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Nonlinear fractal interpolation functions on the Koch curve

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Abstract. Nonlinear fractal interpolation functions defined on the Koch curve by replacing the usual Banach contractions used to define (linear) fractal interpolation functions with Rakotch contractions are considered. This allows to take care of the nonlinear term in the associated Read–Bajraktarević operator, respectively, the underlying iterated function system. Moreover, we present an extremely explicit example to demonstrate the effectiveness of the obtained results.

1. Introduction

The *Koch curve*, or KC for short, appeared in a 1904 paper entitled "On a Continuous Curve Without Tangents, Constructible from Elementary Geometry" by the Swedish mathematician Helge von Koch. It is an example of a compact curve with infinite length; see [5].

The concept of *iterated function system*, or IFS for short, was introduced in [4] and popularised in [2] as a natural generalisation of the well-known Banach fixed-point theorem (also known as the contraction mapping theorem or contractive mapping theorem). The concept of *fractal interpolation function*, or FIF for short, was introduced in [1] on the basis of the theory of IFSs. In the development of fractal interpolation theory, many researchers have generalised the notion in different ways; see [9, 11, 13].

Interpreting the polynomials of degree 1 as classical harmonic functions on an interval and replacing them on the KC by harmonic functions, the authors of [7] obtained an analogue of [2, Theorem VI.2.2] for the KC. The existence of FIFs on the KC follows from Banach fixed-point theorem. As far as we know, the first significant generalisation of that principle was obtained in [8]. A method to generate nonlinear FIFs by using the Rakotch fixed-point theorem instead of the Banach fixed-point theorem in presented in [9].

The results of [7] and [9] inspire us to find potential contractions (not necessarily Banach contractions) for the existence of FIFs on the KC. In this article, in order to

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obtain nonaffine FIFs on the KC, we use Rakotch instead of Banach contractions. Furthermore, we give an explicit illustrative example to demonstrate the effectiveness of the obtained results. The rest of this article is organised as follows and can be generally seen as an extension of [12]. In Section 2, we recall some results needed in constructing nonlinear FIFs on the KC. In Section 3, we introduce a new type of IFSs that will be used in our discussion and give a nonlinear FIF on the KC as the fixed point of certain Read–Bajraktarević operator. The construction is based on previous generalisations of fractal interpolation by some of the authors. We also give an explicit illustrative example to demonstrate the effectiveness of the preceding theory.

2. Preliminaries

Firstly, we introduce nonlinear FIFs by recalling some already known results; see also [11] or [10] and the references therein.

Theorem 2.1. *Let X be a complete metric space and $f\colon X \to X$ be a Rakotch contraction. Then there is a unique fixed point $x_f \in X$ of f, and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x_f$.*

Let N be a positive integer greater than one and $I = [x_0, x_N] \subset \mathbb{R}$. Let a set of *interpolation points* $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \ldots, N\}$ be given, where $x_0 < x_1 < \cdots < x_N$ and $y_0, y_1, \ldots, y_N \in \mathbb{R}$. Set $I_n = [x_{n-1}, x_n] \subset I$ and define, for all $n = 1, 2, \ldots, N$, contractive homeomorphisms $L_n\colon I \to I_n$ by $L_n(x) = a_n x + b_n$, where the real numbers a_n, b_n are chosen to ensure that $L_n(I) = I_n$. Let $\varphi\colon (0, +\infty) \to (0, +\infty)$ be a nondecreasing continuous function such that for any $t > 0$, $\alpha(t) = \varphi(t)/t < 1$ and the function $(0, +\infty) \ni t \mapsto \varphi(t)/t$ is nonincreasing.

Consider an IFS of the form $\{I \times \mathbb{R}; w_n, n = 1, 2, \ldots, N\}$ in which the maps are nonlinear transformations of the special structure

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_n(x) \\ F_n(x, y) \end{pmatrix} = \begin{pmatrix} a_n x + b_n \\ c_n x + s_n y + e_n \end{pmatrix},$$

where the transformations are constrained by the data according to

$$w_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \quad \text{and} \quad w_n \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

for $n = 1, 2, \ldots, N$, and s_n are some Rakotch contractions.

