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An upper bound for the intermediate dimensions of Bedford–McMullen carpets

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Abstract. The intermediate dimensions of a set Λ, elsewhere denoted by $\dim_\theta \Lambda$, interpolate between its Hausdorff and box dimensions using the parameter $\theta \in [0,1]$. For a Bedford–McMullen carpet Λ with distinct Hausdorff and box dimensions, we show that $\dim_\theta \Lambda$ is strictly less than the box dimension of Λ for every $\theta < 1$. Moreover, the derivative of the upper bound is strictly positive at $\theta = 1$. This answers a question of Fraser; however, determining a precise formula for $\dim_\theta \Lambda$ still remains a challenging problem.

1. Introduction and main result

In fractal geometry, perhaps the most studied notions of dimension of a subset F of $\mathbb{R}^d$ are its Hausdorff and box dimensions. Both quantities can be formulated by means of covers of the set F. A finite or countable collection of sets $\{U_i\}$ is a *cover* of F if $F \subseteq \bigcup_i U_i$. Throughout, the diameter of a set F is denoted by $|F|$.

The Hausdorff dimension of F is

$$\dim_{\mathrm{H}} F = \inf\Big\{s \geq 0 : \text{for all } \varepsilon > 0, \text{ there exists a cover } \{U_i\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \varepsilon\Big\},$$

see [12, Section 3.2], while the (lower) box dimension is

$$\underline{\dim}_{\mathrm{B}} F = \inf\Big\{s \geq 0 : \text{ for all } \varepsilon > 0, \text{ there exists a cover } \{U_i\} \text{ of } F \text{ such that } |U_i| = |U_j| \text{ for all } i,j \text{ and } \sum_i |U_i|^s \leq \varepsilon\Big\},$$

see [12, Chapter 2]. We commonly refer to the quantity $\sum_i |U_i|^s$ as the *s-cost* of the cover $\{U_i\}$.

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The main difference between the two is that while there are *no* restrictions on the diameters of the covering sets for the Hausdorff dimension, for the box dimension we are restricted to coverings using sets of the *same* diameter. In particular, if $\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F$, then F has an optimal covering strategy where each covering contains sets with equal diameter. However, if $\dim_{\mathrm{H}} F < \underline{\dim}_{\mathrm{B}} F$, then it is natural to ask what different diameters are used in an optimal covering strategy for $\dim_{\mathrm{H}} F$? To obtain such finer information, the discussion above suggests a way to interpolate between $\dim_{\mathrm{H}} F$ and $\underline{\dim}_{\mathrm{B}} F$.

Falconer, Fraser and Kempton [14] introduced a continuum of *intermediate dimensions* that achieve this interpolation by imposing increasing restrictions on the relative sizes of covering sets governed by a parameter $0 \leq \theta \leq 1$. The Hausdorff and box dimensions are the two extreme cases when $\theta = 0$ and 1, respectively.

Definition 1.1. For $0 \leq \theta \leq 1$, the *lower θ-intermediate dimension* of a bounded set $F \subset \mathbb{R}^d$ is defined by

$$\underline{\dim}_{\theta} F = \inf\Big\{ s \geq 0 : x \text{ for all } \varepsilon > 0 \text{ and all } \delta_0 > 0, \text{ there exists } 0 < \delta \leq \delta_0 \text{ and}$$
$$\text{a cover } \{U_i\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ and } \sum_i |U_i|^s \leq \varepsilon \Big\},$$

while its *upper θ-intermediate dimension* is given by

$$\overline{\dim}_{\theta} F = \inf\Big\{ s \geq 0 : \text{ for all } \varepsilon > 0 \text{ there exists } \delta_0 > 0 \text{ such that for all } 0 < \delta \leq \delta_0,$$
$$\text{there is a cover } \{U_i\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ and } \sum_i |U_i|^s \leq \varepsilon \Big\}. \quad (1.1)$$

For a given θ, if the values of $\underline{\dim}_{\theta} F$ and $\overline{\dim}_{\theta} F$ coincide, then the common value is called the *θ-intermediate dimension* and is denoted by $\dim_{\theta} F$.

Thus, the restriction is to only consider covering sets with diameter in the range $[\delta^{1/\theta}, \delta]$. As $\theta \to 0$, the θ-intermediate dimension gives more insight into which scales are used in the optimal cover to reach the Hausdorff dimension. For $\theta < 1$, a natural covering strategy to improve on the exponent given by the box dimension is to use covering sets with diameter of the two permissible extremes, i.e., either $\delta^{1/\theta}$ or δ. It turns out that this strategy is already optimal for the case of elliptical polynomial spirals [8], for concentric spheres [25] and also for the family of countable convergent sequences [14]

$$F_p = \Big\{ 0, \frac{1}{1^p}, \frac{1}{2^p}, \frac{1}{3^p}, \dots \Big\}, \quad \text{where } p > 0.$$

Very recent preprints [4, 5] show examples where the use of more than two scales is necessary.

Intermediate dimensions can also be formulated using capacity-theoretic methods [9], which were used to compute the almost-sure value of the intermediate dimension of the image of Borel sets under index-α fractional Brownian motion [7] and under more general Rosenblatt processes [10]. Knowledge of intermediate dimensions can lead to results that do not follow from other notions of dimension. For example, in [7, 9], continuity at $\theta = 0$ was used to relate the box dimensions of the projections of a set to the Hausdorff dimension of the set. Other applications include gaining information about the Hölder distortion of maps between sets [3, 8] or deciding whether two sets are Lipschitz equivalent [4, Example 2.12].

A similar concept of dimension interpolation between the (quasi-)Assouad dimension and the upper box dimension, called the *Assouad spectrum* was initiated in [18, 19]. Recent surveys [13, 17] contain additional background and references in the topic of dimension interpolation.

Another large, well-known class of sets with distinct Hausdorff and box dimension are self-affine planar carpets. They are dynamically defined as the attractor of an iterated function system and already in the simplest case of Bedford–McMullen carpets, obtaining a precise formula for the intermediate dimensions seems to be a very challenging problem [14, 17]. The current bounds are rather crude and far apart; in particular, the upper bound improves on the trivial bound of the box dimension only for very small values of θ. This is in contrast to the aforementioned Assouad spectrum of Bedford–McMullen carpets, which was determined in [18].

1.1. Main contribution

By properly adapting the strategy of using the two extreme scales $\delta^{1/\theta}$ and δ, we show that the upper intermediate dimension of a Bedford–McMullen carpet (provided it has distinct Hausdorff and box dimension) is strictly smaller than its box dimension for every $\theta < 1$, moreover, the derivative of the upper bound is strictly positive at $\theta = 1$. This answers a question of Fraser [17, Question 2.1]. Using this upper bound, we construct an example which shows that the θ-intermediate dimension is not concave for the whole range of $\theta \in [0, 1]$; see Figure 4.1. This is a new feature compared to all previous known examples.

