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Fourier decay for homogeneous self-affine measures

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Abstract. We show that for Lebesgue almost all d-tuples $(\theta_1, \dots, \theta_d)$, with $|\theta_j| > 1$, any self-affine measure for a homogeneous non-degenerate iterated function system $\{Ax + a_j\}_{j=1}^m$ in $\mathbb{R}^d$, where A^{-1} is a diagonal matrix with the entries $(\theta_1, \dots, \theta_d)$, has power Fourier decay at infinity.

1. Introduction

For a finite positive Borel measure μ on $\mathbb{R}^d$, consider the Fourier transform

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle}\, d\mu(x).$$

We are interested in the decay properties of $\widehat{\mu}$ at infinity. The measure μ is called *Rajchman* if

$$\lim_{|\xi| \to \infty} \widehat{\mu}(\xi) = 0,$$

where $|\xi|$ is a norm (say, the Euclidean norm) of $\xi \in \mathbb{R}^d$. Whereas absolutely continuous measures are Rajchman by the Riemann–Lebesgue lemma, it is a subtle question to decide which singular measures are such; see, e.g., the survey of Lyons [14]. A much stronger property, which is useful for many applications, is the following.

Definition 1.1. For $\alpha > 0$, let

$$\mathcal{D}_d(\alpha) = \big\{\nu \text{ finite positive measure on } \mathbb{R}^d : |\widehat{\nu}(t)| = O_\nu(|t|^{-\alpha}) \text{ as } |t| \to \infty\big\},$$

and define $\mathcal{D}_d = \bigcup_{\alpha > 0} \mathcal{D}_d(\alpha)$. A measure ν is said to have *power Fourier decay* if $\nu \in \mathcal{D}_d$.

Many recent papers have been devoted to the question of Fourier decay for classes of “fractal” measures; see, e.g., [1–3, 9, 11–13, 17, 19, 23, 25]. Here, we continue this line of research, focusing on the class of *homogeneous self-affine measures* in $\mathbb{R}^d$. A measure μ is called *self-affine* if it is the invariant measure for a self-affine iterated

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function system (IFS) $\{f_j\}_{j=1}^m$, with $m \ge 2$, where $f_j(x) = A_j x + a_j$, the matrices $A_j : \mathbb{R}^d \to \mathbb{R}^d$ are invertible linear contractions (in some norm), and $a_j \in \mathbb{R}^d$ are "digit" vectors. This means that for some probability vector $\boldsymbol{p} = (p_j)_{j\le m}$ holds

$$\mu = \sum_{j=1}^m p_j(\mu \circ f_j^{-1}). \tag{1.1}$$

It is well known that this equation defines a unique probability Borel measure. The self-affine IFS is *homogeneous* if all A_j are equal to each other: $A = A_j$ for $j \le m$. Denote the digit set by $\mathcal{D} := \{a_1, \dots, a_m\}$ and the corresponding self-affine measure by $\mu(A, \mathcal{D}, \boldsymbol{p})$. We will write $\boldsymbol{p} > 0$ if all $p_j > 0$. Following [8], we say that the IFS is *affinely irreducible* if the attractor is not contained in a proper affine subspace of $\mathbb{R}^d$. It is easy to see that this is a necessary condition for the self-affine measure to be Rajchman, so this will always be our assumption. By a conjugation with a translation, we can always assume that $0 \in \mathcal{D}$. In this case, affine irreducibility is equivalent to the digit set $\mathcal{D}$ being a *cyclic family* for A, that is, $\mathbb{R}^d$ being the smallest A-invariant subspace containing $\mathcal{D}$.

The IFS is *self-similar* if all A_j are contracting similitudes, that is, $A_j = \lambda_j \mathcal{O}_j$ for some $\lambda_j \in (0, 1)$ and orthogonal matrices $\mathcal{O}_j$. In many aspects, "genuine" (i.e., non-self-similar) self-affine and self-similar IFS are very different; of course, the distinction exists only for $d \ge 2$.

Every homogeneous self-affine measure can be expressed as an infinite convolution product

$$\mu(A, \mathcal{D}, \boldsymbol{p}) = \mathop{\ast}_{n=0}^{\infty} \sum_{j=1}^m p_j \delta_{A^n a_j}, \tag{1.2}$$

and for every $\boldsymbol{p} > 0$, it is supported on the attractor (self-affine set)

$$K_{A,\mathcal{D}} := \Big\{x \in \mathbb{R}^d : x = \sum_{n=0}^{\infty} A^n b_n,\ b_n \in \mathcal{D}\Big\}.$$

By the definition of the self-affine measure,

$$\widehat{\mu}(\xi) = \sum_{j=1}^m p_j \int e^{-2\pi i\langle \xi, Ax+a_j\rangle}\, d\mu = \Big(\sum_{j=1}^m p_j e^{-2\pi i \langle \xi, a_j\rangle}\Big)\widehat{\mu}(A^t\xi),$$

where A^t is the matrix transpose of A. Iterating, we obtain

$$\widehat{\mu}(\xi) = \prod_{n=0}^{\infty}\Big(\sum_{j=1}^m p_j e^{-2\pi i\langle (A^t)^n\xi, a_j\rangle}\Big) = \prod_{n=0}^{\infty}\Big(\sum_{j=1}^m p_j e^{-2\pi i\langle \xi, A^n a_j\rangle}\Big), \tag{1.3}$$

where the infinite product converges, since $\|A^n\| \to 0$ exponentially fast as $n \to \infty$.

