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A fractal interpolation scheme for a possible sizeable set of data

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Abstract. In this paper, we propose a new fractal interpolation scheme. More precisely, we consider $a, b \in \mathbb{R}$, $a < b$, and $A \subseteq \mathbb{R}$ such that $\{a, b\} \subseteq A = \overline{A} \subseteq [a, b]$ and $\mathring{A} = \emptyset$ and prove that for every continuous function $f : A \to \mathbb{R}$, there exist a continuous function $g^* : [a, b] \to \mathbb{R}$ such that $g^*_{|A} = f$ and a possible infinite iterated function system whose attractor is the graph of g^*. If A is finite, we obtain the classical Barnsley's interpolation scheme and for $A = \{x_n \mid n \in \mathbb{N}\} \cup \{b\}$, where $x_1 = a$, $\lim_{n\to\infty} x_n = b$ and $x_n \in [a, b]$ for every $n \in \mathbb{N}$, we obtain a countable scheme due to N. Secelean. Our interpolation scheme permits A to be uncountable as it is the case for the Cantor ternary set.

1. Introduction

The aim of interpolation is to recover a function when some points of its graph are available. The usual interpolation techniques (using polynomial, exponential, rational, trigonometric or spline functions) yield piecewise differentiable interpolation functions, which are not appropriate for many real situations exhibiting a lack of smoothness in their traces. In order to deal even with interpolants which are not differentiable in a dense subset of their domain, M. Barnsley (see [1]) introduced the fractal interpolation functions which are more flexible for interpolation of irregular data, providing a large range of interpolants (from nowhere differentiable to infinitely differentiable ones). More precisely, given a finite subset A of $\mathbb{R}$ and a function $f : A \to \mathbb{R}$, he proved the existence of a continuous function $F : [\min A, \max A] \to \mathbb{R}$ having the following two properties:

(a) $F_{|A} = f$;

(b) there exists an iterated function system such that its attractor is the graph of F.

The function F is called a *fractal interpolation function* (FIF) corresponding to the set of data $\{(a, f(a)) \mid a \in A\}$.

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In order to get more flexible and diverse fractal interpolation functions, Barnsley's original concept has been generalized in different directions. Let us mention some of them:

(a) multivariable fractal interpolation functions, which are obtained via higher-dimensional or recurrent iterated function systems (see [3]);

(b) hidden-variable fractal interpolation functions involving the projection of the attractors of vector-valued iterated function systems to some lower-dimensional spaces (see [2, 6, 8, 15, 27]);

(c) Hermite or spline fractal interpolation functions (see [21, 29]);

(d) bilinear fractal interpolants, which are based on bilinear functions (see [5]);

(e) fractal splines which combine fractal functions and splines (see [4, 13]);

(f) fractal interpolation surfaces (see [7, 11, 14, 16, 31, 34]);

(g) generalization of Barnsley's fractal interpolation technique for a countable set of data (see [12, 23–26, 28]);

For very nice and useful expository accounts of fractal interpolation, one can consult [16, 20].

Fractal interpolation functions have applications in image compression (see [10]), structural mechanic (see [32]), image data reconstruction (see [9]), signal processing (see [18, 33]), theory of Schauder bases (see [19, 22]), etc.

In this paper, we extend Barnsley's fractal interpolation technique. We consider $a, b \in \mathbb{R}$, $a < b$, and $A \subseteq \mathbb{R}$ such that $\{a, b\} \subseteq A = \overline{A} \subseteq [a, b]$ and $\overset{\circ}{A} = \emptyset$. Our main result says that for every continuous function $f : A \to \mathbb{R}$, there exist a continuous function $g^* : [a, b] \to \mathbb{R}$ and a possible infinite iterated function system whose attractor is the graph of g^* (see Theorem 3.13) and such that $g^*_{|A} = f$ (see Remark 3.9), i.e., there exists a fractal interpolation function corresponding to the set of data $\{(a, f(a)) \mid a \in A\}$. If A is finite, we obtain the classical Barnsley's interpolation scheme (see [1]) and for $A = \{x_n \mid n \in \mathbb{N}\} \cup \{b\}$, where $x_1 = a$, $\lim_{n\to\infty} x_n = b$, and $x_n \in [a, b]$ for every $n \in \mathbb{N}$, we obtain the interpolation scheme presented in [25]. We stress the fact that our interpolation scheme permits A to be uncountable as the case of the Cantor ternary set shows.

Let us mention that the main tool used to overcome the difficulties concerning the step between countable and uncountable data is the theorem concerning the structure of the open subsets of $\mathbb{R}$. It provides a sequence $(I_n)_{n\in\mathbb{N}}$ of disjoint open intervals having the property that $[a, b] \setminus A = \bigcup_{n\in\mathbb{N}} I_n$. Then, via this sequence, we consider an operator $T : \mathcal{C}([a, b]) \to \mathcal{C}([a, b])$, where $\mathcal{C}([a, b]) = \{g : [a, b] \to \mathbb{R} \mid g$ is continuous, $g(a) = a$ and $g(b) = b\}$. The most difficult issue that arises in the framework of uncountable data is to prove that T is well defined, i.e., that $T(g) \in \mathcal{C}([a, b])$ for each $g \in \mathcal{C}([a,b])$ (see Proposition 3.7) because one has to overcome some technical obstacles.

