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On the Hausdorff dimension of the recurrent sets induced from endomorphisms of free groups

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Abstract. We show that F. Dekking's recurrent sets in $\mathbb{R}^2$, which correspond to Markov partitions for conformally expanding maps of the 2-torus, have Hausdorff dimension strictly greater than one. This is a counterpart to the classical result of R. Bowen on the non-smoothness of the Markov partitions for Anosov diffeomorphisms of the 3-torus. We also present a non-conformal example where the recurrent set is a parallelogram and hence its Hausdorff dimension is one.

1. Introduction and main results

This paper investigates the Hausdorff dimension of recurrent sets in $\mathbb{R}^2$. The notion of recurrent sets has been introduced by Dekking [5] as a method to construct fractal tilings of the Euclidean spaces. To grasp the idea of his construction, let us start with a particular example.

Let $G = \langle a, b \rangle$ be the free group generated by two elements a and b. We first associate vectors in $\mathbb{R}^2$ to the generators as $f(a) := (1, 0)$, $f(b) := (0, 1)$, $f(a^{-1}) := (-1, 0)$ and $f(b^{-1}) := (0, -1)$. Then for a reduced word $c = c_1 \cdots c_k \in G$ (where each c_j is either a, b, a^{-1} or b^{-1}), we define the *geometric realization* of c denoted by $K[c]$ to be the broken line in $\mathbb{R}^2$ obtained as the successive concatenation of the arrows $f(c_1), \dots, f(c_k)$. For example, when $c = aba^{-1}b^{-1}$, its geometric realization $K[aba^{-1}b^{-1}]$ is the unit square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

Let $\theta : G \to G$ be the endomorphism of G given by

$$\begin{aligned} a &\mapsto ab^{-1}, \\ b &\mapsto a^{-1}b^{-2}. \end{aligned}$$

Then $K[\theta^n(aba^{-1}b^{-1})]$ becomes a broken loop in $\mathbb{R}^2$ for every $n \geq 0$. Let $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the invertible linear map given by

$$L_\theta(x, y) = (x - y, -x - 2y).$$

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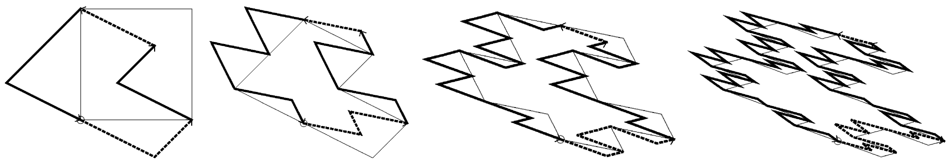


Figure 1. $L_\theta^{-n} K[\theta^n(aba^{-1}b^{-1})]$ for $n = 1, 2, 3$ and 4 (from left to right).

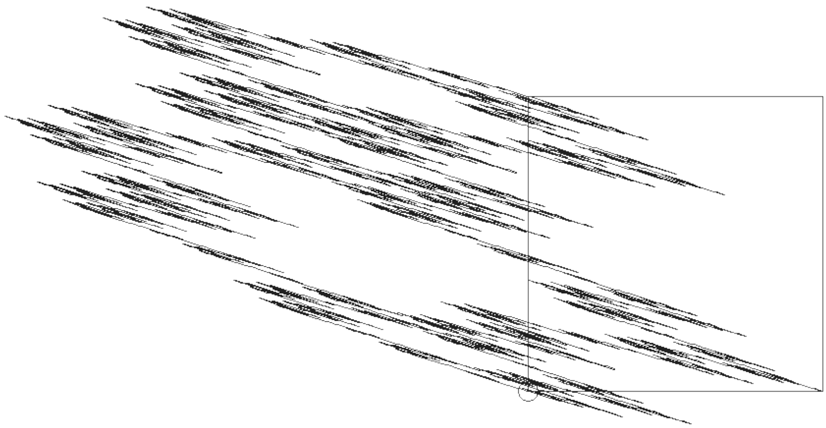


Figure 2. The recurrent set K_θ.

This map is introduced so that its inverse iteration L_θ^{-n} "rescales" the size of $K[\theta^n(aba^{-1}b^{-1})]$ (see Figure 1 for $n = 1, 2, 3$ and 4). Moreover, it can be shown [1,8] that the limit

$$K_\theta = \lim_{n\to\infty} L_\theta^{-n} K\big[\theta^n(aba^{-1}b^{-1})\big]$$

exists in the sense of the Hausdorff topology (see, e.g., [6] for the definition of the Hausdorff topology); we call it the *recurrent set* of θ (see Figure 2).

The purpose of this paper is to show that the recurrent set K_θ for an endomorphism θ of the free group of rank two always forms a fractal set under certain assumptions on θ. In Section 3, we introduce three kinds of assumptions on θ: Assumptions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$. Assumption $\mathfrak{A}$ requires that the rescaling map L_θ is expanding, Assumption $\mathfrak{B}$ requires that the rescaling map L_θ is conformal and expanding (hence Assumption $\mathfrak{B}$ is stronger than Assumption $\mathfrak{A}$), and Assumption $\mathfrak{C}$ requires the primitivity of a matrix which represents certain reduction of θ.

Our first main result is the following.

Theorem 1.1. *If $\theta : G \to G$ satisfies Assumptions $\mathfrak{B}$ and $\mathfrak{C}$, the Hausdorff dimension of its recurrent set K_θ is strictly greater than* 1.

Indeed, Bedford [1] constructed a Markov partition for the expanding map of the 2-torus induced from the linear map $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ in terms of Dekking's formalism (which is different from the classical construction by Sinaĭ [9] and Bowen [2]) and pointed out that the recurrent set K_θ forms the boundary of the Markov partition. Therefore, Theorem 1.1 can be seen as a counterpart to the classical result of Bowen [3] on the non-smoothness of the Markov partitions for Anosov diffeomorphisms of the 3-torus.

We also show that the conformality condition in Assumption $\mathfrak{B}$ is essential in Theorem 1.1.

Theorem 1.2. *Let m be a positive integer and define $\theta : G \to G$ by*

$$\begin{aligned} a &\mapsto ab^{-1}, \\ b &\mapsto a^{-m}b^{-1}. \end{aligned}$$

Then its recurrent set K_θ is a parallelogram with vertices $(0,0)$, $(1,0)$, $(0,1)$ *and* $(-1,1)$. *In particular, the Hausdorff dimension of K_θ is equal to* 1.

We notice that the endomorphism θ in Theorem 1.2 satisfies Assumptions $\mathfrak{A}$ and $\mathfrak{C}$ but not Assumption $\mathfrak{B}$, i.e., the induced linear map L_θ is expanding but not conformal. Note also that a result of Cawley [4] states the existence of a Markov partition with piecewise smooth boundary for an Anosov diffeomorphism of the n-torus for every $n \geq 4$.