Let us denote by $C(I)$ the set of all real-valued continuous functions defined on I, i.e., $C(I) = \{f\colon I \to \mathbb{R} \mid f \text{ continuous}\}$ and by $C^*(I) \subset C(I)$ the set of continuous functions $f\colon I \to \mathbb{R}$ such that $f(x_0) = y_0$ and $f(x_N) = y_N$, i.e., $C^*(I) :=$

$\{f \in C(I) : f(x_0) = y_0, f(x_N) = y_N\}$. Let $C^{**}(I) \subset C^*(I) \subset C(I)$ be the set of continuous functions that pass through the given data points $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \ldots, N\}$, that is, $C^{**}(I) := \{f \in C^*(I) : f(x_i) = y_i, i = 0, 1, \ldots, N\}$. Define a metric $d_{C(I)}$ on $C(I)$ by $d_{C(I)}(g, h) := \max_{x \in [x_0, x_N]} |g(x) - h(x)|$ for all $g, h \in C(I)$. For all $f \in C^*(I)$, define a mapping $T : C^*(I) \to C(I)$ by

$$Tf(x) := F_n(L_n^{-1}(x), f(L_n^{-1}(x))) = c_n L_n^{-1}(x) + s_n(f(L_n^{-1}(x))) + e_n$$

for $x \in [x_{n-1}, x_n]$ and $n = 1, 2, \ldots, N$. Notice that the above equation can be rewritten as $Tf(L_n(x)) = s_n(f(x)) + c_n x + e_n$ for $x \in [x_0, x_N]$ and $n = 1, 2, \ldots, N$.

Theorem 2.2. *Let $\{I \times \mathbb{R}; w_n, n = 1, 2, \ldots, N\}$ denote the IFS defined above. Then, there is a unique continuous function $f : I \to \mathbb{R}$, which is a fixed point of T, such that $f(x_i) = y_i$ for $i = 0, 1, \ldots, N$. If $G \subset I \times \mathbb{R}$ is the graph of f, then*

$$G = \bigcup_{n=1}^{N} w_n(G).$$

Secondly, we give the definition of the KC, the harmonic functions on the KC and the fractal interpolation theorem for the KC. Let $V_0 := \{p_1 = (p_1^1, p_1^2) := (0, 0), p_2 = (p_2^1, p_2^2) := (1, 0)\}$ (see [3]). Let $K_0 := [0, 1] \times \{0\} \subset \mathbb{R}^2$ (see [7]). Consider for $i = 1, 2, 3, 4$, $u_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\begin{aligned}
u_1(x, y) &= (u_1^1(x, y), u_1^2(x, y)) := \left(\frac{x}{3}, \frac{y}{3}\right) = \frac{1}{3}((x, y) + (0, 0)),\\
u_2(x, y) &= (u_2^1(x, y), u_2^2(x, y)) := \left(\frac{x}{6} - \frac{\sqrt{3}y}{6} + \frac{1}{3}, \frac{\sqrt{3}x}{6} + \frac{y}{6}\right),\\
u_3(x, y) &= (u_3^1(x, y), u_3^2(x, y)) := \left(\frac{x}{6} + \frac{\sqrt{3}y}{6} + \frac{1}{2}, -\frac{\sqrt{3}x}{6} + \frac{y}{6} + \frac{\sqrt{3}}{6}\right),\\
u_4(x, y) &= (u_4^1(x, y), u_4^2(x, y)) := \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right) = \frac{1}{3}(x, y) + \left(\frac{2}{3}, 0\right).
\end{aligned}$$

Then for all $i = 1, 2, 3, 4$, $u_i : \mathbb{R}^2 \to \mathbb{R}^2$ are Banach contractions, since for all (x', y'), $(x'', y'') \in \mathbb{R}^2$,

$$\|u_i(x, y) - u_i(x', y')\|_{\mathbb{R}^2} = \frac{1}{3}\|(x, y) - (x', y')\|_{\mathbb{R}^2}.$$