Let us also mention that our main result, namely Theorem 3.13, provides a larger framework than the results existing in the literature. In particular, let us compare our interpolation scheme with the one presented in [30], by considering the set $A = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 2\} \cup \{0\} \subseteq [0,1]$ and the continuous function $f : A \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{\ln n}, & \text{if } x = \frac{1}{n}, \text{ where } n \in \mathbb{N}, n \geq 2, \\ 0, & \text{if } x = 0. \end{cases}$$

Theorem 3.13 ensures the existence of a possible infinite iterated function system $\mathcal{S}$ and of a function $g^* : [0,1] \to \mathbb{R}$ such that $g^*_{|A} = f$ and $G_{g^*} = A_{\mathcal{S}}$. We claim that it does not exist any iterated function system with variable parameters $\mathcal{S}'$ (see [30, pp. 3–4] for details concerning this type of systems) having the property that $G_{g^*} = A_{\mathcal{S}'}$. Indeed, if this is not the case, then, in view of [30, Corollary 3.1], there exist $M > 0$ and $\tau \in (0,1]$ such that $|g^*(x) - g^*(x')| \leq M|x - x'|^{\tau}$ for all $x, x' \in [0,1]$. In particular, we get $|g^*(\frac{1}{n}) - g^*(0)| \leq M\frac{1}{n^{\tau}}$, so $1 \leq M\frac{\ln n}{n^{\tau}}$ for every $n \in \mathbb{N}$, $n \geq 2$. By passing to the limit as n goes to ∞ in the previous inequality, we get the contradiction that $1 \leq 0$.

2. Preliminaries

In the sequel, $\mathbb{N}$ denotes the set $\{1, 2, \ldots\}$.

For a function $f : A \to B$, we use G_f to denote the graph of f, i.e., the set $\{(a, f(a)) \mid a \in A\}$.

For a function $f : X \to X$ and $n \in \mathbb{N}$, we denote the composition of f with itself n times by $f^{[n]}$.

For a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we shall use the following notation:

- $\operatorname{diam}(A) := \sup_{x,y \in A} d(x,y)$;
- $P_{\mathrm{b}}(X) := \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is bounded}\}$;
- $P_{\mathrm{cl}}(X) := \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is closed}\}$;
- $P_{\mathrm{b,cl}}(X) := P_{\mathrm{b}}(X) \cap P_{\mathrm{cl}}(X)$;
- $P_{\mathrm{cp}}(X) := \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\}$.

If, in addition, $A, B \in P_{\mathrm{b}}(X)$ and $x \in X$, we shall also use the following notation:

- $d(x, A) := \inf_{x \in A} d(x, a)$;
- $d(A, B) := \sup_{a \in A} d(a, B)$.

Definition 2.1. For a function $f : X \to X$, where (X, d) is a metric space,

$$\operatorname{lip}(f) := \sup_{x,y\in X; x\neq y} \frac{d\big(f(x), f(y)\big)}{d(x, y)} \in [0, +\infty]$$

is called the *Lipschitz constant* of f.

If $\operatorname{lip}(f) < +\infty$, then f is called *Lipschitz* and if $\operatorname{lip}(f) < 1$, then f is called a *contraction*.

Definition 2.2. Given a metric space (X, d), the function

$$h : P_{\mathrm{b,cl}}(X) \times P_{\mathrm{b,cl}}(X) \to [0, \infty)$$

described by

$$h(A, B) = \max\big\{d(A, B), d(B, A)\big\}$$

for every $A, B \in P_{\mathrm{b,cl}}(X)$, which turns out to be a metric, is called the *Hausdorff–Pompeiu metric* on X.

Definition 2.3. A *possible infinite iterated function system* (PIIFS) is a pair $\mathcal{S} := ((X, d), (f_i)_{i\in I})$, where:

- (X, d) is a complete metric space;
- $f_i : X \to X$ are contractions such that $\sup_{i\in I} \operatorname{lip}(f_i) < 1$;
- the family of functions $(f_i)_{i\in I}$ is bounded, i.e., $\bigcup_{i\in I} f_i(A) \in P_{\mathrm{b}}(X)$ for every $A \in P_{\mathrm{b}}(X)$.

One can associate to $\mathcal{S}$ the function $F_{\mathcal{S}} : P_{\mathrm{b,cl}}(X) \to P_{\mathrm{b,cl}}(X)$ called the *fractal operator* associated to $\mathcal{S}$, given by

$$F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$$

for all $B \in P_{\mathrm{b,cl}}(X)$.

Theorem 2.4 ([17, Theorem 2.2]). *For each $\mathcal{S} =((X, d), (f_i)_{i\in I})$, there exists a unique $A_{\mathcal{S}} \in P_{\mathrm{b,cl}}(X)$, called the* attractor *of $\mathcal{S}$, such that*

$$F_{\mathcal{S}}(A_{\mathcal{S}}) = A_{\mathcal{S}}.$$

In addition, we have

$$\lim_{n\to\infty} h\big(F_{\mathcal{S}}^{[n]}(B), A_{\mathcal{S}}\big) = 0$$

for every $B \in P_{\mathrm{b,cl}}(X)$.

3. The main result

Remark 3.1. Let us consider $a, b \in \mathbb{R}$, $a < b$, and $A \subseteq \mathbb{R}$ having the following properties:

(i) $\{a, b\} \subseteq A = \overline{A} \subseteq [a, b]$;

(ii) $\overset{\circ}{A} = \emptyset$.

Then there exists a sequence $(I_n)_{n\in\mathbb{N}}$ of disjoint open intervals such that

$$[a, b] \setminus A = \bigcup_{n\in\mathbb{N}} I_n,$$

where

$$I_n = (\alpha_n, \beta_n)$$

for every $n \in \mathbb{N}$.