The organization of this paper is as follows. In Section 2, we review the precise formulation of Dekking's recurrent sets. In Section 3, Assumptions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ mentioned above are presented and the reduction technique of θ due to Ito and Ohtsuki [8] is examined. Section 4 is devoted to the proof of Theorem 1.1 and Section 5 is devoted to the proof of Theorem 1.2. Finally, in Appendix A, we present Wielandt's theorem which is a key fact in the proof of Theorem 1.1.

2. Dekking's recurrent sets

Let G be a free group generated by two elements a, b. Namely, G is considered as the quotient set of the free semigroup S^* generated by $S := \{a, b, a^{-1}, b^{-1}\}$ by the equivalence relation $\sim$, where for $W, V \in S^*$, we define $W \sim V$ if their reduced words coincide. Therefore, an element of G can be identified with a unique reduced word.

We first construct the *canonical homomorphism* $f : G \to \mathbb{Z}^2 \subset \mathbb{R}^2$ as follows. We set

$$f(a) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f(b) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f(a^{-1}) := \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad f(b^{-1}) := \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

as in Section 1. Then we extend f to G by the relation

$$f(VW) = f(V) + f(W)$$

for reduced words $V, W \in G$, where $VW \in G$ is the reduced word obtained from the concatenation of V and W.

Let $\mathcal{K}[\mathbb{R}^2]$ be the set of all non-empty compact sets in $\mathbb{R}^2$. We next define a map $K : G \to \mathcal{K}[\mathbb{R}^2]$ which assigns a broken line to each reduced word. First, we set

$$\widetilde{K}[s] := \{\alpha f(s) : 0 \leq \alpha \leq 1\}$$

for $s \in S$, which is a unit segment. For a reduced word $W = s_1 \cdots s_m \in G$ ($s_i \in S$), we set

$$\widetilde{K}[W] := \bigcup_{i=1}^{m} \big(\widetilde{K}[s_i] + f(s_1 \cdots s_{i-1})\big),$$

where $A + z = \{a + z : a \in A\}$ for $A \subset \mathbb{R}^2$ and $z \in \mathbb{R}^2$. When $W \in G$ satisfies $f(W) \neq (0,0)$, we set $K[W] := \widetilde{K}[W]$. When $W \in G$ satisfies $f(W) = (0,0)$, we set

$$K[W] := \widetilde{K}[W'] + f(A),$$

where A is the longest word such that $W = AW'A^{-1}$ and $AW'A^{-1}$ has no cancellations.

Given an endomorphism θ of G, there is a linear map $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ so that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\theta} & G \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \mathbb{R}^2 & \xrightarrow{L_\theta} & \mathbb{R}^2 \end{array}$$

Definition 2.1. We say that θ has *short range cancellations* if for any reduced word of the form $stu \in G$ ($s, t, u \in S$), reducing $\theta(s)\theta(t)\theta(u)$ does not erase all letters of any of the subwords $\theta(s), \theta(t), \theta(u) \in G$.

To clarify this definition, let us consider the following two examples.

Example 2.2. Let θ be the endomorphism given by

$$a \mapsto ab^{-1},$$
$$b \mapsto a^{-2}b^{-1}.$$

It is then easy to check that some letters of any subwords $\theta(s)$ ($s \in S$) appeared in $\theta(a^{-1}ba^{-1}) = \theta(a^{-1})\theta(b)\theta(a^{-1})$ and $\theta(b^{-1}ab^{-1}) = \theta(b^{-1})\theta(a)\theta(b^{-1})$ are not erased by the cancellations. Therefore, θ has short range cancellations.

Example 2.3. Let θ be the endomorphism given by

$$\begin{aligned} a &\mapsto aba^2, \\ b &\mapsto ab^2a^2. \end{aligned}$$

Then, we see

$$\begin{aligned} \theta(b^{-1}ab^{-1}) &= \theta(b^{-1})\theta(a)\theta(b^{-1}) \\ &= \underbrace{a^{-2}b^{-2}a^{-1}}_{\theta(b^{-1})}\underbrace{aba^2}_{\theta(a)}\underbrace{a^{-2}b^{-2}a^{-1}}_{\theta(b^{-1})} \\ &= \underbrace{a^{-2}b^{-1}}_{\theta(b^{-1})}\underbrace{b^{-2}a^{-1}}_{\theta(b^{-1})}. \end{aligned}$$

Since all letters in the subwords $\theta(a)$ are erased by the cancellations, θ does not have short range cancellations.

Having short range cancellations yields the existence of the recurrent set. Namely, we have the following result due to Dekking.

Theorem 2.4 (Dekking [5]). *If θ has short range cancellations, the limit*

$$K_\theta := \lim_{n\to\infty} L_\theta^{-n} K\big[\theta^n(aba^{-1}b^{-1})\big]$$

exists with respect to the Hausdorff topology in $\mathcal{K}[\mathbb{R}^2]$.

Let us call the limit K_θ in Theorem 2.4 the *recurrent set* of θ.

3. Reductions of Ito and Ohtsuki

In this section, we review the reduction technique of θ due to Ito and Ohtsuki [8].

Definition 3.1. We say that a broken loop $K[\theta(aba^{-1}b^{-1})]$ is *double point free* if no pairs of edges in the loop intersect topologically transversally.

Here, *topological transversality* means that the intersection persists under small perturbations of the broken loop. Figure 3 represents two examples of an overlap of two edges in a broken loop which are double point free and Figure 4 represents two examples which are not double point free.

We say that the linear map L_θ is *expanding* if the absolute values of its eigenvalues are both strictly greater than 1. The next assumption is identical to [8, Assumption 1].

Figure 3. Double point free broken loop.

Figure 4. Not double point free broken loop.

Definition 3.2 (Assumption $\mathfrak{A}$). If an endomorphism θ of G satisfies the following conditions:

(i) θ has short range cancellations;

(ii) $K[\theta(aba^{-1}b^{-1})]$ is double point free;

(iii) L_θ is expanding;

then we say that θ satisfies *Assumption* $\mathfrak{A}$.

We say that the linear map L_θ is *conformally expanding* if L_θ is a rotation followed by a scalar multiplication by λ_θ with absolute value strictly greater than 1. The next assumption is identical to [8, Assumption 1′].

Definition 3.3 (Assumption $\mathfrak{B}$). If an endomorphism θ of G satisfies the following conditions:

(i) θ has short range cancellations;

(ii) $K[\theta(aba^{-1}b^{-1})]$ is double point free;

(iii) L_θ is conformally expanding;

then we say that θ satisfies *Assumption* $\mathfrak{B}$.

Obviously, Assumption $\mathfrak{A}$ is weaker than Assumption $\mathfrak{B}$.

The *adjoint* $\theta_W : G \to G$ of θ with respect to $W \in G$ is defined by

$$\theta_W(V) := W\theta(V)W^{-1}$$

for any $V \in G$. We set

$$P_\theta := \{\theta(s)\theta(t) : (s,t) = (a,b), (b,a^{-1}), (a^{-1},b^{-1}), (b^{-1},a)\}$$

for an endomorphism θ.

