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Asymptotic solution of Bowen equation for perturbed potentials on shift spaces with countable states

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Abstract. In this paper, we study the asymptotic expansions for the zero of the pressure function $s \mapsto P(s\varphi(\varepsilon,\cdot) + \xi(\varepsilon,\cdot))$ for perturbed potentials $\varphi(\varepsilon,\cdot)$ and $\xi(\varepsilon,\cdot)$ defined on the shift space with countable state space. In our main result, we give a sufficient condition for the solution $s = s(\varepsilon)$ of $P(s\varphi(\varepsilon,\cdot) + \xi(\varepsilon,\cdot)) = 0$ to have the n-order asymptotic expansion for the small parameter ε. In addition, we also obtain the case where the order of the expansion of the solution $s = s(\varepsilon)$ is less than the order of the expansion of the perturbed potentials. Our results can be applied to problems concerning asymptotic behaviours of Hausdorff dimensions given by the Bowen formula: conformal graph directed Markov systems and other concrete examples.

Contents

1. Introduction and Main results

Let E^∞ be a shift space with countable (finite or infinite) state, and $g(\varepsilon,\cdot)$ and $\psi(\varepsilon,\cdot)$ be two real-valued functions defined on E^∞ with $\psi(\varepsilon,\cdot) > 0$ and with a small parameter $\varepsilon > 0$. We study the asymptotic solution of the pressure function

$$s \mapsto P(s\varphi(\varepsilon,\cdot) + \xi(\varepsilon,\cdot))$$

with the perturbed potentials $\varphi(\varepsilon,\cdot) := \log|g(\varepsilon,\cdot)|$ and $\xi(\varepsilon,\cdot) := \log\psi(\varepsilon,\cdot)$, where P is the topological pressure defined in (A.1).

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The main result of this paper is the following: we show that if a shift space E^∞ is finitely irreducible, and the real-valued functions $g(\varepsilon, \cdot)$ and $\psi(\varepsilon, \cdot)$ on E^∞ have asymptotic expansions

$$g(\varepsilon, \cdot) = g + g_1\varepsilon + \cdots + g_n\varepsilon^n + \tilde{g}_n(\varepsilon, \cdot)\varepsilon^n \tag{1.1}$$

$$\psi(\varepsilon, \cdot) = \psi + \psi_1\varepsilon + \cdots + \psi_n\varepsilon^n + \tilde{\psi}_n(\varepsilon, \cdot)\varepsilon^n \tag{1.2}$$

with $\|\tilde{g}_n(\varepsilon, \cdot)\|_\infty \to 0$, $\psi > 0$ and $\|\tilde{\psi}_n(\varepsilon, \cdot)\|_\infty \to 0$ as $\varepsilon \to 0$, and suitable conditions are satisfied, then the solution $s = s(\varepsilon)$ of the equation

$$P(s \log |g(\varepsilon, \cdot)| + \log \psi(\varepsilon, \cdot)) = p_0$$

with a fixed real number p_0 has an n-order asymptotic behaviour

$$s(\varepsilon) = s_0 + s_1\varepsilon + \cdots + s_n\varepsilon^n + \tilde{s}_n(\varepsilon)\varepsilon^n \tag{1.3}$$

with $\tilde{s}_n(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see Theorem 1.1), where $P(f)$ is the topological pressure of f defined in (A.1). In particular, each coefficient s_k and the small order part $\tilde{s}_n(\varepsilon)$ are explicitly determined (see (3.17), (3.18), and (3.19)).

In application, this result can be applied directly to the asymptotic behaviour of the dimension obtained by a Bowen type formula (Section 2). We will demonstrate asymptotic expansions for the Hausdorff dimension of the limit set of the perturbed conformal graph directed Markov system for some concrete examples. We will also give an example of perturbed linear countable IFS that the dimension of this limit set has asymptotic behaviour with the order $n - 1$ but does not have the order n. Moreover, the coefficient and the remainder of the solution $s(\varepsilon)$ can be numerically calculated (Section 2.4). Note that though the functions $\psi(\varepsilon, \cdot)$ may be equal to $\psi(\varepsilon, \cdot) \equiv 1$ in our examples of this paper, we shall treat the case of $\psi(\varepsilon, \cdot) \not\equiv 1$ for the study of a multifractal analysis of a perturbed system in future works. Another promising direction of future study is to estimate the dimensions of limit sets of *non-conformal* graph directed iterated function systems with infinite state ([11, 18], for example).

Our main result is an infinite state version of our previous result [24, Theorem 2.6]. However, the proof of the finite state version given in [24] cannot be applied directly to the infinite state version. Indeed, a difficult point between the finite state case and the infinite state case is that in the infinite state case, even if the potential $g(\varepsilon, \cdot)$ has an asymptotic expansion with order $n + 1$ or more with natural regularity coefficients and remainder, the order of the asymptotic solution may have only length n or less. Furthermore, the remainder of $s(\varepsilon)$ can become any small fractional order even if $\tilde{g}_n(\varepsilon, \cdot) \equiv 0$ and $\psi(\varepsilon, \cdot) \equiv 1$ (see Theorem 2.5(2)). This fact suggests that the general analytic perturbation theory cannot be applied to the asymptotic behaviour of the solution $s(\varepsilon)$ in the infinite state case. Therefore, we need to introduce additional conditions for expansions with an order of length n (see (g.3)–(g.5) and (ψ.3)–(ψ.4)).

By the generalization of asymptotic perturbation theory of linear operators in [24] and by developing the method of asymptotic solution of the pressure function in the finite state version, the main result is proved.

In order to state our main results precisely, we introduce some notions of a symbolic system below. Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph endowed with countable vertex set V, countable edge set E, and two maps, $i(\cdot)$ and $t(\cdot)$ from E to V. For each $e \in E$, $i(e)$ is called the initial vertex of e, and $t(e)$ is called the terminal vertex of e. Denote by E^∞ the one-sided shift space

$$E^\infty = \left\{\omega = \omega_0\omega_1 \cdots \in \prod_{k=0}^{\infty} E : t(\omega_n) = i(\omega_{n+1}) \text{ for any } n \geq 0\right\}$$

endowed with the shift transformation $\sigma: E^\infty \to E^\infty$ defined as $(\sigma\omega)_n = \omega_{n+1}$ for any $n \geq 0$. For $\theta \in (0,1)$, a metric d_θ on E^∞ is given by $d_\theta(\omega, \upsilon) = \theta^{\inf\{n\geq 0: \omega_n \neq \upsilon_n\}}$. The *incidence* matrix A of E^∞ is defined by $A = (A(ee'))_{E\times E}$ with $A(ee') = 1$ if $t(e) = i(e')$, and $A(ee') = 0$ if $t(e) \neq i(e')$. The matrix A is *finitely irreducible* if there exists a finite subset F of $\bigcup_{n=1}^\infty E^n$ such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. A function $f: E^\infty \to \mathbb{K}$ is called *weakly d_θ-Lipschitz continuous* if the number $\sup_{e\in E}\sup_{\omega,\upsilon\in[e]:\omega\neq\upsilon} |f(\omega) - f(\upsilon)|/d_\theta(\omega, \upsilon)$ is finite. A function $f: E^\infty \to \mathbb{K}$ is a *weakly Hölder continuous* function if it is weakly d_θ-Lipschitz continuous for some $\theta \in (0,1)$. Denote by $\|\cdot\|_\infty$ the supremum norm defined as $\|f\|_\infty = \sup_{\omega\in E^\infty} |f(\omega)|$.

To state our main result, we introduce some conditions for potentials. Let n be a nonnegative integer. We consider conditions (g.1)–(g.5) below for the function $g(\varepsilon, \cdot): E^\infty \to \mathbb{R}$ with a small parameter $\varepsilon \in (0,1)$:

- (g.1) The function $g(\varepsilon, \cdot): E^\infty \to \mathbb{R}$ has the form (1.1) for some real-valued weakly Hölder continuous functions $g, g_1, \ldots, g_n, \tilde{g}_n(\varepsilon, \cdot)$ with
$$\lim_{\varepsilon\to 0} \|\tilde{g}_n(\varepsilon, \cdot)\|_\infty = 0.$$
- (g.2) $g(\omega) \neq 0$ for each $\omega \in E^\infty$ and $\|g\|_\infty < 1$.
- (g.3) $|g(\omega) - g(\upsilon)| \leq c_1|g(\omega)|d_\theta(\omega, \upsilon)$ for $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$ for some $c_1 > 0$, $\theta \in (0,1)$.
- (g.4) $|g_k(\omega)| \leq c_2|g(\omega)|^{t_k}$ and $|g_k(\omega) - g_k(\upsilon)| \leq c_3|g(\omega)|^{t_k} d_\theta(\omega, \upsilon)$ for any $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$ for some constants $c_2, c_3 > 0$ and $t_k \in (0,1]$ for $k = 1, 2, \ldots, n$.
- (g.5) $|\tilde{g}_n(\varepsilon, \omega)| \leq c_4(\varepsilon)|g(\omega)|^{\tilde{t}}$ for any $\omega \in E^\infty$ for some constants $\tilde{t} \in (0,1]$ and $c_4(\varepsilon) > 0$ with $c_4(\varepsilon) \to 0$.

Moreover, we assume that the function $\psi(\varepsilon, \cdot): E^\infty \to \mathbb{R}$ satisfies conditions (ψ.1)–(ψ.4) below:

(ψ.1) The function $\psi(\varepsilon,\cdot)\colon E^\infty \to \mathbb{R}$ has the form (1.2) for some real-valued weakly Hölder continuous functions $\psi, \psi_1, \ldots, \psi_n, \tilde{\psi}_n(\varepsilon,\cdot)$ with

$$\lim_{\varepsilon\to 0} \|\tilde{\psi}_n(\varepsilon,\cdot)\|_\infty = 0.$$

(ψ.2) $\psi(\omega) > 0$ for any $\omega \in E^\infty$.

(ψ.3) $|\psi_k(\omega)| \le c_5|\psi(\omega)|$ and $|\psi_k(\omega) - \psi_k(\upsilon)| \le c_6|\psi(\omega)|d_\theta(\omega,\upsilon)$ for any $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$ and for some $c_5, c_6 > 0$ for $k = 1, 2, \ldots, n$.