Remark 3.2. (a) If $x \in A$ is not an accumulation point of $A \cap (x, \infty)$, then there exists $n \in \mathbb{N}$ such that

$$x = \alpha_n.$$

Indeed, there exists $\varepsilon > 0$ such that $(x, x + \varepsilon) \subseteq \bigcup_{n\in\mathbb{N}} I_n$ so, as $x \in A$, there exists $n \in \mathbb{N}$ such that $x = \alpha_n$.

(b) Similarly, if x is not an accumulation point of $A \cap (-\infty, x)$, then there exists $n \in \mathbb{N}$ such that

$$x = \beta_n.$$

Remark 3.3. (a) If $x \in A$ is an accumulation point of $A \cap (x, \infty)$, then for every sequence $(x_k)_{k\in\mathbb{N}} \subseteq (x, b) \setminus A$ having the property that $\lim_{k\to\infty} x_k = x$, there exists a sequence $((\alpha_{n_k}, \beta_{n_k}))_{k\in\mathbb{N}}$ of elements from the family $\{(\alpha_n, \beta_n) \mid n \in \mathbb{N}\}$ such that:

(i) $x < \alpha_{n_k} < \beta_{n_k}$ for every $k \in \mathbb{N}$;

(ii) $x_k \in (\alpha_{n_k}, \beta_{n_k})$ for every $k \in \mathbb{N}$;

(iii) the set $\{x_k \mid k \in \mathbb{N}\} \cap (\alpha_{n_k}, \beta_{n_k})$ is finite for every $k \in \mathbb{N}$;

(iv) $\lim_{k\to\infty} \alpha_{n_k} = \lim_{k\to\infty} \beta_{n_k} = x$.

Indeed, there exists a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that

$$\lim_{n\to\infty} a_n = x$$

and

$$x < a_{n+1} < a_n$$

for every $n \in \mathbb{N}$. As $(a_{n+1}, a_n) \setminus A$ is an at most countable union of intervals I_n, the conclusion follows.

(b) Similarly, one could prove that if $x \in A$ is an accumulation point of the set $A \cap (-\infty, x)$, then for every sequence $(x_k)_{k\in\mathbb{N}} \subseteq (a, x) \setminus A$ having the property that $\lim_{k\to\infty} x_k = x$, there exists a sequence $((\alpha_{n_k}, \beta_{n_k}))_{k\in\mathbb{N}}$ of elements from the family $\{(\alpha_n, \beta_n) \mid n \in \mathbb{N}\}$ such that:

(i) $\alpha_{n_k} < \beta_{n_k} < x$ for every $k \in \mathbb{N}$;

(ii) $x_k \in (\alpha_{n_k}, \beta_{n_k})$ for every $k \in \mathbb{N}$;

(iii) the set $\{x_k \mid k \in \mathbb{N}\} \cap (\alpha_{n_k}, \beta_{n_k})$ is finite for every $k \in \mathbb{N}$;

(iv) $\lim_{k\to\infty} \alpha_{n_k} = \lim_{k\to\infty} \beta_{n_k} = x$.

We consider the functions $l_n : [a, b] \to [\alpha_n, \beta_n]$ given by

$$l_n(x) = \frac{\beta_n - \alpha_n}{b-a}x + \frac{\alpha_n b - \beta_n a}{b-a}$$

for every $x \in [a, b]$ and every $n \in \mathbb{N}$. With the notation

$$a_n := \frac{\beta_n - \alpha_n}{b-a} \quad \text{and} \quad b_n := \frac{\alpha_n b - \beta_n a}{b-a},$$

we have

$$l_n(x) = a_n x + b_n$$

for every $x \in [a, b]$ and every $n \in \mathbb{N}$.

Remark 3.4. The function l_n has the following properties:

(a) $l_n(a) = \alpha_n$ *and* $l_n(b) = \beta_n$ for every $n \in \mathbb{N}$;

(b) $l_n^{-1} : [\alpha_n, \beta_n] \to [a, b]$ is given by

$$l_n^{-1}(x) = \frac{b-a}{\beta_n - \alpha_n}x + \frac{\beta_n a - \alpha_n b}{\beta_n - \alpha_n}$$

for every $x \in [\alpha_n, \beta_n]$ and every $n \in \mathbb{N}$;

(c) $l_n^{-1}(\alpha_n) = a$ and $l_n^{-1}(\beta_n) = b$ for every $n \in \mathbb{N}$.

For a continuous function $f : A \to \mathbb{R}$, one can consider the functions $g_n : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g_n(x, y) = \Big(\frac{f(\beta_n) - f(\alpha_n)}{b-a} - d_n \frac{f(b) - f(a)}{b-a}\Big)x + d_n y + \frac{bf(\alpha_n) - af(\beta_n)}{b-a} - d_n \frac{bf(a) - af(b)}{b-a}$$

for every $(x, y) \in \mathbb{R}^2$ and every $n \in \mathbb{N}$, where $(d_n)_{n\in\mathbb{N}} \subseteq [0, 1)$ is such that

$$\lim_{n\to\infty} d_n = 0.$$

With the notation

$$c_n := \frac{f(\beta_n) - f(\alpha_n)}{b-a} - d_n \frac{f(b) - f(a)}{b-a}$$

and

$$e_n := \frac{bf(\alpha_n) - af(\beta_n)}{b-a} - d_n \frac{bf(a) - af(b)}{b-a},$$

we have

$$g_n(x, y) = c_n x + d_n y + e_n$$

for every $(x, y) \in \mathbb{R}^2$ and every $n \in \mathbb{N}$.

Remark 3.5. The function g_n satisfies that

$$g_n(a, f(a)) = f(\alpha_n) \quad \text{and} \quad g_n(b, f(b)) = f(\beta_n)$$

for every $n \in \mathbb{N}$.