1.2. Bedford–McMullen carpets

Independently of each other, Bedford [6] and McMullen [24] were the first to study non-self-similar planar carpets. They split $R = [0, 1]^2$ into m columns of equal width and n rows of equal height for some integers $n > m \geq 2$ and considered orientation preserving maps of the form

$$f_{(i,j)}(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} i/m \\ j/n \end{pmatrix}$$

for the index set $(i, j) \in \mathcal{A} \subseteq \{0, \dots, m-1\} \times \{0, \dots, n-1\}$. It is well known that associated to the iterated function system (IFS) $\mathcal{F} = \{f_{(i,j)}\}_{(i,j)\in\mathcal{A}}$ there exists a unique non-empty compact subset $\Lambda_{\mathcal{F}} = \Lambda$ of R called the *attractor*, such that

$$\Lambda = \bigcup_{(i,j)\in\mathcal{A}} f_{(i,j)}(\Lambda).$$

We call Λ a *Bedford–McMullen carpet* and refer the interested reader to the recent survey [16] for further references. Figure 1.1 shows the simplest possible example for a Bedford–McMullen carpet with distinct Hausdorff and box dimensions.

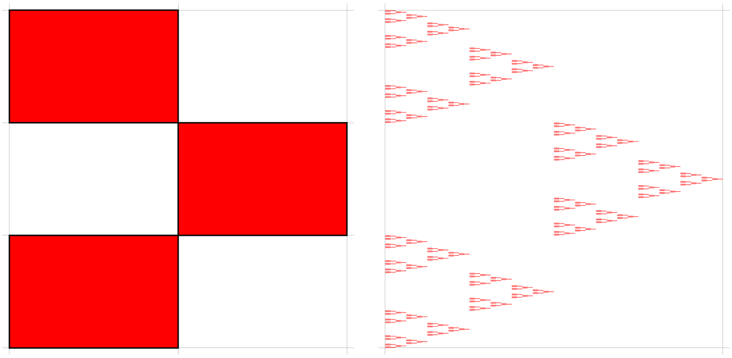


Figure 1.1. A Bedford–McMullen carpet with non-uniform vertical fibres. Left: the images of $[0, 1]^2$ under the maps of $\mathcal{F}$. Right: the attractor Λ.

Notation. Let Λ be the Bedford–McMullen carpet associated to the IFS $\mathcal{F}$. For the remainder of the paper, we index the maps of $\mathcal{F}$ by $i \in \{1, \dots, N\}$. We frequently use the abbreviation $[N] := \{1, \dots, N\}$. We can partition $[N]$ into $1 < M \le m$ sets $\mathcal{I}_1, \dots, \mathcal{I}_M$ with cardinality $\#\mathcal{I}_{\hat{\jmath}} = N_{\hat{\jmath}} > 0$ so that

$$\mathcal{I}_1 = \{1, \dots, N_1\} \quad \text{and} \quad \mathcal{I}_{\hat{\jmath}} = \{N_1 + \ldots + N_{\hat{\jmath}-1} + 1, \dots, N_1 + \ldots + N_{\hat{\jmath}}\}$$

for $\hat{\jmath} = 2, \dots, M$. Moreover, this partition satisfies that

$$i \in \mathcal{I}_{\hat{\jmath}} \quad \Longleftrightarrow \quad f_i \text{ maps } R \text{ to the } \hat{\jmath}\text{-th non-empty column.} \tag{1.2}$$

Formally, to keep track of this, we use the function

$$\phi : \{1,2,\dots,N\} \to \{1,2,\dots,M\}, \quad \phi(i) := \hat{\jmath}, \quad \text{if } i \in \mathcal{I}_{\hat{\jmath}}.$$

Throughout, i is an index from $[N]$, while $\hat{\jmath}$ with the hat is an index corresponding to a column from $[M] := \{1,\dots,M\}$; see Section 2.1 for details on symbolic notation. In Figure 1.1, we have $N = 3$, $M = 2$, $\mathcal{I}_1 = \{1,2\}$, $\mathcal{I}_2 = \{3\}$, $\phi(1) = \phi(2) = 1$ and $\phi(3) = 2$. Let

$$\boldsymbol{N} := (N_1,\dots,N_M).$$

The uniform probability vectors on $[N]$ and $[M]$ are denoted by

$$\widetilde{\boldsymbol{p}} := \Big(\frac{1}{N},\dots,\frac{1}{N}\Big) \quad \text{and} \quad \widetilde{\boldsymbol{q}} := \Big(\frac{1}{M},\dots,\frac{1}{M}\Big).$$

The entropy of a probability vector $\boldsymbol{p}$ is $H(\boldsymbol{p}) = -\sum_i p_i \log p_i$. In particular, $H(\widetilde{\boldsymbol{p}}) = \log N$ and $H(\widetilde{\boldsymbol{q}}) = \log M$.

We say that Λ has *uniform vertical fibres* if and only if

$$\boldsymbol{N} = \Big(\frac{N}{M},\dots,\frac{N}{M}\Big),$$

i.e., each non-empty column has the same number of maps. Bedford and McMullen showed that the Hausdorff dimension of Λ is equal to

$$\dim_{\mathrm{H}} \Lambda = \frac{H(\widehat{\boldsymbol{p}})}{\log n} + \Big(1 - \frac{\log m}{\log n}\Big)\frac{H(\widehat{\boldsymbol{q}})}{\log m}, \tag{1.3}$$

where $\widehat{\boldsymbol{p}} = (\widehat{p}_1,\dots,\widehat{p}_N)$ and $\widehat{\boldsymbol{q}} = (\widehat{q}_1,\dots,\widehat{q}_M)$ are equal to

$$\widehat{p}_k = N_{\hat{\imath}}^{\frac{\log m}{\log n}-1} \cdot \Bigg(\sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\frac{\log m}{\log n}}\Bigg)^{-1} \quad \text{and} \quad \widehat{q}_{\hat{\imath}} = N_{\hat{\imath}} \cdot \widehat{p}_k, \quad \text{if } k \in \mathcal{I}_{\hat{\imath}}.$$

Bedford and McMullen also showed a similar formula for the box dimension

$$\dim_{\mathrm{B}} \Lambda = \frac{H(\widetilde{\boldsymbol{p}})}{\log n} + \Big(1 - \frac{\log m}{\log n}\Big)\frac{H(\widetilde{\boldsymbol{q}})}{\log m}. \tag{1.4}$$

In particular, $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$ if and only if Λ has uniform vertical fibres, in which case $\dim_\theta \Lambda \equiv \dim_{\mathrm{H}} \Lambda$ for all $\theta \in [0,1]$. Therefore,

we always assume that Λ has *non*-uniform vertical fibres.