Let $V_1 := u_1(V_0) \cup u_2(V_0) \cup u_3(V_0) \cup u_4(V_0) \subset \mathbb{R}^2$. Then

$$V_1 = \left\{(0, 0), \left(\frac{1}{3}, 0\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(\frac{2}{3}, 0\right), (1, 0)\right\}.$$

Also for all $n \geq 1$,

$$K_n := u_1(K_{n-1}) \cup u_2(K_{n-1}) \cup u_3(K_{n-1}) \cup u_4(K_{n-1}) \subset \mathbb{R}^2,$$

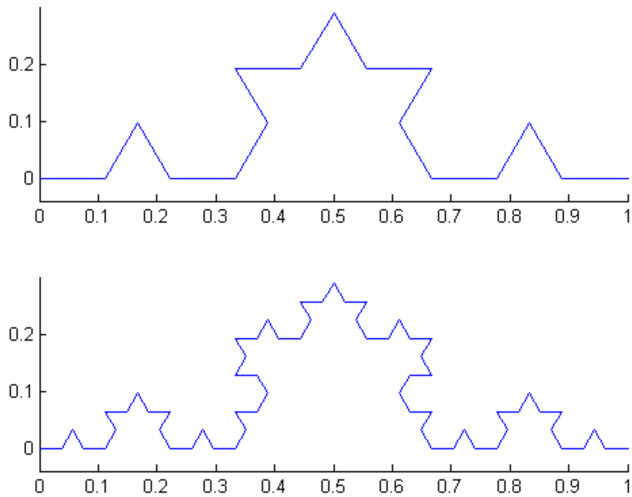

Figure 1. K_2 and K_3.

and $KC = \lim_{n\to\infty} K_n \subset \mathbb{R}^2$ (see [7]). So, $KC \subset \mathbb{R}^2$ is the attractor of $KC = u_1(KC) \cup u_2(KC) \cup u_3(KC) \cup u_4(KC)$. K_2 and K_3 are illustrated in Figure 1.

Fix a number $n \in \mathbb{N}$ and consider the iterations $u_w = u_{w_1} u_{w_2} \cdots u_{w_n} \colon \mathbb{R}^2 \to \mathbb{R}^2$ for any sequence $w = (w_1, w_2, \ldots, w_n) \in \{1, 2, 3, 4\}^n$. Let $V_n \subset \mathbb{R}^2$ be the union of the images of $V_0 \subset \mathbb{R}^2$ under these iterations. Given any function $h \colon \mathbb{R}^2 \supset V_n \to \mathbb{R}$, there is an operator H_n, defined by $H_n(h) \colon \mathbb{R}^2 \supset V_n \to \mathbb{R}$, where

$$H_n(h)(p) = \sum_{q \in N_{p,n}} (h(q) - h(p)),$$

and $N_{p,n} \subset \mathbb{R}^2$ denotes the “neighbourhood” of p in $V_n \subset \mathbb{R}^2$, the set of “next neighbours” of p in $V_n \subset \mathbb{R}^2$, two for $p \in V_n \backslash V_1 \subset \mathbb{R}^2$ and one or two for $p \in V_1 \subset \mathbb{R}^2$. Then $h \colon \mathbb{R}^2 \supset V_n \to \mathbb{R}$ is called *harmonic on* $V_n \subset \mathbb{R}^2$ if $H_n(h)(p) = 0$ for all $p \in V_n \backslash V_1 \subset \mathbb{R}^2$. A continuous function $h \colon \mathbb{R}^2 \supset KC \to \mathbb{R}$ is called *harmonic* if its restriction to $V_n \subset \mathbb{R}^2$ is harmonic for all $n \in \mathbb{N}$ (see [7]).