Let us consider

$$\mathcal{C}([a,b]) = \{g : [a,b] \to \mathbb{R} \mid g \text{ is continuous, } g(a) = f(a) \text{ and } g(b) = f(b)\}$$

and, for $g \in \mathcal{C}([a,b])$, consider the function $T_g : [a,b] \to \mathbb{R}$ given by

$$T_g(x) = \begin{cases} f(x), & \text{if } x \in A, \\ c_n l_n^{-1}(x) + d_n g(l_n^{-1}(x)) + e_n, & \text{if } x \in (\alpha_n, \beta_n). \end{cases}$$

Remark 3.6. (a) One has

$$T_g(\alpha_n) = c_n l_n^{-1}(\alpha_n) + d_n g(l_n^{-1}(\alpha_n)) + e_n$$

for every $g \in \mathcal{C}([a,b])$ and every $n \in \mathbb{N}$.

Indeed, we have

$$\begin{aligned} T_g(\alpha_n) &\overset{(1)}{=} f(\alpha_n) \overset{(2)}{=} g_n(a, f(a)) \overset{(3)}{=} g_n(a, g(a)) \\ &= c_n a + d_n g(a) + e_n \overset{(4)}{=} c_n l_n^{-1}(\alpha_n) + d_n g(l_n^{-1}(\alpha_n)) + e_n, \end{aligned}$$

where (1) holds because $\alpha_n \in A$, (2) is due to Remark 3.5, (3) uses that $g \in \mathcal{C}([a,b])$, and (4) is due to Remark 3.4 (c).

(b) In a similar way, one can prove that

$$T_g(\beta_n) = c_n l_n^{-1}(\beta_n) + d_n g(l_n^{-1}(\beta_n)) + e_n$$

for every $g \in \mathcal{C}([a,b])$ and every $n \in \mathbb{N}$.

Proposition 3.7. *In the above framework, we have*

$$T_g \in \mathcal{C}([a,b])$$

for every $g \in \mathcal{C}([a,b])$.

Proof. As $a, b \in A$, we have

$$T_g(a) = f(a) \quad \text{and} \quad T_g(b) = f(b)$$

by the definition of T_g.

From the definition of T_g, as l_n is a homeomorphism and g is continuous, we infer that T_g is continuous on $[a,b] \setminus A$.

Now let us consider $x_0 \in A$.

If x_0 is not an accumulation point of $A \cap (x_0, \infty)$, in view of Remark 3.2, there exists $n \in \mathbb{N}$ such that $x_0 = \alpha_n$. Consequently,

$$\begin{aligned}\lim_{x \to x_0^+} T_g(x) &\overset{(1)}{=} \lim_{x \to x_0^+} \left(c_n l_n^{-1}(x) + d_n g(l_n^{-1}(x)) + e_n\right) \\ &\overset{(2)}{=} c_n l_n^{-1}(\alpha_n) + d_n g(l_n^{-1}(\alpha_n)) + e_n \overset{(3)}{=} c_n a + d_n g(a) + e_n \\ &\overset{(4)}{=} c_n a + d_n f(a) + e_n = g_n(a, f(a)) \\ &\overset{(5)}{=} f(\alpha_n) = f(x_0) \overset{(6)}{=} T_g(x_0),\end{aligned}$$

where (1) follows from the definition of T_g, (2) uses the continuity of g and l_n^{-1}, (3) follows from Remark 3.4 (c), (4) uses the fact that $g \in \mathcal{C}([a,b])$, (5) follows from Remark 3.5, and (6) uses that $x_0 \in A$ together with the definition of T_g. So

$$\lim_{x \to x_0^+} T_g(x) = T_g(x_0). \tag{3.1}$$

In a similar manner, we get that if x_0 is not an accumulation point of $A \cap (-\infty, x_0)$, then

$$\lim_{x \to x_0^+} T_g(x) = T_g(x_0). \tag{3.2}$$

If $x_0 \in A$ is an accumulation point of $A \cap (x_0, \infty)$, let us consider a fixed but arbitrarily chosen sequence $(x_k)_{k \in \mathbb{N}} \subseteq (x_0, b]$ with the property that $\lim_{k\to\infty} x_k = x_0$.

We are going to prove that

$$\lim_{k \to \infty} T_g(x_k) = T_g(x_0). \tag{3.3}$$

If $\{k \in \mathbb{N} \mid x_k \notin A\}$ is finite, then, according to the definition of T_g, (3.3) takes the form $\lim_{k\to\infty} f(x_k) = f(x_0)$ whose validity is ensured by the continuity of f.

If $\{k \in \mathbb{N} \mid x_k \in A\}$ is finite, according to Remark 3.3, we can suppose that there exists a sequence $((\alpha_{n_k}, \beta_{n_k}))_{k\in\mathbb{N}}$ of elements from the family $\{(\alpha_n, \beta_n) \mid n \in \mathbb{N}\}$ such that:

(i) $x_0 < \alpha_{n_k} < \beta_{n_k}$ for every $k \in \mathbb{N}$;

(ii) $x_k \in (\alpha_{n_k}, \beta_{n_k})$ for every $k \in \mathbb{N}$;

(iii) the set $\{x_k \mid k \in \mathbb{N}\} \cap (\alpha_{n_k}, \beta_{n_k})$ is finite for every $k \in \mathbb{N}$;

(iv) $\lim_{k\to\infty} \alpha_{n_k} = \lim_{k\to\infty} \beta_{n_k} = x_0$.