Formulas (1.3) and (1.4) extend to more general compact $(\times m, \times n)$-invariant sets [20, 21].

1.3. Main result

Before stating our new result, we summarize the results of Falconer, Fraser and Kempton [14]. The authors of [14] proved that for any non-empty bounded set $F \subset \mathbb{R}^d$ the functions $\theta \mapsto \underline{\dim}_\theta F$ and $\theta \mapsto \overline{\dim}_\theta F$ are continuous for $\theta \in (0, 1]$. In addition, for Bedford–McMullen carpets, they gave an upper bound for $\overline{\dim}_\theta \Lambda$, which implies continuity also at $\theta = 0$. However, this bound only improves on the trivial upper bound of $\dim_B \Lambda$ for very very small values of θ. In particular, it remained open whether $\dim_\theta \Lambda < \dim_B \Lambda$ for all $\theta < 1$. They also give a linear lower bound which shows that $\underline{\dim}_\theta \Lambda > \dim_H \Lambda$ for every $\theta \in (0, 1]$, and moreover, a general lower bound which reaches $\dim_B \Lambda$ at $\theta = 1$. In essence, their results concentrate on the behaviour of $\dim_\theta \Lambda$ near $\theta = 0$, while we concentrate more on the behaviour of $\dim_\theta \Lambda$ near $\theta = 1$.

The concavity of the logarithm function and our standing assumption of non-uniform vertical fibres imply that

$$\overline{\log \boldsymbol{N}} := \frac{1}{M} \sum_{\hat{\jmath}=1}^{M} \log N_{\hat{\jmath}} < \log\Big(\frac{N}{M}\Big). \tag{1.5}$$

Let X denote a uniformly distributed random variable on the set $\{\log N_1, \ldots, \log N_M\}$. Then $\overline{\log \boldsymbol{N}}$ is the expected value of X. The large deviation rate function of X is

$$I(x) = \sup_{\lambda \in \mathbb{R}} \left\{ \lambda x - \log\left(\frac{1}{M} \sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\lambda} \right) \right\}. \tag{1.6}$$

It is a convex function, $I\left(\overline{\log \boldsymbol{N}}\right) = 0$, it is non-decreasing for $x \geq \overline{\log \boldsymbol{N}}$, and on this range of x, the supremum over $\lambda \in \mathbb{R}$ is equivalent to taking $\lambda \geq 0$; see [11, Lemma 2.2.5]. Now we state our main result.

Theorem 1.2. *Let Λ be a Bedford–McMullen carpet with non-uniform vertical fibres. Then for every $\theta \in [\log_n m, 1)$,*

$$\overline{\dim}_\theta \Lambda \leq \dim_B \Lambda - \frac{\Delta_0(\theta)}{\log n}(1-\theta) < \dim_B \Lambda,$$

where $\Delta_0(\theta) \in (0, \log(N/M) - \overline{\log \boldsymbol{N}})$ is the unique solution of

$$(1-\theta) \cdot \Delta_0(\theta) = \Big(\frac{1}{\theta} - 1\Big) \cdot I\Big(\log\Big(\frac{N}{M}\Big) - \Delta_0(\theta) \Big). \tag{1.7}$$

In particular, the derivative of the upper bound remains strictly positive as $\theta \to 1$.

Moreover, since $\overline{\dim}_\theta \Lambda$ is non-decreasing, for every $\theta \in [0, \log_n m)$,

$$\overline{\dim}_\theta \Lambda \leq \dim_B \Lambda - \frac{\Delta_0(\log_n m)}{\log n}(1 - \log_n m) < \dim_B \Lambda.$$

Remark 1.3. In explicit examples, $\Delta_0(\theta)$ can be numerically calculated; see Figure 4.1.

Loosely speaking, the upper bound is obtained by constructing a cover of Λ using the two extreme scales δ and $\delta^{1/\theta}$. The cost of each part of the cover is upper bounded so that, with a properly chosen exponent s, it can be made arbitrarily small. Then for a fixed θ, condition (1.7) defining $\Delta_0(\theta)$ ensures that the order of magnitude of the two parts of the cover are equal. The bound in Theorem 1.2 is not the best possible; see Claim 4.1. However, this is simpler to state and already demonstrates the behaviour we wanted to show, namely that $\overline{\dim}_\theta \Lambda < \dim_B \Lambda$.

Remark 1.4. An unpublished, extended version of this paper on the arXiv [22] also contains results about the lower bound for $\underline{\dim}_\theta \Lambda$. For most carpets it improves on current lower bounds, however there are also examples when it does not. Since it is just an incremental improvement and does not show new qualitative behaviour, we have decided to omit it from this paper.

Remark 1.5. Well after the initial submission of the paper, Banaji and the author obtained an explicit formula for $\dim_\theta \Lambda$ for the complete range of $\theta \in [0, 1]$ in the preprint [4]. The same rate function defined in (1.6) appears in their significantly more complicated formula.

Structure of paper. Section 2 introduces additional notation, defines approximate squares and outlines the covering strategy for the upper bound. Section 3 contains the proof of Theorem 1.2. In Section 4, we comment on how to improve the upper bound and raise a number of questions for further research.

2. Preliminaries

In this section, we collect important notation and outline our strategy for proving the upper bound.

2.1. Symbolic notation

Let $\mathcal{F} = \{f_i\}$ be an IFS generating a Bedford–McMullen carpet Λ. The map f_i is indexed by $i \in \{1, 2, \dots, N\}$. Recall from (1.2) that we partitioned $\{1, 2, \dots, N\}$ into non-empty disjoint index sets $\mathcal{I}_1, \dots, \mathcal{I}_M$ to indicate which column f_i maps to. To keep track of this, we introduced the function

$$\phi : \{1, 2, \dots, N\} \to \{1, 2, \dots, M\}, \quad \phi(i) := \hat{\imath}, \quad \text{if } i \in \mathcal{I}_{\hat{\imath}}.$$

For compositions of maps, we use the standard notation $f_{i_1\dots i_n} := f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}$, where $i_\ell \in \{1, 2, \dots, N\}$.