(ψ.4) $|\tilde{\psi}_n(\varepsilon,\omega)| \le c_7(\varepsilon)|\psi(\omega)|$ for any $\omega \in E^\infty$ for some $c_7(\varepsilon) > 0$ with $c_7(\varepsilon) \to 0$.

Let

$$\underline{p} = \inf\{p \ge 0\colon P(p\log|g| + \log\psi) < +\infty\}, \tag{1.4}$$

where $P(f)$ means the topological pressure of f which is defined by (A.1). Put

$$p(n) = \begin{cases} \underline{p}/\tilde{t} & \text{if } n = 0, \\ \max\bigl(\underline{p} + n(1-t_1), \underline{p} + n(1-t_2)/2, \ldots, \underline{p} + n(1-t_n)/n, & \\ \qquad \underline{p}/t_1, \underline{p}/t_2, \ldots, \underline{p}/t_n, \underline{p} + 1 - \tilde{t}, \underline{p}/\tilde{t}\bigr) & \text{if } n \ge 1. \end{cases} \tag{1.5}$$

Now we are in a position to state our main result.

Theorem 1.1. *Fix a nonnegative integer n. Assume that the incidence matrix of E^∞ is finitely irreducible and the conditions (g.1)–(g.5) and (ψ.1)–(ψ.4) are satisfied. Choose any $s(0) \in (p(n), +\infty)$ and any compact neighborhood $I \subset (p(n), +\infty)$ of $s(0)$. Let $p_0 = P(s(0)\log|g| + \log\psi)$. Then there exist numbers $\varepsilon_0 > 0$, $s_1, \ldots, s_n \in \mathbb{R}$ such that the equation $P(s\log|g(\varepsilon,\cdot)| + \log\psi(\varepsilon,\cdot)) = p_0$ has a unique solution $s = s(\varepsilon) \in I$ for each $0 < \varepsilon < \varepsilon_0$, and $s(\varepsilon)$ forms the asymptotic expansion*

$$s(\varepsilon) = s(0) + s_1\varepsilon + \cdots + s_n\varepsilon^n + \tilde{s}_n(\varepsilon)\varepsilon^n, \tag{1.6}$$

and $|\tilde{s}_n(\varepsilon)| \to 0$ as $\varepsilon \to 0$. In particular,

$$\tilde{s}_n(\varepsilon) = \begin{cases} -\frac{\nu(\varepsilon,\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon,h))}{\nu(h\log|g|)} + O(\varepsilon) & \text{if } n \ge 1, \\ -\frac{\nu(\varepsilon,\tilde{\mathcal{L}}_{0,s(\varepsilon)}(\varepsilon,h))}{\nu(h\log|g|)} + o(\|\tilde{\mathcal{L}}_{0,s(\varepsilon)}(\varepsilon,h)\|_\infty) & \text{if } n = 0, \end{cases} \tag{1.7}$$

where h is the Perron eigenfunction of the eigenvalue e^{p_0} of the Ruelle operator of $s(0)\log|g| + \log\psi$, ν is the Perron eigenvector of this dual operator with $\nu(h) = \nu(E^\infty) = 1$, and $\nu(\varepsilon,\cdot)$ is the Perron eigenvector of the dual of the Ruelle operator of $s(\varepsilon)\log|g(\varepsilon,\cdot)| + \log\psi(\varepsilon,\cdot)$. Moreover, $\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon,\cdot)$ is an operator of the remainder of the expansion of the Ruelle operator (see Lemma 3.6 for details).

Note that the coefficients s_k and the remainder $\tilde{s}_n(\varepsilon)$ are precisely given in (3.17) and (3.19), respectively. The following result is an immediate consequence of this theorem.

Corollary 1.2. *Under the same conditions of the above theorem, assume also that there exists* $s(0) > p(n)$ *such that* $P(s(0)\log|g| + \log\psi) = 0$. *Then the equation* $P(s\log|g(\varepsilon,\cdot)| + \log\psi(\varepsilon,\cdot)) = 0$ *for* $s \in \mathbb{R}$ *has a unique solution* $s = s(\varepsilon)$ *for any small* $\varepsilon > 0$, *and* $s(\varepsilon)$ *forms the* n*-order asymptotic expansion as* (1.6).

Next, we give sufficient conditions for a situation when $s(\varepsilon)$ does not have $(n+1)$-order asymptotic expansion though it is n-order asymptotic behaviour. We introduce the following conditions:

(g.6) E is infinitely countable.

(g.7) $\tilde{g}_n(\varepsilon,\cdot) \equiv 0$ (i.e., $\tilde{t} = 1$) and $\psi(\varepsilon,\cdot) \equiv 1$.

(g.8) There exist $t_0 \in [t_1, (nt_1+1)/(n+1))$ and $c_8 > 0$ such that $p(n) < \underline{p} + (1-t_0)(n+1)$ and $g_1(\omega)\,\mathrm{sign}(g(\omega)) \geq c_8|g(\omega)|^{t_0}$ for any $\omega \in E^\infty$.

(g.9) The numbers $t_2,\ldots,t_n$ satisfy $t_0 \leq t_k$ for any $k = 2,\ldots,n$.

Proposition 1.3. *Assume that the conditions* (g.1)–(g.9) *with* $n \geq 1$ *are satisfied. Choose any* $s(0) \in (p(n), \underline{p} + (n+1)(1-t_0)) \setminus \{1,2,\ldots,n\}$ *and put* $p_0 = P(s(0)\log|g|)$. *Then the unique solution* $s = s(\varepsilon)$ *of the equation* $P(s\log|g(\varepsilon,\cdot)|) = p_0$ *has the form* $s(\varepsilon) = s(0) + s_1\varepsilon + \cdots + s_n\varepsilon^n + \tilde{s}_n(\varepsilon)\varepsilon^n$ *with* $\lim_{\varepsilon\to 0}|\tilde{s}_n(\varepsilon)|/\varepsilon = +\infty$.

In Section 2, we will illustrate asymptotic perturbations of Hausdorff dimensions of limit sets from conformal graph directed Markov systems, e.g., continued fractions and Kleinian groups of Schottky type. Furthermore, we will demonstrate an example of linear countable IFS such that the coefficients and the remainder of the solution are explicitly calculated (Section 2.4). In Section 3, we present the proofs of all of our results. In the appendices, we shall introduce some facts necessary for the proof of the main theorem. In Appendix A, we recall the notion of thermodynamic formalism and the Ruelle operators acting on a suitable function space in the infinite graph. In particular, a version of the Ruelle–Perron–Frobenius Theorem on these operators is described (Theorem A.2). We state in Appendix B the general theory of asymptotic behaviours of the eigenvalue and the corresponding eigenvector of bounded linear operators. This result is obtained by generalizing the results of [24, Theorem 2.1]. Finally, we shall give an upper bound of an intermediate point of the binomial expansion in Appendix C which plays an important role in giving the proof of our results.

2. Examples

2.1. Conformal graph directed Markov systems

Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph for which V is finite and E is countable. In this section, we consider the asymptotic behaviours of the Hausdorff dimensions of the limit sets of perturbed graph directed Markov systems introduced in [17]. We begin with the definition of this system. Let D be a positive integer, $\beta \in (0,1]$ and $r \in (0,1)$. We introduce a set $(G,(J_v),(O_v),(T_e))$ satisfying conditions (i)–(v) below:

(i) For each $v \in V$, J_v is a compact and connected subset of $\mathbb{R}^D$ satisfying that the interior int J_v of J_v is not empty, and int J_v and int $J_{v'}$ are disjoint for $v' \in V$ with $v \neq v'$.

(ii) For each $v \in V$, O_v is a bounded, open and connected subset of $\mathbb{R}^D$ containing J_v.

(iii) For each $e \in E$, a function $T_e\colon O_{t(e)} \to T_e(O_{t(e)}) \subset O_{i(e)}$ is a $C^{1+\beta}$-conformal diffeomorphism with $T_e(\operatorname{int} J_{t(e)}) \subset \operatorname{int} J_{i(e)}$ and $\sup_{x\in O_{t(e)}}\|T_e'(x)\| \le r$, where $\|T_e'(x)\|$ means the operator norm of $T_e'(x)$. Moreover, for any $e, e' \in E$ with $e \neq e'$ and $i(e') = i(e)$, $T_e(\operatorname{int} J_{t(e)}) \cap T_{e'}(\operatorname{int} J_{t(e')}) = \emptyset$, namely, the open set condition (OSC) is satisfied.

(iv) (Bounded distortion) There exists a constant $c_9 > 0$ such that for any $e \in E$ and $x, y \in O_{t(e)}$, $\big|\|T_e'(x)\| - \|T_e'(y)\|\big| \le c_9\|T_e'(x)\||x-y|^\beta$, where $|\cdot|$ means a norm of any Euclidean space.

(v) (Cone condition) If $\#E = \infty$, then there exist $\gamma, l > 0$ with $\gamma < \pi/2$ such that for any $v \in V$, $x \in J_v$, there is a $u \in \mathbb{R}^D$ with $|u| = 1$ so that the set $\{y \in \mathbb{R}^D : 0 < |y-x| < l \text{ and } (y-x,u) > |y-x|\cos\gamma\}$ is in int J_v, where $(y-x,u)$ denotes the inner product of $y-x$ and u.

Under conditions (i)–(v), we call the set $(G,(J_v),(O_v),(T_e))$ a graph directed Markov system (GDMS for short). The Hausdorff dimension of the limit set of this system has been studied by many authors [7, 16, 17, 19, 20].