1.1. Background

We start with the known results on Fourier decay for classical Bernoulli convolutions ν_λ, namely, self-similar measures on the line, corresponding to the IFS $\{\lambda x, \lambda x + 1\}$, with $\lambda \in (0,1)$ and probabilities $(\frac{1}{2}, \frac{1}{2})$ (often the digits ± 1 are used instead; it is easy to see that taking any two distinct digits results in the same measure, up to an affine change of variable). Erdős [5] proved that $\widehat{\nu}_\lambda(t) \not\to 0$ as $t \to \infty$ when $\theta = 1/\lambda$ is a Pisot number. Recall that a *Pisot number* is an algebraic integer greater than one, whose algebraic (Galois) conjugates are all less than one in modulus. Salem [18] showed that if $1/\lambda$ is not a Pisot number, then $\widehat{\nu}_\lambda$ is a Rajchman measure. In the other direction, Erdős [6] proved that for any $[a,b] \subset (0,1)$, there exists $\alpha > 0$ such that $\nu_\lambda \in \mathcal{D}_1(\alpha)$ for a.e. $\lambda \in [a,b]$. Later, Kahane [10] indicated that Erdős' argument actually gives that $\nu_\lambda \in \mathcal{D}_1$ for all $\lambda \in (0,1)$ outside a set of zero Hausdorff dimension. (We should mention that very few specific λ are known, for which ν_λ has power Fourier decay; see Dai, Feng, and Wang [4].) In the original papers of Erdős and Kahane, there were no explicit quantitative bounds; this was done in the survey [15], where the expression "Erdős–Kahane argument" was used first. The general case of a homogeneous self-similar measure on the line is treated analogously to that of Bernoulli convolutions: the self-similar measure is still an infinite convolution and the Erdős–Kahane argument on power Fourier decay goes through with minor modifications; see [4, 22]. Although one of the main motivations for the study of the Fourier transform has been the question of absolute continuity/singularity of ν_λ, here we do not discuss it but refer the reader to the recent survey [24].

Next, we turn to the non-homogeneous case on the line. Li and Sahlsten [13] proved that if μ is a self-similar measure on the line with contraction ratios $\{r_i\}_{i=1}^m$ and there exist $i \neq j$ such that $\log r_i / \log r_j$ is irrational, then μ is Rajchman. Moreover, they showed logarithmic decay of the Fourier transform under a Diophantine condition. A related result for self-conformal measures was recently obtained by Algom, Rodriguez Hertz, and Wang [1]. Brémont [3] obtained an (almost) complete characterization of (non-)Rajchman self-similar measures in the case when $r_j = \lambda^{n_j}$ for $j \le m$. To be non-Rajchman, it is necessary for $1/\lambda$ to be Pisot. For "generic" choices of the probability vector $\boldsymbol{p}$, assuming that $\mathcal{D} \subset \mathbb{Q}(\lambda)$ after an affine conjugation, this is also sufficient, but there are some exceptional cases of positive co-dimension. Varjú and Yu [25] proved logarithmic decay of the Fourier transform in the case when $r_j = \lambda^{n_j}$ for $j \le m$ and $1/\lambda$ is algebraic, but not a Pisot or Salem number. In [23], we showed that outside a zero Hausdorff dimension exceptional set of parameters, all self-similar measures on $\mathbb{R}$ belong to $\mathcal{D}_1$; however, the exceptional set is not explicit.

Turning to higher dimensions, we mention the recent paper by Rapaport [17], where he gives an algebraic characterization of self-similar IFS for which there exists a probability vector yielding a non-Rajchman self-similar measure. Li and Sahlsten [12] investigated self-affine measures in $\mathbb{R}^d$ and obtained power Fourier decay under some algebraic conditions, which never hold for a homogeneous self-affine IFS. Their main assumptions are total irreducibility of the closed group generated by the contraction linear maps A_j and non-compactness of the projection of this group to $\mathrm{PGL}(d,\mathbb{R})$. For $d = 2, 3$ they showed that this is sufficient.

1.2. Statement of the results

We assume that A is a matrix diagonalizable over $\mathbb{R}$. Then we can reduce the IFS, via a linear change of variable, to one where A is a diagonal matrix with real entries. Given $A = \mathrm{Diag}[\theta_1^{-1}, \dots, \theta_d^{-1}]$, with $|\theta_j| > 1$, a set of digits $\mathcal{D} = \{a_1, \dots, a_m\} \subset \mathbb{R}^d$, and a probability vector $\boldsymbol{p}$, we write $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)$ and denote by $\mu(\boldsymbol{\theta}, \mathcal{D}, \boldsymbol{p})$ the self-affine measure defined by (1.1). Our main motivation is the class of measures which can be viewed as "self-affine Bernoulli convolutions", with $A = \mathrm{Diag}[\theta_1^{-1}, \dots, \theta_d^{-1}]$ a diagonal matrix with distinct real entries and $\mathcal{D} = \{0, (1, \dots, 1)\}$. In this special case, we denote the self-affine measure by $\mu(\boldsymbol{\theta}, \boldsymbol{p})$.

