simplest cases is the unification of vertices of $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geq 0\}$. For $a \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we identify 2^n edges $\{(2^n(a-1), 2^n(a-1)+1), (2^n(a-1)+1, 2^n(a-1)+2), \dots, (2^n a - 1, 2^n a)\} =: K_{(n,a)}$ and consider the unified graphs $\{G_n, E_n\}_{n\geq 0}$, where $G_n = \{(n,a) \mid a \in \mathbb{N}\}$ and E_n is the set of links between (n,a) and $(n,a+1)$. Let $(n,a) \sim (m,b)$ if $n - m = 1$ and $K_{(n,a)} \supset K_{(m,b)}$, or $m - n = 1$ and $K_{(n,a)} \subset K_{(m,b)}$. Consider $T := \bigcup_{n,a}(n,a)$ as a tree given by $\sim$, then we obtain a correspondence between $\{G_n, E_n\}_{n\geq 0}$ and T (see Figure 1). We call such a correspondence between unified graphs and a tree, a partition (see Definition 4.4, and note that we construct K by unification of edges but we treat K as a subsets of vertices because of technical reasons). Thus, we relate a division of noncompact space to a tree whose root is an infinite ray. Such an idea has been considered in the continuous cases, for example, in [7] or [8].

In this paper, we characterize the ARC dimension with a partition. For a given partition, we can define an upper p-energy $\overline{\mathcal{E}}_p$ of the partition as a certain limit of p-energies on unification graphs, see Definition 2.10, which is based on definitions of [12]. The p-energy enjoys a phase transition when p varies, that is, there exists a $p_0 > 0$ such that $\overline{\mathcal{E}}_p > 0$, if $p < p_0$, and $\overline{\mathcal{E}}_p = 0$, if $p > p_0$. We can also define a lower p-energy $\underline{\mathcal{E}}_p$, as well.

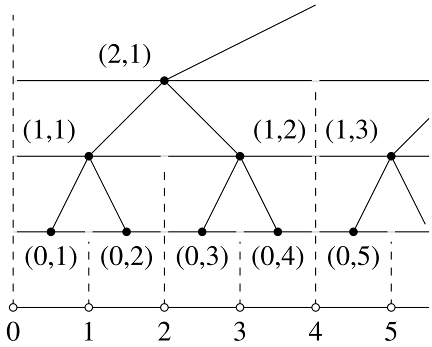

Figure 1. A partition of $\mathbb{Z}_+$

The main result of this paper is the following.

Theorem 1.4 (Theorem 4.15 (1)). *Let (G, E) be a graph and d be a metric on G. Under some conditions about d and for partitions within a certain class,*

$$\dim_{\mathrm{AR}}(G, d) = \inf\{p \mid \overline{\mathcal{E}}_p = 0\} = \inf\{p \mid \underline{\mathcal{E}}_p = 0\}.$$

For the detailed conditions, see Theorem 4.15. Let us give an interesting example of the ARC dimension for an unbounded metric space.

Example 1.5. Let $f(n) : \mathbb{Z}_+ \to \mathbb{Z}_+$ be such that $f(n) \leq n$ for any n. For $n \geq 0$, divide $[2^n, 2^{n+1}] \times [0, 2^n]$ into $2^{f(n)} \times 2^{f(n)}$ blocks and call them G_n, and consider

$$G = \bigcup_{n \geq 0} G_n \cup \{(0, 0)\}$$

as a subgraph of $\mathbb{Z}^2$ (see Figure 2, and Example 5.1 for the precise definition).

Using Theorem 1.4, we obtain the following.

Proposition 1.6 (Proposition 5.2).

(1) *If* $\limsup_{n\to\infty} f(n) = \infty$, *then* $\dim_{\mathrm{AR}}(G, d) = 2$.

(2) *If* $\limsup_{n\to\infty} f(n) < \infty$, *then* $\dim_{\mathrm{AR}}(G, d) = 1$.

It is remarkable that only $\limsup_{n\to\infty} f(n) = \infty$ implies that $\dim_{\mathrm{AR}}(G, d) = 2$, although the size of the boxes, $2^{n-f(n)}$, may diverge.

We also compare the Ahlfors conformal dimension with the spectral dimension. For $p > 0$, we can define the upper and lower p-spectral dimension, $\overline{d}^S_p$ and $\underline{d}^S_p$, of a partition (see Definition 2.13).

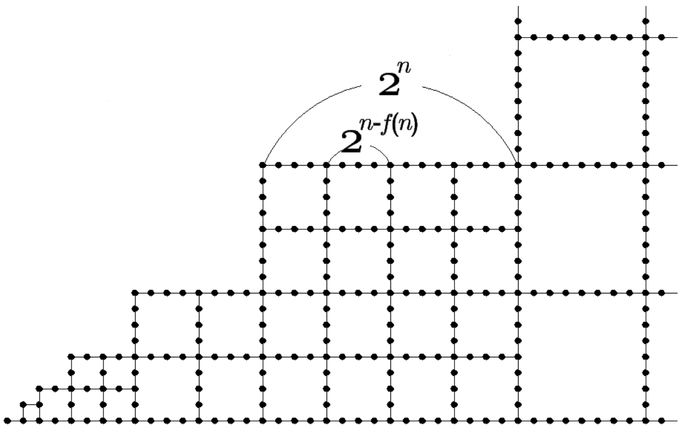

Figure 2. Example 1.5

We can further obtain the following.

Theorem 1.7 (Theorem 4.15 (2) and (3)). *Let (G, E) be a graph and d is a metric on G. Under the same conditions as in Theorem 1.4,*

- *if* $\dim_{\mathrm{AR}}(G, d) < p$, *then*

$$\dim_{\mathrm{AR}}(G, d) \le \underline{d}_p^S \le \overline{d}_p^S < p;$$

- *if* $\dim_{\mathrm{AR}}(G, d) \ge p$, *then*

$$\dim_{\mathrm{AR}}(G, d) \ge \overline{d}_p^S \ge \underline{d}_p^S \ge p.$$

When $p = 2$, the p-spectral dimension coincides in many examples with the notion of the spectral dimension of random walks, where the latter is defined as follows:

$$\overline{d}_S(G) = 2 \limsup_{n\to\infty} \frac{-\log p_{2n}(x,x)}{\log n}, \quad \underline{d}_S(G) = 2 \liminf_{n\to\infty} \frac{-\log p_{2n}(x,x)}{\log n},$$

where $p_n(x, y)$ is the transition density of the associated random walk. Hence, where this occurs, the latter theorem will also relate the ARC dimension and the spectral dimension of random walks. See Theorem 4.27 and Corollary 4.30 for a sufficient condition that the 2-spectral dimension and the spectral dimension of random walks coincide. However, we prove that they can also be different; see Example 5.4 for an example where this is the case.

This paper is based on my master's thesis [17] in Kyoto University. We omit the proof of some statements; for the full proof, see [17]. The outline of this paper is as follows. In Section 2, we give the framework and results of [12] for compact spaces without isolated points, on which the main result of this paper is based. In Section 3, we extend the results of [12] to σ-compact spaces without isolated points. Section 4 is the main part of this paper, which is devoted to proving our main results. In Section 5, we give examples that illustrate some properties of the ARC dimension of graphs.

Notation. In this paper, we use the following notation.

- Let A be a set and F be a map from A to itself. Then F^n denotes the composition

  $$\overbrace{F \circ \cdots \circ F}^{n},$$

  for $n > 0$, and id_A, for $n = 0$. Moreover, A^n denotes the product

  $$\overbrace{A \times \cdots \times A}^{n}.$$

- Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of sets, then $\bigsqcup_{\lambda \in \Lambda} A_\lambda$ denotes $\bigcup_{\lambda \in \Lambda} A_\lambda$ in the case that $A_\lambda \cap A_\tau = \emptyset$ for any $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$.
- Let f and g be functions with variables $x_1, \dots, x_n$. We say that $f \asymp g$ for any $(x_1, \dots, x_n) \in A$ if there exists $C > 0$ such that

  $$C^{-1} f(x_1, \dots, x_n) \le g(x_1, \dots, x_n) \le C f(x_1, \dots, x_n),$$

  for any $(x_1, \dots, x_n) \in A$.
- Let (X, d) be a metric space and suppose that μ is a (Borel) measure on (X, d), then we write

  $$B_d(x, r) = \{y \mid d(x, y) < r\}, \quad V_{d,\mu}(x, r) = \mu(B_d(x, r)).$$

  Whenever no confusion may occur, we omit d and/or μ.
- We also use the notation

  $$[n, m]_{\mathbb{Z}} = \{k \in \mathbb{Z} \mid n \le k \le m\},$$

  for $n, m \in \mathbb{Z}$, and

  $$a \vee b = \max\{a, b\}, \quad a \wedge b = \min\{a, b\}.$$

In the following, whenever $d \sim_{\mathrm{QS}} \rho$ for metrics d, ρ on a space, θ denotes a homeomorphism such that d is θ-quasisymmetric to ρ, if no confusion may occur. We will use the same notation for a vertex and its equivalent class.

2. Kigami's results for compact metric spaces

As the preparation for this paper, we introduce the results of [12] on which the results of this paper are based. For this purpose, we first give the notation we use for graphs.

Definition 2.1 (Graph, tree). Let T be an (at most) countable set and let $\mathcal{A} \subset T \times T$ be such that

- for any $w \in T$, $(w, w) \notin \mathcal{A}$;
- $(w, v) \in A$ if $(v, w) \in \mathcal{A}$.

We call (T, A) a *simple graph*. We write $w \sim v$ if $(w, v) \in \mathcal{A}$.

(1) A simple graph (T, A) is called *locally finite* if $\#(\{y \mid y \sim x\}) < \infty$ for any $x \in T$. We say that (T, A) has *bounded degree* if $\sup_{x \in T} \#(\{y \mid y \sim x\}) < \infty$.

(2) Let $n \geq 0$. We call $(w_0, w_1, \ldots, w_n) \in T^n$ an *n-path* (between w_0 and w_n) if $w_i \sim w_{i-1}$ for any $i \in [1, n]_{\mathbb{Z}}$. Furthermore, we call $(w_0, w_1, \ldots, w_n)$ an *n-simple path* (between w_0 and w_n) if it is an n-path and $w_i \neq w_j$ whenever $i \neq j$.
We call $(w_0, w_1, \ldots, w_n)$ a *path* if it is an n-path for some $n \geq 0$, and a *simple path* if it is an n-simple path for some $n \geq 0$.

(3) We call $(T, \mathcal{A})$ *connected* if there exists a path between w and v for any $w, v \in T$. Moreover, we call $(T, \mathcal{A})$ a *tree* if there exists an unique simple path between w and v for any $w, v \in T$.

(4) Let $(T, \mathcal{A})$ be a simple graph. We define $l_{\mathcal{A}}$ by

$$l_{\mathcal{A}}(w, v) = \min\bigl\{n \bigm| \text{there exists an } n\text{-path between } w \text{ and } v\bigr\}.$$

If $(T, \mathcal{A})$ is connected, then $l_{\mathcal{A}}$ is called the *graph metric* of $(T, \mathcal{A})$.

In this paper, we will consider only simple graphs.

Definition 2.2 (Rooted tree). Let $(T, \mathcal{A})$ be a tree and let $\phi \in T$. We call the triple $(T, \mathcal{A}, \phi)$ a *rooted tree*.

(1) For $w \in T$, define $|w| = l_{\mathcal{A}}(\phi, w)$ and $(T)_n = \{w \mid |w| = n\}$ for any $n \geq 0$, and define $\pi : T \to T$ by

$$\begin{aligned}\pi(w) &= \pi_{(T,A,\phi)}(w) \\ &= \begin{cases} w_{n-1}, & \text{if } w \neq \phi \text{ and } (\phi = w_0, \ldots, w_{n-1}, w_n = w) \\ & \qquad\qquad \text{is the unique simple path between } \phi \text{ and } w, \\ \phi, & \text{if } w = \phi. \end{cases}\end{aligned}$$

We use S to denote the inverse of π (excluding ϕ), namely

$$S(A) = \{w \in T \setminus \{\phi\} \mid \pi(w) \in A\}$$

for any $A \subset X$, and we write $S(w)$ instead of $S(\{w\})$. We call $|w|$ (resp. $\pi(w)$ and $S(w)$) the *depth* (resp. *parent* (or *predecessor*) and *children* (or *successors*)) of w.

(2) For $w, v \in T$, we say that w is an *ancestor* of v and v is a *descendant* of w if there exists $n \geq 1$ such that $w = \pi^n(v)$. Then T_w denotes the subtree consisting of all descendants of w and w itself.

(3) Define the *geodesics* of T (from ϕ) by

$$\Sigma = \{\omega = (\omega_n)_{n \geq 0} \mid \omega_n \in (T)_n,\ \pi(\omega_{i+1}) = \omega_i \text{ for all } i \geq 0\},$$

and the geodesics passing through w by $\Sigma_w = \{\omega \in \Sigma \mid \omega_{|w|} = w\}$ for any $w \in T$.

Throughout this section, $\mathcal{T} = (T, \mathcal{A}, \phi)$ is a locally finite rooted tree.

Definition 2.3 (Partition). Let $(X, \mathcal{O})$ be a compact metrizable space having no isolated points, and let $\mathcal{C}(X, \mathcal{O})$ be the collection of nonempty compact subsets of $(X, \mathcal{O})$ without single points. A map $K : T \to \mathcal{C}(X, \mathcal{O})$, where we write K_w instead of $K(w)$ for ease of notation, is called a *partition* of $(X, \mathcal{O})$ parametrized by $\mathcal{T}$ if it satisfies the following conditions.

(P1) $K_\phi = X$ and for any $w \in T$,

$$\bigcup_{v \in S(w)} K_v = K_w.$$

(P2) For any $\omega \in \Sigma$, $\bigcap_{m \geq 0} K_{\omega_m}$ is a single point.

We will consider the following notion.