Lemma 2.3. *For two given numbers α and β, there exists a unique harmonic function h on the KC satisfying $h(p_1) = \alpha$ and $h(p_2) = \beta$.*

Proof. The proof is similar to that in [5] and it is thus omitted. ■

Let for $n \geq 1$, $v \colon \mathbb{R}^2 \supset V_n \to \mathbb{R}$ be any given function (not necessarily a harmonic function on $V_n \subset \mathbb{R}^2$). Let $h_w \colon \mathbb{R}^2 \supset KC \to \mathbb{R}$ be a harmonic function on the KC for $w \in \{1, 2, 3, 4\}^n$.

Theorem 2.4 (See [7, Theorem 1]). *For any given numbers* α_w $(w \in \{1,2,3,4\}^n)$ *with* $0 < |\alpha_w| < 1$, *there exists a unique continuous function* $f\colon \mathbb{R}^2 \supset KC \to \mathbb{R}$ *such that* $f|_{V_n} = v\colon \mathbb{R}^2 \supset V_n \to \mathbb{R}$ *and* $f(u_w(x,y)) = \alpha_w f(x,y) + h_w(x,y)$ *for* $(x,y) \in KC \subset \mathbb{R}^2$.

3. Main result

We assume that $V_0, V_n \subset \mathbb{R}^2$, $u_i\colon \mathbb{R}^2 \to \mathbb{R}^2$, $v\colon \mathbb{R}^2 \supset V_n \to \mathbb{R}$, $\varphi\colon (0,+\infty) \to (0,+\infty)$ and $h_w\colon \mathbb{R}^2 \supset KC \to \mathbb{R}$ keep their meaning from the previous section. Consider an IFS of the form $\{\mathbb{R}^2 \times \mathbb{R} \supset KC \times \mathbb{R}; w_i, i = 1,2,3,4\}$ in which the maps are transformations of the special structure

$$w_i \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_i^1(x,y) \\ u_i^2(x,y) \\ F_i(x,y,z) \end{pmatrix} = \begin{pmatrix} u_i^1(x,y) \\ u_i^2(x,y) \\ s_i(z) + h_i(x,y) \end{pmatrix},$$

where $(x,y) \in \mathbb{R}^2$, $z \in \mathbb{R}$ and $s_i\colon \mathbb{R} \to \mathbb{R}$ are Rakotch contractions (with the same function φ). We assume that for $i = 1,2,3,4$, the transformations are constrained by the data according to

$$w_i \begin{pmatrix} p_1^1 \\ p_1^2 \\ v(p_1^1, p_1^2) \end{pmatrix} = w_i \begin{pmatrix} 0 \\ 0 \\ v(0,0) \end{pmatrix} = \begin{pmatrix} u_i^1(0,0) \\ u_i^2(0,0) \\ v(u_i^1(0,0), u_i^2(0,0)) \end{pmatrix},$$

$$w_i \begin{pmatrix} p_2^1 \\ p_2^2 \\ v(p_2^1, p_2^2) \end{pmatrix} = w_i \begin{pmatrix} 1 \\ 0 \\ v(1,0) \end{pmatrix} = \begin{pmatrix} u_i^1(1,0) \\ u_i^2(1,0) \\ v(u_i^1(1,0), u_i^2(1,0)) \end{pmatrix},$$

that is, for $i = 1,2,3,4$,

$$\begin{aligned} & s_i(v(0,0)) + h_i(0,0) = v(u_i^1(0,0), u_i^2(0,0)) \\ & \qquad \Leftrightarrow \quad h_i(0,0) = v(u_i^1(0,0), u_i^2(0,0)) - s_i(v(0,0)), \end{aligned}$$

and

$$\begin{aligned} & s_i(v(1,0)) + h_i(1,0) = v(u_i^1(1,0), u_i^2(1,0)) \\ & \qquad \Leftrightarrow \quad h_i(1,0) = v(u_i^1(1,0), u_i^2(1,0)) - s_i(v(1,0)). \end{aligned}$$

Then, for all $(x,y,z'), (x,y,z'') \in KC \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$,

$$\begin{aligned} |F_i(x,y,z') - F_i(x,y,z'')| &= |s_i(z') - s_i(z'')| \\ &\le |s_i(z') - s_i(z'')| \le \varphi(|z' - z''|). \end{aligned}$$

That is, each w_i is chosen so that F_i is a Rakotch contraction with respect to the third variable.

