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A closed graph theorem for hyperbolic iterated function systems

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Abstract. In this note, we introduce the notion of a morphism between two hyperbolic iterated function systems. We prove that the graph of such a morphism is the attractor of an iterated function system, giving a closed graph theorem, and demonstrate how it can be used to approach the topological conjugacy problem for iterated function systems.

1. Introduction

Since Hutchinson's seminal paper [7], iterated function systems have remained close to the heart of fractal geometry. Iterated function systems have continued to be studied and generalised in numerous directions.

In this article, we focus on the dynamics of hyperbolic iterated function systems from a topological viewpoint, rather than geometric or measure theoretic perspectives. Although we work in the hyperbolic setting, we take a viewpoint similar to that of topological iterated function systems [3, 8, 10]. Determining whether two iterated function systems are topologically conjugate (in the sense of Definition 3.1 below) is a subtle problem, as highlighted in Example 4.6. In contrast to symbolic dynamical systems, the topology of both the ambient space and the attractor play a greater role determining conjugacy for iterated function systems.

The main results of this article—Theorem 4.2 and Corollary 4.3—establish a closed graph theorem for an elementary notion of morphism between hyperbolic iterated function systems. Closed graph theorems are prevalent throughout mathematics with the most well-known results being those for continuous linear maps between Banach spaces, and for continuous maps between compact Hausdorff spaces. In the latter case, the closed graph theorem states that a function $f\colon X \to Y$ between compact Hausdorff spaces is continuous if and only if its graph

$$\mathrm{Gr}(f) = \big\{(x, f(x)) \bigm| x \in X\big\}$$

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is a closed subspace of $X \times Y$. An alternate formulation—which is perhaps more relevant to our context—is that $f: X \to Y$ is continuous if and only if $\mathrm{Gr}(f)$ is itself a compact Hausdorff space. Applying a set-theoretic lens to functions—where f is defined as its graph—one can interpret f itself as a compact Hausdorff space.

In Theorem 4.2, we show that the graph of a morphism between two iterated function systems is itself the attractor of an associated iterated function system. We show that by using this closed graph theorem, it is sometimes possible to deduce that two iterated function systems are not topologically conjugate. We also obtain a characterisation of the code map from a labelled Cantor space as the attractor of a certain iterated function system.

In contrast to some of the categorical approaches to self-similarity that have previously appeared—including Leinster's general categorical approach [12] and Sumi's interaction cohomology [15]—we study self-similarity via morphisms *between* iterated function systems, rather than focusing on individual self-similar sets or systems.

Finally, we remark that the results in this article build on work appearing in the author's Ph.D. dissertation [14].

2. Iterated function systems

Although many generalisations of iterated function systems exist in the literature, in this article we restrict our attention entirely to the setting of hyperbolic systems. That is, iterated function systems consisting of contractions.

Definition 2.1. A *(hyperbolic) iterated function system* (X, Γ) consists of a complete metric space (X, d) together with a finite collection Γ of proper contractions on X. That is, for each $\gamma \in \Gamma$, there exists $0 \leq c_\gamma < 1$ such that $d(\gamma(x), \gamma(y)) \leq c_\gamma d(x, y)$ for all $x, y \in X$.

Denote by $\mathcal{K}(X)$ the collection of non-empty compact subsets of X. Recall that the Hausdorff metric d_H on $\mathcal{K}(X)$ is defined by

$$d_H(A, B) = \max\left\{ \sup_{x \in A} \inf_{y \in B} d(x, y),\ \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}$$

for all $A, B \in \mathcal{K}(X)$. It is well known that if (X, d) is a complete metric space, then so is $(\mathcal{K}(X), d_H)$ (see, for example, [11, Proposition 1.1.5]).

Given an iterated function system (X, Γ), we abuse notation and also use Γ to denote the *Hutchinson operator* $\Gamma: \mathcal{K}(X) \to \mathcal{K}(X)$ defined, for all $K \in \mathcal{K}(X)$, by

$$\Gamma(K) = \bigcup_{\gamma \in \Gamma} \gamma(K).$$

Hutchinson [7] showed that Γ is a contraction on $(\mathcal{K}(X), d_H)$ and, consequently, has a unique fixed point $\mathbb{A} \in \mathcal{K}(X)$ by the contraction mapping principle. The fixed-point $\mathbb{A}$ is called the *attractor* of (X, Γ). In particular, $\Gamma(\mathbb{A}) = \mathbb{A}$ and for any $K \in \mathcal{K}(X)$, we have $d_H(\Gamma^k(K), \mathbb{A}) \to 0$ as $k \to \infty$. This result is collectively referred to as Hutchinson's theorem. Hutchinson's original result was stated for $X = \mathbb{R}^n$, however the proof of the result for general hyperbolic systems remains nearly identical (cf. [11, Theorem 1.1.7]).