(1) Let K be a partition of X. We define O_w by

$$O_w = K_w \setminus \Bigg(\bigcup_{v \in (T)_{|w|} \setminus \{w\}} K_v \Bigg).$$

Then K is called *minimal* if $O_w \neq \emptyset$ for any $w \in T$.

(2) For $m \geq 0$, we define $E^h_m \subset (T)_m \times (T)_m$ by

$$J^h_m = \{(w, v) \mid w, v \in (T)_m,\ w \neq v \text{ and } K_w \cap K_v \neq \emptyset\},$$

and $\Gamma_n(w) = \{v \in (T)_{|w|} \mid l_{J^h_m}(w, v) \leq n\}$ for any $w \in T$.

We simply write X for $(X, \mathcal{O})$ if no confusion may occur.

Definition 2.4 (Weight function). A function $g : T \to (0, 1]$ is called a *weight function* if it satisfies the following conditions.

(G1) $g(\phi) = 1$.

(G2) For any $w \in T$, $g(\pi(w)) \geq g(w)$.

(G3) For any $\omega \in \Sigma$, $\lim_{m\to\infty} g(\omega_m) = 0$.

We also consider the following.

(1) For $s > 0$, we define the scale Λ_s^g associated to g by

$$\Lambda_s^g = \begin{cases} \{w \in T \mid g(w) \leq s < g(\pi(w))\}, & \text{if } 0 < s < 1, \\ \{\phi\}, & \text{otherwise.} \end{cases}$$

We also define $E_s^g \subset \Lambda_s^g \times \Lambda_s^g$ by

$$E_s^g = \{(w, v) \mid w, v \in \Lambda_s^g,\ w \neq v \text{ and } K_w \cap K_v \neq \emptyset\}.$$

(2) For $x \in X$, $s > 0$, $M \geq 0$ and $w \in \Lambda_s^g$, we define

$$\Lambda_{s,M}^g(w) = \{v \in \Lambda_s^g \mid l_{E_s^g}(w, v) \leq M\}, \quad \Lambda_{s,M}^g(x) = \bigcup_{\substack{w \in \Lambda_s^g \\ \text{with } x \in K_w}} \Lambda_{s,M}^g(w),$$

and

$$U_M^g(x, s) = \bigcup_{w \in \Lambda_{s,M}^g(x)} K_w.$$

Definition 2.5. Let $(X, \mathcal{O})$ be a compact metrizable space having no isolated point, and K be a partition of X. Define

$$\mathcal{D}(X, \mathcal{O}) = \{d \mid d \text{ is a metric on } X \text{ inducing the topology } \mathcal{O} \text{ and } \operatorname{diam}(X, d) = 1\}.$$

For $d \in \mathcal{D}(X, \mathcal{O})$, define $g_d : T \to (0, 1]$ by $g_d(w) = \operatorname{diam}(K_w, d)$ for any $w \in T$.

Proposition 2.6 ([12, Proposition 2.3.5 (1)]). *Let $(X, \mathcal{O})$ be a compact metrizable space having no isolated point and K be a partition of X. For any $d \in \mathcal{D}(X, \mathcal{O})$, g_d is a weight function.*

We denote g_d by d if no confusion may occur. For example, we will use the notation $U_M^d(x, r)$ (resp. $d(w)$) instead of $U_M^{g_d}(x, r)$ (resp. $g_d(w)$).

For the purpose of stating the main result of [12], we introduce some properties of weight functions and metrics.

For the rest of this section, $(X, \mathcal{O})$ is a compact metrizable space and $d \in \mathcal{D}(X, \mathcal{O})$ (in other words, (X, d) is a compact metric space with $\mathrm{diam}(X, d) = 1$, and $\mathcal{O}$ is the induced topology). Moreover, K is a partition of X.

Definition 2.7. Let g be a weight function. Then

- g is called *uniformly finite* if
$$\sup\{\#(\Lambda^g_{s,1}(w)) \mid s > 0, w \in \Lambda^g_s\} < \infty. \tag{2.1}$$
- g is called *thick (with respect to K)* if there exists $\alpha > 0$ such that for any $w \in T$, $U^g_1(x, \alpha g(\pi(w))) \subset K_w$ for some $x \in K_w$.

Furthermore, d is called *uniformly finite* (resp. *thick*) if g_d is uniformly finite (resp. thick).

Definition 2.8. d is called *adapted* if there exist $\alpha_1, \alpha_2 > 0$ and $M \in \mathbb{N}$ such that
$$B_d(x, \alpha_1 r) \subset U^d_M(x, r) \subset B_d(x, \alpha_2 r)$$
for any $x \in X$ and $r \leq 1$.

Example 2.9 (Sierpiński carpet). Let $\{p_i\}_{i=1}^8 \subset \mathbb{C}$ be such that
$$p_1 = 0, \qquad p_2 = \frac{1}{2}, \qquad p_3 = 1, \qquad p_4 = 1 + \frac{1}{2}i,$$
$$p_5 = 1 + i, \qquad p_6 = \frac{1}{2} + i, \qquad p_7 = i, \qquad p_8 = \frac{1}{2}i,$$
and let $F_i = \frac{1}{3}(z - p_i) + p_i$ for any $i \in [1, 8]_{\mathbb{Z}}$. It is well known that there exists a unique compact set X such that $\bigcup_{i=1}^8 F_i(X) = X$, called the *Sierpiński carpet*. Let $T = \bigcup_{n \geq 0}([1, 8]_{\mathbb{Z}})^n$, where $([1, 8]_{\mathbb{Z}})^0 = \{\phi\}$ and define $\pi : T \setminus \{\phi\} \to T$ by
$$\pi(w) = \begin{cases} (w_1, w_2, \ldots, w_{n-1}), & \text{if } w = (w_1, w_2, \ldots, w_n) \in \bigcup_{n \geq 2}([1, 8]_{\mathbb{Z}})^n, \\ \phi, & \text{if } w \in [1, 8]_{\mathbb{Z}}. \end{cases}$$
We also let $\mathcal{A} = \{(w, v) \mid w = \pi(v) \text{ or } v = \pi(w)\}$, then $\mathcal{T} = (T, \mathcal{A}, \phi)$ is a rooted tree. Moreover, for $w \in T$, define $F_w : \mathbb{C} \to \mathbb{C}$ by
$$F_w = \begin{cases} F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}, & \text{if } w = (w_1, w_2, \ldots, w_n) \in \bigcup_{n \geq 1}[1, 8]^n_{\mathbb{Z}}, \\ \mathrm{id}_{\mathbb{C}}, & \text{if } w = \phi, \end{cases}$$
and define $K : T \to \mathcal{C}(X, \mathcal{O})$ by $K_w = F_w(X)$. Then K is a partition of X parametrized by $\mathcal{T}$. We also let $d(z, w) = \frac{\sqrt{2}}{2}|z - w|$, then $d \in \mathcal{D}(X, \mathcal{O})$ and by Proposition 2.6, g_d is a weight function. We can see that $d(w) = \mathrm{diam}(F_w(X), d) = 3^{-|w|}$, and hence $\Lambda^d_s = (T)_m$ for any $s \in (0, 1)$ and $m \geq 1$ such that $3^{-m} \leq s < 3^{-(m-1)}$. This implies that d is uniformly finite, thick, and adapted (for $M = 1$).

Definition 2.10. For any $N \geq 1$, $N_2 \geq N_1 \geq 0$ and $m \geq 0$, define

$$J^h_{N,m} = \{(w,v) \mid w, v \in (T)_m \text{ are such that } 0 < l_{J^h_m}(w,v) \leq N\},$$

and

$$\mathcal{E}_{p,k,w}(N_1,N_2,N) = \inf\Big\{\frac{1}{2}\sum_{(x,y)\in J^h_{N,|w|+k}} |f(x)-f(y)|^p \;\Big|\; f : (T)_{|w|+k} \to \mathbb{R} \text{ is such that } f|_{S^k(\Gamma_{N_1}(w))} \equiv 1,\ f|_{(S^k(\Gamma_{N_2}(w)))^c} \equiv 0\Big\}.$$

(We remark that $J^h_{1,m} = J^h_m$.) We also define

$$\begin{aligned}
\mathcal{E}_{p,k}(N_1,N_2,N) &= \sup_{w\in T} \mathcal{E}_{p,k,w}(N_1,N_2,N),\\
\overline{\mathcal{E}}_p(N_1,N_2,N) &= \limsup_{k\to\infty} \mathcal{E}_{p,k}(N_1,N_2,N),\\
\underline{\mathcal{E}}_p(N_1,N_2,N) &= \liminf_{k\to\infty} \mathcal{E}_{p,k}(N_1,N_2,N),\\
\overline{I}_{\mathcal{E}}(N_1,N_2,N) &= \inf\{p \mid \overline{\mathcal{E}}_p(N_1,N_2,N) = 0\},\\
\underline{I}_{\mathcal{E}}(N_1,N_2,N) &= \inf\{p \mid \underline{\mathcal{E}}_p(N_1,N_2,N) = 0\}.
\end{aligned}$$

As in [12], we define the basic framework as follows.

Definition 2.11. Let (X,d) be a metric space and assume K is minimal. We say d *satisfies the basic framework (with respect to K)* if the following conditions hold:

- $\sup_{w\in T} \#(S(w)) < \infty$.
- d is uniformly finite, thick and adapted.
- There exists $r \in (0,1)$ such that $d(w) \asymp r^{|w|}$ for any $w \in T$.

Theorem 2.12 ([12, Theorems 4.6.4 and 4.6.9]). *If K is minimal and d satisfies the basic framework, then*

$$\underline{I}_{\mathcal{E}}(N_1,N_2,N) = \overline{I}_{\mathcal{E}}(N_1,N_2,N) = \dim_{\mathrm{AR}}(X,d)$$

for any N, N_1, N_2 such that $N_2 - N_1$ is sufficiently large.

As a corollary of Theorem 2.12, we get the result for comparison between $\dim_{\mathrm{AR}}$ and the "p-spectral" dimension, defined for any $p > 0$, as follows.

Definition 2.13 (p-spectral dimension). Define

$$\overline{N}_* = \limsup_{k\to\infty} \sup_{w\in T} \#(S^k(w))^{1/k},$$

$$\overline{R}_p(N_1, N_2, N) = \limsup_{k\to\infty} \mathcal{E}_{p,k}(N_1, N_2, N)^{1/k},$$
$$\underline{R}_p(N_1, N_2, N) = \liminf_{k\to\infty} \mathcal{E}_{p,k}(N_1, N_2, N)^{1/k},$$

and the *upper p-spectral dimension* $\overline{d}_p^S(N_1, N_2, N)$ and the *lower p-spectral dimension* $\underline{d}_p^S(N_1, N_2, N)$ by

$$\overline{d}_p^S(N_1, N_2, N) = \frac{p \log \overline{N}_*}{\log \overline{N}_* - \log \overline{R}_p(N_1, N_2, N)},$$
$$\underline{d}_p^S(N_1, N_2, N) = \frac{p \log \overline{N}_*}{\log \overline{N}_* - \log \underline{R}_p(N_1, N_2, N)}.$$

Corollary 2.14 ([12, Theorem 4.7.9]). *Assume K is minimal and d satisfies the basic framework. Let $N, N_1, N_2 \geq 0$ be such that $N_2 - N_1$ is sufficiently large.*

(1) *If $\underline{R}_p(N_1, N_2, N) < 1$, then*

$$\dim_{\mathrm{AR}}(X, d) \leq \underline{d}_p^S(N_1, N_2, N) \leq \overline{d}_p^S(N_1, N_2, N) < p.$$

(2) *If $\overline{R}_p(N_1, N_2, N) \geq 1$, then*

$$\dim_{\mathrm{AR}}(X, d) \geq \overline{d}_p^S(N_1, N_2, N) \geq \underline{d}_p^S(N_1, N_2, N) \geq p.$$

3. Extension to σ-compact metric spaces

In order to obtain the former results for infinite graphs, we first extend these theorems to σ-compact spaces. To do that, we introduce the notion of a bi-infinite tree.

Definition 3.1. Let T be a countable set and $\pi_T : T \to T$ be a map which satisfies the following.

(T1) For any $w, v \in T$, there exist $n, m \geq 0$ such that $\pi^n(w) = \pi^m(v)$.

(T2) For any $n \geq 1$ and $w \in T$, $\pi^n(w) \neq w$.

Then we define $\mathcal{A}_\pi = \{(w, v) \mid \pi(w) = v \text{ or } \pi(v) = w\}$ and consider the simple graph $(T, \mathcal{A}_\pi)$. We call (T, π) a *bi-infinite tree*.

As for rooted trees, we denote the inverse of π by S and write $S(w)$ instead of $S(\{w\})$. Moreover, we define the subtree $T_w = \{v \in T \mid \pi^n(v) = w \text{ for some } n \geq 0\}$.

Lemma 3.2. *Let (T, π) be a bi-infinite tree and fix $w, v \in T$. Then $n - m$ is constant if $\pi^n(w) = \pi^m(v)$.*

Proof. Assume $\pi^{n_i}(w) = \pi^{m_i}(v)$ $(i = 1, 2)$. Without loss of generality, we may assume $n_1 \leq n_2$. Then

$$\pi^{m_2}(v) = \pi^{n_2}(w) = \pi^{n_2-n_1} \circ \pi^{n_1}(w) = \pi^{n_2-n_1} \circ \pi^{m_1}(v).$$

This together with (T2) shows $m_2 = (n_2 - n_1) + m_1$, which means that $m_2 - n_2 = m_1 - n_1$. ∎

Definition 3.3. Let (T, π) be a bi-infinite tree.

(1) Let $\phi \in T$. We call the triple (T, π, ϕ) a *bi-infinite tree with a reference point* ϕ, and for any $w \in T$ we define the *height* of vertices by $[w] = [w]_\phi = n - m$, where $\pi^n(w) = \pi^m(\phi)$. We also define $(T)_n = \{w \in T \mid [w] = n\}$ for any $n \in \mathbb{Z}$.

