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Invariant measures for iterated function systems with inverses

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Abstract. We consider iterated function systems that contain inverses in the overlapping case. We focus on the parameterized families of iterated function systems with inverses, satisfying the transversality condition. We show that the invariant measure is absolutely continuous for a.e. parameter when the random walk entropy is greater than the Lyapunov exponent. We also show that if the random walk entropy does not exceed the Lyapunov exponent, then their ratio gives the Hausdorff dimension of the invariant measure for a.e. parameter value.

1. Introduction

A finite collection of strictly contractive maps on the real line is called an *iterated function system* (IFS). Let $\Phi = \{\varphi_a\}_{a\in\Lambda}$ be an IFS, and let $p = (p_a)_{a\in\Lambda}$ be a probability vector. Then it is well-known that there exists a unique Borel probability measure ν, called the *invariant measure*, such that

$$\nu = \sum_{a\in\Lambda} p_a \cdot \varphi_a \nu,$$

where $\varphi_a \nu$ is the push-forward of ν under the map $\varphi_a : \mathbb{R} \to \mathbb{R}$. When the construction does not involve complicated overlaps (say, under the open set condition), the invariant measure is relatively easy to understand. For example, if the open set condition holds, then the dimension of ν is given by

$$\dim \nu = \frac{h}{\chi},$$

where $h = h(p)$ is the *entropy* and $\chi = \chi(\Phi, p)$ is the *Lyapunov exponent*. However, the situation is dramatically more difficult in the overlapping case. Let Φ^t be a parameter family of IFS with overlaps, and let ν_t be the associated invariant measure.

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Using the so-called transversality method, in [14] the authors showed that under the transversality condition the measure $\nu_{\boldsymbol{t}}$ satisfies

$$\dim \nu_{\boldsymbol{t}} = \min\left\{1, \frac{h}{\chi_{\boldsymbol{t}}}\right\}$$

for a.e. $\boldsymbol{t}$, and is absolutely continuous for a.e. $\boldsymbol{t}$ in

$$\left\{\boldsymbol{t} \in U : \frac{h}{\chi_{\boldsymbol{t}}} > 1\right\}.$$

For the transversality method, see also [2, 9, 10, 12, 15].

In this paper, we extend the above result. Namely, we consider the case that the collection of maps contains inverses. We call such system an *IFS with inverse*. For the proof, we need to employ the ideas from random walks on free groups.

The paper is organized as follows. The next section contains definitions and the statement of the main result. Section 3 is devoted to preliminaries. In Section 4, we prove the main result. In Appendix A, we discuss the Furstenberg measure, and see how IFS with inverses arise naturally.

2. Definitions and the main result

2.1. Notations and setup

Let G be the free group of rank $r \geq 2$, and let S be a free generating set of G. Let Λ be a set that satisfies

$$S \subset \Lambda \subset S \cup S^{-1},$$

where $S^{-1} = \{a^{-1}\}_{a \in S}$. Let $\Omega^* = \bigcup_{n \geq 1} \Lambda^n$ and $\Omega = \Lambda^{\mathbb{N}}$. For $\omega = \omega_0\omega_1\cdots$, we write $\omega|_n = \omega_0 \cdots \omega_n$ (of course, this definition is not usual, but for our purpose, it is more convenient to define it in this way). For $\omega, \tau \in \Omega \cup \Omega^*$, we denote by $\omega \wedge \tau$ their common initial segment. For $\omega \in \Omega^*$ and $\tau \in \Omega \cup \Omega^*$, we say that ω *precedes* τ if $\omega \wedge \tau = \omega$.

Let $p = (p_a)_{a \in \Lambda}$ be a probability vector, and let μ be the associated Bernoulli measure on Ω. We assume that $p_a \neq 0$ for all $a \in \Lambda$. We say that a (finite or infinite) sequence $\omega \in \Omega^* \cup \Omega$ is *reduced* if $\omega_i\omega_{i+1} \neq aa^{-1}$ for all $i \geq 0$ and $a \in \Lambda$. Let Ω^*_{red} (resp. Ω_{red}) be the set of all finite (resp. infinite) reduced sequences. For $\omega \in \Omega^*_{\mathrm{red}}$, we denote the associated cylinder set in Ω_{red} by $[\omega]_{\mathrm{red}}$. Define the map

$$\mathrm{red} : \Omega^* \to \Omega^*_{\mathrm{red}}$$

in the obvious way. Let $\overline{\Omega} \subset \Omega$ be the set of all ω such that the limit

$$\lim_{n \to \infty} \mathrm{red}(\omega|_n) \tag{1}$$

exists. For example, for any $a \in \Lambda$, we have $aaa \cdots \in \overline{\Omega}$ and $aa^{-1}aa^{-1} \cdots \notin \overline{\Omega}$. By abuse of notation, for $\omega \in \overline{\Omega}$, we denote the limit (1) by $\mathrm{red}(\omega)$. Notice that, for $\omega \in \overline{\Omega}$, if $|\mathrm{red}(\omega|_n)| > k$ for all $n > N$, then $\mathrm{red}(\omega|_n)|_k$ precedes $\mathrm{red}(\omega)$ for all $n > N$. The following is well-known (see, e.g., [7, Chapter 14]).

Lemma 2.1. *There exists* $0 < \ell \leq 1$ *(*drift *or* speed*) such that*

$$\lim_{n\to\infty} \frac{1}{n} |\mathrm{red}(\omega|_n)| = \ell \tag{2}$$

for μ*-a.e.* $\omega \in \Omega$. *In particular,* $\overline{\Omega}$ *has full measure.*

It is easy to see that if $\Lambda = S$, then $\ell = 1$, and if $\Lambda = S \cup S^{-1}$ and $p_a \equiv 1/|\Lambda|$, then $\ell = 1 - 1/|S|$.

2.2. IFS with inverses

Define $\Lambda^\star = \{(a,b) \in \Lambda^2 : a \neq b^{-1}\}$.

Definition 2.1. Let $\mathcal{X} = \{X_a\}_{a\in\Lambda}$ be a collection of (not necessarily mutually disjoint) open intervals and $\theta \in (0,1]$. Write $X = \bigcup_{a\in\Lambda} X_a$. Suppose that there exists $0 < \gamma < 1$ such that the following holds: for any $(a,b) \in \Lambda^\star$, the map $\varphi_{ab} : X_b \to X_a$ is $C^{1+\theta}$ and satisfies

(i) $\overline{\varphi_{ab}(X_b)} \subset X_a$;

(ii) $0 < |\varphi'_{ab}(x)| < \gamma$ for all $x \in X_b$;

(iii) $\varphi_{ab}^{-1} : \varphi_{ab}(X_b) \to X_b$ is $C^{1+\theta}$.

We call $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}$ an *IFS with inverse*, and write $\Phi \in \Gamma_{\mathcal{X}}(\theta)$.

For $\omega = \omega_0 \cdots \omega_n \in \Omega^*_{\mathrm{red}}$, we define

$$\varphi_\omega = \varphi_{\omega_0\omega_1} \circ \cdots \circ \varphi_{\omega_{n-1}\omega_n}.$$

Let $\Pi : \overline{\Omega} \to X$ be the natural projection map, i.e.,

$$\Pi(\omega) = \bigcap_{n\geq 1} \varphi_{\mathrm{red}(\omega)|_n}(\overline{X_{\mathrm{red}(\omega)_n}}).$$

For $\omega \in \overline{\Omega}$, it happens that $\Pi(\omega) \notin X_{\omega_0}$. Consider, for example, $\omega = bb^{-1}aaa \cdots$ for $a, b \in \Lambda$ with $a \neq b$. Define the measure ν by $\nu = \Pi\mu$, where $\Pi\mu$ is the push-forward of the measure μ under the map $\Pi : \overline{\Omega} \to X$. It is easy to see that if $\Lambda = S$, then the measure ν is a self-similar measure of the usual IFS.

Notice that in Definition 2.1, we do not have any explicit inverse map. The next example illustrates why we call the system given in Definition 2.1 an IFS with inverse.