Notice, that the function s_i is a substantial generalisation of the function $\alpha_i z$ $(0 \le \alpha_i < 1)$ in the FIF on the KC. Obviously, in [7], $s_i(z) = \alpha_i z$. Hence, each w_i is chosen so that F_i is a Banach contraction with respect to the third variable z. Also, for $i = 1, 2, 3, 4$,

$$\begin{aligned} \alpha_i v(0,0) + h_i(0,0) &= v(u_i^1(0,0), u_i^2(0,0)) \\ \Leftrightarrow \quad h_i(0,0) &= v(u_i^1(0,0), u_i^2(0,0)) - \alpha_i v(0,0), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \alpha_i v(1,0) + h_i(1,0) &= v(u_i^1(1,0), u_i^2(1,0)) \\ \Leftrightarrow \quad h_i(1,0) &= v(u_i^1(1,0), u_i^2(1,0)) - \alpha_i v(1,0). \end{aligned}$$

Secondly, by using Rakotch contractions, we give a FIF as the fixed point of certain Read–Bajraktarević operator (see [6]), and we show that the graph of that FIF is the invariant set of a certain IFS. Let us denote by $C(KC)$ the set of continuous functions $f: KC \to \mathbb{R}$. Let $C^*(KC) \subset C(KC)$ denote the set of continuous functions $f: KC \to \mathbb{R}$ such that $f(p_1) = v(p_1)$ and $f(p_2) = v(p_2)$, that is,

$$C^*(KC) := \{ f \in C(KC) : f(p) = v(p), p \in V_0 \}.$$

Let $C^{**}(KC) \subset C^*(KC) \subset C(KC)$ be the set of continuous functions that pass through the points $v(p)$ $(p \in V_n)$, that is,

$$C^{**}(KC) := \{ f \in C^*(KC) : f(p) = v(p), p \in V_n \}.$$

Define a metric $d_{C(KC)}$ on $C(KC)$ by

$$d_{C(KC)}(g,h) := \max_{(x,y) \in KC} |g(x,y) - h(x,y)|$$

for all $g, h \in C(KC)$. Then $(C(KC), d_{C(KC)})$, $(C^*(KC), d_{C(KC)})$ and $(C^{**}(KC), d_{C(KC)})$ are complete metric spaces. For all $f \in C^*(KC)$, define a mapping

$$T: C^*(KC) \to C(KC)$$

by

$$Tf(u_w(x,y)) := s_w(f(x,y)) + h_w(x,y)$$

for $(x,y) \in KC$, where h_w are harmonic functions on the KC for all $w \in \{1,2,3,4\}^n$. That is,

$$Tf(x,y) := s_w(f(u_w^{-1}(x,y))) + h_w(u_w^{-1}(x,y))$$

for $(x, y) \in u_w(KC)$ and $w \in \{1, 2, 3, 4\}^n$, where h_w are harmonic functions on the KC. The following lemma, similar to [9, Lemma 3.8], is the key element in the proof of Theorem 3.2.

Lemma 3.1. *$Tf \in C^{**}(KC)$ for all $f \in C^*(KC)$, that is,*

$$T: C^*(KC) \to C^{**}(KC).$$

Therefore, $T^n: C^{**}(KC) \to C^{**}(KC)$ for all $n \geq 2$.

Proof. Since $f \in C^*(KC)$, $(x, y) \in u_w(KC)$ and $w \in \{1, 2, 3, 4\}^n$,

$$Tf(x, y) = F_w(u_w^{-1}(x, y), f(u_w^{-1}(x, y))) = s_w(f(u_w^{-1}(x, y))) + h_w(u_w^{-1}(x, y))$$

and for all $i = 1, 2, 3$, $u_i(p_2) = u_{i+1}(p_1)$, we obtain that if $u_{i+1}(p_1) = u_i(p_2) \in [u_i(p_1), u_i(p_2)]$ for $i = 1, 2, 3$, then