Theorem 3.4 ([8, Theorem 2.1]). *Let θ be an endomorphism of G satisfying Assumption $\mathfrak{A}$. Then there exists a word $W \in G$ so that the adjoint θ_W satisfies one of the following conditions:*

(i) *θ_W has cancellations only in $\theta_W(b)\theta_W(a^{-1})$ among P_{θ_W};*

(ii) *θ_W has cancellations only in $\theta_W(a^{-1})\theta_W(b^{-1})$ among P_{θ_W};*

(iii) *θ_W has no cancellations.*

Furthermore, the case (ii) in the previous theorem can be reduced to the case (i). To see this, let η be the automorphism of G defined by $\eta(a) = b$ and $\eta(b) = a^{-1}$.

Theorem 3.5 ([8, Theorem 2.2]). *Let θ an endomorphism satisfying Assumption $\mathfrak{A}$ and let W be the word as in Theorem 3.4. If the adjoint θ_W satisfies (ii) of Theorem 3.4, then $\theta' = \eta\theta_W\eta^{-1}$ has cancellations only in $\theta'(b)\theta'(a^{-1})$ among $P_{\theta'}$.*

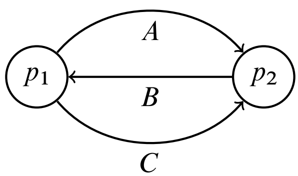
Thanks to Theorems 3.4 and 3.5, we may assume that only $\theta(b)\theta(a^{-1})$ can have cancellations among P_θ for θ satisfying Assumption $\mathfrak{A}$. Then $\theta(a)$ and $\theta(b)$ can be uniquely decomposed as

$$\theta(a) = AB \quad \text{and} \quad \theta(b) = CB \tag{3.1}$$

by some words $A, B, C \in G$, where

$$AB, CB, BC, BA, C^{-1}A, CA^{-1} \text{ are reduced words.} \tag{3.2}$$

For $W \in G$, we can write $\theta(W) = W_1W_2\cdots W_k$, where W_i is either $A^{\pm1}$, $B^{\pm1}$ or $C^{\pm1}$, which we call the *block representation* of $\theta(W)$. Let $\mathcal{G}$ be the directed graph given by



We say that $\theta(W) = W_1W_2\cdots W_k$ is *$\mathcal{G}$-admissible* if $W_1, \ldots, W_k$ forms a path in $\mathcal{G}$.

Proposition 3.6 ([8, Proposition 3.1]). *For any $W \in G$, $\theta(W)$ is $\mathcal{G}$-admissible.*

In order to check whether $\theta(P)\theta(Q)$ has cancellations for $P, Q \in \{A, B, C, A^{-1}, B^{-1}, C^{-1}\}$, we only need to consider the following cases:

$$(P, Q) \in \left\{(A, B), (C, A^{-1}), (B^{-1}, C^{-1}), (B, C), (A^{-1}, B^{-1}), (C^{-1}, A)\right\}$$

thanks to Proposition 3.6. To see this, let us decompose A, B and C as

$$A = vA't, \quad B = yB'x \quad \text{and} \quad C = wC'u, \tag{3.3}$$

where $v, t, y, x, w, u \in S$ and $A', B', C' \in G$. Then, from (3.2), we have the following relations:

$$y \neq t^{-1}, u^{-1}, \quad t \neq u, \quad x \neq v^{-1}, w^{-1} \quad \text{and} \quad v \neq w.$$

Lemma 3.7 ([8, Lemma 4.1]). *Under the assumption that only $\theta(b)\theta(a^{-1})$ has cancellations among P_θ, we have*

(1) *$\theta(A)\theta(B)$ has cancellations if and only if*

$$(t, u, y) \in \left\{(a, a^{-1}, b^{-1}), (a, b^{-1}, b^{-1}), (b, b^{-1}, a^{-1}), (b, a^{-1}, a^{-1})\right\};$$

(2) *$\theta(C)\theta(A^{-1})$ has cancellations if and only if*

$$(t, u, y) \in \left\{(b, a, a), (b, a, b), (a, b, a), (a, b, b)\right\};$$

(3) *$\theta(B^{-1})\theta(C^{-1})$ has cancellations if and only if*

$$(t, u, y) \in \left\{(a^{-1}, a, b^{-1}), (b^{-1}, a, b^{-1}), (a^{-1}, b, a^{-1}), (b^{-1}, b, a^{-1})\right\};$$

(4) *$\theta(B)\theta(C)$ has cancellations if and only if*

$$(x, v, w) \in \left\{(a, a, b^{-1}), (a, b, b^{-1}), (b, a, a^{-1}), (b, b, a^{-1})\right\};$$

(5) *$\theta(A^{-1})\theta(B^{-1})$ has cancellations if and only if*

$$(x, v, w) \in \left\{(a, b^{-1}, a), (a, b^{-1}, b), (b, a^{-1}, a), (b, a^{-1}, b)\right\};$$

(6) *$\theta(C^{-1})\theta(A)$ has cancellations if and only if*

$$(x, v, w) \in \left\{(a^{-1}, a^{-1}, b^{-1}), (b^{-1}, a^{-1}, b^{-1}), (a^{-1}, b^{-1}, a^{-1}), (b^{-1}, b^{-1}, a^{-1})\right\}.$$

Lemma 3.8 ([8, Lemma 4.2]). *Under the assumption of Lemma 3.7, we have*

(1) *the cases (1)–(3) in Lemma 3.7 are mutually exclusive;*

(2) *the cases (4)–(6) in Lemma 3.7 are mutually exclusive.*

Let $\widetilde{G}$ be the free group of rank 3 where the words A, B and C are regarded as generators. Let $i : \widetilde{G} \to G$ be a homomorphism sending the generators A, B, C of $\widetilde{G}$ to the words A, B, C of G, and define an endomorphism Θ of $\widetilde{G}$ as

$$\Theta(W) := \text{the block representation of } \theta(i(W)) \tag{3.4}$$

for $W \in \widetilde{G}$. Then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\Theta} & \widetilde{G} \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\ G & \xrightarrow{\theta} & G \end{array} \tag{3.5}$$

Based on Lemma 3.8, we define the reduced endomorphism $\widehat{\Theta}$ of Θ as follows.

Definition 3.9. We define the reduced endomorphism $\widehat{\Theta}$ of Θ as follows.