Then, we have

$$\begin{aligned}
\left|T_g(x_k) - T_g(x_0)\right| &\overset{(1)}{=} \left|T_g(x_k) - f(x_0)\right| \\
&\leq \left|T_g(x_k) - T_g(\alpha_{n_k})\right| + \left|T_g(\alpha_{n_k}) - f(x_0)\right| \\
&\overset{(2)}{=} \left|T_g(x_k) - T_g(\alpha_{n_k})\right| + \left|f(\alpha_{n_k}) - f(x_0)\right| \\
&\overset{(3)}{=} \big|c_{n_k} l_{n_k}^{-1}(x_k) + d_{n_k} g(l_{n_k}^{-1}(x_k)) + e_{n_k} \\
&\qquad - (c_{n_k} a + d_{n_k} f(a) + e_{n_k})\big| + \left|f(\alpha_{n_k}) - f(x_0)\right| \\
&\leq |c_{n_k}|\left|l_{n_k}^{-1}(x_k) - a\right| + d_{n_k}\left|g(l_{n_k}^{-1}(x_k)) - f(a)\right| \\
&\qquad + \left|f(\alpha_{n_k}) - f(x_0)\right| \\
&\overset{(4)}{=} |c_{n_k}|\left|l_{n_k}^{-1}(x_k) - l_{n_k}^{-1}(\alpha_{n_k})\right| + d_{n_k}\left|g(l_{n_k}^{-1}(x_k)) - g(a)\right| \\
&\qquad + \left|f(\alpha_{n_k}) - f(x_0)\right| \\
&\leq 2(b-a)|c_{n_k}| + 2\operatorname{diam}(\operatorname{Im} g) d_{n_k} \\
&\qquad + \left|f(\alpha_{n_k}) - f(x_0)\right| \qquad (3.4)
\end{aligned}$$

for every $k \in \mathbb{N}$, where (1) follows from the fact that $x_0 \in A$ and the definition of T_g, (2) is due the fact that $\alpha_{n_k} \in A$ and the definition of T_g, (3) follows from Remark 3.6 and the defintion of T_g, and, finally, for (4), we refer to Remark 3.4 (c) and the fact that $g \in \mathcal{C}([a, b])$.

Note that as $\lim_{n\to\infty} d_n = 0$, f is continuous and $\lim_{k\to\infty} \alpha_{n_k} = \lim_{k\to\infty} \beta_{n_k} = x_0$, we infer that $\lim_{k\to\infty} c_{n_k} = 0$, so we get

$$\lim_{k\to\infty} \left(2(b-a)|c_{n_k}| + 2\operatorname{diam}(\operatorname{Im} g) d_{n_k} + \left|f(\alpha_{n_k}) - f(x_0)\right|\right) = 0$$

and therefore, via (3.4), (3.3) is valid.

If the sets $\{k \in \mathbb{N} \mid x_k \notin A\}$ and $\{k \in \mathbb{N} \mid x_k \in A\}$ are infinite, then there exist two subsequences $(u_n)_{n\in\mathbb{N}} \subseteq (x_0, b] \setminus A$ and $(v_n)_{n\in\mathbb{N}} \subseteq (x_0, b] \cap A$ of $(x_k)_{k\in\mathbb{N}}$ such that

$$\left\{x_k \mid k \in \mathbb{N}\right\} = \left\{u_n \mid n \in \mathbb{N}\right\} \cup \left\{v_n \mid n \in \mathbb{N}\right\}.$$

As the previous cases ensure that

$$\lim_{n\to\infty} T_g(u_n) = \lim_{n\to\infty} T_g(v_n) = T_g(x_0),$$

we infer that (3.3) is also valid in this case.

We conclude that if $x_0 \in A$ is an accumulation point of $A \cap (x_0, \infty)$, then

$$\lim_{x\to x_0^+} T_g(x) = T_g(x_0). \tag{3.5}$$

One can similarly prove that if x_0 is an accumulation point of $A \cap (-\infty, x_0)$, then

$$\lim_{x\to x_0^+} T_g(x) = T_g(x_0). \tag{3.6}$$

Relations (3.1), (3.2), (3.5) and (3.6) ensure that T_g is continuous. This concludes the proof of Proposition 3.7. ■

Proposition 3.7 allows us to define the operator $T : \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$ given by

$$T(g) = T_g$$

for every $g \in \mathcal{C}([a,b])$.

Proposition 3.8. *In the above mentioned framework, we have*

$$d_u\big(T(g_1), T(g_2)\big) \le \big(\sup_{n\in\mathbb{N}} d_n\big) d_u(g_1, g_2)$$

for all $g_1, g_2 \in \mathcal{C}([a,b])$, so T is a contraction with respect to the uniform metric d_u.

Proof. We have

$$\begin{aligned} d_u\big(T(g_1), T(g_2)\big) &= \sup_{x\in[a,b]} \big|T(g_1)(x) - T(g_2)(x)\big| \\ &\overset{(1)}{=} \sup_{x\in[a,b]\setminus A} \big|T(g_1)(x) - T(g_2)(x)\big| \\ &\overset{(2)}{=} \sup_{n\in\mathbb{N}, x\in(\alpha_n,\beta_n)} d_n\big|g_1(l_n^{-1}(x)) - g_2(l_n^{-1}(x))\big| \\ &\le \big(\sup_{n\in\mathbb{N}} d_n\big) \sup_{x\in[a,b]} \big|g_1(x) - g_2(x)\big| = \big(\sup_{n\in\mathbb{N}} d_n\big) d_u(g_1, g_2) \end{aligned}$$

for all $g_1, g_2 \in \mathcal{C}([a,b])$, where (1) and (2) follow from the definition of T. ■

As $(\mathcal{C}([a,b]), d_u)$ is a complete metric space, Proposition 3.8, via the Picard–Banach–Caccioppoli principle, ensures that there exists a unique $g^* \in \mathcal{C}([a,b])$ such that

$$T(g^*) = g^*.$$

Remark 3.9. $g^*_{|A} = f$.