We require the following result in the sequel.

Lemma 2.2. *Let (X, d) be a complete metric space. Suppose that $(K_i)_{i=1}^{\infty}$ is a sequence in $\mathcal{K}(X)$ such that $K_i \subseteq K_{i+1}$ for all $i \in \mathbb{N}$ and $d_H(K_i, K) \to 0$ as $i \to \infty$ for some $K \in \mathcal{K}(X)$. Then $K_i \subseteq K$ for all $i \in \mathbb{N}$. In particular, if (X, Γ) is a hyperbolic iterated function system with attractor $\mathbb{A}$ and $K \in \mathcal{K}(X)$ is such that $K \subseteq \Gamma(K)$, then $K \subseteq \mathbb{A}$.*

Proof. Suppose for contradiction that there exists $i \in \mathbb{N}$ and $x_0 \in K_i \setminus K$. Since $x_0 \in K_i \subseteq K_j$ for all $j \geq i$, it follows that

$$d_H(K_j, K) \geq \sup_{x \in K_j} \inf_{y \in K} d(x, y) \geq \inf_{y \in K} d(x_0, y) > 0,$$

where strict positivity is warranted by the compactness of K.

The second statement follows from the first for if $K \subseteq \Gamma(K)$, then

$$\Gamma^k(K) \subseteq \Gamma^{k+1}(K) \quad \text{for all } k \in \mathbb{N},$$

and Hutchinson's theorem implies that $d_H(\Gamma^k(K), \mathbb{A}) \to 0$ as $k \to \infty$. ∎

Remark 2.3. Lemma 2.2 is immediate if one recalls that a d_H-convergent increasing sequence of compact sets $(K_i)_{i \in \mathbb{N}}$ has limit $K = \overline{\bigcup_{i \in \mathbb{N}} K_i}$ (see [6, Proposition 1.3]).

Sets $K \in \mathcal{K}(X)$ satisfying $K \subseteq \Gamma(K)$ go by various names in the literature, including *sub-self-similar sets* [4] and *backward complete sets* [13].

3. Morphisms of iterated function systems

Topological conjugacy of iterated function systems is a well-established notion and the problem of determining whether two systems are topologically conjugate can be approached in numerous ways (see, for example, [9, Corollary 1.27]). When generalising conjugacy, there are choices to be made about selecting an appropriate notion of morphism, and several approaches have been taken previously. Kieninger, for instance, describes semiconjugacy of topological iterated function systems [10, Definition 4.6.3], which is analogous to the corresponding notion in symbolic dynamics.

Another approach is via the fractal homeomorphisms of Barnsley [2], which use shift-invariant sections of a code map.

Expanding on the established definition of conjugacy we use the following—somewhat naïve—notion of morphism, related to Kieninger's semiconjugacies.

Definition 3.1. A *(topological) morphism* from an iterated function system (X, Γ) to (Y, Λ) is a pair (f, α) consisting of a continuous map $f: X \to Y$ and a function $\alpha: \Gamma \to \Lambda$ such that for each $\gamma \in \Gamma$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X \\ f\big\downarrow & & \big\downarrow f \\ Y & \xrightarrow{\alpha(\gamma)} & Y \end{array}$$

commutes. We write $(f, \alpha): (X, \Gamma) \to (Y, \Lambda)$ to mean that (f, α) is a morphism from (X, Γ) to (Y, Λ). We mention some special types of morphism.

(1) (f, α) is an *embedding* if both f and α are injective.

(2) (f, α) is a *semiconjugacy* if both f and α are surjective.

(3) (f, α) is an *isomorphism* or *conjugacy* if f is a homeomorphism and α is bijective. In this case, we say that (X, Γ) is *isomorphic* or *conjugate* to (Y, Λ).

Morphisms may be composed by setting $(f, \alpha) \circ (g, \beta) = (f \circ g, \alpha \circ \beta)$.

In Definition 3.1, we have not made use of the metric space structure of either X or Y, and this has its drawbacks. In particular, if $(f, \alpha): (X, \Gamma) \to (Y, \Lambda)$ is a morphism, then it is not typically true that $(f(X), \alpha(\Gamma))$ is an iterated function system since $f(X)$ is not necessarily a complete metric space. This could be amended by insisting that f is also a closed map or that X is compact (for example, if X is the attractor itself), however, we do not require either assumption in what follows.