Example 2.1. Let $r = 2$, $S = \{0, 1\}$ and $\Lambda = \{0, 1, 1^{-1}\}$. For $0 < k, l < 1$, define

$$f_0(x) = kx, \quad f_1(x) = \frac{(1+l)x + 1 - l}{(1-l)x + 1 + l}.$$

Let $f_{1^{-1}} = f_1^{-1}$. It is easy to see that we have $f_0(0) = 0$, $f_1(-1) = -1$, $f_1(1) = 1$ and $f_0'(0) = k$, $f_1'(1) = l$. It is well known that there exists a unique Borel probability measure ν that satisfies

$$\nu = \sum_{a \in \Lambda} p_a f_a \nu.$$

Let

$$Y_0 = (-k, k), \quad Y_1 = (f_1(-k), 1) \quad \text{and} \quad Y_{-1} = (-1, f_{-1}(k)).$$

Then we have

$$f_a(Y \setminus Y_{a^{-1}}) \subset Y_a,$$

for all $a \in \Lambda$, where $Y = \bigcup_{a \in \Lambda} Y_a$ and $Y_{0^{-1}} = \emptyset$. Notice that the sets $\{Y_a\}_{a \in \Lambda}$ are not mutually disjoint if and only if $k > f_1(-k)$, which is equivalent to

$$\sqrt{l} > \frac{1-k}{1+k}. \tag{3}$$

It is easy to see that there exist open intervals $X_0, X_1, X_{1^{-1}} \subset \mathbb{R}$ such that

$$Y_a \subset X_a \quad \text{and} \quad \overline{f_a(X \setminus X_{a^{-1}})} \subset X_a$$

for all $a \in \Lambda$, where $X = \bigcup_{a \in \Lambda} X_a$ and $X_{0^{-1}} = \emptyset$. In Appendix A, we will show that $\{f_a|_{X_b}\}_{(a,b) \in \Lambda^*}$ is an IFS with inverse, and the associated invariant measure agrees with ν.

2.3. Transversality condition and the main result

Let $U \subset \mathbb{R}^d$ be an open set. Consider a family of IFS with inverse

$$\Phi^{\boldsymbol{t}} = \{\varphi^{\boldsymbol{t}}_{ab}\}_{(a,b) \in \Lambda^\star} \in \Gamma_{\mathcal{X}}(\theta), \quad \boldsymbol{t} \in \overline{U}.$$

Denote by $\Pi_{\boldsymbol{t}} : \overline{\Omega} \to X$ the natural projection map. Let $\nu_{\boldsymbol{t}} = \Pi_{\boldsymbol{t}} \mu$. Assume that for any $(a, b) \in \Lambda^\star$, the maps $\boldsymbol{t} \mapsto \varphi^{\boldsymbol{t}}_{ab}$ and $\boldsymbol{t} \mapsto (\varphi^{\boldsymbol{t}}_{ab})^{-1}$ are continuous, where $\varphi^{\boldsymbol{t}}_{ab}$ and $(\varphi^{\boldsymbol{t}}_{ab})^{-1}$ are equipped with $C^{1+\theta}$ norm. Denote the d-dimensional Lebesgue measure by $\mathcal{L}_d$.

Definition 2.2. We say that $\Phi^{\boldsymbol{t}}$ satisfies the *transversality condition* if the following holds: there exists a constant $C_1 > 0$ such that for all $\omega, \tau \in \Omega_{\mathrm{red}}$ with $\omega_0 = \tau_0$ and $\omega_1 \neq \tau_1$, we have

$$\mathcal{L}_d\big(\{\boldsymbol{t} \in U : |\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| \leq r\}\big) \leq C_1 r \quad \text{for all } r > 0.$$

For an arbitrary positive measure ν on the real line, we define the dimension of ν by

$$\dim \nu = \inf\bigl\{\dim_H(Y) : \nu(\mathbb{R} \setminus Y) = 0\bigr\}.$$

We denote the random walk entropy by $h_{RW} = h_{RW}(p)$ and the Lyapunov exponent of $\Phi^{\boldsymbol{t}}$ by $\chi_{\boldsymbol{t}}$ (for the precise definition, see the next section). Our main result is the following.

Theorem 2.1. *Assume that the transversality condition is satisfied. Then*

(i) *for a.e.* $\boldsymbol{t} \in U$,

$$\dim \nu_{\boldsymbol{t}} = \min\Bigl\{\frac{h_{RW}}{\chi_{\boldsymbol{t}}}, 1\Bigr\};$$

(ii) *the measure* $\nu_{\boldsymbol{t}}$ *is absolutely continuous for a.e.* $\boldsymbol{t}$ *in*

$$U' = \Bigl\{\boldsymbol{t} \in U : \frac{h_{RW}}{\chi_{\boldsymbol{t}}} > 1\Bigr\}.$$

Example 2.2. Let $r = 2$, $S = \{0, 1\}$ and $\Lambda = \{0, 0^{-1}, 1, 1^{-1}\}$. Let $I \supset [0, 1]$ be an open interval. For $a \in \Lambda$, we define $X_a = I \times \{a\}$, which we identify with I. For $(a, b) \in \Lambda^\star$ and $1/3 < \lambda < 1$, we define $\varphi_{ab} : X_b \to X_a$ by

$$\varphi_{ab}(x) = \begin{cases} \lambda x + \frac{1-\lambda}{2}, & \text{if } a = b, \\ \lambda x, & \text{if } (a, b) \in \{(0, 1), (1, 0^{-1}), (0^{-1}, 1^{-1}), (1^{-1}, 0)\}, \\ \lambda x + 1 - \lambda, & \text{if } (a, b) \in \{(1, 0), (0^{-1}, 1), (1^{-1}, 0^{-1}), (0, 1^{-1})\}. \end{cases}$$

It is easy to see that $\{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}$ is an IFS with inverse. Let ν_λ be the associated invariant measure. By [11, Corollary 5.2], the transversality condition holds for $1/3 < \lambda < 1/2$. By [5, Theorems 2 and 5], the random walk entropy $h_{RW}(p)$ and the speed $\ell(p)$ depend continuously on the probability vector p. Furthermore, later in Proposition 3.1, we will see that $\chi(p) = -\ell(p) \log \lambda$. Since $h_{RW}(p) = \chi(p) = \frac{1}{2} \log 3$, if $\lambda = 1/3$ and the weight p is uniform, Theorem 2.1 implies the following.

Theorem 2.2. *If the weight* p *is sufficiently close to the uniform weight, then there exists an interval* $J \subset (1/3, 1/2)$ *such that* ν_λ *is absolutely continuous for a.e.* $\lambda \in J$.

3. Preliminaries

3.1. Random walk entropy and the Lyapunov exponent

Definition 3.1. Let η be the probability measure on Λ associated with the probability vector $p = (p_a)_{a\in\Lambda}$, and let $\eta^{(n)}$ be the push-forward of the product measure $\Pi_{i=0}^n \eta$ under the map

$$\mathrm{red} : \Lambda^{n+1} \to \Omega^*_{\mathrm{red}}.$$

We define the *random walk entropy* (or *asymptotic entropy*) $h_{RW} = h_{RW}(G, p)$ by

$$h_{RW} = \lim_{n\to\infty} \frac{1}{n} H(\eta^{(n)}),$$

where $H(\cdot)$ is the Shannon entropy. For the existence of the limit, see, e.g., [6].

It is well known that if $\Lambda = S$, then the random walk entropy is

$$-\sum_{a\in S} p_a \log p_a,$$

and if $\Lambda = S \cup S^{-1}$ and $p_a \equiv 1/|\Lambda|$, then the random walk entropy is

$$\left(1 - \frac{1}{|S|}\right) \log(2|S| - 1).$$

Lemma 3.1 ([6, Theorem 2.1]). *We have*

$$-\frac{1}{n} \log \mu\big(\{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\}\big) \to h_{RW} \tag{4}$$

for μ-a.e. ω.

We next define the *Lyapunov exponent* $\chi = \chi(\mu, \Phi)$. Due to the presence of inverses, the definition is more complicated than in the case of the usual IFS. For $\omega \in \overline{\Omega}$ and $n \geq 0$, let

$$\mathcal{N}(\omega, n) = n + \max_{k\geq 0}\{|\mathrm{red}\,((\sigma^n\omega)|_k)| = 1\}$$

and

$$\mathcal{P}(\omega, n) = \mathrm{red}(\sigma^n\omega)_0.$$

For example, if $\omega = bb^{-1}bbaaa\cdots$, then we have $\mathcal{N}(\omega, 0) = 2$, $\mathcal{P}(\omega, 0) = b$ and $\mathcal{N}(\omega, 1) = 3$, $\mathcal{P}(\omega, 1) = b$. It is easy to see that $\omega_{\mathcal{N}(\omega,n)} = \mathcal{P}(\omega, n)$. Notice that, if $\omega_n \in S$ for all $n \geq 0$, then we have $\mathcal{N}(\omega, n) = n$ and $\mathcal{P}(\omega, n) = \omega_n$.