The coding map $\pi\colon E^\infty \to \mathbb{R}^D$ is defined by $\pi\omega = \bigcap_{n=0}^\infty T_{\omega_0}\cdots T_{\omega_n}(J_{t(\omega_n)})$ for $\omega \in E^\infty$. Put $K = \pi(E^\infty)$. This set is called the limit set of the GDMS. We define a function $\varphi\colon E^\infty \to \mathbb{R}$ by $\varphi(\omega) = \log\|T_{\omega_0}'(\pi\sigma\omega)\|$. Put $\underline{s} = \inf\{s \ge 0 : P(s\varphi) < +\infty\}$. We call the GDMS *regular* if $P(s\varphi) = 0$ for some $s \ge \underline{s}$. The GDMS is said to be *strongly regular* if $0 < P(s\varphi) < +\infty$ for some $s \ge \underline{s}$ (see [17, 19] for the terminology). It is known that the general Bowen's formula is satisfied:

Theorem 2.1 ([19]). *Let $(G,(J_v),(O_v),(T_e))$ be a graph directed Markov system. Assume that E^∞ is finitely irreducible. Then $\dim_H K = \inf\{t \in \mathbb{R} : P(t\varphi) \le 0\}$. In addition to the above condition, we also assume that the potential φ is regular. In this case, $s = \dim_H K$ if and only if $P(s\varphi) = 0$.*

Now we formulate an asymptotic perturbation of graph directed Markov systems. Fix integers $n \ge 0$, $D \ge 1$ and a number $\beta \in (0,1]$. Consider conditions (G.1)$_n$ and (G.2)$_n$ below:

$(\mathrm{G.1})_n$ The code space E^∞ is finitely irreducible. The set $(G, (J_v), (O_v), (T_e))$ is a strongly regular GDMS on $\mathbb{R}^D$ and the limit set K has positive dimension. Moreover, the function T_e is of class $C^{1+n+\beta}(O_{t(e)})$ for each $e \in E$.

$(\mathrm{G.2})_n$ The set $\{(G, (J_v), (O_v), (T_e(\varepsilon, \cdot))) : \varepsilon > 0\}$ is a GDMS with a small parameter $\varepsilon > 0$ satisfying (i)–(iv) below:

(i) For each $e \in E$, the function $T_e(\varepsilon, \cdot)$ has the n-order asymptotic expansion

$$T_e(\varepsilon, \cdot) = T_e + T_{e,1}\varepsilon + \cdots + T_{e,n}\varepsilon^n + \tilde{T}_{e,n}(\varepsilon, \cdot)\varepsilon^n \ \text{ on } J_{t(e)}$$

for some functions $T_{e,k} \in C^{1+n-k+\beta}(O_{t(e)}, \mathbb{R}^D)$ $(k = 1, 2, \ldots, n)$ and $\tilde{T}_{e,n}(\varepsilon, \cdot) \in C^{1+\beta(\varepsilon)}(O_{t(e)}, \mathbb{R}^D)$ $(\beta(\varepsilon) > 0)$ satisfying

$$\sup_{e \in E} \sup_{x \in J_{t(e)}} |\tilde{T}_{e,n}(\varepsilon, x)| \to 0.$$

(ii) There exist constants $t(l, k) \in (0, 1]$ $(l = 0, 1, \ldots, n,\ k = 1, \ldots, n - l + 1)$ such that the function $x \mapsto T_{e,l}^{(k)}(x)/\|T_e'(x)\|^{t(l,k)}$ is bounded, β-Hölder continuous and its Hölder constant is bounded uniformly in $e \in E$.

(iii) $c_{10}(\varepsilon) := \sup_{e \in E} \sup_{x \in J_{t(e)}} \big(\|\frac{\partial}{\partial x}\tilde{T}_{e,n}(\varepsilon, x)\|/\|T_e'(x)\|^{\tilde{t}_0}\big) \to 0$ as $\varepsilon \to 0$ for some $\tilde{t}_0 \in (0, 1]$.

(iv) $\dim_H K/D > p(n)$, where $p(n)$ is taken from (1.5) with

$$t_k := \min\left\{ \frac{1}{D}\sum_{p=1}^{D} t(i_p, j_p + 1) : \begin{array}{l} i := i_1 + \cdots + i_D \text{ and } j := j_1 + \cdots + j_D \\ \text{satisfy } i = k \text{ and } j = 0 \text{ or} \\ 0 \le i < k \text{ and } 1 \le j \le k - i \end{array} \right\} \tag{2.1}$$

$$\tilde{t} := \min\left\{ t_n, \tilde{t}_0, \frac{\tilde{t}_0}{D} + \frac{D-1}{D}t(1,1), \ldots, \frac{\tilde{t}_0}{D} + \frac{D-1}{D}t(n,1) \right\} \tag{2.2}$$

$$\underline{p} := \underline{s}/D.$$

Note that if the edge set E is finite, then the conditions (ii) and (iv) are always satisfied because $\|T_e'(x)\|$ is uniformly bounded away from zero, and $p(n)$ becomes zero by taking $t(l,k) \equiv 1$. Moreover, $c_{10}(\varepsilon)$ in (iii) can be taken as

$$\sup_{e\in E}\sup_{x\in J_{t(e)}}\left\|\frac{\partial}{\partial x}\tilde{T}_{e,n}(\varepsilon,x)\right\|$$

when E is finite. Let $K(\varepsilon)$ be the limit set of the perturbed GDMS

$$\big(G,(J_v),(O_v),(T_e(\varepsilon,\cdot))\big).$$

Then we obtain the following result:

Theorem 2.2. *Assume that the conditions* (G.1)$_n$ *and* (G.2)$_n$ *are satisfied with a fixed integer $n \geq 0$. Then the perturbed GDMS $(G,(J_v),(O_v),(T_e(\varepsilon,\cdot)))$ is strongly regular for any small $\varepsilon > 0$, and there exist $s_1,\ldots,s_n \in \mathbb{R}$ such that the Hausdorff dimension* $\dim_H K(\varepsilon)$ *of the limit set $K(\varepsilon)$ of the perturbed system has the form* $\dim_H K(\varepsilon) = \dim_H K + s_1\varepsilon + \cdots + s_n\varepsilon^n + o(\varepsilon^n)$ *as $\varepsilon \to 0$.*

Remark 2.3. Roy and Urbański [19] considered continuous perturbation of infinitely conformal iterated function systems given as a special GDMS. They also studied analytic perturbation of GDMS with $D \geq 3$ in [20]. We investigated an asymptotic perturbation of GDMS with finite graph in [25]. Theorem 2.2 is an infinite graph version of this previous result in [25].

2.2. Real continued fractions

Consider a graph G with singleton vertex set $V = \{v\}$ and with infinite edge set $E \subset \{k \in \mathbb{Z}: k \geq 2\}$. Put $J_v = [0,1]$ and $O_v = (-\eta, 1+\eta)$ for a fixed small number $\eta > 0$. Fix $a \in \mathbb{R}$ with $a \neq 0$. We define a perturbed map of continued fractions

$$T_e(\varepsilon,x) = \frac{1}{e+x+a\varepsilon}$$

for $e \in E$, $\varepsilon > 0$ and $x \in \mathbb{R}$. Consider a GDMS $(G, J_v, O_v, (T_e(\varepsilon,\cdot)))$ such that this unperturbed GDMS is strongly regular. Note that such a system exists ([16]) by choosing edge set E. The function $T_e(\varepsilon,x)$ has the expansion

$$T_e(\varepsilon,x) = \frac{1}{e+x} - \frac{a}{(e+x)^2}\varepsilon + \cdots + (-1)^n\frac{a^n}{(e+x)^{n+1}}\varepsilon^n + \tilde{T}_n(\varepsilon,x)\varepsilon^n \quad (2.3)$$

with

$$T_{e,k}(x) = (-a)^k(e+x)^{-k-1} \quad \text{and} \quad \tilde{T}_n(\varepsilon,x) = \frac{\varepsilon(-1)^{n+1}a^{n+1}}{(e+x)^{n+1}(e+x+a\varepsilon)}.$$

It is not hard to check that the conditions (G.1)$_n$ and (G.2)$_n$ are fulfilled from $\underline{p} = 0$, $t(l,k) = \tilde{t} = 1$, and therefore $p(n) = 0$. Thus, we have the n-order asymptotic expansion of the Hausdorff dimension of the limit set of the GDMS by Theorem 2.2.

2.3. Complex continued fractions

In this section, we consider the complex versions of continued fractions. Let $V = \{v\}$ be a singleton vertex set, $E \subset E_* := \{m + n\sqrt{-1} : (m,n) \in \mathbb{Z} \times \mathbb{Z}, m \geq 1\}$ a nonempty countable edge set, $X_v = \overline{B(1/2, 1/2)}$ the closed ball in $\mathbb{C}$ with center $1/2$ and radius $1/2$, and $O_v = B(1/2, 3/4)$ the open ball in $\mathbb{C}$ with center $1/2$ and radius $3/4$. For $e \in E$, we define a function $T_e : O_v \to O_v$ by $T_e(z) = 1/(e + z)$. Then the system $(G = (V, E), (J_v), (O_v), (T_e))$ is a conformal GDMS excluding $T_1'(0) = 1$. Note that $T_e \circ T_{e'}$ becomes a contraction mapping uniformly in $ee' \in E^2$. Such systems associated to complex continued fractions with arbitrary alphabet were investigated in [3–5].

Now we give a perturbed map $T_e(\varepsilon, z)$. Assume the following conditions:

(F.1) The edge set E is a nonempty subset of E_* and the GDMS

$$(G, (J_v), (O_v), (T_e))$$

is strongly regular.

(F.2) A perturbed function $T_e(\varepsilon, z)$ is defined by

$$T_e(\varepsilon, z) = \frac{1}{e + z + a(e)\varepsilon}$$

with a fixed number $a(e) \in \mathbb{C}$ with $\sup_e |a(e)| < +\infty$. Moreover, assume also that the set $(G, (J_v), (O_v), (T_e(\varepsilon, \cdot)))$ is a GDMS for each $\varepsilon > 0$.

Notice that if $E = E_*$, then $(G, (J_v), (O_v), (T_e))$ becomes strongly regular [4, 15]. Figure 1 shows a subsystem and its perturbed system.