Morphisms in the sense of Definition 3.1 occur fairly naturally.

Example 3.2. Let $\Omega_N := \{1, \dots, N\}^{\mathbb{N}}$. For each $w \in \Omega_N$, we write $w = w_1 w_2 w_3 \cdots$, where each $w_i \in \{1, \dots, N\}$. Equip Ω_N with the metric $d: \Omega_N \times \Omega_N \to [0, \infty)$ defined by

$$d(w, v) = \begin{cases} 2^{1-\min\{k \mid w_k \neq v_k\}}, & \text{if } w \neq v, \\ 0, & \text{if } w = v, \end{cases}$$

for each $w, v \in \Omega_N$, so that Ω_N is a Cantor space. For each $1 \leq i \leq N$, consider the contraction $\sigma_i: \Omega_N \to \Omega_N$ defined by $\sigma_i(w_1 w_2 \cdots) = i w_1 w_2 \cdots$. Then $(\Omega_N, \Sigma_N := \{\sigma_1, \dots, \sigma_N\})$ is a hyperbolic iterated function system. Since Ω_N is invariant under the Hutchinson operator Σ_N, it is necessarily the attractor.

Now suppose that (X, Γ) is an iterated function system with attractor $\mathbb{A}$, and suppose that $L: \{1, \dots, N\} \to \Gamma$ is a bijective labelling of the maps in Γ. Denoting $L(i)$

by γ_i, there is a continuous surjection $\pi_L\colon \Omega_N \to \mathbb{A}$ called the *code map associated to the labelling* L which satisfies $\pi_L \circ \sigma_i = \gamma_i \circ \pi_L$ for all $1 \le i \le N$ [5, Theorem 3.2]. The code map may be defined explicitly by

$$\{\pi_L(w_1 w_2 \cdots)\} = \bigcap_{k=1}^{\infty} \gamma_{w_1} \circ \cdots \circ \gamma_{w_k}(\mathbb{A}).$$

Let $\beta_L\colon \Sigma_N \to \Gamma$ denote the bijection $\beta_L(\sigma_i) = \gamma_i$. Then $(\pi_L, \beta_L)\colon (\Omega_N, \Sigma_N) \to (\mathbb{A}, \Gamma)$ is a semiconjugacy.

Example 3.3. Subsystems of iterated function systems give examples of embeddings. Indeed, if (X, Λ) is an iterated function system and $\Gamma \subseteq \Lambda$, then the pair $(\mathrm{id}_X, \Gamma \hookrightarrow \Lambda)$ is an embedding.

Example 3.4. Consider the iterated function system $(\mathbb{R}, \Gamma = \{\gamma_1, \gamma_2\})$ with $\gamma_1(x) = \frac{x}{2}$ and $\gamma_2(x) = \frac{1+x}{2}$. Then $[0,1]$ is the attractor of $(\mathbb{R}, \Gamma)$. Now let $(\mathbb{R}^2, \Lambda = \{\lambda_1, \lambda_2, \lambda_3\})$ with $\lambda_1(x, y) = (\frac{x}{2}, \frac{y}{2})$, $\lambda_2(x, y) = (\frac{1+x}{2}, \frac{y}{2})$, and $\lambda_3(x, y) = (\frac{2x+1}{4}, \frac{2y+\sqrt{3}}{4})$. The attractor of $(\mathbb{R}^2, \Lambda)$ is a Sierpiński gasket with outer vertices at $(0,0)$, $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be the closed embedding given by $f(x) = (\frac{x}{2}, \frac{\sqrt{3}x}{2})$ and let $\alpha\colon \Gamma \to \Lambda$ be given by $\alpha(\gamma_1) = \lambda_1$ and $\alpha(\gamma_2) = \lambda_3$. Then $(f, \alpha)\colon (\mathbb{R}, \Gamma) \to (\mathbb{R}^2, \Lambda)$ is an embedding.

In the previous examples, α was an injection, but this is not always the case.

Example 3.5. Let (X, Γ) be an iterated function system and consider the product system $(X^2, \Gamma^2 = \{\gamma \times \gamma' \mid \gamma, \gamma' \in \Gamma\})$, where $\gamma \times \gamma'(x, y) = (\gamma(x), \gamma'(y))$ for $(x, y) \in X^2$. The pair $((x, y) \mapsto x, \gamma \times \gamma' \mapsto \gamma)$ defines a semiconjugacy.

Morphisms intertwine the induced dynamics on compact sets.