(I) If exactly one of (1)–(3) in Lemma 3.7 holds but any of (4)–(6) in Lemma 3.7 fails, we define

$$\widehat{\Theta}(A) := \Theta(A)B^{-1}, \quad \widehat{\Theta}(B) := B\Theta(B), \quad \widehat{\Theta}(C) := \Theta(C)B^{-1}.$$

(II) If exactly one of (4)–(6) in Lemma 3.7 holds but any of (1)–(3) in Lemma 3.7 fails, we define

$$\widehat{\Theta}(A) := B\Theta(A), \quad \widehat{\Theta}(B) := \Theta(B)B^{-1}, \quad \widehat{\Theta}(C) := B\Theta(C).$$

(III) If exactly one of (1)–(3) in Lemma 3.7 holds and exactly one of (4)–(6) in Lemma 3.7 holds, we define

$$\widehat{\Theta}(A) := B\Theta(A)B^{-1}, \quad \widehat{\Theta}(B) := B\Theta(B)B^{-1}, \quad \widehat{\Theta}(C) := B\Theta(C)B^{-1}.$$

Theorem 3.10 ([8, Theorem 4.1]). *The endomorphism $\widehat{\Theta} : \widetilde{G} \to \widetilde{G}$ has no cancellations on any $\mathscr{G}$-admissible words.*

Set $X_1 := A$, $X_2 := B$ and $X_3 := C$. Let m_{ij}^{+} (resp. m_{ij}^{-}) be the number of X_i's (resp. X_i^{-1}'s) in $\widehat{\Theta}(X_j)$. Let $m_{ij} := m_{ij}^{+} - m_{ij}^{-}$ and define a matrix $M_{\widehat{\Theta}} = (m_{ij})$. Similarly, let $n_{ij} := m_{ij}^{+} + m_{ij}^{-}$ and define a matrix $N_{\widehat{\Theta}} = (n_{ij})$. Note that we easily see that $|m_{ij}| \leq n_{ij}$. The next assumption is identical to [8, Assumption 2].

Definition 3.11 (Assumption $\mathfrak{C}$). We say that an endomorphism θ satisfies *Assumption* $\mathfrak{C}$ if $N_{\widehat{\Theta}}$ is primitive, i.e., there exists $n \geq 1$ such that all entries of the n-th power $N_{\widehat{\Theta}}^n$ are strictly positive.

Denote by $\lambda_{\widehat{\Theta}}$ (resp. $\Lambda_{\widehat{\Theta}}$) the greatest eigenvalue of $M_{\widehat{\Theta}}$ (resp. $N_{\widehat{\Theta}}$) in modulus. Then the Hausdorff dimension of K_θ is given by the following formula.

Theorem 3.12 ([8, Theorem 6.1 (2)]). *If an endomorphism θ satisfies Assumptions $\mathfrak{B}$ and $\mathfrak{C}$, the Hausdorff dimension of the recurrent set K_θ is given by*

$$\dim_{\mathrm{H}} K_\theta = \frac{\log \Lambda_{\widehat{\Theta}}}{\log |\lambda_{\widehat{\Theta}}|}. \tag{3.6}$$

We note that the conformality condition (iii) in Assumption $\mathfrak{B}$ implies the existence of a probability measure with local scaling property of order

$$\frac{\log \Lambda_{\widehat{\Theta}}}{\log |\lambda_{\widehat{\Theta}}|}.$$

This together with Frostman's lemma (see [6, Mass distribution principle 4.2]) yields that the right-hand side of (3.6) gives a lower bound for $\dim_{\mathrm{H}} K_\theta$.

4. Proof of Theorem 1.1

In this section, we prove Theorem1.1. The proof consists of three steps: first, to find certain restriction on the entries of $M_{\widehat{\Theta}}$ satisfying $|\lambda_{\widehat{\Theta}}| = \Lambda_{\widehat{\Theta}}$ thanks to Wielandt's theorem (see (A.2)); second, to classify such matrices $M_{\widehat{\Theta}}$ into some cases according to Lemma 3.7; and third, to show that these cases do not satisfy the conformality condition in Assumption $\mathfrak{B}$.

Let D_i, $0 \leq i \leq 7$, be the matrices given by

$$D_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$D_4 := -D_0, \quad D_5 := -D_1, \quad D_6 := -D_2 \quad \text{and} \quad D_7 := -D_3.$$

Proposition 4.1. *Suppose that Assumption $\mathfrak{C}$ holds. Then $|\lambda_{\widehat{\Theta}}| = \Lambda_{\widehat{\Theta}}$ if and only if there exists $0 \leq i \leq 7$ such that $M_{\widehat{\Theta}} = \pm D_i N_{\widehat{\Theta}} D_i$.*

Proof. Recall that we have $|m_{ij}| \leq n_{ij}$. Since $N_{\widehat{\Theta}}$ is primitive by Assumption $\mathfrak{C}$, we apply Wielandt's theorem (see (A.1)) to obtain $|\lambda_{\widehat{\Theta}}| \leq \Lambda_{\widehat{\Theta}}$. Moreover, $|\lambda_{\widehat{\Theta}}| = \Lambda_{\widehat{\Theta}}$ holds if and only if there exists $0 \leq i \leq 7$ such that $M_{\widehat{\Theta}} = \pm D_i N_{\widehat{\Theta}} D_i$. ∎

The condition $M_{\widehat{\Theta}} = \pm D_i N_{\widehat{\Theta}} D_i$ in Proposition 4.1 can be expressed more concretely as

$$M_{\widehat{\Theta}} = \pm D_0 N_{\widehat{\Theta}} D_0 = \pm D_4 N_{\widehat{\Theta}} D_4 = \pm \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}, \tag{4.1}$$

$$M_{\widehat{\Theta}} = \pm D_1 N_{\widehat{\Theta}} D_1 = \pm D_5 N_{\widehat{\Theta}} D_5 = \pm \begin{pmatrix} n_{11} & -n_{12} & -n_{13} \\ -n_{21} & n_{22} & n_{23} \\ -n_{31} & n_{32} & n_{33} \end{pmatrix}, \tag{4.2}$$

$$M_{\widehat{\Theta}} = \pm D_2 N_{\widehat{\Theta}} D_2 = \pm D_6 N_{\widehat{\Theta}} D_6 = \pm \begin{pmatrix} n_{11} & -n_{12} & n_{13} \\ -n_{21} & n_{22} & -n_{23} \\ n_{31} & -n_{32} & n_{33} \end{pmatrix}, \tag{4.3}$$

$$M_{\widehat{\Theta}} = \pm D_3 N_{\widehat{\Theta}} D_3 = \pm D_7 N_{\widehat{\Theta}} D_7 = \pm \begin{pmatrix} n_{11} & n_{12} & -n_{13} \\ n_{21} & n_{22} & -n_{23} \\ -n_{31} & -n_{32} & n_{33} \end{pmatrix}. \tag{4.4}$$

In what follows, when the equation (4.1) holds with positive or negative sign, we write (4.1^+) or (4.1^-) respectively (and the same for (4.2)–(4.4)).

The proofs of the following Propositions 4.2 and 4.3 rely on a detailed discussion of the homomorphism $\widehat{\Theta}$ given in Definition 3.9. This definition is based on Lemma 3.7 which analyzes the cancellations among $\theta(A)$, $\theta(B)$ and $\theta(C)$. Therefore, in what follows, it is important to look at (t, u, y), which are initial or final letters of the words A, B and C.