Theorem 1.2. *There exists an exceptional set $E \subset \mathbb{R}^d$, with $\mathcal{L}^d(E) = 0$, such that for all $\boldsymbol{\theta} \in \mathbb{R}^d \setminus E$, with $\min_j |\theta_j| > 1$, for all sets of digits $\mathcal{D}$, such that the IFS is affinely irreducible, and all $\boldsymbol{p} > 0$, holds $\mu(\boldsymbol{\theta}, \mathcal{D}, \boldsymbol{p}) \in \mathfrak{D}_d$.*

The theorem is a consequence of a more quantitative statement.

Theorem 1.3. *Fix $1 < b_1 < b_2 < \infty$ and $c_1, \varepsilon > 0$. Then there exist $\alpha > 0$ and $\mathcal{E} \subset \mathbb{R}^d$, depending on these parameters, such that $\mathcal{L}^d(\mathcal{E}) = 0$ and for all $\boldsymbol{\theta} \notin \mathcal{E}$ satisfying*

$$b_1 \le \min_j |\theta_j| < \max_j |\theta_j| \le b_2 \quad \text{and} \quad |\theta_i - \theta_j| \ge c_1,\ i \ne j,$$

for all digit sets $\mathcal{D}$ such that the IFS is affinely irreducible, and all $\boldsymbol{p}$ such that $\min_j p_j \ge \varepsilon$, we have $\mu(\boldsymbol{\theta}, \mathcal{D}, \boldsymbol{p}) \in \mathfrak{D}_d(\alpha)$.

Reduction of Theorem 1.2 to Theorem 1.3. For $M \in \mathbb{N}$, let $\mathcal{E}^{(M)}$ be the exceptional set obtained from Theorem 1.3 with $b_1 = 1 + M^{-1}, b_2 = M$, and $\varepsilon = c_1 = M^{-1}$. Then the set

$$E = \bigcup_{M=2}^{\infty} \mathcal{E}^{(M)} \cup \{\boldsymbol{\theta} : \exists i \ne j,\ \theta_i = \theta_j\}.$$

has the desired properties. ∎

The proof of Theorem 1.3 uses a version of the Erdős–Kahane technique. We follow the general scheme of [15, 22], but this is not a trivial extension.

In view of the convolution structure, Theorem 1.3 yields some information on absolute continuity of self-affine measures, by a standard argument.

Corollary 1.4. *Fix $1 < b_1 < b_2 < \infty$ and $c_1, \varepsilon > 0$. Then there exist a sequence (n_k) with $n_k \to \infty$ as $n \to \infty$ and a set $\widetilde{\mathcal{E}}_k \subset \mathbb{R}^d$, depending on these parameters, such that $\mathcal{L}^d(\widetilde{\mathcal{E}}_k) = 0$ and for all $\boldsymbol{\theta} \notin \widetilde{\mathcal{E}}_k$, satisfying*

$$b_1 \le \min_j |\theta_j^{n_k}| < \max_j |\theta_j^{n_k}| \le b_2 \quad \text{and} \quad |\theta_i^{n_k} - \theta_j^{n_k}| \ge c_1,\ i \ne j,$$

for all digit sets $\mathcal{D}$ such that the IFS is affinely irreducible, and all $\boldsymbol{p}$ such that $\min_j p_j \ge \varepsilon$, the measure $\mu(\boldsymbol{\theta}, \mathcal{D}, \boldsymbol{p})$ is absolutely continuous with respect to $\mathcal{L}^d$, with a Radon–Nikodym derivative in $C^k(\mathbb{R}^d)$, $k \ge 0$.

Derivation of Corollary 1.4 from Theorem 1.3. Let $n \ge 2$. It follows from (1.2) that

$$\mu(A, \mathcal{D}, \boldsymbol{p}) = \mu(A^n, \mathcal{D}, \boldsymbol{p}) * \mu(A^n, A\mathcal{D}, \boldsymbol{p}) * \cdots * \mu(A^n, A^{n-1}\mathcal{D}, \boldsymbol{p}).$$

It is easy to see that if the original IFS is affinely irreducible, then so are the IFS associated with $(A^n, A^j\mathcal{D})$ and, moreover, these IFS are all affine conjugate to each other. Therefore, if $\mu(A^n, \mathcal{D}, \boldsymbol{p}) \in \mathcal{D}_d(\alpha)$, then $\mu(A, \mathcal{D}, \boldsymbol{p}) \in \mathcal{D}_d(n\alpha)$. As it is well known,

$$\mu \in \mathcal{D}_d(\beta),\ \beta > d + k \quad \Longrightarrow \quad \frac{d\mu}{d\mathcal{L}^d} \in C^k(\mathbb{R}^d),$$

so we can take n_k such that $n_k\alpha > d + k$, and $\widetilde{\mathcal{E}}_k = \{\boldsymbol{\theta} : \boldsymbol{\theta}^{n_k} \in \mathcal{E}\}$, where α and $\mathcal{E}$ are from Theorem 1.3. ∎

Remark 1.5. (a) In general, the power decay cannot hold for all $\boldsymbol{\theta}$; for instance, it is easy to see that the measure $\mu(\boldsymbol{\theta}, \boldsymbol{p})$ is not Rajchman if at least one of θ_k is a Pisot number. Thus, in the most basic case with two digits, the exceptional set has Hausdorff dimension at least $d - 1$.