We define the symbolic spaces

$$\Sigma = \left\{1, 2, \dots, N\right\}^{\mathbb{N}} \quad \text{and} \quad \Sigma_{\mathcal{H}} = \left\{1, 2, \dots, M\right\}^{\mathbb{N}}$$

with elements $\boldsymbol{i} = i_1 i_2 \dots \in \Sigma$ and $\hat{\boldsymbol{\imath}} = \hat{\imath}_1 \hat{\imath}_2 \dots \in \Sigma_{\mathcal{H}}$. The function ϕ naturally induces the map $\Phi : \Sigma \to \Sigma_{\mathcal{H}}$ defined by

$$\Phi(\boldsymbol{i}) := \hat{\boldsymbol{\imath}} = \phi(i_1)\phi(i_2)\dots$$

Finite words of length n are either denoted with a 'bar' like $\bar{\imath} = i_1 \dots i_n \in \Sigma_n$ or as a truncation $\boldsymbol{i}|n = i_1 \dots i_n$ of an infinite word $\boldsymbol{i}$. The length is denoted $|\cdot|$. The set of all finite length words is denoted by $\Sigma^* = \bigcup_n \Sigma_n$ and, analogously, $\Sigma^*_{\mathcal{H}}$. The left shift operator on Σ and $\Sigma_{\mathcal{H}}$ is σ, i.e. $\sigma(\boldsymbol{i}) = i_2 i_3 \dots$ and $\sigma(\hat{\boldsymbol{\imath}}) = \hat{\imath}_2 \hat{\imath}_3 \dots$ Slightly abusing notation, Φ is also defined on finite words: $\Phi(i_1 \dots i_n) = \phi(i_1) \dots \phi(i_n)$.

The longest common prefix of $\boldsymbol{i}$ and $\boldsymbol{j}$ is denoted $\boldsymbol{i} \wedge \boldsymbol{j}$, i.e., its length is $|\boldsymbol{i} \wedge \boldsymbol{j}| = \min\{k : i_k \neq j_k\} - 1$. This is also valid if one of them has or both have finite length. The nth level cylinder set of $\boldsymbol{i} \in \Sigma$ is $[\boldsymbol{i}|n] := \{\boldsymbol{j} \in \Sigma : |\boldsymbol{i} \wedge \boldsymbol{j}| \geq n\}$. Similarly for $\bar{\imath} \in \Sigma_n$ and $\hat{\boldsymbol{\imath}} \in \Sigma_{\mathcal{H}}$. The nth level cylinders corresponding to $\boldsymbol{i}$ on the attractor and $[0,1]^2$ are

$$\Lambda_n(\boldsymbol{i}) := f_{\boldsymbol{i}|n}(\Lambda) \quad \text{and} \quad C_n(\boldsymbol{i}) := f_{\boldsymbol{i}|n}\big([0,1]^2\big).$$

The sets $\{C_n(\boldsymbol{i})\}_{n=1}^{\infty}$ form a nested sequence of compact sets with diameter tending to zero; hence their intersection is a unique point $x \in \Lambda$. This defines the natural projection $\Pi : \Sigma \to \Lambda$

$$\Pi(\boldsymbol{i}) := \bigcap_{n=1}^{\infty} C_n(\boldsymbol{i}) = \lim_{n\to\infty} f_{\boldsymbol{i}|n}(\underline{0}).$$

In particular, $\Pi([\boldsymbol{i}|n]) = \Lambda_n(\boldsymbol{i})$. The coding of a point $x \in \Lambda$ is not necessarily unique, but Π is finite-to-one.

2.2. Approximate squares

The notion of an 'approximate square' is crucial in the study of planar carpets. Essentially, they play the role of balls in a cover of the attractor. Since $m > n$, a cylinder set $C_K(\boldsymbol{i})$ has width m^{-K} exponentially larger than its height n^{-K}.

The correct scales at which to achieve approximately equal width and height is K and

$$L(K) := \lfloor K \cdot \log_n m \rfloor < K.$$

In other words, $L(K)$ is the unique integer such that $n^{-L(K)-1} < m^{-K} \le n^{-L(K)}$. A level K *approximate square* is defined as

$$B_K(\boldsymbol{i}) := \big\{\Pi(\boldsymbol{j}) : |\boldsymbol{i} \wedge \boldsymbol{j}| \ge L(K) \text{ and } |\Phi(\boldsymbol{i}) \wedge \Phi(\boldsymbol{j})| \ge K\big\}.$$

It is essentially a level K column within a level $L(K)$ cylinder set.

Remark 2.1. One can also consider approximate squares to be the balls in the symbolic space Σ with metric, say,

$$d(\boldsymbol{i}, \boldsymbol{j}) := m^{-|\Phi(\boldsymbol{i})\wedge\Phi(\boldsymbol{j})|} + n^{-|\boldsymbol{i}\wedge\boldsymbol{j}|}.$$

See [23, Section 4] in a slightly more general setting.

The choice of $L(K)$ implies that there exists a uniform constant $C > 1$ such that $C^{-1}m^{-K} \le |B_K(\boldsymbol{i})| \le Cm^{-K}$ for every K and $\boldsymbol{i}$. Since this does not influence the behaviour of the s-cost of any cover with approximate squares, we simply neglect C henceforth. Each approximate square can be identified with the unique sequence

$$B_K(\boldsymbol{i}) = \big(i_1, \dots, i_{L(K)}, \hat{\imath}_{L(K)+1}, \dots, \hat{\imath}_K\big), \tag{2.1}$$

where $i_1, \dots, i_{L(K)} \in \{1, \dots, N\}$ and $\hat{\imath}_{L(K)+1}, \dots, \hat{\imath}_K \in \{1, \dots, M\}$. The set of level K approximate squares, denoted by $\mathcal{B}_K$, clearly gives a cover of Λ with cardinality

$$\#\mathcal{B}_K = N^{L(K)} \cdot M^{K-L(K)} = m^{K \dim_{\mathrm{B}} \Lambda},$$

where the second equality follows from (1.4). Moreover, the number of level K cylinder sets within $B_K(\boldsymbol{i})$ is

$$\begin{aligned}\#B_K(\boldsymbol{i}) &:= \#\big\{\Lambda_K(\boldsymbol{j}) : \ \boldsymbol{j}|L(K) = \boldsymbol{i}|L(K) \ \text{ and } \ \Phi(\boldsymbol{j}|K) = \Phi(\boldsymbol{i}|K)\big\} \\ &= \prod_{\ell=L(K)+1}^{K} N_{\phi(i_\ell)}.\end{aligned}$$

2.3. Covering strategy

For a fixed $\delta > 0$, we choose K so that $m^{-K} \le \delta < m^{-(K-1)}$. In our covering strategy, we start from $\mathcal{B}_K$ and decide for each $B_K(\boldsymbol{i}) \in \mathcal{B}_K$ if it is more 'cost efficient' to subdivide it into smaller approximate squares or not. When working with $\dim_\theta \Lambda$, we are allowed to use scales $k = K, \dots, \lfloor K/\theta \rfloor$, corresponding to covering sets of diameter between δ and $\delta^{1/\theta}$. We first determine the number of level k_2 approximate squares within an approximate square of level $k_1 < k_2$. Let $\mathcal{B}_{k_2}^{k_1,\boldsymbol{i}}$ denote the set of level k_2 approximate squares $B_{k_2}^{k_1,\boldsymbol{i}}(\boldsymbol{j})$ within the approximate square $B_{k_1}(\boldsymbol{i})$.