For $\omega \in \overline{\Omega}$ and $n \geq 0$, let

$$\hat{\varphi}_{\omega,n} = \begin{cases} \varphi_{\mathcal{P}(\omega,n)\mathcal{P}(\omega,n+1)}, & \text{if } \mathcal{N}(\omega, n) < \mathcal{N}(\omega, n+1), \\ \varphi^{-1}_{\mathcal{P}(\omega,n+1)\mathcal{P}(\omega,n)}, & \text{if } \mathcal{N}(\omega, n) > \mathcal{N}(\omega, n+1). \end{cases}$$

We then define the Lyapunov exponent by

$$\chi = -\int_{\overline{\Omega}} \log\left|\hat{\varphi}'_{\omega,0}(\Pi(\sigma\omega))\right| \, d\mu(\omega). \tag{5}$$

Since $\hat{\varphi}_{\sigma\omega,n} = \hat{\varphi}_{\omega,n+1}$ and $\Pi(\omega) = \hat{\varphi}_{\omega,0}(\Pi(\sigma\omega))$, by the Birkhoff ergodic theorem, we have

$$\begin{aligned}\int_{\overline{\Omega}} \log\left|\hat{\varphi}'_{\omega,0}(\Pi(\sigma\omega))\right| d\mu(\omega) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\left|\hat{\varphi}'_{\sigma^k\omega,0}(\Pi(\sigma^{k+1}\omega))\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\left|\hat{\varphi}'_{\omega,k}(\Pi(\sigma^{k+1}\omega))\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \log\left|(\hat{\varphi}_{\omega,0} \circ \cdots \circ \hat{\varphi}_{\omega,n-1})'(\Pi(\sigma^n\omega))\right|\end{aligned} \tag{6}$$

for a.e. $\omega \in \overline{\Omega}$.

Lemma 3.2. *Let $\tau^0 \in \Omega^*$, $\tau^1 \in \overline{\Omega}$ and $a \in \Lambda$. Write $\omega^0 = \tau^0 a a^{-1} \tau^1$ and $\omega^1 = \tau^0\tau^1$. Let $n \geq |\tau^0| + 2$. Then we have*

$$\hat{\varphi}_{\omega^0,0} \circ \cdots \circ \hat{\varphi}_{\omega^0,n-1} = \hat{\varphi}_{\omega^1,0} \circ \cdots \circ \hat{\varphi}_{\omega^1,n-3}.$$

Proof. Since $\mathcal{P}(\omega^0, |\tau^0| + 2 + k) = \mathcal{P}(\omega^1, |\tau^0| + k)$ for all $k \geq 0$, we have

$$\hat{\varphi}_{\omega^0,|\tau^0|} \circ \hat{\varphi}_{\omega^0,|\tau^0|+1} = \mathrm{id}$$

and

$$\hat{\varphi}_{\omega^0,|\tau^0|+k+2} = \hat{\varphi}_{\omega^1,|\tau^0|+k}$$

for all $k \geq 0$. The result follows from this. ∎

Fix $x_a \in X_a$ for each $a \in \Lambda$. For $\omega \in \overline{\Omega}$ and $n \in \mathbb{N}$, we write $x_{\omega,n} = x_j$, where $j = j(\omega, n) \in \Lambda$ is the last letter of $\mathrm{red}(\omega|_n)$.

Proposition 3.1. *We have*

$$-\lim_{n\to\infty} \frac{1}{n} \log\left|\varphi'_{\mathrm{red}(\omega|_n)}(x_{\omega,n})\right| = \chi \tag{7}$$

for μ-a.e. ω.

Proof. Let $\omega \in \overline{\Omega}$ be such that (2) and (6) holds. Write

$$n_l = \max_{k\geq 0}\left\{k : |\mathrm{red}(\omega|_k)| = l\right\}$$

for $l \geq 2$. By Lemma 3.2, we have

$$\hat{\varphi}_{\omega,0} \circ \cdots \circ \hat{\varphi}_{\omega,n_l-1} = \varphi_{\mathrm{red}(\omega|_{n_l})}.$$

Therefore, by (6) and the distortion property (for the distortion property, see, e.g., [8, Chapter 4]), we have

$$-\lim_{l\to\infty}\frac{1}{n_l}\log\bigl|\varphi'_{\mathrm{red}(\omega|_{n_l})}(x_{\omega,n_l})\bigr| = \chi. \tag{8}$$

Notice that by (2), we have

$$\lim_{l\to\infty}\frac{n_{l+1}}{n_l} = 1.$$

Together with (8), the result follows. ∎

3.2. Estimate above

For $\varphi \in C^{1+\theta}(I)$, where $I \subset \mathbb{R}$ is an open interval, we write

$$\|\varphi'\|_\theta = \sup\bigl\{|\varphi'(x)-\varphi'(y)|\cdot|x-y|^{-\theta} : x, y \in I\bigr\}.$$

For an IFS with inverse $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}$, write

$$\|\Phi'\|_\theta = \max\bigl\{\|\varphi'_{ab}\|_\theta : (a,b)\in\Lambda^\star\bigr\}.$$

We denote by $\|\cdot\|$ the supremum norm. Given $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}, \Psi = \{\psi_{ab}\}_{(a,b)\in\Lambda^\star} \in \Gamma_{\mathcal{X}}(\theta)$, we write

$$\|\Phi-\Psi\| = \max_{(a,b)\in\Lambda^\star}\|\varphi_{ab}-\psi_{ab}\| \quad\text{and}\quad \|\Phi'-\Psi'\| = \max_{(a,b)\in\Lambda^\star}\|\varphi'_{ab}-\psi'_{ab}\|.$$

The following is well known (see [14, Section 3] and the references therein): for any nonzero Borel measure ν on $\mathbb{R}$, we have

$$\dim\nu = \nu\text{-ess}\sup\Bigl\{\liminf_{r\downarrow 0}\frac{\log\nu[x-r,x+r]}{\log 2r}\Bigr\}, \tag{9}$$

and

$$\dim\nu \ge \sup\Bigl\{\alpha>0 : \iint_{\mathbb{R}^2}\frac{d\nu(x)\,d\nu(y)}{|x-y|^\alpha} < \infty\Bigr\}. \tag{10}$$

For $\Phi \in \Gamma_{\mathcal{X}}(\theta)$, $0 < \gamma, u < 1$ and $M > 0$, we write

$$\Phi \in \Gamma_{\mathcal{X}}(\theta,\gamma,u,M)$$

if we have

(i) $u < |\varphi'_{ab}(x)| < \gamma$ for all $(a,b)\in\Lambda^\star$ and $x \in X_b$;

(ii) $\|\Phi'\|_\theta$, $\|(\Phi^{-1})'\|_\theta < M$.

Proposition 3.2. *We have*

$$\dim \nu \leq \frac{h_{RW}}{\chi}.$$

Proof. Let $x \in X$ be in the support of ν, and let $\omega \in \Omega$ be such that $x = \Pi(\omega)$. We can assume that ω satisfies (2), (4) and (7). Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ in such a way that the following holds for all $n > N$:

$$n(\ell - \varepsilon) < \left|\mathrm{red}(\omega|_n)\right| < n(\ell + \varepsilon),$$

$$n(h_{RW} - \varepsilon) < -\log \mu\big(\{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\}\big) < n(h_{RW} + \varepsilon)$$

and

$$e^{-n(\chi+\varepsilon)} < \left|\varphi'_{\mathrm{red}(\omega|_n)}(x_{\omega,n})\right| < e^{-n(\chi-\varepsilon)}.$$

For $n \in \mathbb{N}$, let $n' \in \mathbb{N}$ be the integer part of $n(\ell - \varepsilon)$. Then $\mathrm{red}(\omega)|_{n'}$ precedes $\mathrm{red}(\omega|_n)$ for all sufficiently large $n \in \mathbb{N}$. By the mean value theorem, we have

$$\mathrm{diam}\big(\Pi\left([\mathrm{red}(\omega)|_{n'}]_{\mathrm{red}}\right)\big) \leq \mathrm{diam}(X) \cdot \left\|\varphi'_{\mathrm{red}(\omega)|_{n'}}\right\| =: r_n$$

for all $n \in \mathbb{N}$. The following claim follows by the Markov property of the random walk.