We are going to prove that there exists a possible infinite iterated function system whose attractor is the graph of g^*.

Let us consider the functions $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_n(x, y) = (a_n x + b_n, c_n x + d_n y + e_n)$$

for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 3.10. (a) $\sup_{n\in\mathbb{N}} a_n < 1$.

Indeed, the set $\{n \in \mathbb{N} \mid a_n > \frac{1}{2}\}$ has at most one element since otherwise there exist $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$, such that $a_{n_1} > \frac{1}{2}$ and $a_{n_2} > \frac{1}{2}$ and we get $1 < a_{n_1} + a_{n_2}$, which is a contradiction: since $(\alpha_{n_1}, \beta_{n_1})$ and $(\alpha_{n_2}, \beta_{n_2})$ are disjoint intervals lying inside (a, b), we have $a_{n_1} + a_{n_2} < 1$. As $a_n < 1$ for every $n \in \mathbb{N}$, the conclusion follows.

(b) The sequence $(c_n)_{n\in\mathbb{N}}$ is bounded.

Indeed, this follows from the compactness of $f(A)$ (note that f is continuous and A is compact) and the boundedness of $(d_n)_{n\in\mathbb{N}}$.

Remark 3.10 allows us to consider

$$\theta \in \Big(0, \frac{1 - \sup_{n\in\mathbb{N}} a_n}{C}\Big),$$

with $C > 0$ being such that

$$|c_n| \leq C \quad \text{for every } n \in \mathbb{N},$$

and the metric ρ, on $\mathbb{R}^2$, given by

$$\rho\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + \theta\,|y_1 - y_2|$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Proposition 3.11. *In the above mentioned framework, the functions f_n are contractions with respect to the metric ρ.*

Proof. We have

$$\begin{aligned}
&\rho\big(f_n(x_1, y_1), f_n(x_2, y_2)\big)\\
&\quad = \rho\big((a_n x_1 + b_n, c_n x_1 + d_n y_1 + e_n), (a_n x_2 + b_n, c_n x_2 + d_n y_2 + e_n)\big)\\
&\quad = \big|(a_n x_1 + b_n) - (a_n x_2 + b_n)\big|\\
&\qquad + \theta\big|(c_n x_1 + d_n y_1 + e_n) - (c_n x_2 + d_n y_2 + e_n)\big|
\end{aligned}$$

$$
\begin{aligned}
&= a_n|x_1 - x_2| + \theta\big|c_n(x_1 - x_2) + d_n(y_1 - y_2)\big| \\
&\leq a_n\,|x_1 - x_2| + \theta\big(|c_n|\,|x_1 - x_2| + d_n\,|y_1 - y_2|\big) \\
&= \big(a_n + \theta|c_n|\big)\,|x_1 - x_2| + \theta d_n\,|y_1 - y_2| \\
&\leq \big(\sup_{n\in\mathbb{N}} a_n + \theta C\big)|x_1 - x_2| + \theta\big(\sup_{n\in\mathbb{N}} d_n\big)|y_1 - y_2| \\
&\leq \max\big\{\sup_{n\in\mathbb{N}} a_n + \theta C, \sup_{n\in\mathbb{N}} d_n\big\}\big(|x_1 - x_2| + \theta|y_1 - y_2|\big) \\
&= \max\big\{\sup_{n\in\mathbb{N}} a_n + \theta C, \sup_{n\in\mathbb{N}} d_n\big\}\rho\big((x_1, y_1), (x_2, y_2)\big)
\end{aligned}
$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $n \in \mathbb{N}$. As

$$
\max\big\{\sup_{n\in\mathbb{N}} a_n + \theta C, \sup_{n\in\mathbb{N}} d_n\big\} < 1,
$$

the proof is complete. ∎

Remark 3.12. (a) $(\mathbb{R}^2, \rho)$ is a complete metric space.

(b) One has

$$
\sup_{n\in\mathbb{N}} \operatorname{lip}(f_n) \leq \max\big\{\sup_{n\in\mathbb{N}} a_n + \theta C,\ \sup_{n\in\mathbb{N}} d_n\big\} < 1.
$$

(c) The family $(f_n)_{n\in\mathbb{N}}$ is bounded since the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ are bounded, as $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are bounded and f is bounded (as a continuous function on a compact subset of $\mathbb{R}$).

In view of Remark 3.12, one can consider the PIIFS $\mathcal{S} = ((\mathbb{R}^2, \rho), (f_n)_{n\in\mathbb{N}})$. Let us also consider

$$
G_{g^*} = \big\{(x, g^*(x)) \,\big|\, x \in [a, b]\big\} =: G.
$$

Theorem 3.13. *In the above framework, we have*

$$
G = A_{\mathcal{S}}.
$$

Proof. As g^* is continuous and $[a, b]$ is compact, we infer that

$$
G \in P_{\mathrm{cp}}(\mathbb{R}^2) \subseteq P_{\mathrm{b,cl}}(\mathbb{R}^2). \tag{3.7}
$$

We have

$$
\begin{aligned}
f_n\big((x, g^*(x))\big) &= \big(a_n x + b_n, c_n x + d_n g^*(x) + e_n\big) \\
&= \big(a_n x + b_n, c_n l_n^{-1}(l_n(x)) + d_n g^*(l_n^{-1}(l_n(x))) + e_n\big) \\
&\overset{(1)}{=} \big(a_n x + b_n, T(g^*)(l_n(x))\big) \\
&\overset{(2)}{=} \big(l_n(x), g^*(l_n(x))\big) \in G
\end{aligned}
$$

for all $x \in [a,b]$ and $n \in \mathbb{N}$, where (1) follows from the definition of $T(g^*)$ and Remark 3.6, and (2) uses that $T(g^*) = g^*$. So

$$\bigcup_{n\in\mathbb{N}} f_n(G) \subseteq G.$$

Consequently, we get

$$F_{\mathcal{S}}(G) = \overline{\bigcup_{n\in\mathbb{N}} f_n(G)} \subseteq \overline{G} \overset{(1)}{=} G, \tag{3.8}$$

where (1) is by (3.7).