$$\begin{aligned} Tf(u_{i+1}(p_1)) = Tf(u_i(p_2)) &= F_i(u_i^{-1}(u_i(p_2)), f(u_i^{-1}(u_i(p_2)))) \\ &= s_i(f(u_i^{-1}(u_i(p_2)))) + h_i(u_i^{-1}(u_i(p_2))) \\ &= s_i(f(p_2)) + h_i(p_2) = F_i(p_2, f(p_2)) \\ &= F_i(p_2, v(p_2)) = v(u_i(p_2)) = v(u_{i+1}(p_1)), \end{aligned}$$

and if $u_{i+1}(p_1) = u_i(p_2) \in [u_{i+1}(p_1), u_{i+1}(p_2)]$ for $i = 1, 2, 3$, then

$$\begin{aligned} Tf(u_i(p_2)) = Tf(u_{i+1}(p_1)) &= F_{i+1}(u_{i+1}^{-1}(u_{i+1}(p_1)), f(u_{i+1}^{-1}(u_{i+1}(p_1)))) \\ &= s_{i+1}(f(u_{i+1}^{-1}(u_{i+1}(p_1)))) + h_{i+1}(u_{i+1}^{-1}(u_{i+1}(p_1))) \\ &= s_{i+1}(f(p_1)) + h_{i+1}(p_1) = F_{i+1}(p_1, f(p_1)) \\ &= F_{i+1}(p_1, v(p_1)) = v(u_{i+1}(p_1)) = v(u_i(p_2)). \end{aligned}$$

So Tf is continuous at each of the points of $V_1 \backslash V_0$. Tf is continuous on the segment $[u_i(p_1), u_i(p_2)]$ for all $i = 1, 2, 3, 4$ by definition. Hence, by induction, $Tf \in C^{**}(KC)$ and $T^n: C^{**}(KC) \to C^{**}(KC)$ for all $n \geq 2$. ∎

Theorem 3.2. *Let $\{KC \times \mathbb{R}; w_i, i = 1, 2, 3, 4\}$ denote the IFS defined above, associated with the points $(p, v(p)) \in \mathbb{R}^3$, where $p \in V_n \subset \mathbb{R}^2$ and $n \geq 1$. Then the operator T is a Rakotch contraction (considered as a map $T: C^*(KC) \to C^*(KC)$). Hence there is a unique continuous function $f: KC \to \mathbb{R}$ which is a fixed point of T, that is,*

$$f(u_w(x, y)) = s_w(f(x, y)) + h_w(x, y)$$

for all $(x, y) \in KC$, where h_w are harmonic functions on the KC for all $w \in \{1, 2, 3, 4\}^n$. In particular, $f(p) = v(p)$ for all $p \in V_n$, that is, $f|_{V_n} = v$. Moreover, the graph G

of f is invariant with respect to $\{KC \times \mathbb{R}; w_i, i = 1,2,3,4\}$, i.e.,

$$G = \bigcup_{i=1}^{4} w_i(G).$$

Proof. For all $g, h \in C^*(KC)$,

$$\begin{aligned} d_{C(KC)}(Tg, Th) &= \max_{(x,y)\in KC} |Tg(x,y) - Th(x,y)| \\ &= \max_{i=1,2,3,4} \max_{(x,y)\in u_i(KC)} |Tg(x,y) - Th(x,y)| \\ &= \max_{i=1,2,3,4} \max_{(x,y)\in u_i(KC)} |s_i(g(u_i^{-1}(x,y))) - s_i(h(u_i^{-1}(x,y)))| \\ &\le \max_{i=1,2,3,4} \sup_{(x,y)\in u_i(KC)} \varphi(|g(u_i^{-1}(x,y)) - h(u_i^{-1}(x,y))|), \end{aligned}$$

where $\varphi : (0,+\infty) \to (0,+\infty)$ is some nondecreasing function such that $\varphi(t) < t$ for $t > 0$ and $t \mapsto \frac{\varphi(t)}{t}$ is nonincreasing. Since $\varphi : (0,+\infty) \to (0,+\infty)$ is a nondecreasing continuous function and $u_i^{-1} : u_i(KC) \to KC$ for all $i = 1,2,3,4$, we obtain that for $i_0 \in \{1,2,3,4\}$ and $(x_{i_0}, y_{i_0}) \in u_{i_0}(KC)$,