Lemma 3.6 (cf. [10, Proposition 4.6.4 (vi)]). *Let $(f, \alpha)\colon (X, \Gamma) \to (Y, \Lambda)$ be a morphism and consider the d_H-continuous map $\overline{f}\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ induced by f. Then $\overline{f}$ is a homomorphism between the single-map dynamical systems $(\mathcal{K}(X), \Gamma)$ and $(\mathcal{K}(Y), \alpha(\Gamma))$ in the sense that $\overline{f} \circ \Gamma = \alpha(\Gamma) \circ \overline{f}$. Here, $\alpha(\Gamma)$ is the Hutchinson operator for the system $(Y, \alpha(\Gamma))$.*

Proof. For $K \in \mathcal{K}(X)$, we simply compute

$$\overline{f}(\Gamma(K)) = \bigcup_{\gamma \in \Gamma} f \circ \gamma(K) = \bigcup_{\gamma \in \Gamma} \alpha(\gamma) \circ f(K) = \alpha(\Gamma)(\overline{f}(K)). \qquad \blacksquare$$

Remark 3.7. A morphism $(f, \alpha)\colon (X, \Gamma) \to (Y, \Lambda)$ is distinct from a homomorphism between the single-map systems $(\mathcal{K}(X), \Gamma)$ and $(\mathcal{K}(Y), \alpha(\Gamma))$. Indeed, if $X = Y$ and $\Gamma(K) = \Lambda(K)$ for all $K \in \mathcal{K}(X)$, then as a single-map system $(\mathcal{K}(X), \Gamma)$ is equal, not just conjugate, to $(\mathcal{K}(X), \Lambda)$.

For example, fix $\gamma \in \Gamma$ and let γ_0 be a distinct copy of γ. Form the disjoint union $\Gamma_0 = \Gamma \sqcup \{\gamma_0\}$ so that γ_0 is a "redundant" map. Then $(\mathrm{id}_X, \Gamma \hookrightarrow \Gamma_0)$ is a morphism that is not a conjugacy as $\Gamma \hookrightarrow \Gamma_0$ does not surject, but the single-map systems $(\mathcal{K}(X), \Gamma)$ and $(\mathcal{K}(X), \Gamma_0)$ are equal.

For hyperbolic systems, morphisms always map attractors to compact subsets of attractors so that the image is a subsystem of the codomain.

Lemma 3.8. *Let (X, Γ) and (Y, Λ) be iterated function systems with attractors $\mathbb{A}$ and $\mathbb{B}$, respectively. If $(f, \alpha)\colon (X, \Gamma) \to (Y, \Lambda)$ is a morphism, then $f(\mathbb{A}) \subseteq \mathbb{B}$. In particular, $(f(\mathbb{A}), \alpha(\Gamma))$ embeds in $(\mathbb{B}, \Lambda)$.*

Proof. Lemma 3.6 implies that

$$f(\mathbb{A}) = f(\Gamma(\mathbb{A})) = \alpha(\Gamma)(f(\mathbb{A})) \subseteq \Lambda(f(\mathbb{A})).$$

Since $f(\mathbb{A})$ is compact, it follows from Lemma 2.2 that $f(\mathbb{A}) \subseteq \mathbb{B}$. ∎

4. A closed graph theorem for morphisms

In this section we prove the main result of this article, a closed graph theorem for morphisms of hyperbolic iterated function systems. We show that, when restricted to attractors, the graph of a morphism is itself the attractor of an iterated function system. To this end, we introduce a fibred system.

Definition 4.1. Let (X, Γ) and (Y, Λ) be hyperbolic iterated function systems and suppose that $\alpha\colon \Gamma \to \Lambda$. For each $\gamma \in \Gamma$ define $\gamma^\alpha\colon X \times Y \to X \times Y$ by

$$\gamma^\alpha(x, y) = (\gamma(x), \alpha(\gamma)(y))$$

and let $\Gamma \times_\alpha \Lambda := \{\gamma^\alpha \mid \gamma \in \Gamma\}$. Equipping $X \times Y$ with the metric

$$d_\infty\big((x_1, y_1), (x_2, y_2)\big) = \max\big\{d_X(x_1, x_2), d_Y(y_1, y_2)\big\},$$

the pair $(X \times Y, \Gamma \times_\alpha \Lambda)$ is a hyperbolic iterated function system which we call the *system fibred over α*[2].

The system $(X \times Y, \Gamma \times_\alpha \Lambda)$ is fibred in the sense that $\Gamma \times_\alpha \Lambda$ is a fibre product in the category of sets and functions. Typically, $(X \times Y, \Gamma \times_\alpha \Lambda)$ itself is not a fibre product in a categorical sense, however the restriction of the system to its attractor is a fibre product in the category of iterated function systems with morphisms (see Corollary 4.9).