(2) We define the *set of (descending) geodesics of* T by $\Sigma^* = \{\omega = (\omega_n)_{n\in\mathbb{Z}} \mid \omega_n \in (T)_n,\ \pi(\omega_{n+1}) = \omega_n \text{ for all } n \in \mathbb{Z}\}$ and the *set of geodesics passing through* w by $\Sigma^*_w = \{\omega \in \Sigma^* \mid \omega_{[w]} = w\}$ for any $w \in T$.

Remark 3.4. The property (T1) and Lemma 3.2 ensure that $[w]$ is well-defined. Moreover, for fixed $w, v \in T$, $[w]_\phi - [v]_\phi$ is constant for every $\phi \in T$ by Lemma 3.2 (that is, the difference of the height of a bi-infinite tree is determined only by π and does not depend on its reference point).

As the name shows, a bi-infinite tree is a tree.

Proposition 3.5. *Let (T, π) be a bi-infinite tree, then $(T, \mathcal{A}_\pi)$ is a tree.*

Proof. For all $w, v \in T$, there exists a path between them by (T1), and also a simple path exists.

Next we prove the uniqueness of a simple path. Fix any reference point $\phi \in T$ and think about $(T, \mathcal{A}, \phi)$. By definition, if $w = \pi(v)$, then $[w] = [v] - 1$. Let $(w_0, w_1, \ldots, w_n)$ be a simple path. If $[w_i] > [w_{i+1}]$, then $\pi(w_i) = w_{i+1}$ and so $w_{i-1} \neq \pi(w_i)$, which means $[w_{i-1}] > [w_i]$. In the same way, we can see $[w_{i+1}] > [w_i]$ if $[w_i] > [w_{i-1}]$.

Now let $(w, \pi(w), \ldots, \pi^{n_1}(w) = \pi^{m_1}(v), \ldots, \pi(v), v)$ and $(w, \pi(w), \ldots, \pi^{n_2}(w) = \pi^{m_2}(v), \ldots, \pi(v), v)$ be simple paths. If $n_1 < n_2$ then $m_1 < m_2$ by Lemma 3.2, so the latter simple path take $\pi^{n_1}(w) = \pi^{m_1}(v)$ two times, which is contradiction. The case $n_1 > n_2$ is the same. If $n_1 = n_2$ then $m_1 = m_2$ by Lemma 3.2, so the paths are equal. Therefore $(T, \mathcal{A})$ is a tree. ∎

The root of a bi-infinite tree does not exist, but "$\pi^\infty(w)$" is thought to be a virtual root.

Now we extend notions of partitions and weight functions to σ-compact spaces. For the rest of this section, let $\mathcal{T} = (T, \pi, \phi)$ be a locally finite bi-infinite tree with a reference point. Note that (T, π) is locally finite if and only if $\#(S(w)) < \infty$ for any $w \in T$.

Definition 3.6 (Partition). Let $(X, \mathcal{O})$ be a σ compact metrizable space having no isolated points, and let $\mathcal{C}(X, \mathcal{O})$ be the collection of nonempty compact subsets of $(X, \mathcal{O})$ without single points. A map $K : T \to \mathcal{C}(X, \mathcal{O})$, where we denote $K(w)$ by k_w for ease of notation, is called a *partition of* $(X, \mathcal{O})$ *parametrized by* $\mathcal{T}$ if it satisfies the following conditions.

(P1) For any $w \in T$,

$$\bigcup_{v \in S(w)} K_v = K_w.$$

(P2) For any $\omega \in \Sigma^*$, $\bigcap_{m \geq 0} K_{\omega_m}$ is a single point.

(P3) $\bigcup_{w \in (T)_0} K(w) = X$.

We say a partition K is *locally finite* if it satisfies that for any $w \in (T)_0$, there exists an open set U_w which satisfies $K_w \subset U_w$ and $\#\{v \in (T)_0 \mid K_v \cap U_w \neq \emptyset\} < \infty$.

We define O_w, J^h_M, $\Gamma_n(w)$ and minimality in the same way as in the compact case, and similarly use X instead of $(X, \mathcal{O})$.

Remark 3.7. Condition (P3) is the counterpart of $K_\phi = X$ in Definition 2.3. The locally finiteness of a partition is used to reduce local properties of partitions of σ-compact spaces to the compact case.

Definition 3.8 (Weight function). A function $g : T \to (0, \infty)$ is called a *weight function* if it satisfies the following conditions.

(G1) $\lim_{n\to\infty} g(\pi^n(\phi)) = \infty$.

(G2) For any $w \in T$, $g(\pi(w)) \geq g(w)$.

(G3) For any $\omega \in \Sigma^*$, $\lim_{m\to\infty} g(\omega_m) = 0$.

For $s > 0$, we define the scale Λ^g_s associated to g by

$$\Lambda^g_s = \{w \in T \mid g(w) \leq s < g(\pi(w))\},$$

and define E^g_s, $\Lambda^g_{s,M}(w)$, $\Lambda^g_{s,M}(x)$, $U^g_M(x, s)$ in the same way as in Definition 2.4.

Definition 3.9. Let $(X, \mathcal{O})$ be a σ-compact metrizable space having no isolated points and K be a partition of X. Define

$$\mathcal{D}_\infty(X, \mathcal{O}) = \{d \mid d \text{ is a metric on } X \text{ inducing the topology } \mathcal{O} \text{ and } \\ \operatorname{diam}(X, d) = \infty\}.$$

For $d \in \mathcal{D}_\infty(X, \mathcal{O})$, define $g_d : T \to (0, \infty)$ by $g_d(w) = \operatorname{diam}(X, d)$ for any $w \in T$.

Similar to the compact case, we can obtain the following.

Proposition 3.10. *Let $(X, \mathcal{O})$ be a σ-compact metrizable space having no isolated point and K be a partition of X, then for any $d \in \mathcal{D}_\infty(X, \mathcal{O})$, g_d is a weight function.*

Remark 3.11. Condition (G1) follows from (P1).

We denote g_d by d if no confusion may occur. For the rest of this section, $(X, \mathcal{O})$ is a σ-compact metrizable space having no isolated points and $d \in \mathcal{D}_\infty(X, \mathcal{O})$. Moreover, K is a partition of X.

We introduce the properties of d and a weight function g in the same way as in Definitions 2.7, 2.8 and 2.11, and we also introduce variables in the same way as in Definitions 2.10 and 2.13. By using bi-infinite trees and these properties, we can extend the theory of [12] to σ-compact spaces. In particular, we get the following result.

Theorem 3.12. *Assume K is locally finite and minimal. If d satisfies the basic framework, then for any N, N_1, N_2 such that $N_2 - N_1$ is sufficiently large,*

(1) $\underline{I}_{\mathcal{E}}(N_1, N_2, N) = \overline{I}_{\mathcal{E}}(N_1, N_2, N) = \dim_{\mathrm{AR}}(X, d)$.

(2) *If* $\underline{R}_p(N_1, N_2, N) < 1$, *then*

$$\dim_{\mathrm{AR}}(X, d) \le \underline{d}_p^S(N_1, N_2, N) \le \overline{d}_p^S(N_1, N_2, N) < p.$$

(3) *If* $\overline{R}_p(N_1, N_2, N) \ge 1$, *then*

$$\dim_{\mathrm{AR}}(X, d) \ge \overline{d}_p^S(N_1, N_2, N) \ge \underline{d}_p^S(N_1, N_2, N) \ge p.$$

Idea of the proof. Most of the results in [12] do not use the property $T = T_\phi$ nor compactness of X. Therefore if K is locally finite, we can prove σ-compact version for most of the statements in [12] line by line in the same way. In order to prove the rest of statements, we essentially need the following lemma.

Lemma 3.13. *Let g be a weight function on T. Then for any $s \in (0, \infty)$, $w \in \Lambda_s^g$, $M \ge 0$, there exists $v \in T$ such that $\Lambda_{s,M}^g(w) \subset T_v$.*

Proof. Let $\tilde{\Lambda}_{s,1}^g(w) := \{\pi^{0 \vee ([v]-[w])}(v) \mid v \in \Lambda_{s,1}^g(w)\}$, then $\#\tilde{\Lambda}_{s,1}^g(w) < \infty$ because K is locally finite. And, inductively, we define

$$\tilde{\Lambda}_{s,n+1}^g(w) := \bigcup \{\tilde{\Lambda}_{g(v),1}^g(v') \mid \text{for some } v \in \tilde{\Lambda}_{s,n}^g(w), v' \in \Lambda_{g(v)}^g \text{ and } v \in T_{v'}\},$$

then $\#\tilde{\Lambda}_{s,M}^g(w) < \infty$ for all M.

Moreover, for any $v \in \Lambda_{s,n}^g(w)$, there exists $v' \in \tilde{\Lambda}_{s,n}^g(w)$ such that $v \in T_{v'}$. Therefore $\Lambda_{s,M}^g(w) \subset \cup_{v \in \tilde{\Lambda}_{s,M}^g(w)} T_v \subset T_u$ for some u because of (T1). ■

Using this lemma, we can consider the problems in subtrees and so can apply the results of the compact case. See [17, Section 3] for details.

4. Ahlfors regular conformal dimension of infinite graphs

In this section, we give results about the ARC dimensions and the spectral dimensions of metrics on infinite graphs, which are the main results of this paper. To get these results, the cable systems of graphs play an important role. Technically, the main contribution of this paper is to show that the ARC dimension and a partition of a graph coincide with those of its cable system. Cable systems do *not* appear in statements of main results, but we use them and adapt the results of former sections and lead results for graphs. Throughout this section, G is a countable (infinite) set, (G, E) is a connected, bounded degree graph and $\mathcal{T} = (T, \pi, \phi)$ is a bi-infinite tree with a reference point.

4.1. Ahlfors regular conformal dimension of metrics on infinite graphs

We first denote a class of metrics on (G, E), which we consider in this paper.

Definition 4.1 (Fitting metric). We say a metric d on G *fits to* (G, E) if it satisfies the following conditions.

(F1) There exists $C > 0$ such that $d(x, y) \leq Cd(x, z)$ for any $x, y, z \in G$ with $x \sim y$ and $x \neq z$.

(F2) For any $\varepsilon > 0$, there exist $r > 0$, $n \geq 1$ and $\{x_i\}_{i=0}^{n} \subset G$ such that

- $x_i \in B_d(x_0, r)$ for any $i \in [0, n-1]_{\mathbb{Z}}$ and $x_n \notin B_d(x_0, r)$,
- $d(x_i, x_{i-1}) \leq \varepsilon r$ and $x_i \sim x_{i-1}$ for any $i \in [1, n]_{\mathbb{Z}}$.

If the graph (G, E) is fixed or clear, we simply say d is *fitting* when d fits to (G, E).

Condition (F1) is a natural condition for metrics on G. For example, the graph distance l_E and "gently weighted" graph distances satisfy (F1). Moreover, the effective resistance of a weighted graph with controlled weight, which we will introduce later, also satisfies (F1). Condition (F2) is a little technical, which is needed to evaluate $\dim_{\mathrm{AR}}(G, d)$.

Example 4.2. Let $G = \mathbb{Z}$ and $E = \{(n, m) \mid |n - m| = 1\}$. For any $k \geq 1$, let $x_i = i$ for $i \in [0, k]_{\mathbb{Z}}$, then $l_E(i + 1, i) \leq \frac{k}{k}$ and $x_k \notin B_{l_E}(x_0, k)$, so l_E satisfies (F2). On the other hand, let $d(n, m) := |2^n - 2^m|$. For any simple path $(n, n + 1, \dots, n + k)$, $d(n + k - 1, n + k) = 2^{n+k-1} \geq \frac{d(n,n+k)}{2}$. Hence in this case d does not satisfy (F2) for $\varepsilon < 1/2$.

We remark that for any (G, E), l_E satisfies (F2).

Lemma 4.3. *Let d, ρ be metrics on G and suppose that $d \sim_{\mathrm{QS}} \rho$. If d fits to (G, E), then ρ fits to (G, E).*

For this lemma and later statements, now we recall basic properties of quasisymmetry. Let (X, d) and (X, ρ) be metric spaces.

(1) Let $\theta : [0, \infty) \to [0, \infty)$ be a homeomorphism, then the following conditions are equal:

  (a) d is θ-quasisymmetric to ρ.

  (b) $\rho(x, z) \leq \theta(t)\rho(x, z)$ whenever $d(x, y) \leq td(x, z)$.

  (c) $\rho(x, z) < \theta(t)\rho(x, z)$ whenever $d(x, y) < td(x, z)$.

(2) If $d \sim_{\mathrm{QS}} \rho$ and $\operatorname{diam}(X, d) = \infty$, then $\operatorname{diam}(X, \rho) = \infty$.

(3) $\sim_{\mathrm{QS}}$ is an equivalence relation between metrics on X.

(4) If $d \sim_{\mathrm{QS}} \rho$, then both (X, d) and (X, ρ) induce the same topology (in other words, id_X is a homeomorphism between (X, d) and (X, ρ)).

Property (1) follows from the monotonicity of θ. For properties (2)–(4), see [6, Section 10], for example.

Proof of Lemma 4.3. Since d satisfies (F1) and $d \sim_{\mathrm{QS}} \rho$, $\rho(x, y) \leq \theta(C)\rho(x, z)$ for any $x, y, z \in G$ with $x \sim y$ and $x \neq z$, so ρ satisfies (F1). Next, we show ρ satisfies (F2). Fix any $\varepsilon > 0$. Let $\delta < 1/2$ such that $\theta(2\delta)\theta(3) < \varepsilon$. Since d satisfies (F2), there exists $\{x_i\}_{i=0}^n \subset G$ such that

- $x_i \in B_d(x_0, r)$ for any $i \in [0, n-1]_{\mathbb{Z}}$ and $x_n \notin B_d(x_0, r)$,
- $d(x_i, x_{i-1}) \leq \delta r$ and $x_i \sim x_{i-1}$ for any $i \in [1, n]_{\mathbb{Z}}$.