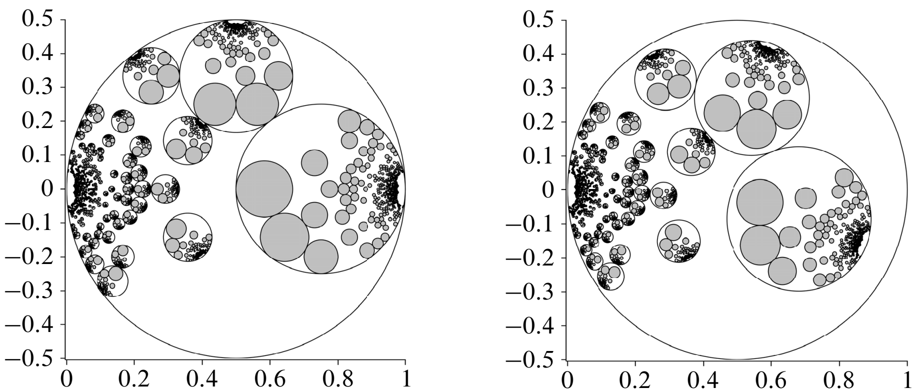

Figure 1. Approximation limit sets of a certain subsystem of a complex continued fraction (left) and of its perturbed system (right) with $a(e) \equiv 0.1 + 0.2\sqrt{-1}$ and $\varepsilon = 1$

We have the following:

Theorem 2.4. *Assume that conditions (F.1) and (F.2) are satisfied. Then the Hausdorff dimension of the limit set of the GDMS* $(G, (J_v), (O_v), (T_e(\varepsilon, \cdot)))$ *has an* n*-order asymptotic expansion as* $\varepsilon \to 0$.

2.4. Linear countable IFS (1)

In this section, we will give the coefficient and the estimate of the remainder for a concrete GDMS. Let $a > 1$ and $E = \{1, 2, \dots\}$. We take an infinite graph $G = (\{v\}, E, i(\cdot), t(\cdot))$ with $i(e) = t(e) = v$ for $e \in E$, $J_v = [0, 1]$ and $O_v = (-\eta, 1 + \eta)$ for a small $\eta > 0$. For $e \in E$ and $\varepsilon \geq 0$, we define a function $T_e(\varepsilon, \cdot)$ by

$$T_e(\varepsilon, x) = \Big(\frac{1}{5^e} + \frac{1}{a^e}\varepsilon\Big)x + b(e). \tag{2.4}$$

Here we choose $b(e)$ so that the set $(G, (J_v), (O_v), T_e(\varepsilon, \cdot))$ satisfies the open set condition for any small $\varepsilon > 0$. Note that when we define a function f_ε from $\bigcup_{e\in E} T_e(\varepsilon, J_{t(e)})$ to $[0, 1]$ by $f_\varepsilon(x) = T_e(\varepsilon, x)^{-1}$ if $x \in T_e(\varepsilon, J_{t(e)})$, this is a piecewise linear expanding (hyperbolic) interval map. Moreover, the limit set $K(\varepsilon)$ of the GDMS becomes the (non-compact) repeller of f_ε. It is not hard to check that the condition $(G.2)_n$ (i), (iii) are valid with $T_e(x) = x/5^e + b(e)$, $T_{e,1}(x) = x/a^e$, $T_{e,k} \equiv 0$ $(k \geq 2)$ and $\tilde{T}_{e,n}(\varepsilon, \cdot) \equiv 0$. To see $(G.2)_n$ (ii), (iv), we remark that $|T'_{1,e}(x)|/|T'_e(x)|^t = (5^t/a)^e$ is bounded uniformly in e if and only if $t \leq \log a/\log 5$. Therefore, we put $t(1, 1) = \min(\log a/\log 5, 1)$, and otherwise $t(l, k) = 1$ for any $(l, k) \neq (1, 1)$ when $n \geq 1$. Let $\varphi(\omega) = \log(1/5^{\omega_0})$. Moreover, $\underline{s} = \inf\{s \geq 0\colon P(s\varphi) < +\infty\}$ is equal to 0. Thus, $p(n) = n\big(1 - \min(\log a/\log 5, 1)\big)$ for any $n \geq 0$. We see that $P(s(0)\varphi) = 0$ if and only if $\sum_{e\in E}(1/5^e)^{s(0)} = 1$ if and only if $s(0) = \dim_H K = \log 2/\log 5$ by the Bowen's formula. Then we obtain the following:

Theorem 2.5. *Take the function* $T_e(\varepsilon, \cdot)$ *defined by* (2.4).

(1) *If* $a \geq 5$, *then the Hausdorff dimension* $s(\varepsilon) = \dim_H K(\varepsilon)$ *of the limit set of this GDMS has the* n*-order asymptotic expansion* $s(\varepsilon) = \log 2/\log 5 + s_1\varepsilon + \cdots + s_n\varepsilon^n + \tilde{s}_n(\varepsilon)\varepsilon^n$ *with* $\tilde{s}_n(\varepsilon) \to 0$ *for any* $n \geq 0$. *Each coefficient* s_k $(k = 1, 2, \dots, n)$ *is defined as*

$$s_k = \frac{1}{2\log 5}\sum_{\substack{0\leq v\leq k\\ 0\leq q\leq k-v:\\ (v,q)\neq(0,1)}}\sum_{j=0}^{\min(v,q)} s_{q,k-v}\frac{a_{v,j,s(0)}}{(q-j)!}(-\log 5)^{q-j}\sum_{e=1}^{\infty} e^{q-j}\Big(\frac{5^v}{2a^v}\Big)^e, \tag{2.5}$$

where constants $s_{q,k-v}$ *and* $a_{v,j,s(0)}$ *are defined by* (3.24) *and* (3.16), *respectively.*

(2) *If* $1 < a < 5$, *then take the largest integer* $k \geq 0$ *satisfying* $a \leq 5/2^{1/(k+1)}$. *In this case,* $s(\varepsilon)$ *has the form*

$$s(\varepsilon) = \begin{cases} s(0) + s_1\varepsilon + \cdots + s_k\varepsilon^k + \hat{s}(\varepsilon)\varepsilon^{\frac{\log 2}{\log(5/a)}} & \text{if } a < 5/2^{1/(k+1)}, \\ s(0) + s_1\varepsilon + \cdots + s_k\varepsilon^k + \hat{s}(\varepsilon)\varepsilon^{k+1}\log\varepsilon & \text{if } a = 5/2^{1/(k+1)} \end{cases}$$

with $|\hat{s}(\varepsilon)| \asymp 1$ *as* $\varepsilon \to 0$, *where* $s_1, \ldots, s_k$ *are given by* (2.5) *and where* $b(\varepsilon) \asymp c(\varepsilon)$ *means* $d^{-1}c(\varepsilon) \leq b(\varepsilon) \leq dc(\varepsilon)$ *for any small* $\varepsilon > 0$ *for some constant* $d \geq 1$. *Note that* $k < \log 2/\log(5/a) \leq k+1$ *is satisfied.*

In particular, the numbers s_1 and s_2 are given by

$$s_1 = \frac{\log 2}{(\log 5)^2}\frac{5}{4a-10}$$

$$s_2 = \frac{25\log 2}{(\log 5)^3}\left(\frac{1}{2(2a-5)^2} - \frac{a\log 2}{(2a-5)(4a^2-5)^2} + \frac{\log(2/5)}{8a^2-100}\right).$$

2.5. Linear countable IFS (2)

Using the same notation for G, V, E, J_v, O_v as in Theorem 2.5, we define a concrete function $T_e(\varepsilon, \cdot)$ by

$$T_e(\varepsilon, x) = \left(\frac{1}{5^e} + \frac{1}{4^e}\varepsilon + \frac{1}{3^e}\varepsilon^2\right)x + b(e),$$

where $b(e)$ is suitably chosen so that the OSC is satisfied. By virtue of Theorem 1.1, we see the following:

Proposition 2.6. *Under the above function, the Hausdorff dimension of the limit set* $K(\varepsilon)$ *of* $\big(G, (J_v), (O_v), (T_e(\varepsilon, \cdot))\big)$ *has at least* 2*-order asymptotic expansion for* ε.

2.6. Kleinian groups of Schottky type

We consider perturbations of Kleinian groups of Schottky type given in [17, Example 5.1.5]. Fix integers $D \geq 1$ and $d \geq 1$. Let $V = \{1, 2, \ldots, d\}$ be a vertex set, and for $v \in V$, $J_v = \overline{B(a(v), r(v))}$, the mutually disjoint closed balls. Consider the inversion map f_v with respect to J_v, namely,

$$f_v(x) = r(v)^2\frac{x - a(v)}{|x - a(v)|^2} + a(v)$$

for $x \in \mathbb{R}^D \cup \{\infty\}$, where we set $f_v(a(v)) = \infty$ and $f_v(\infty) = a(v)$. Then the group $H := \langle\{f_v\}_{v\in V}\rangle$ generated by $\{f_v : v \in V\}$ is called a *Kleinian group of Schottky type*.

Recall that the limit set $L(H)$ of the Kleinian group H is defined by

$$L(H) = \left\{ \lim_{n\to\infty} \xi_n(z) \colon \xi_n \in H \text{ mutually disjoint} \right\}$$

for any fixed $z \in \mathbb{R}^D$. It is known that letting $E_0 = V^2 \setminus \{(v,v) \colon i \in V\}$, $O_v = B(J_v, \eta)$ for any small fixed number η, and $T_e := f_{i(e)}|_{O_{t(e)}} \colon O_{t(e)} \to O_{i(e)}$ for $e \in E_0$, the set $(G = (V, E_0), \{J_v\}, \{O_v\}, \{T_e\})$ is a finite conformal GDMS, except that T_e need not be uniform contractions. Nevertheless, the limit set K of this system is well-defined since there exists an $n \geq 1$ such that any finite path $w \in E_0^n$ of G, $T_w := T_{w_1} \circ T_{w_2} \circ \cdots \circ T_{w_n}$ is uniformly contracting. If no confusion can arise, we also call $(G, \{J_v\}, \{O_v\}, \{T_e\})$ a GDMS. Consider a subgroup Γ of H. Define the set

$$\Gamma_0 = \left\{ \xi \in \Gamma \colon \text{has an irreducible form in } \Gamma \right\},$$

where ξ has an irreducible form in Γ if $\xi \neq \xi_1 \circ \xi_2$ for any two non-identity maps $\xi_1, \xi_2 \in \Gamma$. We define a subset E of paths with finite length by

$$E = \left\{ w \in \bigcup_{n=1}^{\infty} E_0^n \colon f_{i(w_1)} \circ \cdots \circ f_{i(w_m)} \in \Gamma_0 \right\}. \tag{2.6}$$

Then $G = (V, E, i, t)$ is a directed multigraph with finite vertex set and countable edge set, and $(G, (J_v), (O_v), (T_w))$ is a GDMS. In particular, this system satisfies an open set condition [17]. It is known from [17, Theorem 5.1.7] that the limit set K of the GDMS $(G, (J_v), (O_v), (T_w))$ has the equation $\overline{K} = L(\Gamma)$. Moreover, if Γ_0 is finite, then $K = L(\Gamma)$.