If $x \in [a,b] \setminus A = \bigcup_{n\in\mathbb{N}}(\alpha_n, \beta_n)$, then there exists $n \in \mathbb{N}$ such that $x \in (\alpha_n, \beta_n)$ and therefore

$$\begin{aligned}(x, g^*(x)) &= \big(l_n(l_n^{-1}(x)), g^*(l_n(l_n^{-1}(x)))\big)\\ &\overset{(1)}{=} \big(l_n(l_n^{-1}(x)), T(g^*)(l_n(l_n^{-1}(x)))\big)\\ &\overset{(2)}{=} \big(l_n(l_n^{-1}(x)), c_n l_n^{-1}(l_n(l_n^{-1}(x))) + d_n g^*(l_n^{-1}(l_n(l_n^{-1}(x)))) + e_n\big)\\ &= \big(a_n l_n^{-1}(x) + b_n, c_n l_n^{-1}(x) + d_n g^*(l_n^{-1}(x)) + e_n\big)\\ &\overset{(3)}{=} f_n\big(l_n^{-1}(x), g^*(l_n^{-1}(x))\big)\\ &\in f_n(G) \subseteq \bigcup_{n\in\mathbb{N}} f_n(G) \subseteq \overline{\bigcup_{n\in\mathbb{N}} f_n(G)} = F_{\mathcal{S}}(G),\end{aligned} \tag{3.9}$$

where (1) is due to the fact that $T(g^*) = g^*$, (2) follows from the definition of $T(g^*)$, and (3) follows from the definition of f_n.

If $x \in A$ is not an accumulation point of $A \cap (x, \infty)$, then taking into account Remark 3.2, there exists $n \in \mathbb{N}$ such that $x = \alpha_n$ and therefore

$$\begin{aligned}(x, g^*(x)) = (\alpha_n, g^*(\alpha_n)) &\overset{(1)}{=} \big(\alpha_n, T(g^*)(\alpha_n)\big) \overset{(2)}{=} (\alpha_n, f(\alpha_n))\\ &\overset{(3)}{=} \big(l_n(a), g_n(a, f(a))\big) = \big(a_n a + b_n, c_n a + d_n f(a) + e_n\big)\\ &= f_n(a, f(a)) \overset{(4)}{=} f_n(a, g^*(a))\\ &\in f_n(G) \subseteq \bigcup_{n\in\mathbb{N}} f_n(G) \subseteq \overline{\bigcup_{n\in\mathbb{N}} f_n(G)} = F_{\mathcal{S}}(G),\end{aligned} \tag{3.10}$$

where (1) follows from the fact that $T(g^*) = g^*$, (2) uses that $\alpha_n \in A$, (3) follows from Remark 3.5, and, finally, (4) uses that $g^* \in \mathcal{C}([a,b])$.

In a similar way, one can prove that if $x \in A$ is not an accumulation point of $A \cap (-\infty, x)$, then

$$(x, g^*(x)) \in F_{\mathcal{S}}(G). \tag{3.11}$$

If $x \in A$ is an accumulation point of $A \cap (x, \infty)$, in view of Remark 3.3, there exists a sequence $((\alpha_{n_k}, \beta_{n_k}))_{k \in \mathbb{N}}$ of elements from the family $\{(\alpha_n, \beta_n) \mid n \in \mathbb{N}\}$ such that

$$\lim_{k \to \infty} \alpha_{n_k} = x.$$

Consequently,

$$(x, g^*(x)) \overset{(1)}{=} \lim_{k \to \infty} \left(\alpha_{n_k}, g^*(\alpha_{n_k})\right) \overset{(2)}{\in} \overline{F_{\mathcal{S}}(G)} = F_{\mathcal{S}}(G), \tag{3.12}$$

where (1) follows from the fact that g^* is continuous, and (2) is by (3.10).

In a similar manner, one can prove that if $x \in A$ is an accumulation point of $A \cap (-\infty, x)$, then

$$(x, g^*(x)) \in F_{\mathcal{S}}(G). \tag{3.13}$$

Relations (3.9)–(3.13) ensure that

$$G \subseteq F_{\mathcal{S}}(G). \tag{3.14}$$

From (3.8) and (3.14), we conclude that

$$F_{\mathcal{S}}(G) = G. \tag{3.15}$$

From (3.7), (3.15) and the uniqueness of the attractor of $\mathcal{S}$ (see Theorem 2.4), we conclude that

$$A_{\mathcal{S}} = G_{g^*}.$$ ∎

Let us summarize the above results.

We considered $a, b \in \mathbb{R}$, $a < b$ and $A \subseteq \mathbb{R}$ such that $\{a, b\} \subseteq A = \overline{A} \subseteq [a, b]$, $\overset{\circ}{A} = \emptyset$ and a continuous function $f : A \to \mathbb{R}$. We proved that there exists a fractal interpolation function corresponding to the set of data $\{(a, f(a)) \mid a \in A\}$. More precisely, we prove that there exist a continuous function $g^* : [a, b] \to \mathbb{R}$ and a possible infinite iterated function system $\mathcal{S} = ((\mathbb{R}^2, \rho), (f_n)_{n \in \mathbb{N}})$ having the following properties:

(a) $g^*_{|A} = f$;

(b) $G_{g^*} = A_{\mathcal{S}}$.