Claim 3.1. *There exists* $0 < C < 1$ *such that for sufficiently large* n*, we have*

$$\mu\big(\{\upsilon \in \Omega : |\Pi(\upsilon) - \Pi(\omega)| \leq r_n\}\big) > C\mu\big(\{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\}\big).$$

Proof of Claim 3.1. Notice that we have

$$\begin{aligned}&\{\upsilon \in \Omega : \mathrm{red}(\upsilon)|_{n'} = \mathrm{red}(\omega)|_{n'}\}\\ &\quad\supset \{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\}\\ &\qquad\cap \left\{\upsilon \in \Omega : \mathrm{red}(\sigma^{n+1}(\upsilon))_0 \neq \left(\mathrm{red}(\omega|_n)_{|\mathrm{red}(\omega|_n)|-1}\right)^{-1}\right\},\end{aligned}$$

for sufficiently large n, where $\sigma(\cdot)$ is the left shift. Therefore, for such n, we have

$$\begin{aligned}&\mu\big(\{\upsilon \in \Omega : \mathrm{red}(\upsilon)|_{n'} = \mathrm{red}(\omega)|_{n'}\}\big)\\ &\quad\geq \mu\big(\{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\}\big)\\ &\qquad\cdot \mu\left(\left\{\upsilon \in \Omega : \mathrm{red}(\upsilon)_0 \neq \left(\mathrm{red}(\omega|_n)_{|\mathrm{red}(\omega|_n)|-1}\right)^{-1}\right\}\right).\end{aligned}$$

The result follows from this. ∎

Therefore, for sufficiently large $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\nu[x-r_n, x+r_n] &= \mu\big(\{\upsilon \in \Omega : |\Pi(\upsilon) - \Pi(\omega)| \le r_n\}\big) \\
&> C\mu\big(\{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\}\big) \\
&> Ce^{-n(h_{RW}+\varepsilon)}.
\end{aligned}
$$

Let $n'' \in \mathbb{N}$ be such that $\mathrm{red}(\omega|_{n''}) = \mathrm{red}(\omega)|_{n'}$. Then we have $n'' > \frac{n'}{\ell+\varepsilon}$. By the distortion property, there exists $C' > 0$ such that

$$
\begin{aligned}
\left\|\varphi'_{\mathrm{red}(\omega)|_{n'}}\right\| &\le C'\left|\varphi'_{\mathrm{red}(\omega|_{n''})}(x_{\omega,n''})\right| \\
&< C'e^{-n''(\chi-\varepsilon)} \\
&< C'e^{-\frac{\ell-\varepsilon}{\ell+\varepsilon}n(\chi-\varepsilon)}.
\end{aligned}
$$

Therefore,

$$
\frac{\log \nu[x-r_n, x+r_n]}{\log 2r_n} < \frac{n(h_{RW}+\varepsilon) - \log C}{\frac{\ell-\varepsilon}{\ell+\varepsilon}n(\chi-\varepsilon) - \log(2C' \cdot \mathrm{diam}(X))}.
$$

It follows that

$$
\begin{aligned}
\liminf_{r\downarrow 0} \frac{\log \nu[x-r, x+r]}{\log 2r} &\le \liminf_{n\to\infty} \frac{\log \nu[x-r_n, x+r_n]}{\log 2r_n} \\
&= \frac{(\ell+\varepsilon)(h_{RW}+\varepsilon)}{(\ell-\varepsilon)(\chi-\varepsilon)}.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have

$$
\liminf_{r\downarrow 0} \frac{\log \nu[x-r, x+r]}{\log 2r} \le \frac{h_{RW}}{\chi}.
$$

The proof of Proposition 3.2 is finished by applying (9). ∎

The following proposition is immediate.

Proposition 3.3. *Assume that the sets $\{\varphi_{ab}(X_b)\}_{(a,b)\in\Lambda^\star}$ are mutually disjoint. Then we have*

$$
\dim \nu = \frac{h_{RW}}{\chi}.
$$

Proof. By arguing analogously as in the proof of Lemma 3.2, we obtain

$$
\lim_{r\downarrow 0} \frac{\log \nu[x-r, x+r]}{\log 2r} = \frac{h_{RW}}{\chi}.
$$

Therefore, by (9), the result follows. ∎

4. Proof of Theorem 2.1

The following two lemmas follow easily by imitating the proof of [14, Lemma 4.1] and [13, Corollary 6.3]. For the readers' convenience, we include the proof.

Lemma 4.1. *Suppose that*

$$\Phi^{\boldsymbol{t}} = \{\varphi^{\boldsymbol{t}}_{ab}\}_{(a,b)\in\Lambda^\star} \in \Gamma_{\mathcal{X}}(\theta), \quad \boldsymbol{t} \in U,$$

is a family of IFS with inverse. Then the function $\boldsymbol{t} \mapsto \chi_{\boldsymbol{t}}$ is continuous on U.

Proof. Recall that

$$\chi_{\boldsymbol{t}} = -\int_{\overline{\Omega}} \log\big|(\hat{\varphi}^{\boldsymbol{t}}_{\omega,0})'(\Pi_{\boldsymbol{t}}(\sigma\omega))\big|\, d\mu(\omega).$$

By retaking U if necessary, we can choose $0 < \gamma,\ u < 1$ and $M > 0$ such that $\Phi^{\boldsymbol{t}} \in \Gamma_{\mathcal{X}}(\theta, \gamma, u, M)$ for all $\boldsymbol{t} \in U$. The desired result follows immediately from the following claim.

Claim 4.1. *Let $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}$ and $\Psi = \{\psi_{ab}\}_{(a,b)\in\Lambda^\star}$ be two IFS with inverse in $\Gamma_{\mathcal{X}}(\theta, \gamma, u, M)$. Then for all $\omega \in \overline{\Omega}$, we have*

$$\begin{aligned}&\left|\log\left|\frac{\hat{\varphi}'_{\omega,0}(\Pi_\Phi(\sigma\omega))}{\hat{\psi}'_{\omega,0}(\Pi_\Psi(\sigma\omega))}\right|\right| \\ &\qquad \le \frac{1}{u}\big(M\|\Phi-\Psi\|^\theta(1-\gamma)^{-\theta} + \max\big\{\|\Phi'-\Psi'\|, \|(\Phi^{-1})'-(\Psi^{-1})'\|\big\}\big).\end{aligned}$$

Proof of Claim 4.1. First we show that

$$|\Pi_\Phi(\omega) - \Pi_\Psi(\omega)| \le \|\Phi-\Psi\|(1-\gamma)^{-1} \tag{11}$$

for all $\omega \in \Omega_{\mathrm{red}}$. Indeed,

$$\begin{aligned}\big|\Pi_\Phi(\omega) - \Pi_\Psi(\omega)\big| &= \big|\varphi_{\omega_0\omega_1}(\Pi_\Phi(\sigma\omega)) - \psi_{\omega_0\omega_1}(\Pi_\Psi(\sigma\omega))\big| \\ &\le \big|\varphi_{\omega_0\omega_1}(\Pi_\Phi(\sigma\omega)) - \psi_{\omega_0\omega_1}(\Pi_\Phi(\sigma\omega))\big| \\ &\qquad + \big|\psi_{\omega_0\omega_1}(\Pi_\Phi(\sigma\omega)) - \psi_{\omega_0\omega_1}(\Pi_\Psi(\sigma\omega))\big| \\ &\le \|\Phi-\Psi\| + \gamma\big|\Pi_\Phi(\sigma\omega) - \Pi_\Psi(\sigma\omega)\big|.\end{aligned}$$

Repeating this inductively we obtain (11). Since

$$\left|\log\left|\frac{x}{y}\right|\right| \le \frac{|x-y|}{\min\{|x|,|y|\}},$$

we have

$$
\begin{aligned}
&\left|\log\left|\frac{\hat{\varphi}'_{\omega,0}(\Pi_\Phi(\sigma\omega))}{\hat{\psi}'_{\omega,0}(\Pi_\Psi(\sigma\omega))}\right|\right| \\
&\quad\le \frac{|\hat{\varphi}'_{\omega,0}(\Pi_\Phi(\sigma\omega)) - \hat{\psi}'_{\omega,0}(\Pi_\Psi(\sigma\omega))|}{u} \\
&\quad\le \frac{1}{u}\Big(\big|\hat{\varphi}'_{\omega,0}(\Pi_\Phi(\sigma\omega)) - \hat{\varphi}'_{\omega,0}(\Pi_\Psi(\sigma\omega))\big| \\
&\qquad\quad + \big|\hat{\varphi}'_{\omega,0}(\Pi_\Psi(\sigma\omega)) - \hat{\psi}'_{\omega,0}(\Pi_\Psi(\sigma\omega))\big|\Big) \\
&\quad\le \frac{1}{u}\Big(M\big|\Pi_\Phi(\sigma\omega) - \Pi_\Psi(\sigma\omega)\big|^\theta + \max\big\{\|\Phi' - \Psi'\|, \|(\Phi^{-1})' - (\Psi^{-1})'\|\big\}\Big) \\
&\quad\le \frac{1}{u}\Big(M\|\Phi - \Psi\|^\theta(1-\gamma)^{-\theta} + \max\big\{\|\Phi' - \Psi'\|, \|(\Phi^{-1})' - (\Psi^{-1})'\|\big\}\Big). \qquad \blacksquare
\end{aligned}
$$