$$\begin{aligned} &\varphi\big(|g(u_{i_0}^{-1}(x_{i_0}, y_{i_0}) - h(u_{i_0}^{-1}(x_{i_0}, y_{i_0}))|\big) \\ &\quad\le \varphi\Big(\max_{(x,y)\in u_{i_0}(KC)} |g(u_{i_0}^{-1}(x,y)) - h(u_{i_0}^{-1}(x,y))|\Big) \\ &\quad\le \varphi\Big(\max_{(x,y)\in KC} |g(x,y) - h(x,y)|\Big) \\ &\quad= \varphi(d_{C(KC)}(g,h)). \end{aligned}$$

Since (x_{i_0}, y_{i_0}) was arbitrary,

$$\sup_{(x,y)\in u_{i_0}(KC)} \varphi\big(|g(u_{i_0}^{-1}(x,y)) - h(u_{i_0}^{-1}(x,y))|\big) \le \varphi(d_{C(KC)}(g,h)),$$

and, since i_0 was arbitrary,

$$\max_{i=1,2,3,4} \sup_{(x,y)\in u_i(KC)} \varphi\big(|g(u_i^{-1}(x,y)) - h(u_i^{-1}(x,y))|\big) \le \varphi(d_{C(KC)}(g,h)).$$

Hence, we obtain

$$\begin{aligned} d_{C(KC)}(Tg, Th) &\le \max_{i=1,2,3,4} \sup_{(x,y)\in u_i(KC)} \varphi\big(|g(u_i^{-1}(x,y)) - h(u_i^{-1}(x,y))|\big) \\ &\le \varphi(d_{C(KC)}(g,h)) = \varphi(d_{C(KC)}(g,h)). \end{aligned}$$

So we conclude that $T\colon C^*(KC) \to C^{**}(KC) \subset C^*(KC)$ is a Rakotch contraction (with the same function φ) on the complete metric space $(C^*(KC), d_{C(KC)})$. By

Theorem 2.1, T possesses a unique fixed point in $C^*(KC)$. That is, there exists a continuous function $f \in C^*(KC)$ such that for all $(x, y) \in KC$, $Tf(x, y) = f(x, y)$. Since $T : C^*(KC) \to C^{**}(KC)$ (by Lemma 3.1), we have $f = Tf \in C^{**}(KC)$. That is, there is a continuous function f that passes through the given points $(p, v(p)) \in \mathbb{R}^3$ $(p \in V_n \subset \mathbb{R}^2)$. By definition of T, we can see that

$$Tf(u_i(x, y)) = f(u_i(x, y)) = s_i(f(x, y)) + h_i(x, y)$$

for $x \in KC$, for $i = 1, 2, 3, 4$. Let G denote the graph of $f \in C^{**}(KC)$, that is, $G := \{(x, y, f(x, y)) : (x, y) \in KC\}$. Since f is a fixed point of the operator T and if $(x, y) \in u_i(KC)$, then

$$Tf(x, y) = F_i(u_i^{-1}(x, y), f(u_i^{-1}(x, y))),$$

we obtain that for all $x \in KC$,

$$\begin{aligned} f(u_i(x, y)) = Tf(u_i(x, y)) &= F_i(u_i^{-1}(u_i(x, y)), f(u_i^{-1}(u_i(x, y)))) \\ &= F_i(x, y, f(x, y)). \end{aligned}$$

Since $w_i(x, y, z) = (u_i(x, y), F_i(x, y, z))$ for all for $i = 1, 2, 3, 4$, we obtain that

$$\begin{aligned} w_i(G) &= w_i(\{(x, y, f(x, y)) : (x, y) \in KC\}) \\ &= \{w_i(x, y, f(x, y)) : (x, y) \in KC\} \\ &= \{(u_i(x, y), F_i(x, y, f(x, y))) : (x, y) \in KC\} \\ &= \{(u_i(x, y), f(u_i(x, y))) : (x, y) \in KC\} \\ &= \{(x, y, f(x, y)) : (x, y) \in u_i(KC)\}. \end{aligned}$$