Proposition 4.2. *Suppose that $|\lambda_{\widehat{\Theta}}| = \Lambda_{\widehat{\Theta}}$ holds. If (t, u, y) is either (a, b^{-1}, b^{-1}), (b, b^{-1}, a^{-1}), (b, a^{-1}, a^{-1}), (a^{-1}, a, b^{-1}), (b^{-1}, a, b^{-1}) or (a^{-1}, b, a^{-1}), then the endomorphism $\widehat{\Theta}$ satisfies none of* ($4.1^{\pm}$), ($4.2^{\pm}$), ($4.3^{\pm}$) *and* ($4.4^{\pm}$).

Proof. We examine each case separately.

- *Case* $(t, u, y) = (a, b^{-1}, b^{-1})$. In this case, we have

$$\begin{aligned} \Theta(A) &= \theta(v)\theta(A')\theta(a) = \cdots AB, \\ \Theta(B) &= \theta(b^{-1})\theta(B')\theta(x) = B^{-1}C^{-1}\cdots, \\ \Theta(C) &= \theta(w)\theta(C')\theta(b^{-1}) = \cdots B^{-1}C^{-1}, \end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\widehat{\Theta}(A) = \Theta(A)B^{-1} = \cdots A,$$
$$\widehat{\Theta}(B) = B\Theta(B) = C^{-1}\cdots,$$
$$\widehat{\Theta}(C) = \Theta(C)B^{-1} = \cdots B^{-1}C^{-1}B^{-1}.$$

If $\widehat{\Theta}(A)$ contains A^{-1}, then $m_{11}^{+} > 0$ and $m_{11}^{-} > 0$. Since

$$m_{11} = m_{11}^{+} - m_{11}^{-} \quad \text{and} \quad n_{11} = m_{11}^{+} + m_{11}^{-},$$

this implies that $|m_{11}| \neq n_{11}$ and none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied by Proposition 4.1. It follows that $\widehat{\Theta}(A)$ contains A but not A^{-1}, and so $m_{11} > 0$. A similar argument shows that $\widehat{\Theta}(B)$ contains C^{-1} but not C, $\widehat{\Theta}(C)$ contains B^{-1} and C^{-1} but not B and C.[1] This yields that $m_{32} < 0$, $m_{23} < 0$ and $m_{33} < 0$.

All together, we have

$$m_{11} > 0, \quad m_{32} < 0, \quad m_{23} < 0 \quad \text{and} \quad m_{33} < 0.$$

One sees that none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied.

Since the proofs of the other cases are similar, below we only outline them.

- *Case* $(t, u, y) = (b, b^{-1}, a^{-1})$. In this case, we have

$$\Theta(A) = \cdots CB,$$
$$\Theta(B) = B^{-1}A^{-1}\cdots,$$
$$\Theta(C) = \cdots B^{-1}C^{-1},$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\widehat{\Theta}(A) = \cdots C,$$
$$\widehat{\Theta}(B) = A^{-1}\cdots,$$
$$\widehat{\Theta}(C) = \cdots B^{-1}C^{-1}B^{-1}.$$

This yields $m_{31} > 0$, $m_{12} < 0$, $m_{23} < 0$ and $m_{33} < 0$. One sees that none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied.

- *Case* $(t, u, y) = (b, a^{-1}, a^{-1})$. In this case, we have

$$\Theta(A) = \cdots CB,$$
$$\Theta(B) = B^{-1}A^{-1}\cdots,$$
$$\Theta(C) = \cdots B^{-1}A^{-1},$$

[1] This argument will appear repeatedly in the rest of this paper.

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\begin{aligned}\widehat{\Theta}(A) &= \cdots C,\\ \widehat{\Theta}(B) &= A^{-1}\cdots,\\ \widehat{\Theta}(C) &= \cdots B^{-1}A^{-1}B^{-1}.\end{aligned}$$

This yields $m_{31} > 0$, $m_{12} < 0$, $m_{13} < 0$ and $m_{23} < 0$. One sees that none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied.

- *Case* $(t, u, y) = (a^{-1}, a, b^{-1})$. In this case, we have

$$\begin{aligned}\Theta(A) &= \cdots B^{-1}A^{-1},\\ \Theta(B) &= B^{-1}C^{-1}\cdots,\\ \Theta(C) &= \cdots AB,\end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\begin{aligned}\widehat{\Theta}(A) &= \cdots B^{-1}A^{-1}B^{-1},\\ \widehat{\Theta}(B) &= C^{-1}\cdots,\\ \widehat{\Theta}(C) &= \cdots A.\end{aligned}$$

This yields $m_{11} < 0$, $m_{21} < 0$, $m_{32} < 0$ and $m_{13} > 0$. One sees that none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied.

- *Case* $(t, u, y) = (b^{-1}, a, b^{-1})$. In this case, we have

$$\begin{aligned}\Theta(A) &= \cdots B^{-1}C^{-1},\\ \Theta(B) &= B^{-1}C^{-1}\cdots,\\ \Theta(C) &= \cdots AB,\end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\begin{aligned}\widehat{\Theta}(A) &= \cdots B^{-1}C^{-1}B^{-1},\\ \widehat{\Theta}(B) &= C^{-1}\cdots,\\ \widehat{\Theta}(C) &= \cdots A.\end{aligned}$$

This yields $m_{21} < 0$, $m_{31} < 0$, $m_{32} < 0$ and $m_{13} > 0$. One sees that none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied.

- *Case* $(t, u, y) = (a^{-1}, b, a^{-1})$. In this case, we have

$$\begin{aligned}\Theta(A) &= \cdots B^{-1}A^{-1},\\ \Theta(B) &= B^{-1}A^{-1}\cdots,\\ \Theta(C) &= \cdots CB,\end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\widehat{\Theta}(A) = \cdots B^{-1}A^{-1}B^{-1},$$
$$\widehat{\Theta}(B) = A^{-1}\cdots,$$
$$\widehat{\Theta}(C) = \cdots C.$$

This yields $m_{11} < 0$, $m_{21} < 0$, $m_{12} < 0$ and $m_{33} > 0$. One sees that none of (4.1$^{\pm}$), (4.2$^{\pm}$), (4.3$^{\pm}$) and (4.4$^{\pm}$) is satisfied.

This concludes the proof of Proposition 4.2. ∎

If θ satisfies Assumption $\mathfrak{B}$, the linear map L_θ is conformal. Then its representation matrix M_θ must be of the form:

$$M_\theta = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{4.5}$$

where α and β are integers.