Claim 2.2. *Let* $K \le k_1 < k_2 \le \lfloor K/\theta \rfloor$.

(i) *If* $\theta \in [\log_n m, 1)$, *then* $\#\mathcal{B}_{k_2}^{k_1,\boldsymbol{i}} = M^{k_2-k_1} \cdot \prod_{\ell=L(k_1)+1}^{L(k_2)} N_{\hat{\imath}_\ell}$.

(ii) *If* $\theta \in (0, \log_n m)$ *and*

(a) $k_2 \in (K, K \cdot \log_m n]$, *then*

$$\#\mathcal{B}_{k_2}^{k_1,\boldsymbol{i}} = M^{k_2-k_1} \cdot \prod_{\ell=L(k_1)+1}^{L(k_2)} N_{\hat{\imath}_\ell};$$

(b) $k_2 \in (K \cdot \log_m n, K/\theta]$, *then*

$$\#\mathcal{B}_{k_2}^{K,\boldsymbol{i}} = N^{L(k_2)-K} \cdot M^{k_2-L(k_2)} \cdot \prod_{\ell=L(K)+1}^{K} N_{\hat{\imath}_\ell}.$$

Proof. Observe that

$$\theta \in [\log_n m, 1) \qquad \Longleftrightarrow \qquad L(k) \le K \text{ for all } k = K, \dots, \lfloor K/\theta \rfloor.$$

In particular, $L(k_2) \le k_1$.

Let us compare the sequences that define $B_{k_1}(\boldsymbol{i})$ and $B_{k_2}^{k_1,\boldsymbol{i}}(\boldsymbol{j})$:

$$\begin{array}{ccc|ccc|ccc|ccc} i_1 & \cdots & i_{L(k_1)} & \hat{\imath}_{L(k_1)+1} & \cdots & \hat{\imath}_{L(k_2)} & \hat{\imath}_{L(k_2)+1} & \cdots & \hat{\imath}_{k_1} & & & \\ j_1 & \cdots & j_{L(k_1)} & j_{L(k_1)+1} & \cdots & j_{L(k_2)} & \hat{\jmath}_{L(k_2)+1} & \cdots & \hat{\jmath}_{k_1} & \hat{\jmath}_{k_1+1} & \cdots & \hat{\jmath}_{k_2}. \end{array}$$

For the first $L(k_1)$ indices, $i_\ell = j_\ell$. For indices $\ell = L(k_1)+1, \dots, L(k_2)$, we require that $\phi(j_\ell) = \hat{\imath}_\ell$, hence the term $\prod_{\ell=L(k_1)+1}^{L(k_2)} N_{\hat{\imath}_\ell}$. For indices $\ell = L(k_2)+1, \dots, k_1$, there is equality again, $\hat{\imath}_\ell = \hat{\jmath}_\ell$. Finally, there is no restriction on $\hat{\jmath}_{k_1+1}, \dots, \hat{\jmath}_{k_2}$, hence the term $M^{k_2-k_1}$.

In case (ii) (a), it is also true that $L(k_2) \le k_1$. As a result, the same formula holds. Case (ii) (b) can be analyzed analogously to get the formula. ■

We say that it is *more cost efficient* to subdivide $B_{k_1}(\boldsymbol{i})$ into level k_2 approximate squares $B_{k_2}^{k_1,\boldsymbol{i}}(\boldsymbol{j})$ if and only if

$$m^{-k_1 s} = |B_{k_1}(\boldsymbol{i})|^s \ge \sum_{B_{k_2}^{k_1,\boldsymbol{i}}(\boldsymbol{j}) \in \mathcal{B}_{k_2}^{k_1,\boldsymbol{i}}} \left|B_{k_2}^{k_1,\boldsymbol{i}}(\boldsymbol{j})\right|^s = \#\mathcal{B}_{k_2}^{k_1,\boldsymbol{i}} \cdot m^{-k_2 s}.$$

In particular, if $k_1 = K$ and $k_2 = \lfloor K/\theta \rfloor$ for some $\theta \in [\log_n m, 1)$, then from Claim 2.2 (i), it follows after algebraic manipulations that it is more cost efficient to subdivide if and only if

$$s \geq \frac{\log M}{\log m} + \frac{1}{\log n}\left(\frac{1}{L(K/\theta) - L(K)} \sum_{\ell=L(K)+1}^{L(K/\theta)} \log N_{\hat{\imath}_\ell}\right).$$

Moreover, at the same time, we want to be able to choose

$$s \leq \dim_{\mathrm{B}} \Lambda - \frac{\Delta_0}{\log n} = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n} - \frac{\Delta_0}{\log n},$$

where the parameter $\Delta_0 = \Delta_0(\theta) \in [0, \log(N/M) - \frac{1}{M}\sum_{\hat{\jmath}=1}^{M} \log N_{\hat{\jmath}})$ will be made explicit later in the proof. Thus, we will subdivide $B_K(\boldsymbol{i})$ if and only if

$$\frac{1}{L(K/\theta) - L(K)} \sum_{\ell=L(K)+1}^{L(K/\theta)} \log N_{\hat{\imath}_\ell} \leq \log\left(\frac{N}{M}\right) - \Delta_0. \tag{2.2}$$

It is important to note that only indices $\hat{\imath}_{L(K)+1}, \ldots, \hat{\imath}_{L(K/\theta)}$ determine whether $B_K(\boldsymbol{i})$ gets subdivided into level $\lfloor K/\theta \rfloor$ approximate squares or not.

3. Proof of Theorem 1.2

The proof goes by constructing a cover of Λ using approximate squares of level K and $\lfloor K/\theta \rfloor$, which correspond to covering sets of diameter δ and $\delta^{1/\theta}$. Recall from (1.5) that

$$\overline{\log \boldsymbol{N}} = \frac{1}{M}\sum_{\hat{\imath}=1}^{M} \log N_{\hat{\imath}} < \log\left(\frac{N}{M}\right).$$

For the remainder of the proof we fix $\theta \in [\log_n m, 1)$ and we choose

$$\Delta_0 = \Delta_0(\theta) \in \left(0, \log\left(\frac{N}{M}\right) - \overline{\log \boldsymbol{N}}\right),$$

which will be optimized at the end of the proof. Based on condition (2.2), we start from the set $\mathcal{B}_K$ of level K approximate squares and partition it into two sets:

$$\mathrm{Good}_K := \left\{B_K(\boldsymbol{i}) \in \mathcal{B}_K : B_K(\boldsymbol{i}) \text{ satisfies (2.2)}\right\} \quad \text{and} \quad \mathrm{Bad}_K := \mathcal{B}_K \setminus \mathrm{Good}_K.$$