Let $i \in [0, n-1]_{\mathbb{Z}}$. Since $d(x_0, x_n) \geq r$ and $x_i, x_n \in B_d(x_0, (1+\delta)r)$,

$$\frac{r}{2} \leq d(x_0, x_i) \vee d(x_i, x_n) < 3r$$

and hence

$$\rho(x_i, x_{i+1}) \leq \theta(2\delta)(\rho(x_0, x_i) \vee \rho(x_i, x_n)) \leq \theta(2\delta)\theta(3)\rho(x_0, x_n).$$

Let $m = \min\{i \mid x_i \notin B_\rho(x_0, \rho(x_0, x_n))\}$, then $r = \rho(x_0, x_n)$ and $\{x_i\}_{i=0}^m$ satisfies (F2) for ε. ∎

Next, we introduce partitions of infinite graphs.

Definition 4.4 (Partition). A map $K : T \to \{A \subset G \mid \#(A) < \infty\}$ is called a *partition of* (G, E) *parametrized by* $\mathcal{T}$ if it satisfies following conditions.

(PG1) $\bigcup_{v \in S(w)} K_v = K_w$ for any $w \in T$.

(PG2) For any $\omega \in \Sigma_*$, there exist $n_0(\omega) \in \mathbb{Z}$ and $x, y \in G$ such that $x \sim y$ and $K_{\omega_n} = \{x, y\}$ for any $n \geq n_0(\omega)$.

(PG3) For any $(x, y) \in E$, there exists $w \in T$ such that $K_w = \{x, y\}$.

For the rest of this section, K is a partition of (G, E) parametrized by $\mathcal{T}$.

Lemma 4.5. *Let* $\Lambda_e = \{w \in T \mid \#(K_w) = 2$ *and* $\#(K_{\pi(w)}) > 2\}$, *then*

$$\bigsqcup_{w \in \Lambda_e} \Sigma^*_w = \Sigma^*.$$

Proof. $\cup_{w \in \Lambda_e} \Sigma^*_w = \Sigma^*$ directly follows from (PG2). By (PG1), $\#(K_{\omega_n})$ is non-increasing for any $\omega \in \Sigma^*$, so there exists a unique $n \in \mathbb{Z}$ such that $\omega_n \in \Lambda_e$. This shows $\Sigma^*_w \cap \Sigma^*_v = \emptyset$ for any $w, v \in \Lambda_e$ with $w \neq v$. ■

Definition 4.6. (1) We denote $\omega_{n_0(\omega)}$ by ω_e where $n_0(\omega)$ is in condition (PG2). We also define T_e by

$$T_e = \bigl\{w \in T \bigm| T_w \cap \Lambda_e \neq \emptyset\bigr\} = \bigl\{w \in T \bigm| \#(K_{\pi(w)}) > 2\bigr\}.$$

For $w \in (T \setminus T_e) \cup \Lambda_e$, we define $w_e \in \Lambda_e$ such that $w \in T_{w_e}$.

(2) K is called *minimal* if $K_w \neq K_v$ for any $w, v \in \Lambda_e$ with $w \neq v$.

Definition 4.7 (Discrete weight function). Recall that K is a partition of (G, E). A function $g : T_e \to (0, \infty)$ is called a *discrete weight function (with respect to* K*)* if it satisfies following conditions.

(GG1) For some $w \in T_e$, $\lim_{n \to \infty} g(\pi^n(w)) = \infty$.

(GG2) For any $w \in T_e$, $g(\pi(w)) \geq g(w)$.

For $s > 0$, we define the scale Λ^g_s associated to g by

$$\Lambda^g_s = \bigl\{w \in T_e \bigm| g(w) \leq s < g(\pi(w))\bigr\},$$

and define E^g_s, $\Lambda^g_{s,M}(w)$, $\Lambda^g_{s,M}(x)$ in the same way as in the compact case. We also define $U^g_M(x, s)$ for $M \geq 0, x \in G$ and $s > 0$ by

$$U^g_M(x, s) = \begin{cases} \{x\}, & \text{if } \Lambda^g_{s,M}(x) = \emptyset, \\ \bigcup_{w \in \Lambda^g_{s,M}(x)} K_w, & \text{otherwise.} \end{cases}$$

Remark 4.8. In contrast to the compact and σ-compact cases, Σ^* is not necessarily equal to $\bigsqcup_{w\in\Lambda_s^g}\Sigma_w^*$ since they are restricted to T_e. The difference also appears in the definition of $U_M^g(x,s)$.

Lemma 4.9. *Define*

$$\mathcal{D}_\infty(G)=\left\{d \,\middle|\, d \textit{ is a metric on } G \textit{ such that } \operatorname{diam}(G,d)=\infty\right\}$$

and let $d\in\mathcal{D}_\infty(G)$. We also define $g_d: T_e\to(0,\infty)$ by $g_d(w)=\max_{x,w\in K_w} d(x,y)$. Then g_d is a discrete weight function.

We denote g_d by d if no confusion may occur.

Definition 4.10. Let g be a discrete weight function.

- g is called *uniformly finite* if (2.1) holds (see Definition 2.7).
- g is called *thick (with respect to K)* if there exists $\alpha>0$ such that for any $w\in T_e$, $\Lambda^g_{\alpha g(\pi(w)),1}(x)\subset T_w$ for some $x\in K_w$.

The metric $d\in\mathcal{D}_\infty(G)$ is called *uniformly finite* and *thick* if g_d is uniformly finite and thick, respectively.

We define adapted in the same way as in the compact case, and define the following properties.

Definition 4.11. Let (G,d) be a metric space and assume K is minimal. We say (G,d) *satisfies the basic framework (with respect to K)* if the following conditions hold.

- $\sup_{w\in T_e\setminus\Lambda_e}\#(S(w))<\infty$.
- d is uniformly finite, thick and adapted.
- There exists $r\in(0,1)$ such that $d(w)\asymp r^{[w]}$ for any $w\in T_e$.

The difference between these definitions and those in the compact case is given by the T_e's in the notation.

Definition 4.12. Let $r\in(0,1)$. For $w\in\Lambda_e$ and $n\geq 0$, let

$$\mathfrak{S}_{w,m}=\left\{\{(x,k),(y,2^{n(m)}-1-k)\}_{w,m} \,\middle|\, k\in[0,2^{n(m)}-1]_{\mathbb{Z}}\right\}$$

where $x,y\in G$, $n(m)\in\mathbb{N}$ such that $K_w=\{x,y\}$ and $2^{-n(m)}\leq r^m<2^{1-n(m)}$. Then we define

$$T_r=T_e\sqcup\left(\bigcup_{w\in\Lambda_e}\bigsqcup_{m\geq 1}\mathfrak{S}_{w,m}\right)$$

and $\pi': T_r\to T_r$ by

$$\pi'(w) = \begin{cases} \pi(w), & \text{if } w \in T_e, \\ v, & \text{if } w \in \mathfrak{S}_{v,1}, \\ \{(x,l),(y,2^{n(m-1)}-1-l)\}_{v,n-1}, & \text{if } w = \{(x,k),(y,2^{n(m)}-1-k)\}_{v,m} \\ & \text{and } \left[\frac{k}{2^{n(m)}}, \frac{k+1}{2^{(n(m))}}\right] \subset \left[\frac{l}{2^{n(m-1)}}, \frac{l+1}{2^{n(m-1)}}\right]. \end{cases}$$

Moreover, we define K' by $K'_w = K_w$ (if $w \in T_e$) and K_v (if $w \in \bigsqcup_{m \geq 1} \mathfrak{S}_{v,m}$).

Lemma 4.13. *(T_r, π') is a bi-infinite tree and K' is a partition of (G, E). Moreover, if we write $[w]'$ for the height of (T_r, π', ϕ') for $\phi' \in T_r$ and Λ'_e for the K' version of Λ_e, then $\Lambda_e = \Lambda'_e$ and we can take $\phi' \in T_r$ such that $[w] = [w]'$.*

Proof. For any $w \in T_r \setminus T_e$, by the definition of π', we have

- $\pi^n(w) \in \Lambda_e$ for some $n > 0$.
- For any $n > 0$, $\pi^n(w) \neq w$.

These and conditions (T1), (T2) of π show conditions (T1), (T2) of π', so (T_r, π') is a bi-infinite tree. Fix $w \in T_e$ and let $\phi' \in S^{[w]}$, then $[w] = [w]'$ and, by Lemma 3.2, $[v] = [v]'$ for any $v \in T_e$ because $\pi = \pi'$ on T_e. The rest of this lemma is clear by definition. ∎

We remark that discrete weight functions and their properties are given only by T_e, so they do not change if we replace (T, π, ϕ) by (T_r, π', ϕ'). For the rest of this section, assume $(T, \pi, \phi) = (T_r, \pi', \phi')$.

Definition 4.14. Now we formally define $\underline{K}$ on T_r by

$$\underline{K}_w = \begin{cases} K_w, & \text{if } w \in T_e, \\ \{x\}, & \text{if } w = \{(x,0),(y,2^{n(m)}-1)\}_{m,v} \text{ for some } m, v, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We also define $J^h_m \subset (T)_m \times (T)_m$ by

$$J^h_m = J^h_m(K) = \Big\{(w,u) \,\Big|\, w,u \in (T)_m,\ \underline{K}_w \cap \underline{K}_v \neq \emptyset \text{ or there exist } v \in T_e, i \geq 0 \\ \text{such that } w = \{(x,i),(y,2^{n(m-[v])}-1-i)\}_{w,m-[v]}, \\ u = \{(x,i-1),(y,2^{n(m-[v])}-i)\}_{w,m-[v]}\Big\}.$$

The edges J^h_m can be constructed as follows: define

$$\hat{J}^h_m = \{(w,v) \mid w,v \in ((T)_m \cap T_e) \cup \Lambda_{e,m}, K_w \cap K_v \neq \emptyset\},$$

where $\Lambda_{e,m} = \{w_e \mid w \in (T)_m \setminus T_e\}$, and we replace each $w \in \Lambda_{e,m}$ by a $2^{n(m-[v])}$-path. We will justify this idea later in the cable system. We define $\underline{I}_{\mathcal{E}}, \overline{I}_{\mathcal{E}}, \underline{R}_p, \overline{R}_p$, etc. in the same way as in Definitions 2.10 and 2.13.

The following is one of the two main theorems of this paper.

Theorem 4.15. *Let K be a minimal partition of (G, E). If $d \in \mathcal{D}_\infty(G)$ satisfies the basic framework in Definition 2.11 and fits to (G, E), then for any N, N_1, N_2 such that $N_2 - N_1$ is sufficiently large,*

(1) $\underline{I}_{\mathcal{E}}(N_1, N_2, N) = \overline{I}_{\mathcal{E}}(N_1, N_2, N) = \dim_{\mathrm{AR}}(G, d)$.

(2) *If $\underline{R}_p(N_1, N_2, N) < 1$, then*

$$\dim_{\mathrm{AR}}(G, d) \leq \underline{d}_p^S(N_1, N_2, N) \leq \overline{d}_p^S(N_1, N_2, N) < p.$$

(3) *If $\overline{R}_p(N_1, N_2, N) \geq 1$, then*

$$\dim_{\mathrm{AR}}(G, d) \geq \overline{d}_p^S(N_1, N_2, N) \geq \underline{d}_p^S(N_1, N_2, N) \geq p.$$

In order to prove the theorem, we introduce the notion of cable system.

Definition 4.16 (Cable system). Let $\simeq$ denote the minimal equivalence relation on $G \times G \times [0, 1]$ which satisfies

- $((x, y), 0) \simeq ((x, z), 0)$ for any $x, y, z \in G$,
- $((x, y), t) \simeq ((y, x), 1 - t)$ for any (x, y) and $t \in [0, 1]$.

Then we define the *cable system* $\mathfrak{C}_G$ of (G, E) by $\mathfrak{C}_G := (E \times [0, 1])/\simeq$.

For $(x, y) \in E$, we also define $\iota(x, y) = (x, y) \times [0, 1]/\simeq$. Moreover, for any $x \in G$, $\tau(x) = ((x, y), 0)/\simeq$ where $(x, y) \in E$, is well-defined because of the definition of $\simeq$. We equate $\tau(x)$ with x and regard G as a subset of $\mathfrak{C}_G$.

Definition 4.17 (Induced cable metric). Let $\alpha \in (0, 1]$ and $d \in \mathcal{D}_\infty(G)$. We define an *induced cable metric* $d_{c,\alpha} : \mathfrak{C}_G \times \mathfrak{C}_G \to [0, \infty)$ by

$$d_{\mathfrak{C},\alpha}(x, y) = \begin{cases} |t - s|^\alpha d(x_0, x_1), & \text{if there exists } (x_0, x_1) \in E \text{ such that } x \simeq ((x_0, x_1), t) \text{ and } y \simeq ((x_0, x_1), s), \\ \min\{t^\alpha d(x_1, x_0) + d(x_0, y_0) + s^\alpha d(y_0, y_1) \mid x \simeq ((x_0, x_1), t) \text{ and } y \simeq ((y_0, y_1), s)\}, & \text{otherwise.} \end{cases}$$

We write $d_{\mathfrak{C}}$ instead of $d_{\mathfrak{C},1}$. Note that $(\mathfrak{C}_G, d_{\mathfrak{C},\alpha})$ is a metric space. We also note that $d(x, y) = d_{\mathfrak{C},\alpha}(x, y)$ for any $x, y \in G$ since d satisfies the triangle inequality.

For the notation of a partition, we write $K_w = \{w^+, w^-\}$ for each $w \in \Lambda_e$.