[2]Any metric for which $(X \times Y, \Gamma \times_\alpha \Lambda)$ is hyperbolic may be used.

In the case that $(X, \Gamma) = (\Omega_N, \Sigma_N)$ and $|\Lambda| = N$, the system fibred over an associated code map is referred to as the *lifted* system by Barnsley, who uses it to describe fractal tops [1, Definition 4.9.1]. We now come to the main result.

Theorem 4.2. *Let* (X, Γ) *and* (Y, Λ) *be hyperbolic iterated function systems with attractors* $\mathbb{A}$ *and* $\mathbb{B}$*, respectively. If* $(f, \alpha)\colon (X, \Gamma) \to (Y, \Lambda)$ *is a morphism, then the attractor of the system* $(X \times Y, \Gamma \times_\alpha \Lambda)$ *is the graph*

$$\mathrm{Gr}(f|_{\mathbb{A}}) := \{(x, f(x)) \in X \times Y \mid x \in \mathbb{A}\}$$

of f *restricted to* $\mathbb{A}$*. Moreover, suppose that* $\beta\colon \Gamma \to \Lambda$ *and* $\mathbb{D}$ *is the attractor of* $(X \times Y, \Gamma \times_\beta \Lambda)$*. Then there is a continuous function* $g\colon \mathbb{A} \to \mathbb{B}$ *making* $(g, \beta)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ *a morphism if and only if* $\mathbb{D}$ *is the graph of a function from* $\mathbb{A}$ *to* $\mathbb{B}$ *(continuity is automatic).*

Proof. Lemma 3.8 implies that $f(\mathbb{A}) \subseteq \mathbb{B}$, so $\mathrm{Gr}(f|_{\mathbb{A}})$ is a subset of the compact set $\mathbb{A} \times \mathbb{B}$. Since f is continuous, $\mathrm{Gr}(f|_{\mathbb{A}})$ is closed in $\mathbb{A} \times \mathbb{B}$, and therefore compact. The iterated function system $(X \times Y, \Gamma \times_\alpha \Lambda)$ has a unique attractor by Hutchinson's theorem. Hence, it suffices to show that

$$\mathrm{Gr}(f|_{\mathbb{A}}) = \bigcup_{\gamma \in \Gamma} \gamma^\alpha(\mathrm{Gr}(f|_{\mathbb{A}})).$$

Since $f \circ \gamma = \alpha(\gamma) \circ f$, it follows that if $(x, f(x)) \in \mathrm{Gr}(f|_{\mathbb{A}})$, then $\gamma^\alpha(x, f(x)) \in \mathrm{Gr}(f|_{\mathbb{A}})$ for all $\gamma \in \Gamma$. For the reverse inclusion, fix $(x, f(x)) \in \mathrm{Gr}(f|_{\mathbb{A}})$. Since $\mathbb{A} = \bigcup_{\gamma \in \Gamma} \gamma(\mathbb{A})$, for each $x \in \mathbb{A}$, there exists $\gamma \in \Gamma$ and $x_0 \in \mathbb{A}$ such that $x = \gamma(x_0)$. Then

$$\gamma^\alpha(x_0, f(x_0)) = (\gamma(x_0), \alpha(\gamma) \circ f(x_0)) = (x, f \circ \gamma(x_0)) = (x, f(x)),$$

so $(x, f(x)) \in \gamma^\alpha(\mathrm{Gr}(f|_{\mathbb{A}}))$. As such, $\mathrm{Gr}(f|_{\mathbb{A}})$ is the attractor of $(X \times Y, \Gamma \times_\alpha \Lambda)$.

The "only if" direction of the second statement follows from the first statement. For the "if" direction, suppose that $\mathbb{D}$ is the graph of a function $g\colon \mathbb{A} \to \mathbb{B}$. Since $\mathbb{D}$ is closed in $\mathbb{A} \times \mathbb{B}$, it follows from the closed graph theorem for compact Hausdorff spaces that g is continuous. If $(x, g(x)) \in \mathbb{D}$, then invariance under the Hutchinson operator implies that for each $\gamma \in \Gamma$, we have $(\gamma(x), \beta(\gamma) \circ g(x)) \in \mathbb{D}$. Since $\mathbb{D} = \mathrm{Gr}(g)$, it follows that $\beta(\gamma) \circ g(x) = g \circ \gamma(x)$. Consequently, $(g, \beta)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ is a morphism. ∎

Restricting to attractors yields the following closed graph theorem.