Now we formulate an asymptotic perturbation of a subgroup of a Kleinian group of Schottky type. We introduce the following conditions:

(K.1) $H = \langle \{f_v\}_{v \in V} \rangle$ is the Kleinian group of Schottky type with finite disjoint closed balls $J_v = \overline{B(a(v), r(v))}$. Assume also that a subgroup $\Gamma \subset H$ satisfies that the corresponding GDMS $(G, (J_v), (O_v), (T_w))$ with the edge set (2.6) has finitely irreducible incidence matrix and is strongly regular.

(K.2) There exists a decomposition $V = V_0 \cup V_1$ with $V_0 \neq \emptyset$ and $V_0 \cap V_1 = \emptyset$ such that the set $\left\{ f = f_{v_1} \circ \cdots \circ f_{v_m} \in \Gamma \colon v_i \in V_0 \text{ for some } 1 \leq i \leq m \right\}$ is finite. Assume also that if $v \in V_0$, then $r(\varepsilon, v) > 0$ and $a(\varepsilon, v) \in \mathbb{R}^D$ satisfy

$$r(\varepsilon, v) = r(v) + r_1(v)\varepsilon + \cdots + r_n(v)\varepsilon^n + o(\varepsilon^n) \tag{2.7}$$

$$a(\varepsilon, v) = a(v) + a_1(v)\varepsilon + \cdots + a_n(v)\varepsilon^n + o(\varepsilon^n) \tag{2.8}$$

as $\varepsilon \to 0$ for some $r_k(v) \in \mathbb{R}$ and $a_k(v) \in \mathbb{R}^D$, and if $v \in V_1$, then we set $r(\varepsilon, v) = r(v)$ and $a(\varepsilon, v) = a(v)$ for all ε.

In other words, condition (K.2) says that the elements of Γ that change by ε are of a finite number. Let $f_v(\varepsilon,\cdot)$ be the inversion map with respect to the disjoint closed ball $J_v(\varepsilon) = \overline{B(a(\varepsilon,v), r(\varepsilon,v))}$ for any small $\varepsilon > 0$. Put the Kleinian group of Schottky type $H(\varepsilon) = \langle\{f_v(\varepsilon,\cdot)\}_{v\in V}\rangle$. For the subgroup $\Gamma \subset H$ in (K.1), we consider the corresponding subgroup $\Gamma(\varepsilon) \subset H(\varepsilon)$, namely,

$$\Gamma(\varepsilon) = \{f_{v_1}(\varepsilon,\cdot)\circ\cdots\circ f_{v_m}(\varepsilon,\cdot) : f_{v_1}\circ\cdots\circ f_{v_m} \in \Gamma\}.$$

Similarly, the corresponding GDMS $(G, (J_v), (O_v), (T_w(\varepsilon,\cdot)))$ is obtained, where $T_w(\varepsilon,\cdot) := T_{w_1}(\varepsilon,\cdot)\circ\cdots\circ T_{w_n}(\varepsilon,\cdot)$ for $w \in E$ and we let $T_e(\varepsilon,\cdot) := f_{i(e)}(\varepsilon,\cdot)|_{O_{t(e)}}$ for $e \in E_0$. We show in Figure 2 the picture of a concrete perturbed Schottky group under $\Gamma = H$.

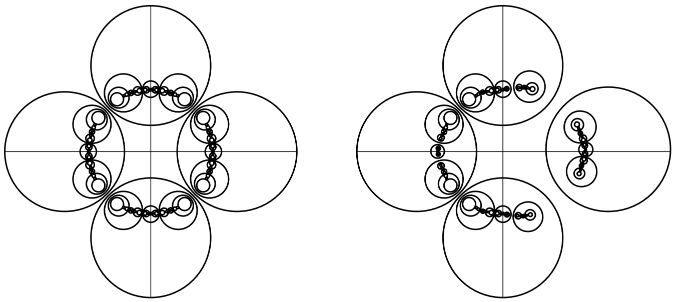

Figure 2. Approximation limit sets of a certain unperturbed Schottky group (left) and of its perturbed group (right)

Then we have the following:

Theorem 2.7. *Assume that the conditions (K.1) and (K.2) are satisfied. Then the Hausdorff dimension of the limit set $K(\varepsilon)$ of the GDMS $(G, (J_v(\varepsilon)), (O_v), (T_w(\varepsilon,\cdot)))$ of the subgroup $\Gamma(\varepsilon)$ has an n-order asymptotic expansion.*

Remark 2.8. Even if $a(\varepsilon, v)$ and $r(\varepsilon, v)$ in the condition (K.2) have the expansions (2.7) and (2.8) for all $v \in V$, respectively, the assertion of Theorem 2.7 is still satisfied. The proof of this assertion is not trivial, but can be proved by showing the conditions $(\mathrm{G.2})_n$ (ii), (iii) with $t(l,k) = \tilde{t} = 1 - s$ for any fixed $s \in (0,1]$. In this case, the condition $(\mathrm{G.2})_n$ (iv) is also fulfilled by taking $s \in (0,1]$ so that $\dim_H K/D > \max(\underline{p} + ns, \underline{p}/s)$.

3. Proofs

In this section, the theorems and propositions in Section 1 and Section 2 are all proved. Recall the notation given in Section 2. For later convenience we introduce some function spaces. Let $\mathbb{K}$ be a numerical space or a complex space. Denote by $C_b(E^\infty, \mathbb{K})$

the set of all $\mathbb{K}$-valued continuous functions f on E^∞ with $\|f\|_\infty < +\infty$, by $F_\theta(E^\infty, \mathbb{K})$ the set of all $\mathbb{K}$-valued weakly d_θ-Lipschitz continuous functions on E^∞, and by $F_{\theta,b}(E^\infty, \mathbb{K})$ the set of all $f \in F_\theta(E^\infty, \mathbb{K})$ with $\|f\|_\theta := \|f\|_\infty + [f]_\theta < +\infty$, where we put $[f]_\theta := \sup_{e\in E} \sup_{\omega,\upsilon\in[e]:\omega\neq\upsilon} |f(\omega) - f(\upsilon)|/d_\theta(\omega, \upsilon)$. If $\mathbb{K}$ is equal to $\mathbb{C}$, then we may drop the notation '$\mathbb{K}$' from these function spaces.

3.1. Proof of Theorem 1.1

To show our main result, we need to prove some auxiliary propositions. We begin with the following short proposition:

Proposition 3.1. *Let (U, d) be a bounded metric space and $(E, \|\cdot\|)$ a Banach algebra. Assume that functions f_k, h_k $(0 \le k \le n)$ from U to E satisfy $\|f_k(x)\| \le c_{11}(k)\|h_k(x)\|$ for any $x \in U$ and $\|f_k(x) - f_k(y)\| \le c_{12}(k)\|h_k(x)\|d(x, y)$ for any $x, y \in U$ for some constants $c_{11}(k), c_{12}(k) > 0$. Then*

$$\left\|\prod_{k=0}^n f_k(x) - \prod_{k=0}^n f_k(y)\right\| \le c_{13} \prod_{i=0}^n \|h_i(x)\| d(x, y)$$

with $c_{13} = \sum_{i=0}^n (\prod_{j=0}^{i-1} c_{11}(j)) c_{12}(i) (\prod_{j=i+1}^n (c_{11}(j) + c_{12}(j) \operatorname{diam} U))$.

Proof. By the assumption, we note

$$\|f_k(y)\| \le \|f_k(x)\| + \|f_k(x) - f_k(y)\| \le (c_{11}(k) + c_{12}(k) \operatorname{diam} U)\|h_k(x)\|.$$

Thus, we have

$$\begin{aligned}\left\|\prod_{k=0}^n f_k(x) - \prod_{k=0}^n f_k(y)\right\| &\le \sum_{i=1}^n \left(\prod_{k=0}^{i-1} \|f_k(x)\|\right) \|f_i(x) - f_i(y)\| \left(\prod_{k=i+1}^n \|f_k(y)\|\right) \\ &\le c_{13} \prod_{s=1}^n \|h_s(x)\| d(x, y).\end{aligned}$$

■

For later convenience, for $p \in \mathbb{R}$ with $p > 0$ and $k, l \in \mathbb{Z}$ with $0 \le k \le n$ and $0 \le l \le k$, we define a function $G^p_{l,k} : E^\infty \to \mathbb{R}$ by

$$G^p_{l,k}(\omega) = \begin{cases} \dfrac{|g(\omega)|^p}{g(\omega)^l} \displaystyle\sum_{\substack{j_1,\ldots,j_k \ge 0:\\ j_1+\cdots+j_k = l\\ j_1+2j_2+\cdots+kj_k = k}} \frac{l!}{j_1!\cdots j_k!} g_1(\omega)^{j_1} \cdots g_k(\omega)^{j_k} & \text{if } k \ge 1, \\ |g(\omega)|^p & \text{if } k = 0. \end{cases} \tag{3.1}$$

This function will be used in the expansion of $|g(\varepsilon, \cdot)|^p$ (see (3.4)). To estimate this function, we will assert the following lemma:

Lemma 3.2. *Assume that* $G = (V, E, i(\cdot), t(\cdot))$ *is a directed multigraph. Assume that* $g\colon E^\infty \to \mathbb{R}$ *satisfies* (g.2) *and* (g.3). *Then for any integer* $l \geq 0$, $q \in \mathbb{R}$, *and* $\omega, \upsilon \in E^\infty$ *with* $\omega_0 = \upsilon_0$, *we have* $\big||g(\omega)|^q/g(\omega)^l - |g(\upsilon)|^q/g(\upsilon)^l\big| \leq c_{14}|g(\omega)|^{q-l} d_\theta(\omega, \upsilon)$ *by putting* $c_{14} = c_{14}(q, l) = (c_1(1 + c_1\theta)^{|q|} + \theta^{-1}l)(1 + c_1\theta)^l$.