Definition 4.18. Define $\mathcal{K} : T = T_r \to \mathfrak{C}_G$ by

$$\mathcal{K}_w = \begin{cases} \bigcup_{\omega \in \Sigma_w^*} \iota(\omega_e^+, \omega_e^-), & \text{if } w \in T_e, \\ (x, y) \times \left[\frac{k}{2^{n(m)}}, \frac{k+1}{2^{n(m)}}\right], & \text{if } w = \left\{(x, k), (y, 2^{n(m)} - 1 - k)\right\}_{w,m}. \end{cases}$$

Lemma 4.19. (1) *$\mathcal{K}$ is a partition of $(\mathfrak{C}_G . d_{\mathfrak{C},\alpha})$.*

(2) *for any $w, v \in T_e$, $\mathcal{K}_w \cap \mathcal{K}_v \neq \emptyset$ if and only if $K_v \cap K_w \neq \emptyset$.*

The following proposition plays the key role in the proof of the main theorem.

Proposition 4.20. *Let $d \in \mathcal{D}_\infty(G)$ and fitting to (G, E). Then*

$$1 \leq \dim_{\mathrm{AR}}(G, d) = \dim_{\mathrm{AR}}(\mathfrak{C}_G, d_{\mathfrak{C}}).$$

Idea of the proof. The proof of this proposition consists of four steps:

(1) $1 \leq \dim_{\mathrm{AR}}(G, d)$,

(2) $\dim_{\mathrm{AR}}(G, d) \leq \dim_{\mathrm{AR}}(\mathfrak{C}_G, d_{\mathfrak{C}})$,

(3) $d \sim_{\mathrm{QS}} \rho$ implies that $d_{\mathfrak{C}} \sim_{\mathrm{QS}} \rho_{\mathfrak{C},1/\alpha}$ for $\alpha \geq 1$, and

(4) $\dim_{\mathrm{AR}}(G, d) \geq \dim_{\mathrm{AR}}(\mathfrak{C}_G, d_{\mathfrak{C}})$.

Property (1) follows from condition (F2). If some α-Ahlfors regular metric ρ_X on $\mathfrak{C}_G$ satisfies $d_C \sim_{\mathrm{QS}} \rho_X$, we can see that $\rho = \rho_X|_{G \times G}$ is α-Ahlfors regular (as a metric on G) and $d \sim_{\mathrm{QS}} \rho$, which implies (2). Property (3) essentially follows from condition (F1). Similar to (2), if a metric ρ on G is α-Ahlfors regular for $\alpha \geq 1$ and $d \sim_{\mathrm{QS}} \rho$, we can see $\rho_{\mathfrak{C},1/\alpha}$ is α-Ahlfors regular. This together with (1) and (3) implies (4).

We can also obtain the following lemma.

Lemma 4.21. *Under the same assumption of Theorem 4.15, $d_{\mathfrak{C}}$ satisfies the assumptions of Theorem 3.12 with respect to the partition $\mathcal{K}_r$.*

Using these, we can prove Theorem 4.15.

Proof of Theorem 4.15. Let $(w, v) \in T_m$ and suppose $\mathcal{K}_w \cap \mathcal{K}_v$ does not intersect with G. Then $\mathcal{K}_w \cap \mathcal{K}_v \neq \emptyset$ if and only if

$$w = \left\{(x, i), (y, 2^{n(m-[w_e])} - 1 - i)\right\}_{w,m-[w_e]} \text{ and}$$
$$v = \left\{(x, j), (y, 2^{n(m-[v_e])} - 1 - j)\right\}_{w,m-[v_e]}$$

with $w_e = v_e$ and $|i - j| = 1$. Therefore

$$J_M^h = J_M^h(K) = J_M^h(\mathcal{K}).$$

Hence, $\underline{I}_{\mathcal{E}}, \overline{I}_{\mathcal{E}}, \underline{d}_p^S$ and $\overline{d}_p^S$ for K and $\mathcal{K}$ coincide respectively. Therefore, by Lemma 4.21 and Theorem 3.12,

$$\underline{I}_{\mathcal{E}}(N_1, N_2, N) = \overline{I}_{\mathcal{E}}(N_1, N_2, N) = \dim_{\mathrm{AR}}(\mathfrak{C}_G, d_{\mathfrak{C}})$$

if $N_2 - N_1$ is sufficiently large. Combining this with Proposition 4.20, we get

$$\underline{I}_{\mathcal{E}}(N_1, N_2, N) = \overline{I}_{\mathcal{E}}(N_1, N_2, N) = \dim_{\mathrm{AR}}(G, d).$$

We also obtain (2) and (3). ■

4.2. Spectral dimension and Ahlfors regular conformal dimension of weighted graphs

In Theorem 4.15, we saw a relation between the ARC dimension and the p-spectral dimension of the associated metrics on graphs. On the other hand, the spectral dimension of the associated random walks on graphs can be determined. In this subsection, we consider the relation between these dimensions. Recall that (G, E) is a connected, bounded degree simple graph and $\mathcal{T} = (T, \pi, \phi) = (T_r, \pi', \phi')$ is a bi-infinite tree with a reference point. Throughout this section, let K be a partition of (G, E) parametrized by $\mathcal{T}$.

Definition 4.22 (Weighted graph). Let μ be a positive symmetric function on E, then we call (G, μ) a *weighted graph* and μ a *conductance* (or a *weight*) on (G, E). Moreover, we treat μ as a measure on G defined by

$$\mu_x := \sum_{y:y\sim x} \mu_{xy}, \quad \mu(A) := \sum_{x\in A} \mu_x$$

for any $x \in G$ and $A \subset G$.

- (Controlled weight). We say (G, μ) has *controlled weight*, or satisfies condition (p_0) if there exists $p_0 > 0$ such that

$$p(x, y) := \frac{\mu_{xy}}{\mu_x} \geq p_0 \quad \text{for any } x, y \in G \text{ with } x \sim y. \tag{p_0}$$

  Note that if (G, μ) has controlled weight, then $\#\{y \mid y \sim x\} \leq \lfloor p_0^{-1} \rfloor$ for any $x \in T$. (It shows that (G, E) must be a bounded degree graph.)
- (Heat kernel). We inductively define

$$p_0(x, y) = \delta_{x,y}, \quad p_n(x, y) = \sum_{z\in G} p_{n-1}(x, z)p(z, y).$$

$p_n(x, y)$ is also thought as transition function of associated random walk; that is,

$$\mathbb{P}^x(X_n = y) = p_n(x, y).$$

Additionally, we define the *heat kernel* of this random walk (with respect to μ) by $h_n(x, y) = p_n(x, y)/\mu_y$. It is easy to see that $h_n(x, y) = h_n(y, x)$.

- (Effective resistance). For $f \in \mathbb{R}^G$, we define

$$\mathcal{E}_\mu(f) = \mathcal{E}(f) := \frac{1}{2} \sum_{(x,y)\in E} (f(x) - f(y))^2 \mu_{xy}.$$

For any $A, B \subset G$, we also define the *effective resistance* of (G, μ) by

$$R(A, B) = \left(\inf\left\{\mathcal{E}(f) \mid f|_A = 1, f|_B = 0\right\}\right)^{-1},$$

where $\inf \emptyset = \infty$. We write $R(x, A)$ (resp. $R(x, y)$) instead of $R(\{x\}, A)$ (resp. $R(\{x\}, \{y\})$).

It is known that the infimum of $R(A, B)^{-1}$ is attained and that $R(x, y)$ is a distance on G (for example, see [9]).

For the rest of this paper, (G, μ) is always a weighted graph and R is the associated effective resistance.

Definition 4.23 (Spectral dimension). Fix $x \in G$ and define

$$\overline{d}_S(G, \mu) = 2 \limsup_{n\to\infty} \frac{\log p_{2n}(x, x)}{\log n}, \quad \underline{d}_S(G, \mu) = 2 \liminf_{n\to\infty} \frac{\log p_{2n}(x, x)}{\log n}.$$

We can see that $\overline{d}_S(G, \mu)$ and $\underline{d}_S(G, \mu)$ are independent of x. We call $\overline{d}_S(G, \mu)$ the *upper spectral dimension* of (G, μ), and $\underline{d}_S(G, \mu)$ the *lower spectral dimension* of (G, μ). If $\overline{d}_S(G, \mu) = \underline{d}_S(G, \mu)$, then we call $d_S(G, \mu) = \overline{d}_S(G, \mu)$ the *spectral dimension* of (G, μ).

We introduce other notions related to the partition.

Definition 4.24. We say K is *connected* if for any $w \in T_e$ and $x, y \in K$, there exists a path between x and y in K_w; in other words, $(K_w, E|_{K_w\times K_w})$ is connected for any $w \in T_e$.

Definition 4.25. We define $\underline{\mathcal{N}}, \overline{\mathcal{N}}, \underline{\mathcal{R}}_p$ by

$$\overline{\mathcal{N}} = \sup_{w\in T} \limsup_{k\to\infty} \#(\{S^k(\pi^k(w))\})^{1/k}, \quad \underline{\mathcal{N}} = \sup_{w\in T} \liminf_{k\to\infty} \#(\{S^k(\pi^k(w))\})^{1/k},$$

$$\underline{\mathcal{R}}_p(N_1, N_2, N) = \sup_{w\in T} \liminf_{k\to\infty} \mathcal{E}_{p,k,\pi^k(w)}(N_1, N_2, N)^{1/k}.$$

We remark that the difference between $\overline{N}_*$ (resp. $\underline{R}_p$) and $\overline{\mathcal{N}}$ (resp. $\overline{\mathcal{R}}_p$) is the order of the supremum over $w \in T$ and the limit as k, the index of scales, approaches infinity. By definition, $\overline{N}_* \geq \overline{\mathcal{N}} \geq \underline{\mathcal{N}}$ and $\underline{R}_p \geq \underline{\mathcal{R}}_p$.

Lemma 4.26. *Assume* $\sup_{w \in T_e \setminus \Lambda_e} \#(S(w)) < \infty$, *then*

(1) $\overline{\mathcal{N}} = \limsup_{k\to\infty} \#(\{S^k(\pi^k(w))\})^{1/k}$, $\underline{\mathcal{N}} = \liminf_{k\to\infty} \#(\{S^k(\pi^k(w))\})^{1/k}$ *for any* $w \in T$.

(2) $\underline{\mathcal{R}}_p(N_1, N_2, N) = \sup_{l\geq 0} \liminf_{k\to\infty} \mathcal{E}_{p,k,\pi^{k+l}(w)}(N_1, N_2, N)^{1/k}$ *for any* $w \in T$.

Idea of the proof. It is easy to see

$$\limsup_{k\to\infty}(\#\{S^k(\pi^k(w))\})^{1/k} = \limsup_{k\to\infty}(\#\{S^k(\pi^k(\pi^l(w)))\})^{1/k},$$

$$\liminf_{k\to\infty}(\mathcal{E}_{p,k,\pi^k(w)}(N_1, N_2, N))^{1/k} \leq \liminf_{k\to\infty}(\mathcal{E}_{p,k,\pi^k(\pi^l(w))}(N_1, N_2, N))^{1/k}.$$

These inequalities and condition (T1) (of the bi-infinite tree) induce this lemma.

We will consider the case that the weight is uniformly bounded. In the following theorem, we evaluate $\overline{d}_S$ and $\underline{d}_S$ by a partition.

Theorem 4.27. *Assume $\mu_{xy} \asymp 1$ for any $(x, y) \in E$ and K is minimal and connected. Let $d \in \mathcal{D}_\infty(G)$, fitting to (G, E) and satisfying the basic framework. Suppose that d satisfies the following for some $\alpha, \beta > 0$:*

- *We have*
$$d(x, y) \asymp 1 \quad \text{for any } (x, y) \in E.$$

- *We have*
$$h_{2n}(x, x) \asymp \frac{c}{V_d(x, n^{1/\beta})} \quad \text{for any } n. \tag{DHK(β)}$$

- *There exist $\lambda, C > 0$ such that*
$$R(B_d(x, \lambda r), B_d(x, r)^c)V(x, r) \geq Cr^\beta \quad \text{for any } r > r_x. \tag{ARL(β)}$$

- *There exists $C' > 0$ such that*
$$R(x, B_d(x, r)^c)V(x, r) \leq C'r^\beta \quad \text{for any } r > r_x. \tag{BRU(β)}$$

- *There exists $C'' > 0$ such that*
$$V_d(x, r) \leq C''(r^\alpha/s^\alpha)V_d(x, s) \quad \text{for any } x \in G \text{ and } r > s > 0. \tag{VG(α)}$$

Then for any $N, N_1 \geq 0$ *and sufficiently large* $N_2 = N_2(N_1)$,

$$\underline{d}_S(G,\mu) = 2\frac{\log \underline{\mathcal{N}}}{\log \underline{\mathcal{N}} - \log \underline{\mathcal{R}_2}(N_1, N_2, N)},$$

$$\overline{d}_S(G,\mu) = 2\frac{\log \overline{\mathcal{N}}}{\log \underline{\mathcal{N}} - \log \underline{\mathcal{R}_2}(N_1, N_2, N)}.$$

The assumption on d seems to be too strong, but we can justify the above assumption in the following way.

Theorem 4.28. *Assume* (G,μ) *and the resistance metric* R *satisfy conditions* (p_0) *and* (VG(α)) *for some* $\alpha = \alpha_* > 0$. *If* $R \in \mathcal{D}_\infty(G)$ *and* $V_R(x,r) < \infty$ *for any* $r > 0$ *and* $x \in G$, *then there exists a fitting metric* d *such that* $d \sim_{\mathrm{QS}} R$, $d(x,y) \asymp 1$ *for any* $(x,y) \in E$ *and satisfy* (DHK(β)), (ARL(β)), (BRU(β)) *and* (VG(α)) *for some* $1 \leq \alpha < \beta$.

This theorem is a discrete version of the corresponding result in [10], and also based on [1]. See [17, Section 6] for the proof.

We remark that for a metric space (X,d) and a measure μ on X, d satisfies (VG(α)) for some $\alpha > 0$ if and only if μ satisfies the following "volume doubling condition", which is used in many papers of heat kernel estimates, including [1] and [10].

Definition 4.29 (Volume doubling condition). Let (X,d) be a metric space and μ be a measure on X. We say μ *satisfies volume doubling condition with respect to* d if there exists $C > 0$ such that

$$V_{d,\mu}(x,2r) \leq C\,V_{d,\mu}(x,r) \quad \text{for any } x \in X \text{ and } r > 0. \tag{VD$_d$}$$

Combining above theorems, we get the following corollary.