Corollary 4.3 (Closed graph theorem). *Let* $(\mathbb{A}, \Gamma)$ *and* $(\mathbb{B}, \Lambda)$ *be hyperbolic iterated function systems with attractors* $\mathbb{A}$ *and* $\mathbb{B}$*, respectively. Suppose* $f\colon \mathbb{A} \to \mathbb{B}$ *and* $\alpha\colon \Gamma \to \Lambda$*. Then* (f, α) *is a morphism if and only if* $\mathrm{Gr}(f)$ *is the attractor of* $(X \times Y, \Gamma \times_\alpha \Lambda)$*.*

Remark 4.4. Set-theoretically, a function *is* its graph. Consequently, Corollary 4.3 may be interpreted as saying that f *is* the attractor of an iterated function system.

Theorem 4.2 implies that morphisms between hyperbolic iterated function systems are rare.

Corollary 4.5 (Morphism rigidity). *Suppose* $(f, \alpha)\colon (X, \Gamma) \to (Y, \Lambda)$ *is a morphism between hyperbolic iterated function systems with attractors* $\mathbb{A}$ *and* $\mathbb{B}$*, respectively. Then the restricted morphism* $(f|_{\mathbb{A}}, \alpha)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ *is determined entirely by* α *in the sense that if* $(g, \alpha)\colon (X, \Gamma) \to (Y, \Lambda)$ *is another morphism, then* $f|_{\mathbb{A}} = g|_{\mathbb{A}}$.

Proof. Theorem 4.2 implies that the graphs $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$ are both attractors of $(X \times Y, \Gamma \times_\alpha \Lambda)$, so the uniqueness of the attractor gives the result. ■

In light of Corollary 4.5, we may use Theorem 4.2 to approach the problem of determining whether there exists a morphism between two hyperbolic iterated function systems. Since a morphism $(f, \alpha)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ is determined by α, to find such a morphism, it suffices to check whether the attractor of $(\mathbb{A} \times \mathbb{B}, \Gamma \times_\alpha \Lambda)$ is the graph of a function for each of the $|\Lambda|^{|\Gamma|}$ possible choices of maps α from Λ to Γ. Moreover, if $|\Gamma| = |\Lambda|$, we can determine whether a conjugacy exists by checking each of the $|\Gamma|!$ possible choices of bijections.

For concrete iterated function systems, the attractor of $(X \times Y, \Gamma \times_\alpha \Lambda)$ may be approximated numerically using the chaos game algorithm. This can be used to determine whether or not morphisms or conjugacies exist as seen in Example 4.6 below.

Example 4.6. Consider the unit interval [0, 1] with the Euclidean metric, and define iterated function systems $([0, 1], \Gamma = \{\gamma_1, \gamma_2\})$ and $([0, 1], \Lambda = \{\lambda_1, \lambda_2\})$, where

$$\gamma_1(x) = \frac{2x}{3}, \quad \gamma_2(x) = \frac{2x}{3} + \frac{1}{3}, \quad \lambda_1(x) = \frac{3x}{4}, \quad \text{and} \quad \lambda_2(x) = \frac{3x}{4} + \frac{1}{4}.$$

At a first glance, the systems $([0, 1], \Gamma)$ and $([0, 1], \Lambda)$ exhibit a similar behaviour. Both have attractor [0, 1], and the sets of overlap are closed intervals given by

$$\gamma_1([0, 1]) \cap \gamma_2([0, 1]) = \left[\frac{1}{3}, \frac{2}{3}\right] \quad \text{and} \quad \lambda_1([0, 1]) \cap \lambda_2([0, 1]) = \left[\frac{1}{4}, \frac{3}{4}\right],$$

respectively. It is natural to ask whether these two systems are conjugate.

In Figure 4.1, a chaos game approximation for the attractor $\mathbb{D}$ of $([0, 1]^2, \Gamma \times_\alpha \Lambda)$ is pictured for the bijection $\alpha\colon \Gamma \to \Lambda$ given by $\alpha(\gamma_1) = \lambda_1$ and $\alpha(\gamma_2) = \lambda_2$. The attractor of the other bijection is given by a horizontal reflection of Figure 4.1. The approximation makes it easy to see that the systems $([0, 1], \Gamma)$ and $([0, 1], \Lambda)$ are not conjugate as $\mathbb{D}$ is not the graph of a bijection.