(b) It is natural to ask what happens if A is not diagonalizable over $\mathbb{R}$. A complex eigenvalue of A corresponds to a 2-dimensional homogeneous self-similar IFS with rotation, or an IFS of the form $\{\lambda z + a_j\}_{j=1}^m$, with $\lambda \in \mathbb{C}$, $|\lambda| < 1$, and $a_j \in \mathbb{C}$. In [21], it was shown that for all λ outside a set of Hausdorff dimension zero, the corresponding self-similar measure belongs to $\mathcal{D}_2$. It may be possible to combine the methods of [21] with those of the current paper to obtain power Fourier decay for a typical A diagonalizable over $\mathbb{C}$. It would also be interesting to consider the case of non-diagonalizable A, starting with a single Jordan block.

(c) In the special case of $d = 2$ and $m = 2$, our system reduces to a planar self-affine IFS, conjugate to $\{(\lambda x, \gamma y) \pm (-1, 1)\}$ for $0 < \gamma < \lambda < 1$. This system has been studied by many authors, especially the dimension and topological properties of its attractor, see [7] and the references therein. For our work, the most relevant is the paper by Shmerkin [20]. Among other results, he proved absolute continuity with a density in L^2 of the self-affine measure (with some fixed probabilities) almost everywhere in some region, in particular, in some explicit neighborhood of $(1, 1)$. He also showed that if $(\lambda^{-1}, \gamma^{-1})$ forms a *Pisot pair*, then the measure is not Rajchman and hence singular.

1.3. Rajchman self-affine measures

The question "when is $\mu(A, \mathcal{D}, \boldsymbol{p})$ Rajchman?" is not addressed here. Recently, Rapaport [17] obtained an (almost) complete characterization of *self-similar* Rajchman measures in $\mathbb{R}^d$. Of course, our situation is vastly simplified by the assumption that the IFS is homogeneous, but still it is not completely straightforward. The key notion here is the following.

Definition 1.6. A collection of numbers $(\theta_1, \ldots, \theta_m)$ (real or complex) is called a *Pisot family* or a *P.V. m-tuple* if

(i) $|\theta_j| > 1$ for all $j \le m$, and

(ii) there is a monic integer polynomial $P(t)$, such that $P(\theta_j) = 0$ for all $j \le m$, whereas every other root θ' of $P(t)$ satisfies $|\theta'| < 1$.

It is not difficult to show, using the classical techniques of Pisot [16] and Salem [18], as well as some ideas from [17, Section 5], that

- if $\mu(A, \mathcal{D}, \boldsymbol{p})$ is not a Rajchman measure and the IFS is affinely irreducible, then the spectrum $\mathrm{Spec}(A^{-1})$ contains a Pisot family;
- if $\mathrm{Spec}(A^{-1})$ contains a Pisot family, then for a "generic" choice of $\mathcal{D}$, with $m \ge 3$, the measure $\mu(A, \mathcal{D}, \boldsymbol{p})$ is Rajchman; however,
- if $\mathrm{Spec}(A^{-1})$ contains a Pisot family, then under appropriate conditions the measure $\mu(A, \mathcal{D}, \boldsymbol{p})$ is not Rajchman. For instance, this holds if there is at least one conjugate of the elements of the Pisot family less than 1 in absolute value, $m = 2$, and A is diagonalizable over $\mathbb{R}$.

We omit the details.

2. Proofs

The following is an elementary inequality.

Lemma 2.1. *Let $\boldsymbol{p} = (p_1, \ldots, p_m) > 0$ be a probability vector and $\alpha_1 = 0$, $\alpha_j \in \mathbb{R}$, $j = 2, \ldots, m$. Denote $\varepsilon = \min_j p_j$ and write $\|x\| = \mathrm{dist}(x, \mathbb{Z})$. Then for any $k \le m$,*

$$\left| \sum_{j=1}^{m} p_j e^{-2\pi i \alpha_j} \right| \le 1 - 2\pi\varepsilon \|\alpha_k\|^2. \tag{2.1}$$

Proof. Fix $k \in \{2, \dots, m\}$. We can estimate

$$\left|\sum_{j=1}^{m} p_j e^{-2\pi \alpha_j}\right| = \left|p_1 + \sum_{j=2}^{m} p_j e^{-2\pi \alpha_j}\right| \le |p_1 + p_k e^{-2\pi i \alpha_k}| + (1 - p_1 - p_k).$$

Assume that $p_1 \ge p_k$; otherwise, write $|p_1 + p_k e^{-2\pi i \alpha_k}| = |p_1 e^{2\pi i \alpha_k} + p_k|$ and repeat the argument. Then observe that

$$\begin{aligned} |p_1 + p_k e^{-2\pi i \alpha_k}| &\le (p_1 - p_k) + p_k |1 + e^{-2\pi i \alpha_k}|, \\ |1 + e^{-2\pi i \alpha_k}| &= 2|\cos(\pi \alpha_k)| \le 2(1 - \pi \|\alpha_k\|^2). \end{aligned}$$