This concludes the proof of Lemma 4.1. ■

Lemma 4.2. *There exists a positive constant*

$$L = L(\mathcal{X}, \theta, \gamma, u, M)$$

such that for any $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}, \Psi = \{\psi_{ab}\}_{(a,b)\in\Lambda^\star} \in \Gamma_{\mathcal{X}}(\theta,\gamma,u,M)$*,* $\omega \in \Omega_{\mathrm{red}}$*,* $n \in \mathbb{N}$ *and* $x \in X_{\omega_n}$*, we have*

$$\frac{|\varphi'_{\omega|_n}(x)|}{|\psi'_{\omega|_n}(x)|} \le \exp\Big(Ln\big(\|\Phi - \Psi\|^\theta + \|\Phi' - \Psi'\|\big)\Big).$$

Proof. Observe that for any $\omega = \omega_0 \cdots \omega_n \in \Omega^*_{\mathrm{red}}$ and $x \in X_{\omega_n}$, we have

$$
\begin{aligned}
&|\varphi_\omega(x) - \psi_\omega(x)| \\
&\quad\le \big|\varphi_{\omega_0\omega_1}(\varphi_{\omega_1\cdots\omega_n}(x)) - \psi_{\omega_0\omega_1}(\varphi_{\omega_1\cdots\omega_n}(x))\big| \\
&\qquad + \big|\psi_{\omega_0\omega_1}(\varphi_{\omega_1\cdots\omega_n}(x)) - \psi_{\omega_0\omega_1}(\psi_{\omega_1\cdots\omega_n}(x))\big| \\
&\quad\le \|\Phi - \Psi\| + \gamma\big|\varphi_{\omega_1\cdots\omega_n}(x) - \psi_{\omega_1\cdots\omega_n}(x)\big| \\
&\quad\le \|\Phi - \Psi\|(1-\gamma)^{-1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\big|\varphi'_{\omega_k\omega_{k+1}}(\varphi_{\omega_{k+1}\cdots\omega_n}(x)) - \psi'_{\omega_k\omega_{k+1}}(\psi_{\omega_{k+1}\cdots\omega_n}(x))\big| \\
&\quad\le \big|\varphi'_{\omega_k\omega_{k+1}}(\varphi_{\omega_{k+1}\cdots\omega_n}(x)) - \psi'_{\omega_k\omega_{k+1}}(\varphi_{\omega_{k+1}\cdots\omega_n}(x))\big| \\
&\qquad + \big|\psi'_{\omega_k\omega_{k+1}}(\varphi_{\omega_{k+1}\cdots\omega_n}(x)) - \psi'_{\omega_k\omega_{k+1}}(\psi_{\omega_{k+1}\cdots\omega_n}(x))\big| \\
&\quad\le \|\Phi' - \Psi'\| + M\big|\varphi_{\omega_{k+1}\cdots\omega_n}(x) - \psi_{\omega_{k+1}\cdots\omega_n}(x)\big|^\theta \\
&\quad\le \|\Phi' - \Psi'\| + M\|\Phi - \Psi\|^\theta(1-\gamma)^{-\theta}.
\end{aligned}
$$

Since $|\log|\frac{x}{y}|| \le \frac{|x-y|}{\min\{|x|,|y|\}}$, we obtain that for each $k \ge 1$,

$$\log\left|\frac{\varphi'_{\omega_k\omega_{k+1}}(\varphi_{\omega_{k+1}\cdots\omega_n}(x))}{\psi'_{\omega_k\omega_{k+1}}(\psi_{\omega_{k+1}\cdots\omega_n}(x))}\right| \le \frac{1}{u}\big(\|\Phi'-\Psi'\| + M\|\Phi-\Psi\|^\theta(1-\gamma)^{-\theta}\big).$$

Therefore,

$$\begin{aligned}\frac{1}{n}\log\left|\frac{\varphi'_{\omega|_n}(x)}{\psi'_{\omega|_n}(x)}\right| &= \frac{1}{n}\sum_{k=0}^{n-1}\log\left|\frac{\varphi'_{\omega_k\omega_{k+1}}(\varphi_{\omega_{k+1}\cdots\omega_n}(x))}{\psi'_{\omega_k\omega_{k+1}}(\psi_{\omega_{k+1}\cdots\omega_n}(x))}\right| \\ &\le \frac{1}{u}\big(\|\Phi'-\Psi'\| + M\|\Phi-\Psi\|^\theta(1-\gamma)^{-\theta}\big),\end{aligned}$$

which implies the lemma. ∎

We will also need the following simple lemma.

Lemma 4.3 ([13, Lemma 3.3]). *Suppose that $\{\Phi^{\boldsymbol{t}}\}_{\boldsymbol{t}\in\overline{U}}$ satisfies the transversality condition. Then for every $0<\alpha<1$, there exists $C_2 = C_2(\alpha)$ such that for all $\omega,\tau \in \Omega_{\mathrm{red}}$ with $\omega_0=\tau_0$ and $\omega_1\ne\tau_1$, we have*

$$\int_U \frac{d\boldsymbol{t}}{|\Pi_{\boldsymbol{t}}(\omega)-\Pi_{\boldsymbol{t}}(\tau)|^\alpha} < C_2.$$

Now we prove Theorem 2.1. The proof follows the scheme of [14].

Proof of Theorem 2.1 (i). It is enough to establish the estimate from below, which follows from the following claim (see [14, Section 4]).

Claim 4.2. *For every $\boldsymbol{t}_0 \in U$ and $\varepsilon'>0$, there exists $\eta>0$ such that*

$$\dim \nu_{\boldsymbol{t}} \ge \min\Big\{\frac{h_{RW}}{\chi_{\boldsymbol{t}}}, 1\Big\} - \varepsilon'$$

for a.e. $\boldsymbol{t} \in B_\eta(\boldsymbol{t}_0)$.

Let $\boldsymbol{t}_0\in U$. Set $\Phi=\Phi^{\boldsymbol{t}_0}$, $\Pi=\Pi_{\boldsymbol{t}_0}$ and $\chi=\chi_{\boldsymbol{t}_0}$. Let $\varepsilon=\frac{1}{2\log r+4}\varepsilon'\chi$. By Lemma 4.2, there exists $\eta>0$ such that for all $\omega\in\Omega_{\mathrm{red}}$, $n\ge1$ and $x\in X_{\omega_n}$, we have

$$|\boldsymbol{t}-\boldsymbol{t}_0|<\eta \quad\Longrightarrow\quad \frac{|\varphi'_{\omega|_n}(x)|}{|(\varphi^{\boldsymbol{t}}_{\omega|_n})'(x)|} < e^{\varepsilon n}. \tag{12}$$

By Egorov's theorem, choose a set $\Omega'\subset\overline{\Omega}$ such that $\mu(\Omega')>0$ and the convergence in (2), (4) and (7) is uniform on Ω'. We can assume that there exists $a\in\Lambda$ such that $\mathrm{red}(\omega)_0 = a$ for all $\omega \in \Omega'$. Write

$$\Omega'_{\mathrm{red}} = \left\{\omega_{\mathrm{red}} : \exists\omega \in \Omega' \text{ s.t. } \mathrm{red}(\omega) = \omega_{\mathrm{red}}\right\}.$$

Define

$$\mu' = \mu|_{\Omega'}, \quad \nu'_{\boldsymbol{t}} = \Pi_{\boldsymbol{t}}\mu' \quad \text{and} \quad \mu'_{\mathrm{red}} = \mathrm{red}\,\mu'.$$

Since $\dim \nu'_{\boldsymbol{t}} \le \dim \nu_{\boldsymbol{t}}$, it suffices to estimate $\dim \nu'_{\boldsymbol{t}}$ from below. Define

$$s = \min\left\{\frac{h_{RW}}{\chi}, 1\right\}.$$

By (10), the claim will follow by showing

$$\mathcal{S} := \int_{B_\eta(\boldsymbol{t}_0)} \iint_{(x,y)\in X_a^2} \frac{d\nu'_{\boldsymbol{t}}(x)d\nu'_{\boldsymbol{t}}(y)}{|x-y|^{s-\varepsilon'}}\,d\boldsymbol{t} < \infty.$$

For a word $\rho \in \Omega^*_{\mathrm{red}}$, we define

$$A_\rho = \left\{(\omega,\tau) \in \Omega'^2_{\mathrm{red}} : \omega \wedge \tau = \rho\right\}.$$