Hence,

$$\begin{aligned} G &= \{(x, y, f(x, y)) : (x, y) \in KC\} \\ &= \bigcup_{i=1}^{4} \{(x, y, f(x, y)) : (x, y) \in u_i(KC)\} \\ &= \bigcup_{i=1}^{4} w_i(G). \end{aligned}$$

■

The function described in Theorem 3.2 generalises the FIF on the KC because in [7], $F_i(x, y, z) = d_i z + h_i(x, y)$ for all $i = 1, 2, 3, 4$, and so for all $t > 0$, $\varphi(t) := \max_{i=1,2,3,4} |d_i| t$, where $|d_i| < 1$ for all $i = 1, 2, 3, 4$. Hence, each function F_i is a Banach contraction with respect to the third variable z because for all (x, y, z'), $(x, y, z'') \in \mathbb{R}^3$,

$$|F_i(x, y, z') - F_i(x, y, z'')| \leq \max_{(x,y) \in KC} |d_i||z' - z''|.$$

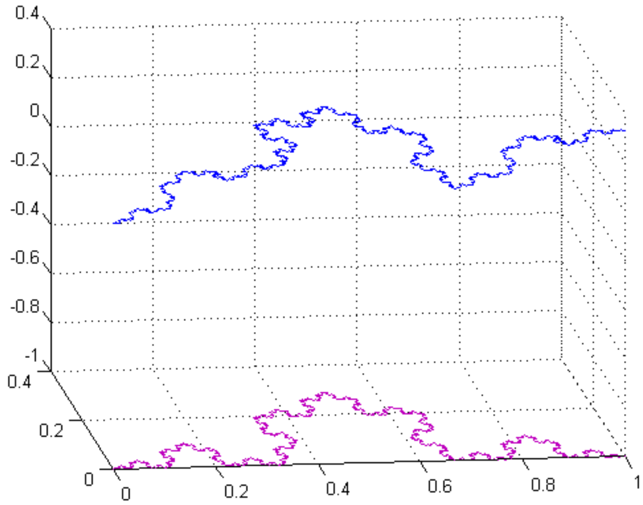

Figure 2. A nonlinear fractal interpolation function on the KC.

We now present an extremely explicit example to demonstrate the effectiveness of the obtained results. Let $\varphi(t) := \frac{t}{1+t}$ for $t \in (0, +\infty)$. Then $\varphi\colon (0, +\infty) \to (0, +\infty)$ is a nondecreasing continuous function and $t \mapsto \frac{\varphi(t)}{t}$ is a nonincreasing continuous function. Let points $(p, v(p)) \in \mathbb{R}^3$ $(p \in V_n \subset \mathbb{R}^2)$ be given. For $z \in [0, +\infty)$ and $i = 1, 2, 3, 4$, let

$$s_i(z) := \frac{z}{1 + iz}.$$

Then, for $z', z'' \in [0, +\infty)$,

$$\begin{aligned} |s_i(z') - s_i(z'')| &= |\frac{z'}{1 + iz'} - \frac{z''}{1 + iz''}| \leq \frac{|z' - z''|}{1 + i|z' - z''|} \\ &\leq \frac{|z' - z''|}{1 + |z' - z''|} = \varphi(|z' - z''|). \end{aligned}$$

That is, each s_i is a Rakotch contraction (with the same function φ) that is not a Banach contraction on $[0, +\infty)$. So, by Theorem 3.2, there exists a continuous function $f\colon KC \to \mathbb{R}$ that interpolates the given $(p, v(p)) \in \mathbb{R}^3$ $(p \in V_n \subset \mathbb{R}^2)$. A graph of a nonlinear fractal interpolation function that is a nonlinear fractal interpolation function on the KC is given in Figure 2.

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