This implies the desired inequality. ■

Recall from (1.3) that

$$\widehat{\mu}(\xi) = \prod_{n=0}^{\infty} \left(\sum_{j=1}^{m} p_j e^{-2\pi i \langle \xi, A^n a_j \rangle} \right).$$

For $\xi \in \mathbb{R}^d$, with $\|\xi\|_\infty \ge 1$, let $\eta(\xi) = (A^t)^{N(\xi)} \xi$, where $N(\xi) \ge 0$ is maximal, such that $\|\eta(\xi)\|_\infty \ge 1$. Then $\|\eta(\xi)\|_\infty \in [1, \|A^{-1}\|_\infty]$ and (1.3) implies

$$\widehat{\mu}(\xi) = \widehat{\mu}(\eta(\xi)) \cdot \prod_{n=1}^{N(\xi)} \left(\sum_{j=1}^{m} p_j e^{-2\pi i \langle \eta(\xi),\, A^{-n} a_j \rangle} \right). \tag{2.2}$$

Proof of Theorem 1.3. First, we show that the case of a general digit set may be reduced to $\mathcal{D} = \{0, (1, \dots, 1)\}$. We start with the formula (2.2), which, under the current assumptions, becomes

$$\widehat{\mu}(\xi) = \widehat{\mu}(\eta(\xi)) \cdot \prod_{n=1}^{N(\xi)} \left(\sum_{j=1}^{m} p_j \exp\left[-2\pi i \sum_{k=1}^{d} \eta_k a_j^{(k)} \theta_k^n \right] \right),$$

where $a_j = (a_j^{(k)})_{k=1}^d$ and $\eta(\xi) = (\eta_k)_{k=1}^d$. Note that $\|\eta(\xi)\|_\infty \in [1, \max_j |\theta_j|] \subset [1, b_2]$. Assume without loss of generality that $a_1 = 0$. Then we have, by (2.1), that for any fixed $j \in \{2, \dots, m\}$,

$$|\widehat{\mu}(\xi)| \le \prod_{n=1}^{N(\xi)} \left(1 - 2\pi \varepsilon \left\| \sum_{k=1}^{d} \eta_k a_j^{(k)} \theta_k^n \right\|^2 \right),$$

where $\|\cdot\|$ denotes the distance to the nearest integer. Further, we can assume that all the coordinates of a_j are non-zero; otherwise, we can work in the subspace

$$\mathcal{H} := \{ x \in \mathbb{R}^d : x_k = 0 \Leftrightarrow a_j^{(k)} = 0 \}$$

and with the corresponding variables θ_k, and then get the exceptional set of zero $\mathcal{L}^d$ measure as a product of a set of zero measure in $\mathcal{H}$ and the entire $\mathcal{H}^\perp$. Finally, apply a linear change of variables, so that $a_j^{(k)} = 1$ for all k, to obtain

$$|\widehat{\mu}(\xi)| \le \prod_{n=1}^{N(\xi)} \left(1 - 2\pi\varepsilon \left\| \sum_{k=1}^{d} \eta_k \theta_k^n \right\|^2\right). \tag{2.3}$$

This is exactly the situation corresponding to the measure $\mu(\boldsymbol{\theta}, \boldsymbol{p})$, and we will be showing (typical) power decay for the right-hand side of (2.3). This completes the reduction.

Next, we use a variant of the Erdős–Kahane argument; see, e.g., [15, 22] for other versions of it. Intuitively, we will get power decay if $\| \sum_{k=1}^{d} \eta_k \theta_k^n \|$ is uniformly bounded away from zero for a set of n's of positive lower density, uniformly in η.

Fix $c_1 > 0$ and $1 < b_1 < b_2 < \infty$, and consider the compact set

$$H = \left\{\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in ([-b_2, -b_1] \cup [b_1, b_2])^d : |\theta_i - \theta_j| \ge c_1,\ i \ne j\right\}.$$

We will use the notation $[N] = \{1, \dots, N\}$ and $[n, N] = \{n, \dots, N\}$. For $\rho, \delta > 0$, we define the "bad set" at scale N:

$$E_{H,N}(\delta, \rho) = \left\{\boldsymbol{\theta} \in H : \max_{\eta:\ 1 \le \|\eta\|_\infty \le b_2} \frac{1}{N} \left| \left\{ n \in [N] : \left\| \sum_{k=1}^{d} \eta_k \theta_k^n \right\| < \rho \right\} \right| > 1 - \delta \right\}. \tag{2.4}$$

Now, we can define the exceptional set

$$\mathcal{E}_H(\delta, \rho) := \bigcap_{N_0=1}^{\infty} \bigcup_{N=N_0}^{\infty} E_{H,N}(\delta, \rho).$$

Theorem 1.3 will immediately follow from the next two propositions.

Proposition 2.2. For any positive ρ and δ, we have $\mu(\boldsymbol{\theta}, \boldsymbol{p}) \in \mathcal{D}_d(\alpha)$ whenever $\boldsymbol{\theta} \in H \setminus \mathcal{E}_H(\delta, \rho)$, where α depends only on δ, ρ, H, and $\varepsilon = \min\{p, 1-p\}$.