Then we have

$$\begin{aligned}
\mathcal{S} &= \iint_{\Omega'_{\mathrm{red}}\times\Omega'_{\mathrm{red}}} \left(\int_{B_\eta(\boldsymbol{t}_0)} \frac{d\boldsymbol{t}}{|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)|^{s-\varepsilon'}}\right) d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega,\tau) \\
&= \sum_{n\ge 0} \sum_{\rho\in\Omega^*_{\mathrm{red}}, |\rho|=n+1} \iint_{A_\rho} \left(\int_{B_\eta(\boldsymbol{t}_0)} \frac{d\boldsymbol{t}}{|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)|^{s-\varepsilon'}}\right) d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega,\tau).
\end{aligned}$$

Let $(\omega,\tau) \in A_\rho$. Then for some $c \in [\Pi_{\boldsymbol{t}}(\sigma^n\omega), \Pi_{\boldsymbol{t}}(\sigma^n\tau)]$, we have

$$\begin{aligned}
\left|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)\right| &= |(\varphi^{\boldsymbol{t}}_{\omega|_n})'(c)| \cdot \left|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)\right| \\
&\ge |\varphi'_{\omega|_n}(c)|e^{-\varepsilon n} \cdot \left|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)\right|,
\end{aligned}$$

where we used (12) in the second step. By the distortion property, there exists a constant $C > 1$ such that

$$|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| \ge \frac{1}{C}\left|\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))\right|e^{-\varepsilon n} \cdot \left|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)\right|.$$

By Lemma 4.3, there exists a constant $C_2 = C_2(s-\varepsilon')$ such that

$$\int_{B_\eta(\boldsymbol{t}_0)} \frac{d\boldsymbol{t}}{|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)|^{s-\varepsilon'}} < C_2.$$

Let us take $N \in \mathbb{N}$ in such a way that for all $\omega \in \Omega'$ and $n > N$, we have

$$e^{-n(h_{RW}+\varepsilon)} < \mu\left(\left\{\upsilon \in \Omega : \mathrm{red}(\upsilon|_n) = \mathrm{red}(\omega|_n)\right\}\right) < e^{-n(h_{RW}-\varepsilon)}, \tag{13}$$

$$e^{-n(\chi+\varepsilon)} < |\varphi'_{\omega|_n}(x_{\omega,n})| < e^{-n(\chi-\varepsilon)}, \tag{14}$$

and

$$n(\ell - \varepsilon) < |\mathrm{red}(\omega|_n)| < n(\ell + \varepsilon). \tag{15}$$

Claim 4.3. *For any $\omega \in \Omega'_{\mathrm{red}}$, we have*

$$\left|\varphi'_{\omega|_n}(\Pi(\sigma^n \omega))\right| > e^{-\frac{n}{\ell-\varepsilon}(\chi+\varepsilon)}$$

for all $n > N$.

Proof. Let $\omega' \in \Omega'$ be such that $\mathrm{red}(\omega') = \omega$. Let n' be the integer part of $\frac{n}{\ell-\varepsilon}$. We have that $\omega|_n$ precedes $\mathrm{red}(\omega'|_{n'})$. Therefore, by (14),

$$|\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))| > |\varphi'_{\omega'|_{n'}}(x_{\omega',n'})| > e^{-\frac{n}{\ell-\varepsilon}(\chi+\varepsilon)}. \qquad \blacksquare$$

Claim 4.4. *For any $\rho \in \Omega^*_{\mathrm{red}}$ with $|\rho| = n+1$ and $n > N$, we have*

$$\mu'_{\mathrm{red}}([\rho]_{\mathrm{red}}) < e^{-\frac{n}{\ell-\varepsilon}(h_{RW}-(2\log r+1)\varepsilon)}.$$

Proof. Let $n', n_{\max} \in \mathbb{N}$ be the integer part of $\frac{n}{\ell-\varepsilon}$ and $(\ell+\varepsilon)n'$, respectively. Notice that, by (15), for any $\upsilon \in \Omega'$, we have $n < |\mathrm{red}(\upsilon|_{n'})| < n_{\max}$. Since

$$\begin{aligned}\#\{\upsilon \in \Omega^*_{\mathrm{red}} : n < |\upsilon| < n_{\max},\ \upsilon|_n = \rho\} &< r + r^2 + \cdots + r^{n_{\max}-n-1} \\ &< r^{n_{\max}-n} = r^{\frac{2\varepsilon}{\ell-\varepsilon}n},\end{aligned}$$

by (13), we have

$$\mu'_{\mathrm{red}}([\rho]_{\mathrm{red}}) < r^{\frac{2\varepsilon}{\ell-\varepsilon}n} \cdot e^{-n'(h_{RW}-\varepsilon)} < e^{\frac{2\varepsilon\log r}{\ell-\varepsilon}n} \cdot e^{-\frac{n}{\ell-\varepsilon}(h_{RW}-\varepsilon)}. \qquad \blacksquare$$

By the above claim, we have

$$(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(A_\rho) \le (\mu'_{\mathrm{red}}([\rho]_{\mathrm{red}}))^2 < \mu'_{\mathrm{red}}([\rho]_{\mathrm{red}})\, e^{-\frac{n}{\ell-\varepsilon}(h_{RW}-(2\log r+1)\varepsilon)}$$

for all $\rho \in \Omega^*_{\mathrm{red}}$ with $|\rho| = n+1$.

Recall that our aim is to show that $\mathcal{S}$ is finite. We have

$$\begin{aligned}&\sum_{n>N}\ \sum_{\rho\in\Omega^*_{\mathrm{red}},|\rho|=n+1} \iint_{A_\rho}\left(\int_{B_\eta(\boldsymbol{t}_0)} \frac{d\boldsymbol{t}}{|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)|^{s-\varepsilon'}}\right) d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega,\tau) \\ &\quad< \sum_{n>N} C_2 C^{s-\varepsilon'} \exp\Big[n\Big\{\varepsilon(s-\varepsilon') + \frac{1}{\ell-\varepsilon}(\chi+\varepsilon)(s-\varepsilon') \\ &\qquad - \frac{1}{\ell-\varepsilon}\big(h_{RW} - (2\log r+1)\varepsilon\big)\Big\}\Big].\end{aligned}$$

Since we have

$$\begin{aligned} & n\Big\{\varepsilon(s-\varepsilon')+\frac{1}{\ell-\varepsilon}(\chi+\varepsilon)(s-\varepsilon')-\frac{1}{\ell-\varepsilon}\big(h_{RW}-(2\log r+1)\varepsilon\big)\Big\} \\ & \quad < \frac{n}{\ell-\varepsilon}\{\varepsilon(s-\varepsilon')+(\chi+\varepsilon)(s-\varepsilon')-(h_{RW}-(2\log r+1)\varepsilon)\} \\ & \quad < \frac{n}{\ell-\varepsilon}\{(s\chi-h_{RW})+(2\log r+3)\varepsilon-\varepsilon'\chi\} < 0, \end{aligned}$$

the above sum converges. ■

Proof of Theorem 2.1 (ii). Note that U' is open by Lemma 4.1. Assume that U' is non-empty, otherwise there is nothing to prove. Fix an arbitrary $\boldsymbol{t}_0 \in U'$. It is enough to show that $\nu_{\boldsymbol{t}}$ is absolutely continuous for a.e. $\boldsymbol{t}$ in some neighborhood of $\boldsymbol{t}_0$. Define $\Phi = \Phi^{\boldsymbol{t}_0}$, $\Pi = \Pi_{\boldsymbol{t}_0}$ and $\chi = \chi_{\boldsymbol{t}_0}$. Fix $\varepsilon > 0$ such that

$$\chi < h_{RW} - (2\log r + 3)\varepsilon.$$

There exists $\eta > 0$ such that for all $\omega \in \Omega_{\mathrm{red}}$, $n \geq 1$ and $x \in X_{\omega_n}$,

$$|\boldsymbol{t}-\boldsymbol{t}_0| < \eta \quad \Longrightarrow \quad \frac{|\varphi'_{\omega|_n}(x)|}{|(\varphi^{\boldsymbol{t}}_{\omega|_n})'(x)|} < e^{\varepsilon n}.$$

By Egorov's theorem, for any $\varepsilon' > 0$ there exists $\Omega' \subset \overline{\Omega}$ such that $\mu(\Omega') > 1-\varepsilon'$ and the convergence in (2), (4) and (7) is uniform on Ω'. Fix $a \in \Lambda$ and write

$$\Omega'_{\mathrm{red}} = \big\{\omega_{\mathrm{red}} \in \Omega_{\mathrm{red}} : \exists \omega \in \Omega' \text{ s.t. } \mathrm{red}(\omega) = \omega_{\mathrm{red}}, (\omega_{\mathrm{red}})_0 = a\big\}.$$