It is more cost efficient to subdivide all $B_K(\boldsymbol{i}) \in \mathrm{Good}_K$ into level $\lfloor K/\theta \rfloor$ approximate squares. Thus, let us define the cover

$$\mathcal{U}_K := \left\{\mathrm{Bad}_K \cup \bigcup_{B_K(\boldsymbol{i}) \in \mathrm{Good}_K} \mathcal{B}_{K/\theta}^{K,\boldsymbol{i}}\right\},$$

where recall that $\mathcal{B}^{K,\boldsymbol{i}}_{K/\theta}$ denotes the set of level $\lfloor K/\theta \rfloor$ approximate squares within $B_K(\boldsymbol{i})$. Claim 2.2 (i) implies that the cost of this cover is

$$\sum_{U_i \in \mathcal{U}_K} |U_i|^s = \#\mathrm{Bad}_K \cdot m^{-Ks} + \sum_{\mathrm{Good}_K} M^{K/\theta - K} \prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \cdot m^{-sK/\theta}. \tag{3.1}$$

The following two lemmas guarantee that for properly chosen s, this cost can be made arbitrarily small for large enough K.

Lemma 3.1. *For every $\theta \in [\log_n m, 1)$,*

$$\frac{\log \#\mathrm{Bad}_K}{K \log m} = \dim_{\mathrm{B}} \Lambda - \frac{I\big(\log(N/M) - \Delta_0\big)}{\log n}\Big(\frac{1}{\theta} - 1\Big) + o(1),$$

where $I(x)$ is the large deviation rate function of the random variable X uniformly distributed on the set $\{\log N_1, \dots, \log N_M\}$; recall (1.6). As a result,

$$\#\mathrm{Bad}_K \cdot m^{-Ks} \to 0 \text{ as } K \to \infty \iff s > \dim_{\mathrm{B}} \Lambda - \frac{I\big(\log(N/M) - \Delta_0\big)}{\log n}\Big(\frac{1}{\theta} - 1\Big).$$

Proof. Recall, the fact that an approximate square $B_K(\boldsymbol{i}) \in \mathrm{Bad}_K$ depends only on the indices $\hat{\imath}_{L(K)+1}, \dots, \hat{\imath}_{L(K/\theta)}$. We introduce

$$\mathcal{D} := \Big\{ (\hat{\imath}_{L(K)+1}, \dots, \hat{\imath}_{L(K/\theta)}) : \frac{1}{L(K/\theta) - L(K)} \sum_{\ell=L(K)+1}^{L(K/\theta)} \log N_{\hat{\imath}_\ell} > \log\Big(\frac{N}{M}\Big) - \Delta_0 \Big\}.$$

Since all other indices of $B_K(\boldsymbol{i})$ can be chosen freely, recall (2.1), we get that

$$\#\mathrm{Bad}_K = N^{L(K)} \cdot M^{K - L(K/\theta)} \cdot \#\mathcal{D}. \tag{3.2}$$

Let $\{I_\ell\}_{\ell=L(K)+1}^{L(K/\theta)}$ be independent uniformly distributed random variables on the discrete set $\{1, \dots, M\}$ and $X_\ell := \log N_{I_\ell}$. Then $\overline{\log \boldsymbol{N}}$ is the expected value of X_ℓ. Introduce

$$\overline{X} := \frac{1}{L(K/\theta) - L(K)} \sum_{\ell=L(K)+1}^{L(K/\theta)} X_\ell.$$

Since all I_ℓ are uniformly distributed, we have that

$$\mathbb{P}\Big(\overline{X} > \log\Big(\frac{N}{M}\Big) - \Delta_0\Big) = \frac{\#\mathcal{D}}{M^{L(K/\theta) - L(K)}}.$$

Hence, combining this with (3.2), we obtain that

$$\#\mathrm{Bad}_K = m^{K \dim_{\mathrm{B}} \Lambda} \cdot \mathbb{P}\Big(\overline{X} > \log\Big(\frac{N}{M}\Big) - \Delta_0\Big). \tag{3.3}$$

Cramér's theorem [11, Theorem 2.1.24] implies that for any $x > \overline{\log \boldsymbol{N}}$,

$$\lim_{K\to\infty} \frac{\log \mathbb{P}(\overline{X} > x)}{L(K/\theta) - L(K)} = -\inf_{y>x} I(y).$$

The infimum is equal to $I(x)$, because I is continuous and non-decreasing. Applying this with $x = \log(N/M) - \Delta_0$ proves the lemma after algebraic manipulations of (3.3). ∎

Lemma 3.2. *For every $\theta \in [\log_n m, 1)$, if $s > \dim_B \Lambda - \frac{\Delta_0}{\log n}(1-\theta)$, then*

$$\sum_{\mathrm{Good}_K} M^{K/\theta - K} \prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \cdot m^{-sK/\theta} \to 0 \quad \textit{as } K \to \infty.$$

Proof. For every $B_K(\boldsymbol{i}) \in \mathrm{Good}_K$, we have the uniform upper bound

$$\prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \le \Big(\frac{N}{M}\cdot e^{-\Delta_0}\Big)^{L(K/\theta)-L(K)}.$$

Moreover, trivially $\#\mathrm{Good}_K \le \#\mathcal{B}_K = N^{L(K)} M^{K-L(K)} = m^{K \dim_B \Lambda}$. Thus,

$$\begin{aligned}\sum_{\mathrm{Good}_K} M^{K/\theta - K} &\prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \cdot m^{-sK/\theta} \\ &\le \#\mathcal{B}_K \cdot M^{K/\theta-K}\cdot\Big(\frac{N}{M}\cdot e^{-\Delta_0}\Big)^{L(K/\theta)-L(K)} \cdot m^{-sK/\theta} \\ &= m^{-K\big((s-\dim_B\Lambda)/\theta + \frac{\Delta_0}{\log n}(1/\theta - 1)\big)},\end{aligned}$$

which tends to 0 as $K \to \infty$ if and only if $s > \dim_B \Lambda - \frac{\Delta_0}{\log n}(1-\theta)$. ∎

Remark 3.3. Lemma 3.1 shows that the bound $\#\mathrm{Good}_K < \#\mathcal{B}_K$ is essentially optimal, because $\#\mathrm{Bad}_K$ grows at an exponentially smaller rate than $\#\mathcal{B}_K$.

The two lemmas also show that choosing $\Delta_0 = 0$ or $\log(N/M) - \overline{\log \boldsymbol{N}}$ would result in a bound $s > \dim_B \Lambda$ for one of the parts of the cover.