Proposition 2.3. There exist $\rho = \rho_H > 0$ and $\delta = \delta_H > 0$ such that

$$\mathcal{L}^d\left(\mathcal{E}_H(\delta, \rho)\right) = 0.$$

Proof of Proposition 2.2. Suppose that $\boldsymbol{\theta} \in H \setminus \mathcal{E}_H(\delta, \rho)$. This implies that there is $N_0 \in \mathbb{N}$ such that $\boldsymbol{\theta} \notin E_{H,N}(\delta, \rho)$ for all $N \ge N_0$. Let $\xi \in \mathbb{R}^d$ be such that $\|\xi\|_\infty > b_2^{N_0}$. Then $N = N(\xi) \ge N_0$, where, as above, $\eta = \eta(\xi) = (A^t)^{N(\xi)}\xi = A^{N(\xi)}\xi$ and $N(\xi)$ is maximal with $\|\eta\|_\infty \ge 1$. From the fact that $\boldsymbol{\theta} \notin E_{H,N}(\delta, \rho)$, it follows that

$$\frac{1}{N} \left| \left\{ n \in [N] : \left\| \sum_{k=1}^{d} \eta_k \theta_k^n \right\| < \rho \right\} \right| \le 1 - \delta.$$

Then, by (2.3),

$$\left|\widehat{\mu}(\boldsymbol{\theta},\boldsymbol{p})(\xi)\right| \le (1-2\pi\varepsilon\rho^2)^{\lfloor \delta N\rfloor}.$$

By the definition of $N = N(\xi)$, we have

$$\|\xi\|_\infty \le b_2^{N+1}.$$

It follows that

$$\left|\widehat{\mu}(\boldsymbol{\theta},\boldsymbol{p})(\xi)\right| = O_{H,\varepsilon}(1)\cdot\|\xi\|_\infty^{-\alpha},$$

for $\alpha = -\delta\log(1-2\pi\varepsilon\rho^2)/\log b_2$, and the proof is complete. ■

Proof of Proposition 2.3. It is convenient to express the exceptional set as a union, according to a dominant coordinate of η (which may be non-unique, of course):

$$E_{H,N}(\delta,\rho) = \bigcup_{j=1}^{d} E_{H,N,j}(\delta,\rho),$$

where

$$E_{H,N,j}(\delta,\rho) := \Bigg\{\boldsymbol{\theta}\in H : \exists\,\eta,\ 1\le|\eta_j| = \|\eta\|_\infty \le b_2,\\ \frac{1}{N}\left|\left\{n\in[N] : \left\|\sum_{k=1}^{d}\eta_k\theta_k^n\right\| < \rho\right\}\right| > 1-\delta\Bigg\}. \quad (2.5)$$

It is easy to see that $E_{H,N,j}(\delta,\rho)$ is measurable. Observe that

$$\mathcal{E}_H(\delta,\rho) := \bigcup_{j=1}^{d}\mathcal{E}_{H,j}(\delta,\rho), \quad \text{where} \quad \mathcal{E}_{H,j}(\delta,\rho) := \bigcap_{N_0=1}^{\infty}\bigcup_{N=N_0}^{\infty} E_{H,N,j}(\delta,\rho).$$

It is, of course, sufficient to show that $\mathcal{L}^d(\mathcal{E}_{H,j}(\delta,\rho)) = 0$ for every $j\in[d]$, for some $\delta,\rho>0$. Without loss of generality, assume that $j=d$. Since $\mathcal{E}_{H,d}(\delta,\rho)$ is measurable, the desired claim will follow if we prove that every slice of $\mathcal{E}_{H,d}(\delta,\rho)$ in the direction of the x_d-axis has zero $\mathcal{L}^1$ measure. Namely, for fixed $\boldsymbol{\theta}' = (\theta_1,\dots,\theta_{d-1})$, let

$$\mathcal{E}_{H,d}(\delta,\rho,\boldsymbol{\theta}') := \left\{\theta_d : (\boldsymbol{\theta}',\theta_d)\in\mathcal{E}_{H,d}(\delta,\rho)\right\}.$$

We want to show that $\mathcal{L}^1(\mathcal{E}_{H,d}(\delta,\rho,\boldsymbol{\theta}')) = 0$ for all $\boldsymbol{\theta}'$. Clearly,

$$\mathcal{E}_{H,d}(\delta,\rho,\boldsymbol{\theta}') := \bigcap_{N_0=1}^{\infty}\bigcup_{N=N_0}^{\infty} E_{H,N,d}(\delta,\rho,\boldsymbol{\theta}'),$$

where

$$E_{H,N,d}(\delta,\rho,\boldsymbol{\theta}') = \Bigg\{\theta_d : (\boldsymbol{\theta}',\theta_d) \in H : \max_{\substack{\eta: 1\le|\eta_d|\le b_2 \\ \|\eta\|_\infty = |\eta_d|}} \frac{1}{N}\left|\left\{n \in [N] : \left\|\sum_{k=1}^d \eta_k \theta_k^n\right\| < \rho\right\}\right| > 1-\delta\Bigg\}. \tag{2.6}$$

Lemma 2.4. *There exists a constant $\rho > 0$ such that for any $N \in \mathbb{N}$ and $\delta \in (0,\frac{1}{2})$, the set $E_{H,N,d}(\delta,\rho,\boldsymbol{\theta}')$ can be covered by* $\exp(O_H(\delta\log(1/\delta)N))$ *intervals of length b_1^{-N}.*