Define

$$\mu' = \mu|_{\Omega'}, \quad \nu'_{\boldsymbol{t}} = \Pi_{\boldsymbol{t}}\mu' \quad \text{and} \quad \mu'_{\mathrm{red}} = \mathrm{red}\,\mu'.$$

It is enough to show that

$$\mathcal{I} = \int_{B_\eta(\boldsymbol{t}_0)}\int_{X_a} \underline{D}(\nu'_{\boldsymbol{t}}, x)\, d\nu'_{\boldsymbol{t}}\, d\boldsymbol{t} < \infty,$$

where

$$\underline{D}(\nu'_{\boldsymbol{t}}, x) = \liminf_{r\downarrow 0} \frac{\nu'_{\boldsymbol{t}}[x-r, x+r]}{2r}$$

is the lower density of the measure $\nu'_{\boldsymbol{t}}$ at the point x. See [14, Section 4]. By applying Fatou's lemma, we obtain

$$\mathcal{I} \leq \liminf_{r\downarrow 0} \int_{B_\eta(\boldsymbol{t}_0)}\int_{X_a} \frac{\nu'_{\boldsymbol{t}}[x-r, x+r]}{2r}\, d\nu'_{\boldsymbol{t}}\, d\boldsymbol{t}.$$

By the change of variable, we have

$$\int_{X_a} \nu'_{\boldsymbol{t}}[x-r, x+r]\, d\nu'_{\boldsymbol{t}} = \iint_{(\omega,\tau)\in\Omega'^2_{\mathrm{red}}} \mathbf{1}_{\{(\omega,\tau)\in\Omega'^2_{\mathrm{red}} : |\Pi_{\boldsymbol{t}}(\omega)-\Pi_{\boldsymbol{t}}(\tau)|\leq r\}}\, d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega,\tau).$$

For a word $\rho \in \Omega^*_{\mathrm{red}}$, we define

$$A_\rho = \{(\omega, \tau) \in \Omega'^2_{\mathrm{red}} : \omega \wedge \tau = \rho\}.$$

Then,

$$\begin{aligned}\mathcal{I} &\le \liminf_{r\downarrow 0} (2r)^{-1} \iint_{(\omega,\tau)\in\Omega'^2_{\mathrm{red}}} \mathcal{L}_d\left(\{\boldsymbol{t} \in B_\eta(\boldsymbol{t}_0) : |\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| \le r\}\right) \\ &\qquad\qquad d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega, \tau) \\ &= \liminf_{r\downarrow 0} (2r)^{-1} \sum_{n\ge 0} \sum_{\rho\in\Omega^*_{\mathrm{red}}, |\rho|=n+1} \iint_{A_\rho} \mathcal{L}_d\left(\{\boldsymbol{t} \in B_\eta(\boldsymbol{t}_0) : |\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| \le r\}\right) \\ &\qquad\qquad d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega, \tau).\end{aligned}$$

Let $(\omega, \tau) \in A_\rho$. Then for some $c \in [\Pi_{\boldsymbol{t}}(\sigma^n\omega), \Pi_{\boldsymbol{t}}(\sigma^n\tau)]$, we have

$$\begin{aligned}|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| &= |(\varphi^{\boldsymbol{t}}_{\omega|_n})'(c)| \cdot \left|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)\right| \\ &\ge |\varphi'_{\omega|_n}(c)| e^{-\varepsilon n} \cdot \left|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)\right|.\end{aligned}$$

By the distortion property, there exists a constant $C > 0$ such that

$$|\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| \ge \frac{1}{C}\left|\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))\right| e^{-\varepsilon n} \cdot \left|\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)\right|.$$

By the transversality condition, we have

$$\begin{aligned}&\mathcal{L}_d\left(\{\boldsymbol{t} \in B_\eta(\boldsymbol{t}_0) : |\Pi_{\boldsymbol{t}}(\omega) - \Pi_{\boldsymbol{t}}(\tau)| \le r\}\right) \\ &\quad\le \mathcal{L}_d\left\{\boldsymbol{t} \in B_\eta(\boldsymbol{t}_0) : |\Pi_{\boldsymbol{t}}(\sigma^n\omega) - \Pi_{\boldsymbol{t}}(\sigma^n\tau)| \le \frac{Ce^{\varepsilon n} r}{|\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))|}\right\} \\ &\quad\le CC_1 e^{\varepsilon n} r \left|\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))\right|^{-1}.\end{aligned}$$

Therefore,

$$\mathcal{I} \le \frac{1}{2} CC_1 \sum_{n\ge 0} e^{\varepsilon n} \sum_{\rho\in\Omega^*_{\mathrm{red}}, |\rho|=n+1} \iint_{A_\rho} |\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))|^{-1} \, d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega, \tau).$$

Let us take $N \in \mathbb{N}$ as in the proof of (i). Then we obtain

$$\begin{aligned}&\sum_{n>N} e^{\varepsilon n} \sum_{\rho\in\Omega^*_{\mathrm{red}}, |\rho|=n+1} \iint_{A_\rho} |\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))|^{-1} \, d(\mu'_{\mathrm{red}} \times \mu'_{\mathrm{red}})(\omega, \tau) \\ &\quad< \sum_{n>N} \exp\left[n\varepsilon + \frac{n}{\ell - \varepsilon}(\chi + \varepsilon) - \frac{n}{\ell - \varepsilon}(h_{RW} - (2\log r + 1)\varepsilon)\right] \\ &\quad< \sum_{n>N} \exp\left[\frac{n}{\ell - \varepsilon}(\chi - h_{RW} + (2\log r + 3)\varepsilon)\right],\end{aligned}$$

which is finite since $\chi - h_{RW} + (2\log r + 3)\varepsilon < 0$. This concludes the proof. ■

A. Application to the Furstenberg measure

A.1. Furstenberg measure

Let $\mathcal{A} = \{A_a\}_{a\in\Lambda}$ be a finite collection of $SL_2(\mathbb{R})$ matrices. The linear action of $\mathcal{A}$ on $\mathbb{R}^2$ induces an action on the projective space $\mathbb{RP}^1$. From now on, we assume that $\mathcal{A}$ generates an unbounded and totally irreducible subgroup (i.e., it does not preserve any finite set in $\mathbb{RP}^1$). Then it is known that there exists a unique probability measure ν on $\mathbb{RP}^1$, called the *Furstenberg measure*, satisfying

$$\nu = \sum_{a\in\Lambda} p_a A_a \nu,$$

where $A_a\nu$ is the push-forward of ν under the action of A_a. See [4]. For $\omega \in \Omega$, we write $A_{\omega|_n} = A_{\omega_0} A_{\omega_1} \cdots A_{\omega_n}$. The following result is classical.

Theorem A.1 (Furstenberg). *For μ-a.e. ω, there exists $z = z(\omega) \in \mathbb{RP}^1$ such that $A_{\omega|_n}\nu$ converges weakly to a random Dirac mass $\delta_{z(\omega)}$, and $\nu = \mathbb{E}(\delta_{z(\omega)})$.*

We denote by $\chi' \geq 0$ the *Lyapunov exponent* of $\mathcal{A} = \{A_a\}_{a\in\Lambda}$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_{\omega|_n}\| = \chi' \tag{16}$$

for μ-a.e. ω.

A.2. Furstenberg measure and IFS with inverses

There is a natural identification between $[0, \pi)$ and the projective space $\mathbb{RP}^1$. From now on, we use this identification freely. We equip $\mathbb{RP}^1$ with the metric induced from the identification with $\mathbb{R}/\pi\mathbb{Z}$.

It is well known that the $SL_2(\mathbb{R})$ action on $\mathbb{RP}^1$ can be expressed in terms of linear fractional transformations. Let $\psi : [0, \pi) \to \mathbb{R}^*$ be such that $\psi(\theta) = \cos\theta/\sin\theta$, where $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),$$

and write $f_A(x) = (ax + b)/(cx + d)$. Denote the action of A on $\mathbb{RP}^1$ by ϕ_A. Then it is easy to see that we have

$$f_A \circ \psi = \psi \circ \phi_A.$$

The next proposition is a direct consequence of [1, Theorem 2.3].