Corollary 4.30. *Assume that* μ *satisfies* (VD$_d$) *with* $d = R$, (G,μ) *satisfies* $\mu_{xy} \asymp 1$ *for any* $(x,y) \in E$, $V_R(x,r) < \infty$ *for any* $r > 0, x \in G$ *and* $\operatorname{diam}(X,R) = \infty$. *If the metric* d *from Theorem* 4.28 *satisfies the basic framework (with respect to some minimal connected partition* K*) and*

$$\frac{\log \underline{R_2}(N_1,N_2,N)}{\log \overline{N}_*} \leq \frac{\log \underline{\mathcal{R}_2}(N_1,N_2,N)}{\log \underline{\mathcal{N}}} \tag{4.1}$$

for some $N, N_1 \geq 0$ *and sufficiently large* $N_2 > N_1$. *Then*

$$\dim_{\mathrm{AR}}(G,R) \leq \underline{d}_S(G,\mu) \leq \overline{d}_S(G,\mu) < 2.$$

Condition (4.1) holds in natural settings, including Sierpiński carpets or n-gaskets. We give an example in Example 5.4 such that all the assumptions of Corollary 4.30 hold except (4.1), and neither $\underline{d}_2^S$ nor $\overline{d}_2^S$ coincides with the spectral dimension $d_S(G,\mu)$. It helps to understand the difference between $\underline{\mathcal{R}}$ and $\underline{R}$.

Proof of Corollary 4.30. Since d satisfies (VG(α)) and (DHK(β)), it follows that

$$\begin{aligned}\underline{d}_S(G,\mu) &\leq 2\limsup_{n\to\infty}\frac{\log V_d(x,n^{1/\beta})}{\log n}\\ &\leq 2\limsup_{n\to\infty}\frac{\log V_d(x,1)+\log n^{\alpha/\beta}}{\log n} = 2\frac{\alpha}{\beta} < 2.\end{aligned}$$

On the other hand, by definition and Theorem 4.27, we have

$$\begin{aligned}\underline{d}_2^S(N_1,N_2,N) &= \left(1-\frac{\log \underline{R}_2(N_1,N_2,N)}{\log \overline{N}_*}\right)^{-1},\\ \underline{d}_s(G,\mu) &= \left(1-\frac{\log \underline{\mathcal{R}}_2(N_1,N_2,n)}{\log \underline{\mathcal{N}}}\right)^{-1},\end{aligned}$$

(note that $\operatorname{diam}(G,d)=\infty$ because $R\sim_{\mathrm{QS}} d$) and hence

$$\underline{d}_2^S(N_1,N_2,N) \leq \underline{d}_S(G,\mu) \leq \overline{d}_S(G,\mu) < 2.$$

Since $\underline{d}_2^S(N_1,N_2,N)<2$, by Theorem 4.15 we have $\dim_{\mathrm{AR}}(G,d)\leq \underline{d}_2^S(N_1,N_2,N)$. Moreover, $\dim_{\mathrm{AR}}(G,R)=\dim_{\mathrm{AR}}(G,d)$ because $R\sim_{\mathrm{QS}} d$, so this shows

$$\dim_{\mathrm{AR}}(G,R) \leq \underline{d}_S(G,\mu) \leq \overline{d}_S(G,\mu) < 2. \qquad \blacksquare$$

In the rest of this section, we prove Theorem 4.27. First, we give general lemmas and a definition.

Lemma 4.31. *Let g be a thick discrete weight function, then for any $M\geq 1$, there exists $\eta=\eta_M>0$ such that for any $w\in T_e$, there exists $x\in K_w$ such that*

$$U_M^d(x,\eta g(\pi(w)))\subset K_w.$$

Lemma 4.32. *Let $d\in\mathcal{D}_\infty(G)$. If d is adapted for $M=M_0$, then d is adapted for any $M\geq M_0$.*

Definition 4.33. For $w\in T_e$ and $M\geq 0$, define $U_M(w)$ by

$$U_M(w)=\bigcup\left\{K_v \;\middle|\; v\in\Gamma_M(w)\cap T_e\right\}.$$

For the rest of this section, we assume d satisfies the basic framework, K is connected, $\mu_{xy} \asymp 1$ and $d(x,y) \asymp 1$ for any $(x,y) \in E$. Let $\eta_0 > 0$ be such that $\eta_0^{-1} r^{[w]} \leq d(w) \leq \eta_0 r^{[w]}$ and write $N_* = \sup_{w \in T} \#(S(w))$ (observe that $N_* \leq \sup_{w \in T_e \setminus \Lambda_e} \#(S(w)) \vee 2r^{-1} < \infty$).

Moreover, since $d(x,y) \asymp 1$ for any $(x,y) \in E$ and $d(w) \asymp r^{[w]}$ for $w \in \Lambda_e$, there exist $m_0, m_1 \in \mathbb{Z}$ such that $m_0 \leq [w] \leq m_1$ for any $w \in \Lambda_e$.

Lemma 4.34. $\sup_{w \in T} \#(\Gamma_1(w)) < \infty$.

We write $L_* = \sup_{w \in T} \#(\Gamma_1(w))$.

Lemma 4.35. *Let $N_1 \geq 0$ and $\lambda \in (0,1)$, then there exist N_2 and $\xi > 0$ such that for any $x \in G$ and $w \in T_e$ such that $x \in K_w$,*

$$U_{N_1}(w) \subset B_d(x, \lambda \xi r^{[w]}) \quad \text{and} \quad B_d(x, \xi r^{[w]}) \subset U_{N_2}(w). \tag{4.2}$$

Using these lemmas, we have the following key lemmas.

Lemma 4.36. *Assume (VG(α)) holds. Then $\#(S^{m_1-[w]}(w)) \asymp V_d(x, r^{[w]})$ for any $w \in T_e$ and $x \in K_w$.*

Proof. Since $\mu \asymp 1$, K is minimal and (G,E) has a bounded degree,

$$\mu(K_w) \asymp \sum_{v \in T_w \cap \Lambda_e} \mu(K_v) \asymp \#\big(\{v \mid v \in T_w \cap \Lambda_e\}\big).$$

Moreover, since $m_0 \leq [v] \leq m_1$ for any $v \in \lambda_e$,

$$\begin{aligned} \#(S^{m_1-[w]}(w)) &\geq \#\big(\{v \mid v \in T_w \cap \Lambda_e\}\big) \\ &\geq \#(S^{(m_0-[w])\vee 0}) \geq N_*^{-2} \#(S^{m_1-[w]}(w)). \end{aligned}$$

On the other hand, let $w \in T_e$ and $x \in K_w$. Then there exist $\eta_1 > 0$ and $M_* \in \mathbb{N}$ such that

$$K_w \subset U^d_{M_*}(x, \eta r^{[w]}) \subset B_d(x, \eta_1 r^{[w]}),$$

because d is adapted. This, together with (VG(α)), shows there exists $C_1 > 0$ such that $\mu(K_w) \leq C_1 V_d(x, r^{[w]})$.

Moreover, since d is thick, there exists $\eta_2, \eta_3 > 0$ and $x' \in K_w$ such that

$$K_w \supset U^d_{M_*}(x', \eta_2 \eta^{-1} r^{[w]-1}) \supset B_d(x', \eta_3 r^{[w]}).$$

Note that $d(x,x') \leq d(w) \leq \eta r^{[w]}$. This, together with (VG(α)), shows there exists $C_2 > 0$ such that

$$V_d(x, r^{[w]}) \leq V_d(x', (1+\eta) r^{[w]}) \leq C_2 V_d(x', \eta_3 r^{[w]}) \leq C_2 \mu(K_w).$$

Therefore

$$\#(S^{m_1-[w]}(w)) \asymp \mu(K_w) \asymp V_d(x, r^{[w]})$$

for any $w \in T_e$ and $x \in K_w$. ∎

Lemma 4.37. *Fix any $w \in T$ such that $[w] \le m_0$. Then*

$$R(U_{N_1}(\pi^k(w)), U_{N_2}(\pi^k(w))^c)^{-1} \asymp \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N)$$

for any $k \ge 0$.

Proof. Let $u, v \in (T)_{[w]}$. If $x \in K_u$, $y \in K_v$ and $x \sim y$, then there exists $\omega \in \Sigma^*$ such that $K_{\omega_e} = \{x, y\}$. Since $[w] \le m_0$, we have $\omega_{[w]} \in T_e$. Then $K_{\omega_{[w]}} \cap K_u \ne \emptyset$ and $K_{\omega_{[w]}} \cap K_v \ne \emptyset$, so we have $l_{J^h_{[w]}}(u, v) \le 1$. Now, let f be a function on $(T)_{[w]}$ such that

$$f \equiv \begin{cases} 1, & \text{on } S^k(\Gamma_{N_1}(\pi^k(w))), \\ 0, & \text{on } S^k(\Gamma_{N_2}(\pi^k(w)))^c. \end{cases}$$

Define $\overline{f}$ on G by $\overline{f}(x) = \max_{w:x\in K_w} f(w)$, then $\overline{f} \equiv 1$ on $U_{N_1}(\pi^k(w))$ and $\overline{f} \equiv 0$ on $U_{N_2}(\pi^k(w))^c$. This is because

- if $x \in U_{N_1}(\pi^k(w))$, then there exists $v \in S^k(\Gamma_{N_1}(\pi^k(w)))$ such that $x \in K_v$ by (PG1),
- if $x \notin U_{N_2}(\pi^k(w))$, then for any $v \in S^k(\Gamma_{N_2}(\pi^k(w)))$, it follows that $x \notin K_v$ by (PG1).

Hence, noting that $\mu_{xy} \asymp 1$, there exists $C > 0$ such that the following holds:

$$\begin{aligned} \frac{1}{2}\sum_{x\sim y} |\overline{f}(x) - \overline{f}(y)|^2 \mu_{xy} &\le \frac{1}{2} C \sum_{x\sim y} \sum_{u:x\in K_u} \sum_{v:y\in K_v} |f(u) - f(v)|^2 \\ &\le \frac{1}{2} C \sum_{(u,v)\in J^h_{[w]}} \sum_{x\in K_u} \sum_{y\in K_v} |f(u) - f(v)|^2 \\ &\le \frac{1}{2} C N_*^{2(m_1-[w])} \sum_{(u,v)\in J^h_{[w]}} |f(u) - f(v)|^2. \end{aligned}$$

This implies

$$R(U_{N_1}(\pi^k(w)), U_{N_2}(\pi^k(w))^c)^{-1} \le C N_*^{2(m_1-[w])} \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N).$$

On the other hand, let h be a function on G such that $h \equiv 1$ on $U_{N_1}(\pi^k(w))$ and $h \equiv 0$ on $U_{N_2}(\pi^k(w))^c$. Define $\underline{h}(w) = \min_{x\in K_w} h(w)$, then similarly we get $\underline{h} \equiv 1$ on $S^k(\Gamma_{N_1}(w))$ and $\underline{h} \equiv 0$ on $S^k(\Gamma_{N_2}(w))^c$.

Let $x, y \in G$. If $x \in K_u$ and $y \in K_v$ for some $(u, v) \in J^h_{[w],N}$, then

$$l_B(x, y) \le (N+1) \sup_{v \in (T)_{[w]}} \#(K_v) \le 2(N+1)N_*^{m_1-[w]},$$

because K is connected.

Therefore,

$$\begin{aligned}
\frac{1}{2} \sum_{(u,v)\in J^h_{[w],n}} |\underline{h}(u) - \underline{h}(v)|^2 &\le \frac{1}{2} \sum_{(u,v)\in J^h_{[w],n}} \sum_{x \in K_u} \sum_{y \in K_v} |h(x) - h(y)|^2 \\
&\le \frac{1}{2} \sum_{x,y: l_E(x,y) \le l_0} \sum_{u: x \in K_u} \sum_{v: y \in K_v} |h(x) - h(y)|^2 \\
&\le \frac{1}{2} N_*^{2(m_1-[w])} \sum_{x,y: l_E(x,y) \le l_0} |h(x) - h(y)|^2,
\end{aligned}$$

where $l_0 = 2(N+1)N_*^{m_1-[w]}$. Note that if $l_E(x, y) \le l_0$, then $|h(x) - h(y)|^2 \le l_0 \sum_{i=1}^n |h(x_{i-1}) - h(x_i)|^2$ for some n-path $\{x_i\}_{i=0}^n$ between x and y with $n \le l_0$, so for $E_{x,y,l_0} = \{(p, q) \mid p \sim q, p \in B_{l_E}(x, l_0), \text{ and } q \in B_{l_E}(y, l_0)\} \subset E$,

$$\begin{aligned}
\sum_{x,y: l_E(x,y) \le l_0} |h(x) - h(y)|^2 &\le l_0 \sum_{x,y: l_E(x,y) \le l_0} \sum_{(p,q) \in E_{x,y,l_0}} |h(p) - h(q)|^2 \\
&\le l_0 \sum_{p \sim q} \sum_{x \in B_{l_E}(x,l_0)} \sum_{y \in B_{l_E}(y,l_0)} |h(p) - h(q)|^2 \\
&\le l_0 (\sup_{x \in G} \#(\{y \mid y \sim x\}))^{2l_0} \sum_{p \sim q} |h(p) - h(q)|^2.
\end{aligned}$$

These inequalities with $\mu_{xy} \asymp 1$ show

$$C' R(U_{N_1}(\pi^k(w)), U_{N_2}(\pi^k(w))^c)^{-1} \ge \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N)$$

for some $C' > 0$. ■

Proof of Theorem 4.27. Fix $N_1, N \ge 0$ and let $w \in T$ such that $[w] \le m_0$. Then, using Lemma 4.35 and the fact that d is adapted, there exist ξ and ζ such that for sufficiently large N_2, for any $x \in K_w$, we have

$$x \in U_{N_1}(w) \subset B_d(x, \lambda \xi r^{[w]})$$

and

$$B_d(x, \xi r^{[w]}) \subset U_{N_2}(w) \subset U^d_{N_2}(\eta r^{[w]}) \subset B_d(x, \zeta r^{[w]}).$$