Proposition 4.3. *Suppose that $|\lambda_{\widehat{\Theta}}| = \Lambda_{\widehat{\Theta}}$ holds. If (t, u, y) is either (a, a^{-1}, b^{-1}), (b, a, a), (b, a, b), (a, b, a), (a, b, b) or (b^{-1}, b, a^{-1}), the linear map L_θ is not conformal.*

Proof. First we determine the possible form of a certain matrix M_Θ. Let us write

$$\begin{pmatrix} m_a \\ m_b \end{pmatrix} := f(A), \quad \begin{pmatrix} n_a \\ n_b \end{pmatrix} := f(B), \quad \begin{pmatrix} l_a \\ l_b \end{pmatrix} := f(C).$$

Since M_θ is the representation matrix of L_θ induced from $\theta : G \to G$ by $f : G \to \mathbb{Z}^2$, and we decompose as $\theta(a) = AB$ and $\theta(b) = CB$, we have

$$M_\theta = \begin{pmatrix} f(\theta(a)) & f(\theta(b)) \end{pmatrix} = \begin{pmatrix} f(AB) & f(CB) \end{pmatrix} = \begin{pmatrix} m_a + n_a & n_a + l_a \\ m_b + n_b & n_b + l_b \end{pmatrix},$$

where each vector $f(*)$ is regarded as a column vector.

Let $\widetilde{f} : \widetilde{G} \to \mathbb{Z}^3 \subset \mathbb{R}^3$ be the canonical homomorphism for Θ determined by

$$\widetilde{f}(A^{\pm 1}) = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widetilde{f}(B^{\pm 1}) = \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \widetilde{f}(C^{\pm 1}) = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}.$$

The representation matrix of the induced linear map $L_\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$M_\Theta = \begin{pmatrix} m_a & n_a & l_a \\ m_a + m_b & n_a + n_b & l_a + l_b \\ m_b & n_b & l_b \end{pmatrix}.$$

Suppose that M_θ takes the form of (4.5). Then we have

$$M_\Theta = \begin{pmatrix} \alpha - n_a & n_a & -\beta - n_a \\ \alpha + \beta - (n_a + n_b) & n_a + n_b & \alpha - \beta - (n_a + n_b) \\ \beta - n_b & n_b & \alpha - n_b \end{pmatrix}. \tag{4.6}$$

Below, we examine each case separately.

- *Case* $(t, u, y) = (a, a^{-1}, b^{-1})$. In this case, we have

$$\begin{aligned} \Theta(A) &= \theta(v)\theta(A')AB, \\ \Theta(B) &= B^{-1}C^{-1}\theta(B')\theta(x), \\ \Theta(C) &= \theta(w)\theta(C')B^{-1}A^{-1}, \end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\begin{aligned} \widehat{\Theta}(A) &= \theta(v)\theta(A')A, \\ \widehat{\Theta}(B) &= C^{-1}\theta(B')\theta(x), \\ \widehat{\Theta}(C) &= \theta(w)\theta(C')B^{-1}A^{-1}B^{-1}. \end{aligned}$$

This yields that $m_{11} > 0$, $m_{32} < 0$, $m_{13} < 0$ and $m_{23} < 0$. The only possible case which satisfies these conditions is (4.4^+).

The part $\theta(v)\theta(A')$ consists of some $\theta(a)$, $\theta(a^{-1})$, $\theta(b)$ and $\theta(b^{-1})$, so it consists of an even number of A, A^{-1}, B, B^{-1}, C and C^{-1}. Since $\widehat{\Theta}(A)$ ends with A, the path in Proposition 3.6 representing $\theta(v)\theta(A')$ should start from the vertex p_1. Therefore, $\theta(v)\theta(A')$ consists of either AB, $B^{-1}A^{-1}$, AC^{-1}, CA^{-1}, CB or $B^{-1}C^{-1}$. Since $\widehat{\Theta}(A)$ contains A, $\theta(v)\theta(A')$ can not contain A^{-1} as in the proof of Proposition 4.2. Suppose that $\theta(v)\theta(A')$ contains C. Then it can not contain C^{-1} by the same reasoning and hence $m_{31} > 0$, contradicting to the condition $m_{31} < 0$ in (4.4^+). Suppose that $\theta(v)\theta(A')$ contains B^{-1}. Then it can not contain B by the same reasoning, and hence $m_{21} < 0$, contradicting to the condition $m_{21} > 0$ in (4.4^+).

The argument above shows that $\theta(v)\theta(A')$ consists of only AB and AC^{-1}. Let p_A be the number of AB and q_A be the number of AC^{-1} in $\theta(v)\theta(A')$. Similarly, $\theta(B')\theta(x)$ consists of only AB and AC^{-1}. Let p_B be the number of AB and q_B be the number of AC^{-1} in $\theta(B')\theta(x)$. Finally, $\theta(w)\theta(C')$ consists of only $B^{-1}A^{-1}$ and CA^{-1}. Let p_C be the number of $B^{-1}A^{-1}$ and q_C be the number of AC^{-1} in $\theta(w)\theta(C')$. Then, we see

$$M_\Theta = \begin{pmatrix} p_A + q_A + 1 & p_B + q_B & -p_C - q_C - 1 \\ p_A + 1 & p_B - 1 & -p_C - 1 \\ -q_A & -q_B - 1 & q_C \end{pmatrix}, \tag{4.7}$$

where $p_A, q_A, p_B, q_B, p_C, q_C \geq 0$.

By comparing the second columns of (4.6) and (4.7), we obtain

$$n_a = p_B + q_B, \quad n_b = -q_B - 1. \tag{4.8}$$

The first columns of (4.6) and (4.7) together with (4.8) yield

$$\alpha = p_A + q_A + p_B + q_B + 1, \quad \beta = -q_B - q_A - 1. \tag{4.9}$$

The third columns of (4.6) and (4.7), together with (4.8), yield

$$\alpha = -q_B + q_C - 1, \quad \beta = -q_A - q_B + p_C + q_C + 1. \tag{4.10}$$

From (4.9) and (4.10), we obtain $p_A + 2q_A + 2q_B + p_C + 4 = 0$, which leads to a contradiction since $p_A, q_A, p_B, q_B, p_C, q_C \geq 0$.

- *Case* $(t, u, y) = (b, a, a)$. In this case, we have

$$\begin{aligned} \Theta(A) &= \theta(v)\theta(A')CB, \\ \Theta(B) &= AB\theta(B')\theta(x), \\ \Theta(C) &= \theta(w)\theta(C')AB, \end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\begin{aligned} \widehat{\Theta}(A) &= \theta(v)\theta(A')C, \\ \widehat{\Theta}(B) &= BAB\theta(B')\theta(x), \\ \widehat{\Theta}(C) &= \theta(w)\theta(C')A. \end{aligned}$$

This yields that $m_{31} > 0$, $m_{12} > 0$, $m_{22} > 0$ and $m_{13} > 0$. The only possible case which satisfies these conditions is (4.1^+). Together with (4.6), we obtain

$$\alpha - n_a \geq 0, \quad \beta - n_b \geq 0, \quad n_a \geq 0, \quad n_b \geq 0, \quad -\beta - n_a \geq 0, \quad \alpha - n_b \geq 0.$$

It is then easy to deduce $n_a = n_b = 0$ from these conditions. Since $\Theta(B)$ is not an empty word, this is a contradiction.

The proofs of the cases $(t, u, y) = (b, a, b), (a, b, a), (a, b, b)$ are similar, and hence we omit them.