Proof. Let $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$. The condition (g.3) yields $|g(\omega)^{-1} - g(\upsilon)^{-1}| \leq |g(\omega)^{-1}|c_1 d_\theta(\omega, \upsilon)$. On the other hand, by virtue of condition (g.3) again, we have

$$(1 + c_1\theta)^{-1}|g(\omega)| \leq |g(\upsilon)| \leq (1 + c_1\theta)|g(\omega)|.$$

Therefore, the Mean Value Theorem and condition (g.3) imply

$$\begin{aligned}
\big||g(\omega)|^q - |g(\upsilon)|^q\big| &= \big|\alpha|g(\omega)| + (1-\alpha)|g(\upsilon)|\big|^{q-1}\big||g(\omega)| - |g(\upsilon)|\big| \\
&\leq \big|\alpha|g(\omega)| + (1-\alpha)|g(\upsilon)|\big|^{q} \frac{\big||g(\omega)| - |g(\upsilon)|\big|}{\min(|g(\omega)|, |g(\upsilon)|)} \\
&\leq \begin{cases} \max(|g(\omega)|, |g(\upsilon)|)^q c_1 d_\theta(\omega, \upsilon) & \text{if } q \geq 0, \\ \min(|g(\omega)|, |g(\upsilon)|)^q c_1 d_\theta(\omega, \upsilon) & \text{if } q < 0 \end{cases} \\
&\leq (1 + c_1\theta)^{|q|} c_1 |g(\omega)|^q d_\theta(\omega, \upsilon)
\end{aligned}$$

for some $\alpha \in [0, 1]$. Choose any $e \in E$. Proposition 3.1 regarding $U = [e]$, $f_0 = h_0 := |g|^q$, $f_1 = \cdots = f_l = h_1 = \cdots = h_l := g^{-1}$, $c_{11}(\cdot) := 1$, $c_{12}(0) := (1 + c_1\theta)^{|q|}c_1$ and $c_{12}(1) = \cdots = c_{12}(l) := c_1$ implies that the assertion holds for the constant $c_{14} = \sum_{i=0}^{l} c_{12}(i) \prod_{j=i+1}^{l}(1 + c_{12}(j)\theta)$. Hence, the proof is complete. ∎

Lemma 3.3. *Assume that the incidence matrix of* E^∞ *is finitely irreducible and the conditions* (g.1)–(g.5) *with fixed nonnegative integer* n *are satisfied. Then for any* $p > p(n)$, $0 \leq k \leq n$ *and* $0 \leq l \leq k$, *the function* $G^p_{l,k}$ *is a weakly Hölder continuous function. In particular,*

$$\begin{aligned}
|G^p_{l,k}(\omega)| &\leq c_{15}|g(\omega)|^{p - \frac{k}{n}p(n) + \frac{k}{n}\underline{p}} \\
|G^p_{l,k}(\omega) - G^p_{l,k}(\upsilon)| &\leq c_{16}|g(\omega)|^{p - \frac{k}{n}p(n) + \frac{k}{n}\underline{p}} d_\theta(\omega, \upsilon)
\end{aligned} \tag{3.2}$$

for any $\omega, \upsilon \in E^\infty$ *with* $\omega_0 = \upsilon_0$ *for some constants* $c_{15} = c_{15}(k, l)$ *and* $c_{16} = c_{16}(k, l) > 0$.

Proof. In the case $k = 0$, the inequality (3.2) is fulfilled from Lemma 3.2.

In the case $k \geq 1$, let $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$. By the condition (g.4), we have $|g_i(\omega)^j| \leq c_2^j |g(\omega)|^{jt_i}$ and $|g_i(\upsilon)^j| \leq (c_2 + c_3\theta)^j |g(\omega)|^{jt_i}$. Then it follows from

Proposition 3.1 that for any positive integer j,

$$|g_i(\omega)^j - g_i(\upsilon)^j| \le j(c_2 + c_3\theta)^{j-1}c_3|g(\omega)|^{jt_i} d_\theta(\omega, \upsilon).$$

Thus, we obtain

$$|g|^{p-l}|g_1^{j_1}\cdots g_k^{j_k}| \le c_2^{j_1+\cdots+j_k}|g|^{p-l+j_1t_1+\cdots+j_kt_k}$$

$$\left|\frac{|g(\omega)|^p}{g(\omega)^l}\prod_{i=1}^{k}(g_i(\omega))^{j_i} - \frac{|g(\upsilon)|^p}{g(\upsilon)^l}\prod_{i=1}^{k}(g_i(\upsilon))^{j_i}\right| \le c_{17}|g(\omega)|^{p-l+j_1t_1+\cdots+j_kt_k} d_\theta(\omega, \upsilon)$$

from Proposition 3.1 again and Lemma 3.2, where

$$c_{17} = c_{17}(p, l) := c_{14}(p, l)(c_2 + c_3\theta)^l + l(c_2 + c_3\theta)^{l-1}$$

using $j_1 + j_2 + \cdots + j_k = l$. When we put $p_k = \underline{p} + (n/k)(1 - t_k)$, then $p_k \le p(n)$, and therefore, $t_k \ge 1 - (k/n)(p(n) - \underline{p})$ are satisfied. We also note that

$$p - l + j_1t_1 + \cdots + j_kt_k \ge p - l + \sum_{i=1}^{k}\Big(j_i - \frac{ij_i}{n}(p(n) - \underline{p})\Big) = p - \frac{k}{n}p(n) + \frac{k}{n}\underline{p}$$

is satisfied using $j_1 + \cdots + j_k = l$ and $j_1 + 2j_2 + \cdots + kj_k = k$. Hence, the lemma is complete by putting $c_{15} = \binom{k-1}{l-1}c_2^l$ and $c_{16} = \binom{k-1}{l-1}c_{17}$. ■

Remark 3.4. When we take $p, \eta > 0$ so that $p > p(n) + \eta$, we see $p - \frac{k}{n}p(n) + \frac{k}{n}\underline{p} \ge (1 - \frac{k}{n})p + \frac{k}{n}\underline{p} + \frac{k}{n}\eta \ge \underline{p} + \eta$. Namely, the series $\sum_{e:t(e)=i(\omega_0)} G_{l,k}^p(e \cdot \omega) f(e \cdot \omega)$ converges for each bounded function f.

Lemma 3.5. *Assume that the incidence matrix of E^∞ is finitely irreducible and conditions (g.1)–(g.5) and (ψ.1)–(ψ.4) with fixed nonnegative integer n are satisfied. Then for any $p > p(n)$, there exist weakly Hölder continuous functions $g_{0,p}$, $g_{1,p}$, ..., $g_{n,p}$, $\tilde{g}_{n,p}(\varepsilon, \cdot)$ and positive constants $c_{18} = c_{18}(k), c_{19} = c_{19}(k)$, and ε' such that for $0 < \varepsilon < \varepsilon'$*

$$|g(\varepsilon, \cdot)|^p = g_{0,p} + g_{1,p}\varepsilon + \cdots + g_{n,p}\varepsilon^n + \tilde{g}_{n,p}(\varepsilon, \cdot)\varepsilon^n \tag{3.3}$$

$$\text{with} \quad g_{k,p}(\omega) = \sum_{l=0}^{k}\binom{p}{l}G_{l,k}^p(\omega) \tag{3.4}$$

satisfying that for any $k = 0, 1, \ldots, n$ and for any $\omega \in E^\infty$

$$|g_{k,p}(\omega)| \le c_{18}|g(\omega)|^{p - \frac{k}{n}p(n) + \frac{k}{n}\underline{p}}, \tag{3.5}$$

$$|g_{k,p}(\omega) - g_{k,p}(\upsilon)| \le c_{19}|g(\omega)|^{p - \frac{k}{n}p(n) + \frac{k}{n}\underline{p}} d_\theta(\omega, \upsilon) \text{ with } \omega_0 = \upsilon_0, \tag{3.6}$$

where $\binom{p}{l}$ is the binomial coefficient. Moreover, for any nonempty compact subset I of the interval $(p(n), +\infty)$, there exist constants $\tilde{\eta} > 0$ and $c_{20}(\varepsilon) > 0$ with $c_{20}(\varepsilon) \to 0$ such that

$$\sup_{p\in I}|\tilde{g}_{n,p}(\varepsilon, \omega)| \le c_{20}(\varepsilon)|g(\omega)|^{\underline{p}+\tilde{\eta}} \tag{3.7}$$

for any small $\varepsilon > 0$ and for any $\omega \in E^\infty$.

Proof. Lemma 3.3 guarantees (3.5) and (3.6) by putting $c_{18} = \sum_{l=0}^{k} |\binom{p}{l}| c_{15}(k,l)$ and $c_{19} = \sum_{l=0}^{k} |\binom{p}{l}| c_{16}(k,l)$. It remains to show that (3.7) holds. Fix $\omega \in E^{\infty}$. For the sake of convenience, we omit 'ω' from the notation, i.e., we write $g = g(\omega)$, $g_k = g_k(\omega)$, $\tilde{g}_n(\varepsilon) = \tilde{g}_n(\varepsilon,\omega)$, $g_{k,p} = g_{k,p}(\omega)$, and $\tilde{g}_{n,p}(\varepsilon) = \tilde{g}_{n,p}(\varepsilon,\omega)$. Put $x(\varepsilon) = \sum_{k=1}^{n} g_k \varepsilon^k$ and $g(\varepsilon) = g + x(\varepsilon) + \tilde{g}_n(\varepsilon)\varepsilon^n$. We also assume that $c_4(\varepsilon)$ satisfies $c_4(\varepsilon)^{\tilde{t}} < 1/2$ by making ε small enough if necessary. We take $\eta \in (0, \min I - p(n))$. Then for any $p \in I$, we see $\max I \geq p \geq \min I > p(n) + \eta \geq \underline{p} + \eta > \underline{p}$. First we check the following claim:

Claim 1. *Assertion* (3.7) *holds for* $n = 0$.

Indeed, for each $\varepsilon > 0$, we will consider two cases: $|g| \leq c_4(\varepsilon)$ and $|g| > c_4(\varepsilon)$.