Proof of Theorem 1.2. Fix $\theta \in [\log_n m, 1)$ and let

$$f_\theta(\Delta_0) := \Delta_0 \cdot (1-\theta) \quad \text{and} \quad g_\theta(\Delta_0) := I\Big(\log\Big(\frac{N}{M}\Big) - \Delta_0\Big)\cdot\Big(\frac{1}{\theta} - 1\Big).$$

Lemmas 3.1 and 3.2 imply that for any $\Delta_0 \in (0, \log(N/M) - \overline{\log \boldsymbol{N}})$, if

$$s > \dim_B \Lambda - \frac{1}{\log n}\min\{f_\theta(\Delta_0), g_\theta(\Delta_0)\}, \tag{3.4}$$

then the cost (3.1) of the cover $\mathcal{U}_K$ can be made arbitrarily small.

Observe that $f_\theta(0) = 0 = g_\theta(\log(N/M) - \overline{\log N})$; moreover, $f_\theta(\Delta_0)$ strictly increases while $g_\theta(\Delta_0)$ strictly decreases as Δ_0 increases. Hence, there is a unique $\Delta_0(\theta)$ such that

$$f_\theta(\Delta_0(\theta)) = g_\theta(\Delta_0(\theta)). \tag{3.5}$$

This is precisely condition (1.7) in Theorem 1.2. It optimizes (3.4) by making the cost of each part of the cover $\mathcal{U}_K$ to have the same order of magnitude. Since f_θ and g_θ are continuous in θ, so is $\theta \mapsto \Delta_0(\theta)$. Furthermore, $\Delta_0(\theta)$ can be extended in a continuous way to be defined for $\theta = 1$. Indeed, let $\theta = 1 - \varepsilon$, then (3.5) becomes

$$\varepsilon \cdot \Delta_0(1-\varepsilon) = \frac{\varepsilon}{1-\varepsilon} \cdot I\Big(\log\Big(\frac{N}{M}\Big) - \Delta_0(1-\varepsilon)\Big).$$

Hence, we define $\Delta_0(1)$ as the unique solution of $\Delta_0(1) = I(\log(N/M) - \Delta_0(1))$, which is clearly strictly positive.

The conclusion of the proof goes by the definition of $\overline{\dim}_\theta \Lambda$, recall (1.1). Fix an arbitrary $\varepsilon > 0$ and $s > \dim_{\mathrm{B}} \Lambda - f_\theta(\Delta_0(\theta))/\log n$. Choose $\delta_0 > 0$ so small that for $K_0 = K_0(\delta_0)$ defined by $m^{-K_0} \leq \delta_0 < m^{-K_0+1}$,

$$m^{-K_0\left(s - \dim_{\mathrm{B}} \Lambda + f_\theta(\Delta_0(\theta))/\log n\right)} < \varepsilon/2.$$

For any $\delta < \delta_0$, we cover Λ with $\mathcal{U}_K$, where $m^{-K} \leq \delta < m^{-K+1}$. Then $\sum_{U \in \mathcal{U}_K} |U|^s < \varepsilon$ and $\delta^{1/\theta} \leq |U| \leq \delta$ for every $U \in \mathcal{U}_K$. Hence,

$$\overline{\dim}_\theta \Lambda \leq \dim_{\mathrm{B}} \Lambda - \frac{f_\theta(\Delta_0(\theta))}{\log n}.$$

Moreover,

$$\lim_{\theta \to 1} \frac{d}{d\theta}\Big(\dim_{\mathrm{B}} \Lambda - \frac{f_\theta(\Delta_0(\theta))}{\log n}\Big) = \frac{\Delta_0(1)}{\log n} > 0. \qquad \blacksquare$$

4. Further discussion

Here we address a few further questions regarding our results. We first claim that the bound in Theorem 1.2 is not optimal.

Claim 4.1. *Even with just the two extreme scales, a better bound can be achieved than the one in Theorem 1.2.*

Proof. As always in the paper, assume $\theta \in [\log_n m, 1)$ and Λ has non-uniform vertical fibres. Let us partition $\mathcal{B}_K$ into

$$\mathrm{Good}_K := \Big\{B_K(\boldsymbol{i}) \in \mathcal{B}_K : \frac{1}{L(K/\theta) - L(K)} \sum_{\ell = L(K)+1}^{L(K/\theta)} \log N_{\hat{\imath}_\ell} \leq \log\Big(\frac{N}{M}\Big) - \Delta_2\Big\},$$

where Δ_2 is chosen to be slightly smaller than $\Delta_0 = \Delta_0(\theta)$, recall (1.7), and

$$\mathrm{Bad}_K := \mathcal{B}_K \setminus \mathrm{Good}_K.$$

An approximate square $B_K(\boldsymbol{i}) \in \mathrm{Good}_K$ gets subdivided to level $\lfloor K/\theta \rfloor$ approximate squares, while $B_K(\boldsymbol{i}) \in \mathrm{Bad}_K$ remains at level K. Then the argument of Lemma 3.1 implies that

$$\#\mathrm{Bad}_K \cdot m^{-Ks} \to 0 \text{ as } K \to \infty \quad \Longleftrightarrow \quad s > \dim_{\mathrm{B}} \Lambda - \frac{I\big(\log(N/M) - \Delta_2\big)}{\log n}\Big(\frac{1}{\theta} - 1\Big),$$

which is a smaller number than the bound obtained with Δ_0 in Theorem 1.2.

Now choose $\Delta_1 > \Delta_0(\theta)$ such that $\Delta_2 + I(\log(N/M) - \Delta_1) > \Delta_0(\theta)$. This can be clearly done if $\Delta_0 - \Delta_2$ is small enough. Define

$$\widetilde{\mathrm{Good}}_K := \Big\{ B_K(\boldsymbol{i}) \in \mathrm{Good}_K : \frac{1}{L(K/\theta) - L(K)} \sum_{\ell=L(K)+1}^{L(K/\theta)} \log N_{\hat{\imath}_\ell} \leq \log\Big(\frac{N}{M}\Big) - \Delta_1 \Big\}.$$