- *Case* $(t, u, y) = (b^{-1}, b, a^{-1})$. In this case, we have

$$\begin{aligned} \Theta(A) &= \theta(v)\theta(A')AB, \\ \Theta(B) &= B^{-1}C^{-1}\theta(B')\theta(x), \\ \Theta(C) &= \theta(w)\theta(C')B^{-1}A^{-1}, \end{aligned}$$

by (3.1) and (3.3). Then, by Definition 3.9 (I), we see

$$\begin{aligned}\widehat{\Theta}(A) &= \theta(v)\theta(A')A,\\ \widehat{\Theta}(B) &= C^{-1}\theta(B')\theta(x),\\ \widehat{\Theta}(C) &= \theta(w)\theta(C')B^{-1}A^{-1}B^{-1}.\end{aligned}$$

This yields that $m_{11} > 0$, $m_{32} < 0$, $m_{13} < 0$ and $m_{23} < 0$. The only possible case which satisfies these conditions is (4.4$^+$).

An argument similar to the case $(t, u, y) = (a, a^{-1}, b^{-1})$ shows that $\theta(v)\theta(A')$ consists of only AC^{-1} and $B^{-1}C^{-1}$. Let p_A be the number of AC^{-1} and q_A be the number of $B^{-1}C^{-1}$ in $\theta(v)\theta(A')$. Similarly, $\theta(B')\theta(x)$ consists of only $B^{-1}A^{-1}$ and CA^{-1}. Let p_B be the number of AB and q_B be the number of AC^{-1} in $\theta(B')\theta(x)$. Finally, $\theta(w)\theta(C')$ consists of only CB and CA^{-1}. Let p_C be the number of CB and q_C be the number of CA^{-1} in $\theta(w)\theta(C')$. Then, we see

$$M_\Theta = \begin{pmatrix} p_A & -p_B - q_B - 1 & -q_C \\ q_A - 1 & -p_B - 1 & p_C + 1 \\ -p_A - q_A - 1 & q_B & p_C + q_C + 1 \end{pmatrix}, \tag{4.11}$$

where $p_A, q_A, p_B, q_B, p_C, q_C \geq 0$.

By comparing the second columns of (4.6) and (4.11), we obtain

$$n_a = -p_B - q_B - 1, \quad n_b = q_B. \tag{4.12}$$

The first columns of (4.6) and (4.11), together with (4.12), yield

$$\alpha = p_A - p_B - q_B - 1, \quad \beta = -p_A - q_A + q_B - 1. \tag{4.13}$$

The third columns of (4.6) and (4.11), together with (4.12), yield

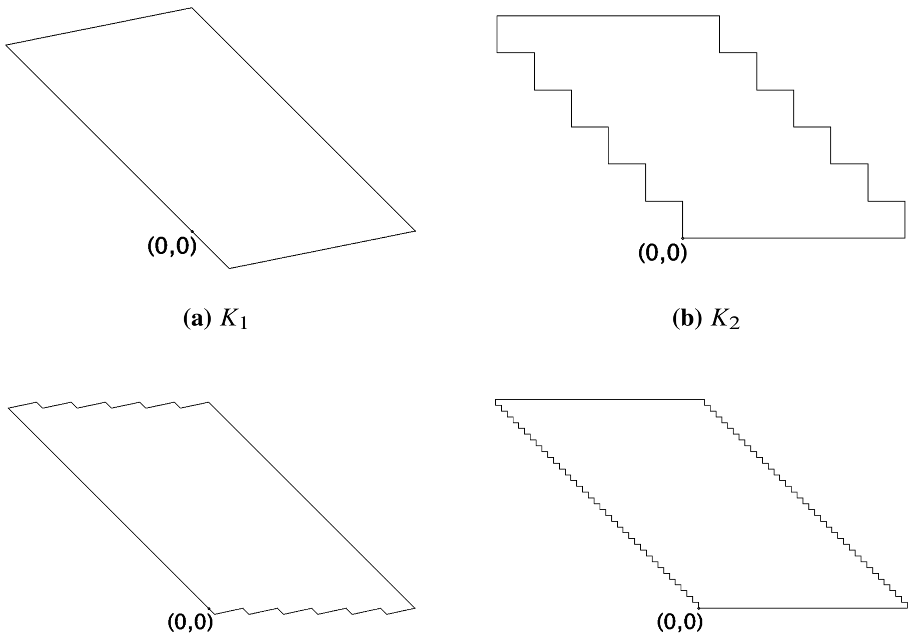
$$\alpha = q_B + p_C + q_C + 1, \quad \beta = p_B + q_B + q_C + 1. \tag{4.14}$$

From (4.13) and (4.14) we obtain $p_B + q_A + 2q_B + p_C + 2q_C + 4 = 0$, which leads to a contradiction since $p_A, q_A, p_B, q_B, p_C, q_C \geq 0$.

This concludes the proof of Proposition 4.3. ∎

Proof of Theorem 1.1. Since $N_{\widehat{\Theta}}$ is primitive by Assumption $\mathfrak{C}$ and since $|m_{ij}| \leq n_{ij}$, we apply Wielandt's theorem (see (A.1)) to obtain $|\lambda_{\widehat{\Theta}}| \leq \Lambda_{\widehat{\Theta}}$. Since θ satisfies Assumptions $\mathfrak{B}$ and $\mathfrak{C}$, we can apply Theorem 3.12 to conclude $\dim_{\mathrm{H}} K_\theta \geq 1$.

Suppose that $\dim_{\mathrm{H}} K_\theta = 1$ holds. Then, by Theorem 3.12, we have $\Lambda_{\widehat{\Theta}} = |\lambda_{\widehat{\Theta}}|$. Proposition 4.1 yields that there exists $0 \leq i \leq 3$ such that $M_{\widehat{\Theta}} = \pm D_i N_{\widehat{\Theta}} D_i$. By Proposition 4.2, the only possibilities are those listed in Proposition 4.3. It follows from Proposition 4.3 that L_θ can not be conformal for these cases, contradicting to Assumption $\mathfrak{B}$. Hence, $\dim_{\mathrm{H}} K_\theta > 1$. ∎



(a) K_1 **(b)** K_2 **(c)** K_3 **(d)** K_4

Figure 5. Pre-limit sets.

5. A piecewise smooth example

In this section, we prove Theorem 1.2. Recall that the endomorphism $\theta : G \to G$ in Theorem 1.2 is given by $\theta(a) = ab^{-1}$ and $\theta(b) = a^{-m}b^{-1}$ for a positive integer m. To clarify the situation, let us first observe the "pre-limit set"

$$K_n = L_\theta^{-n} K\big[\theta^n(aba^{-1}b^{-1})\big]$$

for $1 \le n \le 4$ with $m = 5$ (see Figure 5).

Proof of Theorem 1.2. Let P be the parallelogram with the vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(-1,1)$, in order. We show that the subsequences of K_n for even n and odd n both converge to P.