Case $|g| \leq c_4(\varepsilon)$. We have

$$\begin{aligned}
\big||g(\varepsilon)|^p - |g|^p\big| &\leq (c_{21}-1)|g|^p + c_{21}|\tilde{g}_0(\varepsilon)|^p \\
&\leq (c_{21}-1)c_4(\varepsilon)^{p-\underline{p}-\eta}|g|^{\underline{p}+\eta} + c_{21}c_4(\varepsilon)^p|g|^{p\tilde{t}} \\
&\qquad\qquad (\because |g| \leq c_4(\varepsilon) \text{ and (g.5)}) \\
&\leq \big((c_{21}-1)c_4(\varepsilon)^{p-\underline{p}-\eta} + c_{21}c_4(\varepsilon)^p\big)|g|^{\underline{p}+\tilde{t}\eta} \\
&\qquad\qquad (\because p > p(n) + \eta \geq \underline{p}/\tilde{t} + \eta) \\
&\leq (c_{21}-1+c_{21})c_4(\varepsilon)^{\min I-\underline{p}-\eta}|g|^{\underline{p}+\tilde{t}\eta} = c_4(\varepsilon)^{\min I-\underline{p}-\eta}|g|^{\underline{p}+\tilde{t}\eta}
\end{aligned}$$

with $c_{21} = \max(1, 2^{p-1})$, where the first inequality holds by the basic facts of inequalities (e.g. [14, Corollary 8.1.4.]).

Case $|g| > c_4(\varepsilon)$. In this case, we see $|\tilde{g}_0(\varepsilon)| \leq c_4(\varepsilon)|g|^{\tilde{t}} \leq c_4(\varepsilon)^{\tilde{t}}|g| \leq |g|/2$. Moreover, from $\operatorname{sign}(g)(g + \tilde{g}_0(\varepsilon)) = |g| + \operatorname{sign}(g)\tilde{g}_0(\varepsilon) \geq |g|/2 > 0$, we have $\operatorname{sign}(g) = \operatorname{sign}(g(\varepsilon))$, and therefore, the equation $|g + \tilde{g}_0(\varepsilon)| = |g| + \operatorname{sign}(g)\tilde{g}_0(\varepsilon)$ follows. By using the Mean Value Theorem for the function $a \mapsto (|g| + a)^p$, we get the following estimate:

$$\begin{aligned}
\big||g(\varepsilon)|^p - |g|^p\big| &= p(|g| + \alpha \operatorname{sign}(g)\tilde{g}_0(\varepsilon))^{p-1}|\operatorname{sign}(g)\tilde{g}_0(\varepsilon)| \\
&\leq \begin{cases} p(1/2)^{p-1}|g|^{p-1}|\tilde{g}_0(\varepsilon)| & \text{if } p < 1, \\ p(3/2)^{p-1}|g|^{p-1}|\tilde{g}_0(\varepsilon)| & \text{if } p \geq 1 \end{cases} \\
&\leq \max(1, \max I(3/2)^{\max I-1})c_4(\varepsilon)^{\tilde{t}}|g|^p \quad (\because |\tilde{g}_0(\varepsilon)| \leq c_4(\varepsilon)^{\tilde{t}}|g|) \\
&\leq \max(1, \max I(3/2)^{\max I-1})c_4(\varepsilon)^{\tilde{t}}|g|^{\overline{p}+\eta}.
\end{aligned}$$

Thus (3.7) is valid.

Claim 2. *If* $n \geq 1$ *and* $|g|^{p-\underline{p}-\eta/2} \leq \varepsilon^n$, *then assertion* (3.7) *holds.*

In this case, we have $|g| \leq \varepsilon^{c_{22}n}$ with $c_{22} = 1/(\max I - \underline{p} - \eta/2)$. To estimate the function $\tilde{g}_{n,p}(\varepsilon) = (-|g|^p - \sum_{k=1}^n g_{k,p}\varepsilon^k + |g(\varepsilon)|^p)/\varepsilon^n$, we first consider the function $|g|^p$. We obtain

$$|g|^p = |g|^{\eta/4}|g|^{p-\eta/4} \leq c_{23}(\varepsilon)|g|^{\underline{p}+\eta/4}\varepsilon^n \tag{3.8}$$

by putting $c_{23}(\varepsilon) = \varepsilon^{c_{22}n\eta/4}$. Next, we will calculate $g_{k,p}$ in the expression of $\tilde{g}_{n,p}(\varepsilon)$.

$$\begin{aligned}
|g_{k,p}| &\leq c_{18}(k)|g|^{\frac{n-k}{n}p+\frac{k}{n}\underline{p}+\frac{k}{n}\eta} && (\because \text{ (3.5) and } |g|^{\frac{k}{n}p} \leq |g|^{\frac{k}{n}p(n)+\frac{k}{n}\eta}) \\
&\leq c_{18}(k)|g|^{\frac{n-k}{n}\frac{\eta}{2}+\underline{p}+\frac{k}{n}\eta}\varepsilon^{n-k} && (\because |g|^p \leq |g|^{\underline{p}+\eta/2}\varepsilon^n) \\
&\leq c_{18}(k)c_{24}(\varepsilon)|g|^{\underline{p}+\frac{\eta}{2}}\varepsilon^{n-k}
\end{aligned} \tag{3.9}$$

with $c_{24}(\varepsilon) = \varepsilon^{c_{22}\eta/2}$. Finally, we consider the inequality

$$|g(\varepsilon)|^p \leq c_{25}(|g|^p + |x(\varepsilon)|^p + |\tilde{g}_n(\varepsilon)|^p\varepsilon^{np})$$

with $c_{25} = \max(1, 3^{\max I - 1})$. Let $p_k = \underline{p} + (n/k)(1 - t_k)$. It follows from

$$p > p(n) + \eta \geq p_k + \eta \quad \text{and} \quad t_k = 1 - \left(\frac{k}{n}\right)(p_k - \underline{p})$$

that

$$\begin{aligned}
|x(\varepsilon)|^p &\leq c_{26}c_2^p \sum_{k=1}^n |g|^{t_k p}\varepsilon^{kp} \\
&\leq c_{26}c_2^p \sum_{k=1}^n \begin{cases} |g|^{\underline{p}+t_k\eta}\varepsilon^n & \text{if } kp \geq n \quad (\because p > \underline{p}/t_k + \eta), \\ |g|^{t_k p - \frac{1}{n}(p-\underline{p}-\eta/2)(n-kp)}\varepsilon^n & \text{if } kp < n \\ & (\because |g|^{\frac{1}{n}(p-\underline{p}-\eta/2)(n-kp)} \leq \varepsilon^{n-kp}) \end{cases} \\
&\leq c_{26}c_2^p \sum_{k=1}^n \begin{cases} |g|^{\underline{p}+t_k\eta}\varepsilon^n & \text{if } kp \geq n, \\ |g|^{p-\frac{kp}{n}(p-\underline{p}-\eta)-(p-\underline{p}-\frac{\eta}{2})(1-\frac{kp}{n})}\varepsilon^n & \text{if } kp < n, \\ & (\because t_k > 1 - (k/n)(p - \underline{p} - \eta)) \end{cases} \\
&= c_{26}c_2^p \sum_{k=1}^n \begin{cases} |g|^{\underline{p}+t_k\eta}\varepsilon^n & \text{if } kp \geq n, \\ |g|^{\underline{p}+\frac{\eta}{2}+\frac{\eta}{2}\frac{kp}{n}}\varepsilon^n & \text{if } kp < n \end{cases} \\
&\leq c_{27}(\varepsilon)|g|^{\underline{p}+c_{28}\eta}\varepsilon^n,
\end{aligned}$$

where $c_{26} = \max\{1, n^{\max I - 1}\}$, $c_{27}(\varepsilon) = c_{26}c_2^p n\varepsilon^{nc_{28}c_{22}}$ and $c_{28} = \min(t_1, \ldots, t_n, 1/2)/2$. Similarly, put $\tilde{p} = \underline{p} + 1 - \tilde{t}$. We get

$$|\tilde{g}_n(\varepsilon)|^p\varepsilon^{np} \leq \begin{cases} c_4(\varepsilon)^p|g|^{\underline{p}+\eta\tilde{t}}\varepsilon^n & \text{if } np \geq n \quad (\because p > \underline{p}/\tilde{t} + \eta), \\ |g|^{\tilde{t}p-\frac{1}{n}(p-\underline{p}-\eta/2)(n-np)}\varepsilon^n & \text{if } np < n \end{cases}$$

$$= \begin{cases} c_4(\varepsilon)^p |g|^{\underline{p}+\eta\tilde{t}} \varepsilon^n & \text{if } np \geq n, \\ |g|^{\underline{p}+\eta/2+p(p-\tilde{p}-\eta/2)} \varepsilon^n & \text{if } np < n \quad (\because \tilde{t} = 1 - \tilde{p} + \underline{p}) \end{cases}$$
$$\leq \max(c_4(\varepsilon)^{\min I}, c_{23}(\varepsilon))|g|^{\underline{p}+\eta \min(\tilde{t},1/4)} \varepsilon^n.$$

Thus, $|g(\varepsilon)|^p$ is estimated by

$$|g(\varepsilon)|^p \leq c_{29}(\varepsilon)|g|^{\underline{p}+c_{30}\eta} \varepsilon^n \tag{3.10}$$

with $c_{29}(\varepsilon) = c_{25} \max\{c_{23}(\varepsilon), c_{27}(\varepsilon), c_4(\varepsilon)^{\min I}\}$ and $c_{30} = \min\{1/4, c_{28}, \tilde{t}\}$. Consequently, inequalities (3.8), (3.9) and (3.10) imply $|\tilde{g}_{n,p}(\varepsilon)| \leq c_{31}(\varepsilon)|g|^{\underline{p}+c_{30}\eta}$ with $c_{31}(\varepsilon) = c_{23}(\varepsilon) + \sum_{k=1}^n c_{18}(k)c_{24}(\varepsilon) + c_{29}(\varepsilon)$. Hence, the assertion is valid.

Put $c_{32}(\varepsilon) = nc_2\varepsilon^{c_{33}} + c_4(\varepsilon)$ and $c_{33} = (\eta/2)/(\min I - \underline{p} - \eta/2) > 0$.