Proposition A.1. *The following are equivalent:*

(i) *there exist $c > 0$ and $r > 1$ such that*

$$\|A_{\omega|_n}\| > cr^n \quad \text{for all } \omega \in \Omega_{\mathrm{red}};$$

(ii) *there exist non-empty open sets $X_a \subset \mathbb{RP}^1$ for each $a \in \Lambda$, with $\overline{X_a} \neq \mathbb{RP}^1$, and such that $\overline{\phi_{A_a}(X_b)} \subset X_a$ for all $(a,b) \in \Lambda^\star$.*

Proposition A.2. *Let $\{X_a\}_{a\in\Lambda}$ be a collection of non-empty open sets in $\mathbb{RP}^1$ with $\overline{X_a} \neq \mathbb{RP}^1$. Assume that $\overline{\phi_{A_a}(X_b)} \subset X_a$ for all $(a,b) \in \Lambda^\star$. Then $\{\phi_{A_a}|_{X_b}\}_{(a,b)\in\Lambda^\star}$ is an IFS with inverse, and the associated invariant measure agrees with the Furstenberg measure.*

Proof. By Proposition A.1, $\{\phi_{A_a}|_{X_b}\}_{(a,b)\in\Lambda^\star}$ is an IFS with inverse. It is easy to see that for $\omega \in \overline{\Omega}$, we have $\Pi(\omega) = z(\omega)$. Therefore, by Theorem A.1, the associated invariant measure agrees with the Furstenberg measure. ∎

By Proposition A.2, we immediately obtain the following.

Proposition A.3. *Let $\{X_a\}_{a\in\Lambda}$ be a collection of non-empty open sets in $\mathbb{R}^*$ with $\overline{X_a} \neq \mathbb{R}^*$. Assume that $\overline{f_{A_a}(X_b)} \subset X_a$ for all $(a,b) \in \Lambda^\star$. Then $\{f_{A_a}|_{X_b}\}_{(a,b)\in\Lambda^\star}$ is an IFS with inverse, and the associated invariant measure agrees with the push-forward of the Furstenberg measure under the action of $\psi : \mathbb{RP}^1 \to \mathbb{R}^*$.*

Example A.1. Let $r = 2$, $S = \{0, 1\}$ and $\Lambda = \{0, 0^{-1}, 1, 1^{-1}\}$. Let $\{X_a\}_{a\in\Lambda}$ be mutually disjoint connected open components in $\mathbb{RP}^1$ such that $\pi/2 \in X_0$, $0 \in X_{0^{-1}}$, $\pi/4 \in X_1$ and $3\pi/4 \in X_{1^{-1}}$. Let

$$A_0 = \frac{1}{\sqrt{k}}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \frac{1}{2\sqrt{l}}\begin{pmatrix} 1+l & 1-l \\ 1-l & 1+l \end{pmatrix}$$

for sufficiently small $0 < k, l < 1$. We suppress the dependence of k and l from the notation. Let $A_{a^{-1}} = A_a^{-1}$ for $a \in S$. Write $\phi_a = \phi_{A_a}$ for $a \in \Lambda$. It is easy to see that $\phi_0(0) = 0$, $\phi_0(\pi/2) = \pi/2$, $\phi_1(\pi/4) = \pi/4$ and $\phi_1(3\pi/4) = 3\pi/4$. Since $k, l > 0$ are sufficiently small, we have

$$\overline{\phi_a(X \setminus X_{a^{-1}})} \subset X_a$$

for all $a \in \Lambda$. The proof of the following claim is essentially the same as the proof of [16, Lemma 3.3], so we omit the proof.

Claim A.1. *We have*

$$\chi = 2\chi'.$$

Therefore, by Propositions 3.3 and A.2, we have

$$\dim \nu = \frac{h_{RW}}{2\chi'}.$$

Remark A.1. Consider the IFS with inverse given in Example 2.1. Let $U \subset \mathbb{R}^2$ be the set of (k, l) such that (3) holds. Write $\boldsymbol{t} = (k, l)$, and let $\nu_{\boldsymbol{t}}$ be the associated invariant measure on $\mathbb{R}$. It seems that by following the scheme of [2] it is possible to find non-empty open sets $U_1, U_2 \subset U$ such that

(i) for a.e. $\boldsymbol{t} \in U_1$,

$$\dim \nu_{\boldsymbol{t}} = \min\Big\{\frac{h_{RW}}{\chi_{\boldsymbol{t}}}, 1\Big\};$$

(ii) $\nu_{\boldsymbol{t}}$ is absolutely continuous for a.e. $\boldsymbol{t} \in U_2$.

We do not pursue this in this paper.

Remark A.2. From the viewpoint of random walks on groups, it is natural to consider the Furstenberg measure in the case that the collection of $SL_2(\mathbb{R})$ matrices is symmetric (a collection of $SL_2(\mathbb{R})$ matrices S is symmetric if it satisfies $S = S^{-1}$). In [3], relying on a deep result of additive combinatorics, Bourgain constructed a collection of symmetric $SL_2(\mathbb{R})$ matrices that has absolutely continuous Furstenberg measure. It would be interesting to construct a parameter family of $SL_2(\mathbb{R})$ matrices that has absolutely continuous Furstenberg measure for a.e. parameter. However, this seems to be very difficult to establish and is well beyond the scope of our method.

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References

[1] A. Avila, J. Bochi, and J.-C. Yoccoz, Uniformly hyperbolic finite-valued SL(2, $\mathbb{R}$)-cocycles. *Comment. Math. Helv.* **85** (2010), no. 4, 813–884 Zbl 1201.37032 MR 2718140

[2] B. Bárány, M. Pollicott, and K. Simon, Stationary measures for projective transformations: the Blackwell and Furstenberg measures. *J. Stat. Phys.* **148** (2012), no. 3, 393–421 Zbl 1284.37007 MR 2969625

[3] J. Bourgain, Finitely supported measures on $SL_2(\mathbb{R})$ which are absolutely continuous at infinity. In *Geometric aspects of functional analysis*, pp. 133–141, Lecture Notes in Math. 2050, Springer, Heidelberg, 2012 Zbl 1272.60004 MR 2985129

[4] H. Furstenberg, Noncommuting random products. *Trans. Amer. Math. Soc.* **108** (1963), 377–428 Zbl 0203.19102 MR 163345

[5] V. A. Kaĭmanovich and A. G. Èrshler, Continuity of asymptotic characteristics for random walks on hyperbolic groups. *Funktsional. Anal. i Prilozhen.* **47** (2013), no. 2, 84–89 Zbl 1284.60013 MR 3113872

[6] V. A. Kaĭmanovich and A. M. Vershik, Random walks on discrete groups: boundary and entropy. *Ann. Probab.* **11** (1983), no. 3, 457–490 Zbl 0641.60009 MR 704539

[7] R. Lyons and Y. Peres, *Probability on trees and networks*. Camb. Ser. Stat. Probab. Math. 42, Cambridge University Press, New York, 2016 Zbl 1376.05002 MR 3616205

[8] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*. Cambridge Stud. Adv. Math. 35, Cambridge University Press, Cambridge, 1993 Zbl 0790.58014 MR 1237641

[9] Y. Peres and W. Schlag, Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions. *Duke Math. J.* **102** (2000), no. 2, 193–251 Zbl 0961.42007 MR 1749437

[10] Y. Peres and B. Solomyak, Absolute continuity of Bernoulli convolutions, a simple proof. *Math. Res. Lett.* **3** (1996), no. 2, 231–239 Zbl 0867.28001 MR 1386842

[11] Y. Peres and B. Solomyak, Self-similar measures and intersections of Cantor sets. *Trans. Amer. Math. Soc.* **350** (1998), no. 10, 4065–4087 Zbl 0912.28005 MR 1491873

[12] M. Pollicott and K. Simon, The Hausdorff dimension of λ-expansions with deleted digits. *Trans. Amer. Math. Soc.* **347** (1995), no. 3, 967–983 Zbl 0831.28005 MR 1290729

[13] K. Simon, B. Solomyak, and M. Urbański, Hausdorff dimension of limit sets for parabolic IFS with overlaps. *Pacific J. Math.* **201** (2001), no. 2, 441–478 Zbl 1046.37014 MR 1875903

[14] K. Simon, B. Solomyak, and M. Urbański, Invariant measures for parabolic IFS with overlaps and random continued fractions. *Trans. Amer. Math. Soc.* **353** (2001), no. 12, 5145–5164 Zbl 0987.37075 MR 1852098

[15] B. Solomyak, On the random series $\sum \pm\lambda^n$ (an Erdős problem). *Ann. of Math. (2)* **142** (1995), no. 3, 611–625 Zbl 0837.28007 MR 1356783

[16] B. Solomyak and Y. Takahashi, Diophantine property of matrices and attractors of projective iterated function systems in $\mathbb{RP}^1$. *Int. Math. Res. Not. IMRN* (2021), no. 16, 12639–12669 Zbl 07471394 MR 4300232

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