We bound separately the cost of the cover with the subdivided $B_K(\boldsymbol{i}) \in \widetilde{\mathrm{Good}}_K$ and $B_K(\boldsymbol{i}) \in \mathrm{Good}_K \setminus \widetilde{\mathrm{Good}}_K$. On one hand, the same argument as in Lemma 3.2 yields that the sum

$$\sum_{\widetilde{\mathrm{Good}}_K} M^{K/\theta - K} \prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \cdot m^{-sK/\theta} \to 0 \quad \text{as } K \to \infty,$$

if we choose

$$s > \dim_{\mathrm{B}} \Lambda - \frac{\Delta_1}{\log n}(1-\theta),$$

which again is a smaller number than the bound obtained with Δ_0. On the other hand, to bound the sum

$$\sum_{\mathrm{Good}_K \setminus \widetilde{\mathrm{Good}}_K} M^{K/\theta - K} \prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \cdot m^{-sK/\theta} \tag{4.1}$$

from above, we use that for every $B_K(\boldsymbol{i}) \in \mathrm{Good}_K \setminus \widetilde{\mathrm{Good}}_K$,

$$\prod_{\ell=L(K)+1}^{L(K/\theta)} N_{\hat{\imath}_\ell} \leq \Big(\frac{N}{M}\Big)^{L(K/\theta) - L(K)} \cdot e^{-\Delta_2 \left(L(K/\theta) - L(K)\right)},$$

and, from Lemma 3.1, we can use that

$$\#\big(\mathrm{Good}_K \setminus \widetilde{\mathrm{Good}}_K\big) \leq \#\big(\mathcal{B}_K \setminus \widetilde{\mathrm{Good}}_K\big) \leq m^{K\left(\dim_{\mathrm{B}} \Lambda - \frac{I(\log(N/M) - \Delta_1)}{\log n}(1/\theta - 1) + o(1)\right)}.$$

Substituting these back into (4.1), we obtain that the sum tends to 0 as $K \to \infty$, if

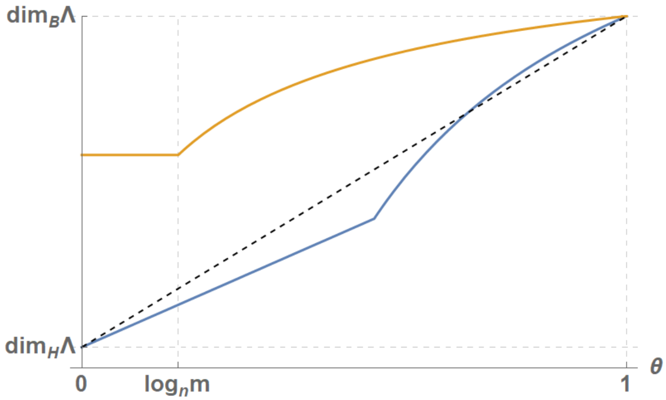
$$s > \dim_{\mathrm{B}} \Lambda - \frac{\Delta_2 + I\big(\log(N/M) - \Delta_1\big)}{\log n}(1-\theta),$$

which is also smaller than $\dim_{\mathrm{B}} \Lambda - (1-\theta)\Delta_0/\log n$ by the choice of Δ_1. Hence, we were able to improve on the upper bound in Theorem 1.2 for all parts of the cover. ∎

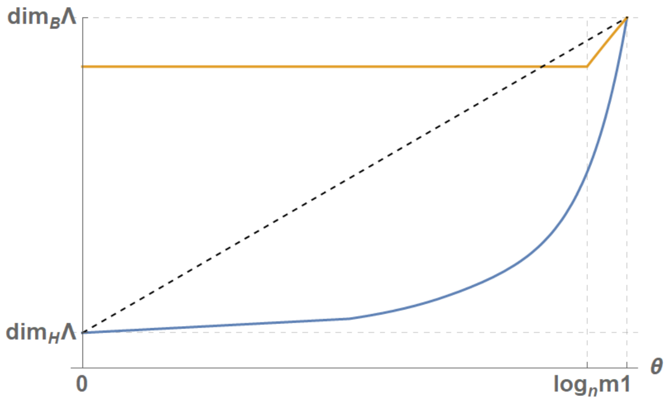
We strongly believe that for $\theta \geq \log_n m$, the upper bound is close to capturing the real value of $\dim_\theta \Lambda$. Moving beyond Claim 4.1, it is natural to ask what is the best achievable bound using just the two extreme scales? Could a matching lower bound be proved for that? It is clear that in an optimal covering strategy, different scales are present and tracking the optimal place to subdivide individual approximate squares seems hard to deal with. The question is, in the optimal covering, whether the squares at the two extreme scales determine the order of magnitude of the s-cost or do the scales in between play a role as well?

The next natural thing to ask is how can the argument be extended to $\theta < \log_n m$? Would it converge to $\dim_{\mathrm{H}} \Lambda$? Claim 2.2 shows that the number of approximate squares within a given approximate square behaves differently for $\theta < \log_n m$; thus it is not clear what could take the place of condition (2.2). Heuristically, if $\theta \in ((\log_n m)^{\ell+1}, (\log_n m)^{\ell})$, one could try to extend the argument to a cover in which 'almost all' approximate squares are at level $\lfloor K/\theta \rfloor$ and there are some 'left over' squares at levels $(\log_n m)^k$ for $k = 0, 1, \ldots, \ell$. It would be an interesting new behaviour if it is true that an unbounded number of scales are necessary as $\theta \to 0$.

Further interesting questions concern the form of $\dim_\theta \Lambda$. It has already been asked whether $\dim_\theta \Lambda$ is strictly increasing, differentiable, or analytic [17, Question 2.1]. A common feature of almost all sets E whose intermediate dimensions are known, $\dim_\theta E$ is strictly concave for the range of θ where $\dim_\theta E > \dim_{\mathrm{H}} E$. Bedford–McMullen carpets show a stark contrast to this. The general lower bound of [2, 14] is a concave function between $(0,0)$ and $(1, \dim_{\mathrm{B}} \Lambda)$, which enables to construct an example which shows that $\dim_\theta \Lambda$ is not convex in general; see Figure 4.1 (a). In addition, with the new bound of Theorem 1.2, we can also construct an example which shows that $\dim_\theta \Lambda$ is *neither concave* in general; see Figure 4.1 (b). In Figure 4.1, the (orange) plot depicting the upper bound comes from Theorem 1.2, while the (blue) plot depicting the lower bound comes from a combination of results in [2, 14, 22]. The ratio $\log_n m$ has an important role in projection and slicing results about Bedford–McMullen carpets [1, 15] and the Assouad spectrum has a phase transition here [18]. These together with Claim 2.2 strongly suggest to us that $\dim_\theta \Lambda$ may have a phase transition at $\log_n m$. Can $\dim_\theta \Lambda$ have additional phase transitions at other integer powers of $\log_n m$? Is it piecewise concave on the intervals in between phase transitions?



(a) $n = 50, m = 2 = M$ and $N = \{1, 50\}$.



(b) $n = 12, m = 10 = M$ and $N = \{12, 2, \ldots, 2\}$.

Figure 4.1. Examples showing that $\dim_\theta \Lambda$ is neither convex (a) nor concave (b) for the whole range of θ.

We remark that the very recent preprint [4] settles all these questions regarding the form of $\dim_\theta \Lambda$. The intermediate dimensions of sets can have surprisingly highly varied behaviour; see the very recent preprint of Banaji and Rutar [5], who give a complete characterisation of the possible functions that can be realised as the intermediate dimensions of a set.

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