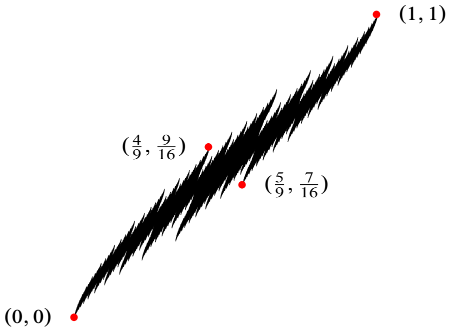


Figure 4.1. A chaos game approximation of the attractor $\mathbb{D}$ of the system $([0,1]^2, \Gamma \times_\alpha \Lambda)$ from Example 4.6.

Formalising the observation, $(0,0)$ and $(1,1)$ belong to $\mathbb{D}$ as the respective fixed points of γ_1^α and γ_2^α. Then $(\frac{5}{9}, \frac{7}{16}) = \gamma_2^\alpha \circ \gamma_2^\alpha(0,0)$ and $(\frac{4}{9}, \frac{9}{16}) = \gamma_1^\alpha \circ \gamma_1^\alpha(1,1)$ also belong to $\mathbb{D}$. If $\mathbb{D}$ were the graph of a (necessarily continuous) function f, then $f(0) = 0$, $f(1) = 1$, $f(\frac{4}{9}) = \frac{9}{16}$, and $f(\frac{5}{9}) = \frac{7}{16}$. The intermediate value theorem implies that f could not be injective.

Corollary 4.3 also provides an alternative definition of the code map.

Example 4.7. Consider an iterated function system $(\mathbb{A}, \Gamma)$ with attractor $\mathbb{A}$. Let $L\colon \{1, \dots, N\} \to \Gamma$ be a bijective labelling of the maps in Γ and let $\beta_L\colon \Sigma_N \mapsto \Gamma$ denote the induced map as in Example 3.2. Then the code map $\pi_L\colon \Omega_N \to \mathbb{A}$ is uniquely determined as the function whose graph is the attractor of the fibred system $(\Omega_N \times \mathbb{A}, \Sigma_N \times_{\beta_L} \Gamma)$.

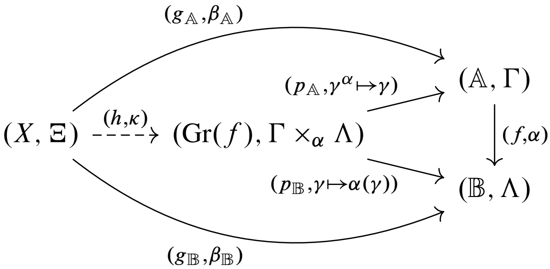
Restricting to attractors, the fibred system associated to a morphism is always conjugate to the domain.

Corollary 4.8. *Suppose that* $(f, \alpha)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ *is a morphism of hyperbolic iterated function systems with respective attractors* $\mathbb{A}$ *and* $\mathbb{B}$. *Then* $(\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$ *is conjugate to* $(\mathbb{A}, \Gamma)$.

Proof. Theorem 4.2 implies that $\mathrm{Gr}(f)$ is the attractor of $(\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$. Let $p\colon \mathrm{Gr}(f) \to \mathbb{A}$ denote the projection onto the first factor. Since $\mathrm{Gr}(f)$ is the graph of a function, p is a continuous bijection from a compact space to a Hausdorff space, and therefore a homeomorphism. It follows that $(p, \gamma^\alpha \mapsto \gamma)$ is the desired conjugacy. ∎

As a consequence of Corollary 4.8, the topological properties of the system $(\mathbb{A}, \Gamma)$ are shared by $(\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$. For example, the covering dimension of $\mathbb{A}$ is equal to the covering dimension of $\mathrm{Gr}(f)$. The relationship at the level of geometric properties is not so clear, because geometric properties of $\mathrm{Gr}(f)$ depend on a choice of metric on $\mathrm{Gr}(f)$. With Corollary 4.8, we obtain, somewhat trivially, the following universal property of the system $(\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$.

Corollary 4.9. *Let $(f, \alpha)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ be a morphism between hyperbolic iterated function systems with attractors $\mathbb{A}$ and $\mathbb{B}$, respectively. Let $p_\mathbb{A}\colon \mathrm{Gr}(f) \to \mathbb{A}$ and $p_\mathbb{B}\colon \mathrm{Gr}(f) \to \mathbb{B}$ be the projections onto the components of $\mathrm{Gr}(f)$. Then the iterated function system $(\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$ is universal in the following sense: if (X, Ξ) is another iterated function and $(g_\mathbb{A}, \beta_\mathbb{A})\colon (X, \Xi) \to (\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$ and $(g_\mathbb{B}, \beta_\mathbb{B})\colon (X, \Xi) \to (\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$ are morphisms such that $f \circ g_\mathbb{A} = g_\mathbb{B}$ and $\alpha \circ \beta_\mathbb{A} = \beta_\mathbb{B}$, then there exists a unique morphism $(h, \kappa)\colon (X, \Xi) \to (\mathrm{Gr}(f), \Gamma \times_\alpha \Lambda)$ witnessing commutativity of the diagram*