Hence,

$$R(x, B_d(x, \zeta r^{[w]})^c) \geq R(U_{N_1}(w), U_{N_2}(w)^c) \geq R(B_d(x, \lambda\xi r^{[w]}), B_d(x, \xi r^{[w]})^c).$$

These, with Lemmas 4.36 and 4.37, show that for any $k \geq 0$, there exist C_1, C_2 such that

$$\begin{aligned} &C_1 R(x, B_d(x, \zeta r^{[w]-k})^c) V_d(x, \zeta r^{[w]-k}) \\ &\quad \geq (\mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N))^{-1} \#(S^{m_1-[w]+k}(\pi^k(w))) \\ &\quad \geq C_2 R(B_d(x, \lambda\xi r^{[w]}), B_d(x, \xi r^{[w]})^c) V_d(x, \xi r^{[w]-k}). \end{aligned}$$

This, together with (ARL(β)), (BRU(β)), implies that there exist $\delta > 0$ such that

$$\begin{aligned} -k\beta \log r - \delta &\leq \log \#(S^{m_1-[w]+k}(\pi^k(w))) - \log \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N) \\ &\leq -k\beta \log r + \delta \end{aligned}$$

and hence, by Lemma 4.26,

$$\log \underline{\mathcal{N}} + \beta \log r = \liminf_{k\to\infty} \log \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N)^{1/k}$$

because

$$N_*^{m_1-[w]} \#(S^k(\pi^k(w))) \geq \#(S^{m_1-[w]+k}(\pi^k(w))) \geq \#(S^k(\pi^k(\pi^l(w)))).$$

This equation also holds for $\pi^l(w)$ with $l \geq 0$, so again using Lemma 4.26, we obtain

$$\begin{aligned} \log \underline{\mathcal{N}} + \beta \log r &= \sup_{l\geq 0} \liminf_{k\to\infty} \log \mathcal{E}_{2,k,\pi^k(\pi^l(w))}(N_1, N_2, N)^{1/k} \\ &= \log \underline{\mathcal{R}_2}(N_1, N_2, N). \end{aligned}$$

Now, by (DHK(β)) and (VG(α)),

$$\frac{\underline{d}_S}{2} = \liminf_{n\to\infty} \frac{\log V_d(x, n^{1/\beta})}{\log n} = \liminf_{r\to\infty} \frac{\log V_d(x, r^{1/\beta})}{\log r} = \liminf_{k\to\infty} \frac{\log V_d(x, r^{-k})}{\log r^{-\beta k}}$$

and by Lemma 4.36,

$$\underline{d}_S = 2\frac{\liminf_{k\to\infty} \frac{1}{k} \log V_d(x, r^{-k})}{-\beta \log r} = 2\frac{\log \underline{\mathcal{N}}}{\log \underline{\mathcal{N}} - \log \underline{\mathcal{R}_2}(N_1, N_2, N)}.$$

In the same way, we also get

$$\overline{d}_S = 2\frac{\log \overline{\mathcal{N}}}{\log \underline{\mathcal{N}} - \log \underline{\mathcal{R}_2}(N_1, N_2, N)}. \qquad \blacksquare$$

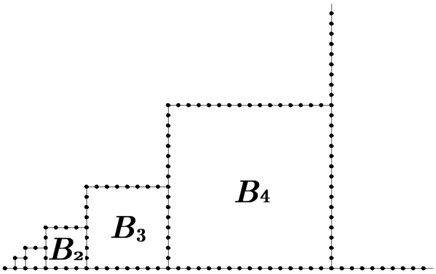

Figure 3. $f(n) \equiv 0$

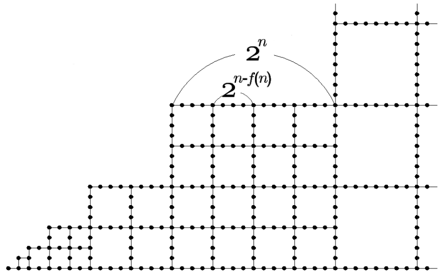

Figure 4. $f(n) = \lfloor \frac{n}{2} \rfloor$

5. Examples

We first give an example that the ARC dimension can be calculated using Theorem 4.15.

Example 5.1. Let $f(n) : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f(n) \leq n$ for any n. For $n \geq 0$, define $B_n, L_n, X_n \in \mathbb{R}^2$ by

$$
\begin{aligned}
B_n &= [2^n, 2^{n+1}] \times [0, 2^n], \\
L_n &= \bigcup_{j \in \mathbb{Z}} \left(\left\{ (x, y) \,\middle|\, x = 2^{n-f(n)} j \right\} \cup \left\{ (x, y) \,\middle|\, y = 2^{n-f(n)} j \right\} \right), \\
X_n &= B_n \cap L_n.
\end{aligned}
$$

We also define X, G, E by

$$
\begin{aligned}
X &= \left\{ (t, 0) \,\middle|\, 0 \leq t \leq 1 \right\} \cup \left(\bigcup_{n \geq 0} X_n \right), \\
G &= X \cap \mathbb{Z}^2, \\
E &= \left\{ (p, q) \in G \times G \,\middle|\, d_2(p, q) = 1 \right\},
\end{aligned}
$$

where d_2 is the Euclidean metric in $\mathbb{R}^2$. See Figure 3 or Figure 4. Next we introduce a partition of (G, E). For $m, a, b \in \mathbb{Z}$, define

$$
\begin{aligned}
\mathcal{S}_{m,a,b} &= \left\{ (x, y) \,\middle|\, 2^m a \leq x + y \leq 2^m (a + 1),\ 2^m b \leq x - y \leq 2^m (b + 1) \right\}, \\
T_{-m} &= \left\{ \mathcal{S}_{m,a,b} \,\middle|\, \mathrm{int}(\mathcal{S}_{m,a,b}) \cap X \neq \emptyset \right\}, \\
T &= \bigcup_{m \in \mathbb{Z}} T_m,
\end{aligned}
$$

and for $w \in T_m$, define $\pi(w)$ as the unique element in T_{m-1} such that $w \subset \pi(w)$ as subsets of $\mathbb{R}^2$. Then (T, π) is a bi-infinite tree, and $T_m = (T)_m$ by taking $\phi = \mathcal{S}_{0,0,0}$.

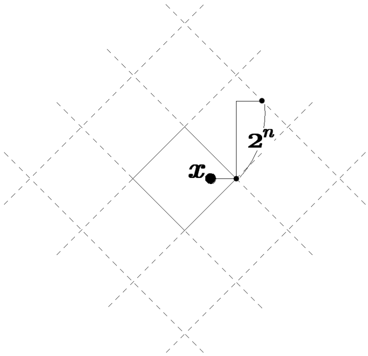

Figure 5. $B_d(x, 2^n) \subset U_1^d(x, 2^n)$

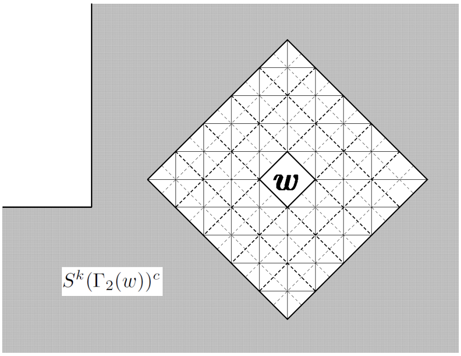

Figure 6. w satisfying (5.1)

We also define $K : T \to \{\text{finite subsets of } G\}$ by

$$K_w = \begin{cases} w \cap G \ \text{(as subsets of } \mathbb{R}^2), & \text{if } [w] \leq 0, \\ \pi^{[w]}(w) \cap G \text{(as subsets of } \mathbb{R}^2), & \text{if } [w] > 0. \end{cases}$$

Then K_w is a partition of (G, E). Moreover, $\Lambda_e = (T)_0$ and $T = T_{1/2}$.

Now we let $d = l_E$, and calculate $\dim_{\mathrm{AR}}(G, d)$.

Proposition 5.2.

(1) *If* $\limsup_{n\to\infty} f(n) = \infty$, *then* $\dim_{\mathrm{AR}}(G, d) = 2$.

(2) *If* $\limsup_{n\to\infty} f(n) < \infty$, *then* $\dim_{\mathrm{AR}}(G, d) = 1$.

Proof. It is easy to check that d, K satisfy the assumptions of Theorem 4.15.

(1) We adapt Theorem 4.15 and show $\dim_{\mathrm{AR}}(G, d) = 2$.

The first step is to show $\dim_{\mathrm{AR}}(G, d) \geq 2$. Since $\sup_n f(n) = \infty$, for any $k \geq 0$, there exists $m \in \mathbb{N}$ and $w = \mathcal{S}_{m,a,b} \in (T)_{-m}$ such that

$$\begin{aligned} \bigl\{ S_{m-k,i,j} \bigm| i &\in [2^k(a-2) - 1, 2^k(a+2) + 1]_{\mathbb{Z}}, \\ j &\in [2^k(b-2) - 1, 2^k(b+2) + 1]_{\mathbb{Z}} \bigr\} \subset (T)_{-(m-k)}. \end{aligned} \tag{5.1}$$

Let g be a function on $(T)_{-(m-k)}$ such that $g \equiv 1$ on $S^k(\Gamma_0(w))$ and $g \equiv 0$ on $S^k(\Gamma_2(w))^c$. We also let $\tilde{g} = (g \vee 0) \wedge 1$, then for any $p \geq 1$, there exists $C_p > 0$ such that

$$\begin{aligned} &\sum_{(u,v)\in E^h_{-(m-k)}} |g(u) - g(v)|^p \\ &\quad \geq \sum_{(u,v)\in E^h_{-(m-k)}} |\tilde{g}(u) - \tilde{g}(v)|^p \end{aligned}$$

$$\geq \sum_{i\in[2^k a,2^k(a+1)]_{\mathbb{Z}}} \sum_{j\in[2^k(b-2),2^k b]_{\mathbb{Z}}} |\tilde{g}(\mathcal{S}_{-(m-k),i,j}) - \tilde{g}(\mathcal{S}_{-(m-k),i,j-1})|^p$$
$$\geq \sum_i (2^{k+1}+1)^{1-p} \geq C2^{(2-p)k}.$$

(This follows from Jensen's inequality, together with $\tilde{g}(\mathcal{S}_{-(m-k),i,2^k b}) = 1$ and $\tilde{g}(\mathcal{S}_{-(m-k),i,2^k(b-2)-1}) = 0$ for any $i \in [2^k a, 2^k(a+1)]_{\mathbb{Z}}$.) Moreover, for $p < 1$,

$$\sum_{(u,v)\in E^h_{-(m-k)}} |g(u)-g(v)|^p \geq \sum_{(u,v)\in E^h_{-(m-k)}} |\tilde{g}(u)-\tilde{g}(v)|^p$$
$$\geq \sum_{j\in[2^k(b-2),2^k b]_{\mathbb{Z}}} |\tilde{g}(\mathcal{S}_{-(m-k),2^k a,j}) - \tilde{g}(\mathcal{S}_{-(m-k),2^k a,j-1})| \geq 1.$$

Therefore $\lim_{k\to\infty} \mathcal{E}_{p,k}(0,2,1) > 0$ for any $p \leq 2$, hence $\dim_{\mathrm{AR}}(G,d) \geq 2$.

On the other hand, define $g = g_w$ on $E^h_{-(m-k)}$ by

$$g(\mathcal{S}_{m-k,i,j}) = \Big(\frac{(2^k(a+2)-i)\wedge(i-2^k(a-2))\wedge(2^k(b+2)-j)\wedge(j-2^k(b-2))}{2^k}\Big) \vee 0 \wedge 1.$$

Then $f \equiv 1$ on $S^k(\Gamma_0(w))$, $f \equiv 0$ on $S^k(\Gamma_2(w))^c$ and

$$\sum_{(u,v)\in E^h_{-(m-k)}} |f(u)-f(v)|^p \leq \sum_{v\in S^k(\Gamma_2(w))} 8\cdot 2^{-kp} \leq C'2^{(2-p)k}$$

for some $C' > 0$, hence $\mathcal{E}_{p,k,w}(0,2,1) \leq C'2^{(2-p)k}$. Moreover, for any $v \in T$, this upper bound holds by the definition of T and K. Therefore $\lim_{k\to\infty} \mathcal{E}_{p,k}(0,1,2) = 0$ for any $p > 2$ and hence $\dim_{\mathrm{AR}}(G,d) = 2$.

(2) Let $\mathfrak{m}(A) = \#(A)$ for any $A \subset G$, and $G_n = X_n \cap G$ for any $n \geq 0$. Then

$$\mathfrak{m}(B_d(x,r)\cap G_n) \leq 2(2^{f(n)}+1)\big(\mathrm{diam}(B_d(x,r)\cap G_n, d)+1\big),$$

because G_n consists of $2(2^{f(n)}+1)$ segments whose length are 2^n. Hence there exists $\bar{C}$ such that for any $x \in G$ and $r \geq 1$,

$$r \leq V_d(x,r) \leq 1 + \sum_{n\geq 0} \mathfrak{m}(B_d(x,r)\cap G_n) \leq \bar{C}r,$$

because $\sum_{n\geq 0} \mathrm{diam}(B_d(x,r)\cap G_n, d) \leq 2r$ and $\sup_n f(n) < \infty$. Therefore d is 1-Ahlfors regular. On the other hand, $\dim_{\mathrm{AR}}(G,d) \geq 1$ by Proposition 4.20 and hence $\dim_{\mathrm{AR}}(G,d) = 1$. ∎

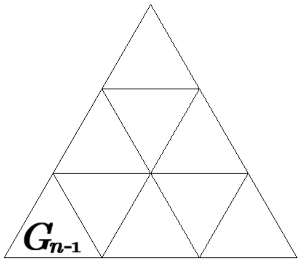

Figure 7. G_n (if $f(n) = 1$)

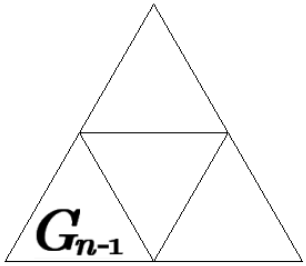

Figure 8. G_n (if $f(n) = 0$)

Remark 5.3. If we use a partition parallel to axes, that is, a partition K' defined by $S'_{m,a,b} = [2^m a, 2^m(a+1)] \times [2^m b, 2^m(b+1)]$ in a similar way to K, then K' is not minimal. For example, both $S'_{0,0,0}$ and $S'_{0,1,0}$ include a edge $((1,0),(1,1)) \in E$. So we need some modification to apply Theorem 4.15 to d, K'.