Put $A := a$, $B := b^{-1}$ and $C := a^{-m}$. Then $\Theta(A) = AB$, $\Theta(B) = B^{-1}C^{-1}$ and $\Theta(C) = (B^{-1}A^{-1})^m$. We see that θ belongs to (I) of Definition 3.9 and hence $\widehat{\Theta}$ is given by $\widehat{\Theta}(A) = A$, $\widehat{\Theta}(B) = C^{-1}$ and $\widehat{\Theta}(C) = (B^{-1}A^{-1})^m B^{-1}$. One can then check that

$$\widehat{\Theta}^2(AB) = (AB)^{m+1},$$
$$\widehat{\Theta}^2(AC^{-1}) = (AC^{-1})^{m+1},$$
$$\widehat{\Theta}^2(CB) = (CA^{-1})^m CB(AB)^m.$$

From this, we have

$$\widehat{\Theta}^{2n}(ABCA^{-1}B^{-1}C^{-1}) = (AC^{-1})^{(m+1)^n}(B^{-1}A^{-1})^{(m+1)^{n+1}}(CA^{-1})^{(m+1)^n-1} CB(AB)^{(m+1)^{n+1}-1}.$$

By (3.5), we see $i(\widehat{\Theta}^{n-1}(ABCA^{-1}B^{-1}C^{-1})) = \theta^n(aba^{-1}b^{-1})$, and hence

$$\theta^{2(n+1)}(aba^{-1}b^{-1}) = (AC^{-1})^{(m+1)^n}(B^{-1}A^{-1})^{(m+1)^{n+1}}(CA^{-1})^{(m+1)^n-1} CB(AB)^{(m+1)^{n+1}-1}. \tag{5.1}$$

On the other hand, we have

$$M_\theta^{-2(n+1)} = \frac{1}{(m+1)^{n+1}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

See Figure 6 for the shape of $K_{2(n+1)}$. The length of an edge of the "staircase" part of $K_{2(n+1)}$ is $(m+1)^{-(n+1)}$, hence we obtain $P \subset (K_{2(n+1)})_\varepsilon$ for $\varepsilon \geq (m+1)^{-(n+1)}$ and $K_{2(n+1)} \subset P_{\varepsilon'}$ for $\varepsilon' \geq \frac{1}{\sqrt{2}}(m+1)^{-(n+1)}$ (see Figure 6 again), where A_ε denotes the ε-neighborhood of $A \subset \mathbb{R}^2$ (see [6]). From this, we conclude

$$d_{\mathrm{H}}(P, K_{2(n+1)}) \leq \frac{1}{(m+1)^{n+1}} \to 0$$

as $n \to \infty$, where d_{H} denotes the Hausdorff distance.

In the case that n is odd, we have

$$\widehat{\Theta}^{2(n-1)}(ABCA^{-1}B^{-1}C^{-1}) = (AB)^{(m+1)^n}(CA^{-1})^{(m+1)^n}(B^{-1}A^{-1})^{(m+1)^n-1} B^{-1}C^{-1}(AC^{-1})^{(m+1)^n-1},$$

and hence

$$\theta^{2n+1}(aba^{-1}b^{-1}) = (AB)^{(m+1)^n}(CA^{-1})^{(m+1)^n}(B^{-1}A^{-1})^{(m+1)^n-1} B^{-1}C^{-1}(AC^{-1})^{(m+1)^n-1}.$$

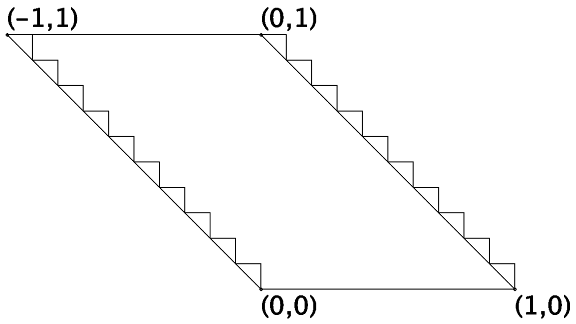


Figure 6. Comparison between $K_{2(n+1)}$ and P.

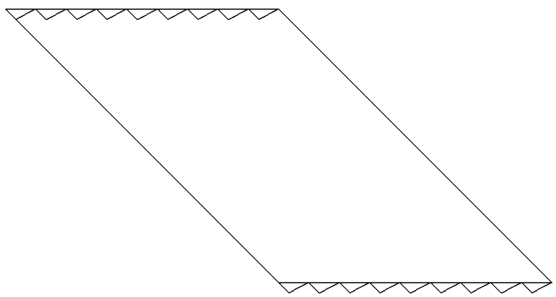


Figure 7. Comparison between K_{2n+1} and P.

On the other hand, we have

$$M_\theta^{-(2n+1)} = \frac{1}{(m+1)^{n+1}} \begin{pmatrix} -1 & m \\ 1 & 1 \end{pmatrix}.$$

See Figure 7 for the shape of K_{2n+1}. Similarly, in the case that n is even, we see that

$$P \subset (K_{2n+1})_\varepsilon \quad \text{for } \varepsilon \geq \sqrt{2}(m+1)^{-(n+1)},$$
$$K_{2n+1} \subset P_{\varepsilon'} \quad \text{for } \varepsilon' \geq (m+1)^{-(n+1)}$$

(see Figure 7 again). From this, we conclude

$$d_{\mathrm{H}}(P, K_{2n+1}) \leq \frac{\sqrt{2}}{(m+1)^{n+1}} \to 0.$$

as $n \to \infty$. Therefore, $K_\theta = P$. ∎

A. Wielandt's theorem

In this appendix, we explain Wielandt's theorem following Gantmacher [7, p. 69, Lemma 2]. The zero matrix is denoted as $\boldsymbol{O}$. For two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, we write $A > B$ (resp. $A \geq B$) if $a_{ij} > b_{ij}$ (resp. $a_{ij} \geq b_{ij}$) for all i, j. A matrix A is said to be *positive* (resp. *non-negative*) if $A > \boldsymbol{O}$ (resp. $A \geq \boldsymbol{O}$). Give a matrix A, let A^+ be obtained by replacing every element of A with its absolute value, i.e., $A^+ = (|a_{ij}|)$.

Theorem A.1 (Wielandt's theorem). *Let A and C be two square matrices of the same size. Assume that A is primitive, and A and C satisfy $C^+ \leq A$. Then the Perron–Frobenius eigenvalue r of A satisfies*

$$|\gamma| \leq r \tag{A.1}$$

for arbitrary eigenvalue γ of C. In addition, γ is an eigenvalue of C satisfying the equality in (A.1) *if and only if*

$$C = \exp(i\varphi) D A D^{-1}, \tag{A.2}$$

where $\exp(i\varphi) = \gamma / r$ *and D is a diagonal matrix such that D^+ is equal to the identity matrix. When moreover C is a real matrix, the diagonal elements of D are equal to* ± 1.

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