Proof. As $(p_\mathbb{A}, \gamma^\alpha \mapsto \gamma)$ is a conjugacy, it suffices to set $h = p_\mathbb{A}^{-1} \circ g_\mathbb{A}$ and $\kappa = (\gamma \mapsto \gamma^\alpha) \circ \beta_\mathbb{A}$. ■

We finish by using Corollary 4.3 to deduce that morphisms of iterated function systems lift uniquely to morphisms between code spaces.

Proposition 4.10. *Let $(\mathbb{A}, \Gamma = \{\gamma_1, \ldots, \gamma_N\})$ and $(\mathbb{B}, \Lambda = \{\lambda_1, \ldots, \lambda_M\})$ be hyperbolic iterated function systems with chosen labellings of the contractions, and attractors $\mathbb{A}$ and $\mathbb{B}$, respectively. Let $(\pi_\Gamma, \beta_\Gamma)\colon (\Omega_N, \Sigma_N) \to (\mathbb{A}, \Gamma)$ and $(\pi_\Lambda, \beta_\Lambda)\colon (\Omega_N, \Sigma_N) \to (\mathbb{B}, \Lambda)$ denote the morphisms induced by the corresponding code maps. For any morphism $(f, \alpha)\colon (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ there exists a unique morphism $(\widetilde{f}, \widetilde{\alpha})\colon (\Omega_N, \Sigma_N) \to (\Omega_M, \Sigma_M)$ making the diagram*

$$\begin{array}{ccc} (\Omega_N, \Sigma_N) & \xrightarrow{(\widetilde{f}, \widetilde{\alpha})} & (\Omega_M, \Sigma_M) \\ {\scriptstyle(\pi_\Gamma, \beta_\Gamma)}\big\downarrow & & \big\downarrow{\scriptstyle(\pi_\Lambda, \beta_\Lambda)} \\ (\mathbb{A}, \Gamma) & \xrightarrow{(f, \alpha)} & (\mathbb{B}, \Lambda) \end{array} \tag{4.1}$$

commute.

Proof. Since β_Λ is a bijection, we can define $\widetilde{\alpha} := \beta_\Lambda^{-1} \circ \alpha \circ \beta_\Gamma$. Then $\widetilde{\alpha}$ induces a function $h\colon \{1, \ldots, N\} \to \{1, \ldots, M\}$ defined by $\widetilde{\alpha}(\sigma_i) = \sigma_{h(i)}$ and h induces a continuous map $\widetilde{f}: \Omega_N \to \Omega_M$ defined by $\widetilde{f}(w_1 w_2 w_3 \cdots) = h(w_1)h(w_2)h(w_3)\cdots$ for all $w_k \in \{1, \ldots, N\}$. In particular, $\widetilde{f} \circ \sigma_i = \widetilde{\alpha}(\sigma_i) \circ \widetilde{f}$ for all $i \in \{1, \ldots, N\}$. The uniqueness of $\widetilde{f}$ follows from Corollary 4.3. ■

Remark 4.11. It is not the case that every morphism $(\widetilde{f}, \widetilde{\alpha}): (\Omega_N, \Sigma_N) \to (\Omega_M, \Sigma_M)$ descends to a morphism $(f, \alpha): (\mathbb{A}, \Gamma) \to (\mathbb{B}, \Lambda)$ making (4.1) commute. Indeed, if $M = N$, $(\mathbb{B}, \Lambda) = (\Omega_N, \Sigma_N)$, and $(\widetilde{f}, \widetilde{\alpha}) = (\pi_\Lambda, \beta_\Lambda) = (\mathrm{id}_{\Omega_N}, \mathrm{id}_{\widetilde{\Sigma_N}})$, then the commutativity of (4.1) would imply that such an f is an inverse for π_Γ—regardless of $\mathbb{A}$—which is absurd.

As a final remark, we note that the notion of morphism introduced in Definition 3.1 generalises readily to the topological iterated function systems of Kameyama [8] or Kieninger [10]. Although we do not pursue it here, the author believes that it would be interesting to see how the collection of invariant sets in the fibred system affects the existence of morphisms for more general topological iterated function systems.

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