We first complete the proof of the proposition assuming the lemma. By Lemma 2.4,

$$\mathcal{L}^1\Bigg(\bigcup_{N=N_0}^\infty E_{H,N,d}(\delta,\rho,\boldsymbol{\theta}')\Bigg) \le \sum_{N=N_0}^\infty \exp\big(O_H(\delta\log(1/\delta)N)\big)\cdot b_1^{-N} \to 0$$

as $N_0 \to \infty$, provided $\delta > 0$ is so small that $\log b_1 > O_H(\delta\log(1/\delta))$. Thus, we have $\mathcal{L}^1(\mathcal{E}_{H,d}(\delta,\rho,\boldsymbol{\theta}')) = 0$. ■

Proof of Lemma 2.4. Fix $\boldsymbol{\theta}'$ in the projection of H to the first $(d-1)$ coordinates and $\eta \in \mathbb{R}^d$, with $1 \le |\eta_d| = \|\eta\|_\infty \le b_2$. Below, all the constants implicit in the $O(\cdot)$ notation are allowed to depend on H and d. Let θ_d be such that $(\boldsymbol{\theta}',\theta_d) \in H$ and write

$$\sum_{k=1}^d \eta_k \theta_k^n = K_n + \varepsilon_n, \quad n \ge 0,$$

where $K_n \in \mathbb{Z}$ is the nearest integer to the expression in the left-hand side, so that $|\varepsilon_n| \le \frac{1}{2}$. We emphasize that K_n depends on η and on θ_d. Define $A_n^{(0)} = K_n$, $\widetilde{A}_n^{(0)} = K_n + \varepsilon_n$, and then for all n, inductively:

$$A_n^{(j)} = A_{n+1}^{(j-1)} - \theta_j A_n^{(j-1)}, \quad \widetilde{A}_n^{(j)} = \widetilde{A}_{n+1}^{(j-1)} - \theta_j \widetilde{A}_n^{(j-1)}, \quad j = 1,\dots,d-1. \tag{2.7}$$

It is easy to check by induction that

$$\widetilde{A}_n^{(j)} = \sum_{i=j+1}^d \eta_i \prod_{k=1}^j (\theta_i - \theta_k)\theta_i^n, \quad j = 1,\dots,d-1,$$

hence

$$\widetilde{A}_n^{(d-1)} = \eta_d \prod_{k=1}^{d-1}(\theta_d - \theta_k)\theta_d^n, \quad \theta_d = \frac{\widetilde{A}_{n+1}^{(d-1)}}{\widetilde{A}_n^{(d-1)}}, \quad n \in \mathbb{N}. \tag{2.8}$$

We have $\|\eta\|_\infty \le b_2$ and $|\widetilde{A}_n^{(0)} - A_n^{(0)}| \le |\varepsilon_n|$, and then by induction, by (2.7),

$$\big|\widetilde{A}_n^{(j)} - A_n^{(j)}\big| \le (1+b_2)^j \max\big\{|\varepsilon_n|, \dots, |\varepsilon_{n+j}|\big\}, \quad j = 1, \dots, d-1. \tag{2.9}$$

Another easy calculation gives

$$\begin{aligned} K_{n+d+1} &= \theta_1 K_{n+d} + A_{n+d}^{(1)} \\ &= \big[\theta_1 K_{n+d} + \theta_2 A_{n+d-1}^{(1)} + \cdots + \theta_{d-1} A_{n+2}^{(d-2)}\big] + A_{n+2}^{(d-1)}. \end{aligned} \tag{2.10}$$

Since

$$\frac{A_{n+2}^{(d-1)}}{A_{n+1}^{(d-1)}} \approx \frac{\widetilde{A}_{n+1}^{(d-1)}}{\widetilde{A}_n^{(d-1)}} = \theta_d,$$

we have

$$\begin{aligned} K_{n+d+1} &\approx \big[\theta_1 K_{n+d} + \theta_2 A_{n+d-1}^{(1)} + \cdots + \theta_{d-1} A_{n+2}^{(d-2)}\big] + \frac{(A_{n+1}^{(d-1)})^2}{A_n^{(d-1)}} \\ &=: R_{\theta_1,\dots,\theta_{d-1}}(K_n, \dots, K_{n+d}), \end{aligned} \tag{2.11}$$

where $R_{\theta_1,\dots,\theta_{d-1}}(K_n, \dots, K_{n+d})$ is a rational function, depending on the (fixed) parameters $\theta_1, \dots, \theta_{d-1}$. To make the approximate equality precise, note that, by (2.8) and our assumptions,

$$\big|\widetilde{A}_n^{(d-1)}\big| \ge c_1^{d-1} b_1^n,$$

where $b_1 > 1$, and $|\widetilde{A}_n^{(d-1)} - A_n^{(d-1)}| \le (1+b_2)^{d-1}/2$ by (2.9). Hence

$$\big|A_n^{(d-1)}\big| \ge c_1^{d-1} b_1^n/2 \quad \text{for } n \ge n_0 = n_0(H), \tag{2.12}$$

and so

$$\left|\frac{A_{n+1}^{(d-1)}}{A_n^{(d-1)}}\right| \le O(1), \quad n \ge n_0.$$

In the next estimates we assume that $n \ge n_0(H)$. In view of the above, especially (2.9) for $j = d-1$,