In the next example, $d_s(G,\mu) \neq \underline{d}_2^S = \overline{d}_2^S$ although d satisfies (DHK(β)), (ARL(β)) and (BRU(β)).

Example 5.4. Let $f : \mathbb{N} \to \{0,1\}$, $G_0 = \{0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2} i\} \in \mathbb{C}$ and $E_0 = \{(x,y) \in G_0 \times G_0 \mid x \neq y\}$. For $n \in \mathbb{N}$, we inductively define $|G_{n-1}|_\infty = \max_{z \in G_{n-1}} |z|$, and

$$F_{n,1}(z) = z, \qquad F_{n,2}(z) = z + |G_{n-1}|_\infty,$$

$$F_{n,3}(z) = z + \Big(\frac{1}{2} + \frac{\sqrt{3}}{2} i\Big)|G_{n-1}|_\infty, \qquad F_{n,4}(z) = z + 2|G_{n-1}|_\infty,$$

$$F_{n,5}(z) = z + \Big(1 + \frac{1}{2} + \frac{\sqrt{3}}{2} i\Big)|G_{n-1}|_\infty,$$

$$F_{n,6}(z) = z + 2\Big(\frac{1}{2} + \frac{\sqrt{3}}{2} i\Big)|G_{n-1}|_\infty,$$

$$F_n(z) = \begin{cases} \bigcup_{i=1}^3 F_{n,i}, & \text{if } f(n) = 0, \\ \bigcup_{i=1}^6 F_{n,i}, & \text{if } f(n) = 1. \end{cases}$$

Now define $G_n = F_n(G_{n-1})$ and

$$E_n = \Big\{(x,y) \in G_n \times G_n \;\Big|\; \text{there exist } x', y' \in G_{n-1} \text{ and } i \geq 0 \text{ such that } (x',y') \in E_{n-1} \text{ and } x = F_{n,i}(x'),\ y = F_{n,i}(y')\Big\}.$$

Note that $|G_n|_\infty = 2^{n-m(n)} \cdot 3^{m(n)}$ where $m(n) = \#(\{k \mid k \leq n, f(k) = 1\})$. Let $G = \bigcup_{n \geq 0} G_n$ and $E = \bigcup_{n \geq 0} E_n$. We also let $\mu \equiv 1$ on E and consider the effective resistance R of (G,μ).

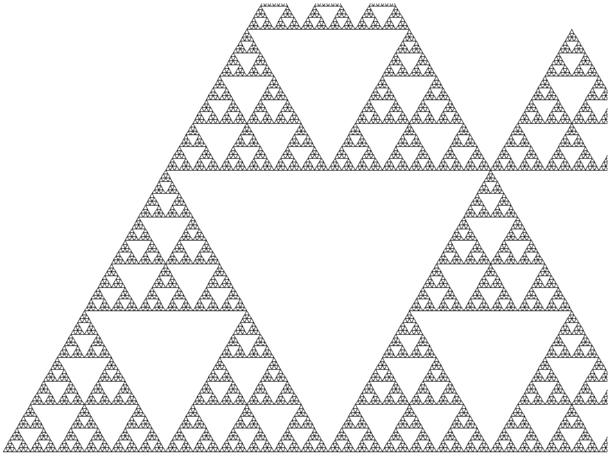

Figure 9. (G, E) (for some f)

Note that

$$\begin{cases} R(x,y)^{-1} \geq 1, & \text{for any } (x,y) \in E, \\ R(x,y)^{-1} \leq \mathcal{E}(\mathbf{1}_{\{x\}}) \leq 6, & \text{for any } x, y \in G \text{ with } x \neq y, \end{cases}$$

so R fits to (G, E).

We will check properties of R in order to apply Theorem 4.28. For the purpose, we first introduce a partition. For $n \geq 0$ and $a, b \in \mathbb{Z}$, define

$$\begin{aligned} \triangle_{0,0,0} &= \left\{ s + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) t \;\middle|\; s \geq 0,\ t \geq 0,\ s + t \leq 1 \right\}, \\ \triangle_{n,a,b} &= \left(\triangle_{0,0,0} + a + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) b \right) |G_n|_\infty, \\ T_{-n} &= \left\{ \triangle_{n,a,b} \;\middle|\; \triangle_{n,a,b} \subset \bigcup_{m \geq n} F_m \circ F_{m-1} \circ \cdots \circ F_n(\triangle_{n,0,0}) \right\}, \\ K_{n,a,b} &= \triangle_{n,a,b} \cap G \text{ (as subsets of } \mathbb{C}). \end{aligned}$$

For any $n \geq 1$, we let $T_n = \bigcup_{w \in T_0} \bigcup_{x,y \in K_w} \{x, y\}$ and $K_w = w$ for any $w \in T_n$. Define $T = \bigsqcup_{n \in \mathbb{Z}} T_n$ and $\pi(w)$ for $w \in T_n$ as the unique elements in T_{n-1} such that $K_w \subset K_{\pi(w)}$. Then (T, π) is a bi-infinite tree, $(T)_n = T_n$ with $\phi = \triangle_{0,0,0}$, K is minimal connected partition and $\Lambda_e = (T)_1$. If necessary, we replace T, K by T_r, K' for $r \in (0,1)$ in the way of Definition 4.12.

By the method of resistance on finite sets, we have the following lemma. See [17, Lemma 5.4] for the proof.

Lemma 5.5. *Let* $\mathfrak{R}(n) = \left(\frac{5}{3}\right)^{n-m(n)} \left(\frac{15}{7}\right)^{m(n)}$ *for any* $n \geq 0$ *and let*

$$n(x, y) = \min\left\{n \geq 0 \;\middle|\; \begin{array}{l} \text{there exist } w, v \in (T)_{-n} \text{ such that} \\ \qquad\qquad x \in K_w, y \in K_v \text{ and } K_w \cap K_v \neq \emptyset \end{array}\right\}$$

for $(x, y) \in G$. *Then* $R(x, y) \asymp \mathfrak{R}(n(x, y))$ *for any* $x, y \in G$ *with* $x \neq y$.

This lemma also implies that R is adapted (for $M = 1$) and $V(x, R(x, y)) \asymp \mathfrak{V}(n(x, y))$ where $\mathfrak{V}(n) = 3^{n-m(n)} \cdot 6^{m(n)}$. This inequality also shows (VD_d) with $d = R$, and (G, μ) satisfies the conditions of Theorem 4.28.

Next, we modify T in order to satisfy $d(w) \asymp r^{[w]}$ for some $r \in (0, 1)$, where d is the metric obtained by Theorem 4.28. For $j \geq 0$, let $n(j) \geq 0$ be such that $\mathfrak{R}(n(j))\mathfrak{V}(n(j)) \leq \left(\frac{90}{7}\right)^j < \mathfrak{R}(n(j)+1)\mathfrak{V}(n(j)+1)$ and for $j < 0$, set $n(j) = j$. We consider $\bar{T} = \bigcup_{j\in\mathbb{Z}}(T)_{-n(j)}$, and $\bar{\pi}(w) = \pi^{n(j+1)-n(j)}$ for $w \in (T)_{-n(j)}$. Then $(\bar{T}, \bar{\pi}, \triangle_{0,0,0})$ is a bi-infinite tree with a reference point, $(\bar{T})_j = (T)_{n(j)}$ and $K|_{\bar{T}}$ is minimal, connected partition, $\Lambda_e = (T)_1$. Moreover, for any $w \in \bigcup_{j\geq 0}(T)_{-n(j)}$,

$$\sup_{x,y\in K_w} R(x, y)V(x, R(x, y)) \asymp \mathfrak{R}(n(j))\mathfrak{V}(n(j)) \asymp \left(\frac{90}{7}\right)^j$$

and hence $d(w) \asymp \left(\frac{7}{90}\right)^{[w]/\beta}$ for any $w \in T_e$ where β is the constant in Theorem 4.28. Comparing with $R(x, y)V(x, R(x, y))$, we can also see that d is uniformly finite, thick and adapted because of Lemma 5.5. Now we let

$$f(n) = \begin{cases} 1, & \text{if } l(l^2-1) < n \leq l^3 \text{ for some } l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have the following.

Proposition 5.6. $d_S(G, \mu) = 2\log 3/\log 5$ *and* $\underline{d}_2^S(0, 2, 1) = \overline{d}_2^S(0, 2, 1) = 2\log 6/(\log 90 - \log 7)$.

Proof. Let $w = \triangle_{n(j),0,0}$ for some $j \geq 0$. With the Δ-Y transform (see [9, Lemma 2.1.15]), we can see that $\mathcal{E}_{p,k,\pi^k(w)}(0, 2, 1) \asymp \mathfrak{R}(n(k+j))/\mathfrak{R}(n(j))$ (see Figure 10), so

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{k} \log \mathcal{E}_{p,k,\pi^k(w)}(0, 2, 1) &= \lim_{k\to\infty} \frac{1}{k} \log \frac{\mathfrak{R}(n(k+j))}{\mathfrak{R}(n(j))} = \lim_{k\to\infty} \frac{1}{k} \log \mathfrak{R}(n(k)) \\ &= \lim_{k\to\infty} \frac{1}{k}\Big(n(k) \log \frac{3}{5} + m(n(k)) \log \frac{7}{15}\Big). \end{aligned}$$

Now we consider $\lim_{k\to\infty} n(k)/k$. By definition, we obtain

$$k\frac{\log 90 - \log 7}{\log 5} \geq n(k) \geq k\frac{\log 90 - \log 7}{\log 5} - m(n(k)) - C$$

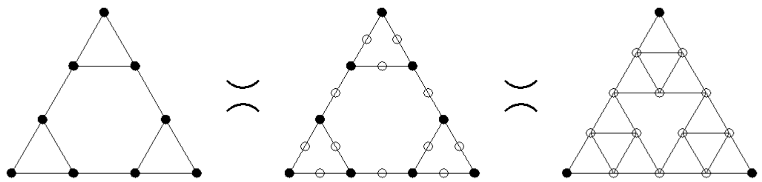

Figure 10. $\mathcal{E}_{p,k,\pi^k(w)}(0,2,1) \asymp \Re(n(k+j))/\Re(n(j))$

for some $C > 0$. Note that $\lim_{k\to\infty} m(k)/k = \lim_{k\to\infty} k^{-1/3} = 0$ because $m(k^3) = \sum_{j=1}^{k} j = k(k-1)/2$, hence $\lim_{k\to\infty} k/n(k) = \log 5/(\log 90 - \log 7)$. Therefore, by Lemma 4.26,

$$\underline{\mathcal{R}}_2(0,2,1) = \sup_{j\geq 0} \lim_{k\to\infty} \frac{1}{k} \log \mathcal{E}_{p,k,\pi^k(\triangle_{n(j),0,0})}(0,2,1) = \frac{\log 90 - \log 7}{\log 5} \log \frac{3}{5}.$$

Similarly, we get

$$\begin{aligned}
\underline{\mathcal{N}} = \overline{\mathcal{N}} &= \lim_{k\to\infty} \frac{1}{k} \log \#\big(S^k(\pi^k(\triangle_{0,0,0}))\big) \\
&= \lim_{k\to\infty} \frac{1}{k}\big(n(k)\log 3 + m(n(k))\log 6\big) \\
&= \frac{\log 90 - \log 7}{\log 5} \log 3.
\end{aligned}$$

Therefore, by Theorem 4.27, $d_S(G,\mu) = 2\log 3/(\log 3 - \log \frac{3}{5}) = 2\log 3/\log 5$.

On the other hand, since

$$\sup\left\{k \;\middle|\; \text{there exist } a \in \mathbb{N} \text{ such that } f(b) = 1 \text{ for any } b \in [a, a+k]_{\mathbb{Z}}\right\} = \infty,$$

it follows that

$$\log \overline{N_*} = \lim_{k\to\infty} \frac{1}{k}\left(\log 6^k \vee \log 3^{(\log \frac{90}{7}/\log 5)k}\right) = \log 6,$$

because $\log_{10} 6 > 0.77 > 0.76 > \frac{\log 90 - \log 7}{\log 5} \log_{10} 3$. Similarly,

$$\log \overline{R}_2(0,1,2) = \log \underline{R}_2(0,1,2) = \log \frac{7}{15} \vee \frac{\log 90 - \log 7}{\log 5} \log \frac{3}{5} = \log \frac{7}{15}.$$

Therefore,

$$\overline{d}_2^S(0,2,1) = \underline{d}_2^S(0,2,1) = 2\frac{\log 6}{\log 6 - \log \frac{7}{15}} = 2\frac{\log 6}{\log 90 - \log 7}. \qquad \blacksquare$$

Remark 5.7. In the same way, we can prove $d_S(G,\mu) = \underline{d}_2^S(0,2,1) = \overline{d}_2^S(0,2,1) = 2\log 3/\log 5$ if $f \equiv 0$ (Sierpiński gasket graph) and $d_S(G,\mu) = \underline{d}_2^S(0,2,1) = \overline{d}_2^S(0,2,1) = 2\log 6/(\log 90 - \log 7)$ if $f \equiv 1$. Clearly the assumptions of Corollary 4.30 hold in these cases.

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