Let us present some examples of sets A satisfying the above mentioned conditions.

If A is finite, we obtain the classical Barnsley's interpolation scheme (see [1]).

For

$$A = \left\{x_n \;\middle|\; n \in \mathbb{N}\right\} \cup \{b\},$$

where $x_1 = a$, $\lim_{n \to \infty} x_n = b$, and $x_n \in [a, b]$ for every $n \in \mathbb{N}$; we obtain the interpolation scheme presented in [25].

We can also choose

$$A = \{x_n \mid n \in \mathbb{N}\} \cup \{y_n \mid n \in \mathbb{N}\} \cup \{a, b\},$$

where $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, $x_n \in [a, \frac{a+b}{2}]$, and $y_n \in [\frac{a+b}{2}, b]$ for every $n \in \mathbb{N}$.

An important example is to consider A to be the Cantor ternary set which is not countable. Hence our scheme is a genuine generalization of the one presented in [25].

Finally, we present a result, namely Theorem 3.15, showing that, for every $g \in \mathcal{C}([a,b])$, we can approximate, as close as we want, the graph of the interpolation function by the graph of $T^{[n]}(g)$, if n is big enough.

Theorem 3.14. *In the above framework, we have*

$$F_S(G_g) = G_{T(g)}$$

for every $g \in \mathcal{C}([a,b])$.

Proof. Let g be an arbitrarily chosen, but fixed, element of $\mathcal{C}([a,b])$.

Let us note that since g is continuous,

$$G_g \in P_{\mathrm{cp}}(\mathbb{R}^2) \subseteq P_{\mathrm{b,cl}}(\mathbb{R}^2).$$

The relation

$$F_S(G_g) \subseteq G_{T(g)} \tag{3.16}$$

could be proven as in [23, Theorem 6].

The relation

$$\big(x, T(g)(x)\big) \in F_S(G_g) \tag{3.17}$$

for every $x \in [a,b] \setminus A$ could also be proven as in [23, Theorem 6].

If $x \in A$ is not an accumulation point of $A \cap (x, \infty)$, in view of Remark 3.2, there exists $n \in \mathbb{N}$ such that $x = \alpha_n$ and we have

$$\begin{aligned} (x, T(g)(x)) &= \big(\alpha_n, T(g)(\alpha_n)\big) \\ &\overset{(1)}{=} \big(a_n a + b_n, c_n l_n^{-1}(\alpha_n) + d_n g(l_n^{-1}(\alpha_n)) + e_n\big) \\ &\overset{(2)}{=} \big(a_n a + b_n, c_n a + d_n g(a) + e_n\big) = f_n(a, g(a)) \\ &\in f_n(G_g) \subseteq \bigcup_{n\in\mathbb{N}} f_n(G_g) \subseteq \overline{\bigcup_{n\in\mathbb{N}} f_n(G_g)} = F_S(G_g), \end{aligned} \tag{3.18}$$

where (1) follows from Remarks 3.4 (a) and 3.6, and (2) follows from Remark 3.4 (c).

If $x \in A$ is not an accumulation point of $A \cap (-\infty, x)$, one can prove in a similar manner that

$$\big(x, T(g)(x)\big) \in F_S(G_g). \tag{3.19}$$

If $x \in A$ is an accumulation point of $A \cap (x, \infty)$, in view of Remark 3.3, there exists a sequence $((\alpha_{n_k}, \beta_{n_k}))_{k\in\mathbb{N}}$ of elements from the family $\{(\alpha_n, \beta_n) \mid n \in \mathbb{N}\}$ such that $\lim_{k\to\infty} \alpha_{n_k} = x$ and therefore

$$\big(x, T(g)(x)\big) \overset{(1)}{=} \lim_{k\to\infty} \big(\alpha_{n_k}, T(g)(\alpha_{n_k})\big) \overset{(2)}{\in} \overline{F_S(G_g)} = F_S(G_g), \tag{3.20}$$

where (1) uses that $T(g)$ is continuous, and (2) is by (3.18).

In a similar way, one can prove that

$$\big(x, T(g)(x)\big) \in F_S(G_g), \tag{3.21}$$

if $x \in A$ is an accumulation point of $A \cap (-\infty, x)$.

From (3.17)–(3.21), we deduce that

$$G_{T(g)} \subseteq F_S(G_g). \tag{3.22}$$

Relations (3.16) and (3.22) complete the proof. ∎

Theorem 3.15. *In the above framework, we have*

$$\lim_{n\to\infty} h\big(G_{T^{[n]}(g)}, G\big) = 0$$

for every $g \in \mathcal{C}([a, b])$.

Proof. Note that

$$G_g \in P_{\mathrm{cp}}(\mathbb{R}^2) \subseteq P_{\mathrm{b,cl}}(\mathbb{R}^2). \tag{3.23}$$

By mathematical induction, one can prove that

$$F_S^{[n]}(G_g) = G_{T^{[n]}(g)} \tag{3.24}$$

for every $g \in \mathcal{C}([a, b])$ and every $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} h\big(G_{T^{[n]}(g)}, G\big) \overset{(1)}{=} \lim_{n\to\infty} h\big(F_S^{[n]}(G_g), A_{\mathcal{S}}\big) \overset{(2)}{=} 0,$$

where (1) is by (3.24) and Theorem 3.13, and (2) is by (3.23) and Theorem 2.4. ∎

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