$$\begin{aligned} \left|\frac{A_{n+1}^{(d-1)}}{A_n^{(d-1)}} - \theta_d\right| &= \left|\frac{A_{n+1}^{(d-1)}}{A_n^{(d-1)}} - \frac{\widetilde{A}_{n+1}^{(d-1)}}{\widetilde{A}_n^{(d-1)}}\right| \\ &\le \left|\frac{A_{n+1}^{(d-1)} - \widetilde{A}_{n+1}^{(d-1)}}{A_n^{(d-1)}}\right| + \big|\widetilde{A}_{n+1}^{(d-1)}\big| \cdot \left|\frac{1}{A_n^{(d-1)}} - \frac{1}{\widetilde{A}_n^{(d-1)}}\right| \\ &\le O(1) \cdot \max\big\{|\varepsilon_n|, \dots, |\varepsilon_{n+d}|\big\} \cdot \big|A_n^{(d-1)}\big|^{-1}. \end{aligned}$$

It follows that, on the one hand,

$$\left|\frac{A_{n+1}^{(d-1)}}{A_n^{(d-1)}} - \theta_d\right| \le O(1) \cdot b_1^{-n}; \tag{2.13}$$

and, on the other hand,

$$\left|\frac{(A_{n+1}^{(d-1)})^2}{A_n^{(d-1)}} - A_{n+2}^{(d-1)}\right| \le O(1) \cdot \max\big\{|\varepsilon_n|, \dots, |\varepsilon_{n+d+1}|\big\}. \tag{2.14}$$

Note that $A_n^{(j)}$, for $j \in [d-1]$, is a linear combination of $K_n, K_{n+1}, \dots, K_{n+j}$ with coefficients that are polynomials in the (fixed) parameters $\theta_1, \dots, \theta_{d-1}$; hence the inequality (2.13) shows that

$$\text{given } K_n, \dots, K_{n+d}, \text{ we have an } O(1) \cdot b_1^{-n}\text{-approximation of } \theta_d. \tag{2.15}$$

The inequality (2.14) yields, using (2.11) and (2.10), that, for $n \ge n_0$,

$$\big|K_{n+d+1} - R_{\theta_1,\dots,\theta_{d-1}}(K_n, \dots, K_{n+d})\big| \le O(1) \cdot \max\big\{|\varepsilon_n|, \dots, |\varepsilon_{n+d+1}|\big\}.$$

Thus we have the following.

(i) Given $K_n, \dots, K_{n+d}$, there are at most $O(1)$ possible values for K_{n+d+1}, uniformly in η and $\theta_1, \dots, \theta_{d-1}$. There are also $O(1)$ possible values for $K_1, \dots, K_{n_0}$ since $\|\eta\|_\infty$ and $\|\boldsymbol{\theta}\|$ are bounded above by b_2.

(ii) There is a constant $\rho = \rho(H) > 0$ such that if $\max\{|\varepsilon_n|, \dots, |\varepsilon_{n+d+1}|\} < \rho$, then $K_n, \dots, K_{n+d}$ uniquely determine K_{n+d+1}, as the nearest integer to $R_{\theta_1,\dots,\theta_{d-1}}(K_n, \dots, K_{n+d})$, again independently of η and $\theta_1, \dots, \theta_{d-1}$.

Fix an N sufficiently large. We claim that for each fixed set $J \subset [N]$ with $|J| \ge (1-\delta)N$, the set

$$\left\{(K_n)_{n\in[N]} : \varepsilon_n = \left\|\sum_{k=1}^{d} \eta_k \theta_k^n\right\| < \rho \text{ for some } \theta_d, \eta \text{ and all } n \in J\right\}$$

has cardinality $\exp(O(\delta N))$. Indeed, fix such a J and let

$$\widetilde{J} = \big\{i \in [n_0 + (d+1), N] : i, i-1, \dots, i-(d+1) \in J\big\}.$$

We have $|\widetilde{J}| \ge (1-(d+2)\delta)N - n_0 - (d+1)$. If we set

$$\Lambda_j = (K_i)_{i\in[j]},$$

then (i) and (ii) above show that $|\Lambda_{j+1}| = |\Lambda_j|$, if $j \in \widetilde{J}$, and $|\Lambda_{j+1}| = O(|\Lambda_j|)$, otherwise. Thus, $|\Lambda_N| \le O(1)^{(d+2)\delta N}$, as claimed.

The number of subsets A of $[N]$ of size greater than or equal to $(1-\delta)N$ is bounded by $\exp(O(\delta \log(1/\delta)N))$ (using, e.g., Stirling's formula), so we conclude that there are

$$\exp\Big(O\Big(\delta \log\Big(\frac{1}{\delta}\Big)N\Big)\Big) \cdot \exp(O(\delta N)) = \exp\Big(O\Big(\delta \log\Big(\frac{1}{\delta}\Big)N\Big)\Big)$$

sequences $K_1, \dots, K_N$ such that $|\varepsilon_n| < \rho$ for at least $(1-\delta)N$ values of $n \in [N]$. Hence, by (2.15), the set (2.6) can be covered by $\exp(O_H(\delta \log(1/\delta) N))$ intervals of length b_1^{-N}, as desired. ■

The proof of Theorem 1.3 is now complete. ■

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