Claim 3. *If $n \geq 1$, $|g|^{p-\underline{p}-\eta/2} > \varepsilon^n$ and $c_{32}(\varepsilon) < 1/2$ are satisfied, then the inequality*

$$|x(\varepsilon)| + |\tilde{g}_n(\varepsilon)\varepsilon^n| \leq c_{32}(\varepsilon)|g|$$

holds. In this case, we have $\operatorname{sign}(g(\varepsilon)) = \operatorname{sign}(g + x(\varepsilon)) = \operatorname{sign}(g)$.

Indeed, note that the number $c_{33} = 1 - (\max I - \underline{p} - \eta)/(\max I - \underline{p} - \eta/2)$ is less than 1. We obtain

$$|x(\varepsilon)| + |\tilde{g}_n(\varepsilon)\varepsilon^n| \leq \sum_{k=1}^n c_2|g|^{t_k}\varepsilon^k + c_4(\varepsilon)|g|^{\tilde{t}}\varepsilon^n.$$

We have the estimates

$$|g|^{t_k-1}\varepsilon^{k-c_{33}} \leq |g|^{-\frac{k}{n}(p_k-\underline{p})+\frac{k}{n}(p-\underline{p}-\eta/2)-\frac{c_{33}}{n}(\max I-\underline{p}-\eta/2)}$$
$$= |g|^{\frac{1}{n}(k(p-\underline{p}-\eta/2)-\eta/2)} = |g|^{\frac{1}{n}(k(p-\underline{p}-\eta)+\eta(k-1)/2)} \leq 1$$

and $|g|^{\tilde{t}-1}\varepsilon^n \leq |g|^{p-\tilde{p}-\eta/2} \leq 1$, where p_k and $\tilde{p}$ are given in Claim 2.

Finally, we prove the last assertion. Choose any $\varepsilon > 0$ so that $c_{32}(\varepsilon) < 1/2$. We have

$$\operatorname{sign}(g)g(\varepsilon) = |g| + \operatorname{sign}(g)x(\varepsilon) + \operatorname{sign}(g)\tilde{g}_n(\varepsilon)$$
$$\geq |g| - (|x(\varepsilon)| + |\tilde{g}_n(\varepsilon)|) \geq |g|/2 > 0.$$

This means that $\operatorname{sign}(g) = \operatorname{sign}(g(\varepsilon))$. Similarly,

$$\frac{g}{g + x(\varepsilon)} = \frac{|g|}{|g| + \operatorname{sign}(g)x(\varepsilon)} \geq \frac{|g|}{|g| + |x(\varepsilon)|} \geq \frac{2}{3} > 0.$$

Hence, we get $\operatorname{sign}(g) = \operatorname{sign}(g + x(\varepsilon))$.

Claim 4. *If* $n \geq 1$, $|g|^{p-\underline{p}-\eta/2} > \varepsilon^n$ *and* $c_{32}(\varepsilon) < 1/2$ *are satisfied, then*

$$\left|g + \sum_{k=1}^{n} g_k \varepsilon^k + \tilde{g}_n(\varepsilon)\varepsilon^n\right|^p = \left|g + \sum_{k=1}^{n} g_k \varepsilon^k\right|^p + Y(\varepsilon)$$

and $|Y(\varepsilon)| \leq c_{34}(\varepsilon)|g|^{p-1+\tilde{\iota}}\varepsilon^n$ *with* $c_{34}(\varepsilon) \to 0$ *are valid.*

Note the form $|g + x(\varepsilon) + \tilde{g}_n(\varepsilon)\varepsilon^n| = |g + x(\varepsilon)| + \operatorname{sign}(g)\tilde{g}_n(\varepsilon)\varepsilon^n$ from the above claim for any small $\varepsilon > 0$ so that $c_{32}(\varepsilon) < 1/2$. By virtue of the Mean Value Theorem, we have $Y(\varepsilon) = p(|g + x(\varepsilon)| + \alpha \operatorname{sign}(g)\tilde{g}_n(\varepsilon)\varepsilon^n)^{p-1} \operatorname{sign}(g)\tilde{g}_n(\varepsilon)\varepsilon^n$ for some $\alpha \in [0, 1]$. Thus, it follows from the above claim again that

$$\begin{aligned}|Y(\varepsilon)| &\leq \begin{cases} p(|g| - |x(\varepsilon)| - |\tilde{g}_n(\varepsilon)\varepsilon^n|)^{p-1}|\tilde{g}_n(\varepsilon)|\varepsilon^n & \text{if } p < 1,\\ p(|g| + |x(\varepsilon)| + |\tilde{g}_n(\varepsilon)\varepsilon^n|)^{p-1}|\tilde{g}_n(\varepsilon)|\varepsilon^n & \text{if } p \geq 1\end{cases}\\ &\leq 2^{|1-p|}p|g|^{p-1}c_4(\varepsilon)|g|^{\tilde{\iota}}\varepsilon^n \leq c_{34}(\varepsilon)|g|^{\underline{p}+\eta}\end{aligned}$$

with $c_{34}(\varepsilon) = \max(2, 2^{\max I-1})(\max I)c_4(\varepsilon)$.

Claim 5. *If* $n \geq 1$, $|g|^{p-\underline{p}-\eta/2} > \varepsilon^n$ *and* $c_{32}(\varepsilon) < 1/2$ *are satisfied, then* $|\tilde{g}_{n,p}(\varepsilon)| \leq c_{35}(\varepsilon)|g|^{\underline{p}+\tilde{\iota}\eta}$ *holds for some constant* $c_{35}(\varepsilon) > 0$ *with* $c_{35}(\varepsilon) \to 0$.

Now we apply the Taylor expansion to the function $F\colon \varepsilon \mapsto |g + x(\varepsilon)|^p = (|g| + \operatorname{sign}(g)x(\varepsilon))^p$:

$$|g + x(\varepsilon)|^p = F(\varepsilon) = F(0) + \sum_{k=1}^{n} \frac{F^{(k)}(0)}{k!}\varepsilon^k + \left(\frac{F^{(n)}(\alpha\varepsilon)}{n!} - \frac{F^{(n)}(0)}{n!}\right)\varepsilon^n$$

for some $\alpha \in [0, 1]$. By virtue of the Faà di Bruno formula [6], we obtain the equation $F^{(k)}(0)/k! = g_{k,p}$. We will show that the remainder $|F^{(n)}(\alpha\varepsilon) - F^{(n)}(0)|$ is bounded by $c_{36}(\varepsilon)|g|^{\underline{p}+\eta}$ with some constant $c_{36}(\varepsilon) \to 0$. Since the function F is the composition of the two functions $G\colon y \mapsto y^p$ and $H\colon \varepsilon \mapsto |g| + \operatorname{sign}(g)x(\varepsilon)$, we have

$$\begin{aligned}&\frac{F^{(n)}(\alpha\varepsilon)}{n!} - \frac{F^{(n)}(0)}{n!}\\ &\quad = \sum_{j=1}^{n}\sum \binom{p}{j} j!\left(H(\alpha\varepsilon)^{p-j}\prod_{i=1}^{n}\frac{H^{(i)}(\alpha\varepsilon)^{\lambda_i}}{\lambda_i!(i!)^{\lambda_i}} - H(0)^{p-j}\prod_{i=1}^{n}\frac{H^{(i)}(0)^{\lambda_i}}{\lambda_i!(i!)^{\lambda_i}}\right),\end{aligned}$$

where the second summation is taken over all nonnegative integers $\lambda_1, \ldots, \lambda_n$ so that $\sum_{k=1}^{n}\lambda_k = j$ and $\sum_{k=1}^{n}k\lambda_k = n$. Put $A(\varepsilon) = H(\alpha\varepsilon)^{p-j}$ and $B_i(\varepsilon) = H^{(i)}(\alpha\varepsilon)^{\lambda_i}$ $(1 \leq i \leq n)$. We begin with the estimate $A(\varepsilon) - A(0)$:

$$\begin{aligned}|A(\varepsilon) - A(0)| &= |(|g| + \operatorname{sign}(g)x(\alpha\varepsilon))^{p-j} - |g|^{p-j}|\\ &\leq 2^{1+|p-j|}|g|^{p-j}c_{32}(\varepsilon)|p-j|\\ &\leq c_{37}(\varepsilon)|g|^{p-j}\end{aligned}$$

with $c_{37}(\varepsilon) = 2^{1+\max(|\max I-1|,|\min I-n|)} c_{32}(\varepsilon)\max(|\max I - 1|, |\min I - n|)$, where the second inequality uses Claim 3 and the basic inequality $|(1+b)^p - 1| \leq 2^{1+|p|}|bp|$ whenever $b,\ p \in \mathbb{R}$ with $|b| < 1/2$. Moreover, it follows from $\varepsilon^l < |g|^{\frac{l}{n}(p-\underline{p}-\eta/2)}$ that for $\varepsilon \geq 0$,

$$\begin{aligned} |H^{(i)}(\alpha\varepsilon)| &= \left|\sum_{l=0}^{n-i} \frac{(l+i)!}{i!} g_{l+i}\alpha^l\varepsilon^l\right| \\ &\leq c_2 \sum_{l=0}^{n-i} \frac{(l+i)!}{l!}|g|^{1-\frac{l+i}{n}(p_{l+i}-\underline{p})+\frac{l}{n}(p-\underline{p}-\eta/2)} \\ &\leq c_{38}(i)|g|^{1-\frac{i}{n}(p-\underline{p}-\eta)} \end{aligned}$$

by putting $c_{38}(i) = c_2\sum_{l=0}^{n-i}(l+i)!/l!$ and by using the fact $p_{l+i} = \underline{p} + \frac{n}{l+i}(1-t_{l+i}) \leq p(n) < p - \eta$. Therefore,

$$\begin{aligned} &|(A(\varepsilon) - A(0))B_1(0)\cdots B_n(0)| \\ &\quad\leq c_{37}(\varepsilon)c_{38}(1)^{\lambda_1}\cdots c_{38}(n)^{\lambda_n}|g|^{p-j+\lambda_1+\cdots+\lambda_n-\frac{\lambda_1+2\lambda_2+\cdots+n\lambda_n}{n}(p-\underline{p}-\eta)} \\ &\quad= c_{37}(\varepsilon)c_{38}(1)^{\lambda_1}\cdots c_{38}(n)^{\lambda_n}|g|^{\underline{p}+\eta} \end{aligned} \tag{3.11}$$

by choosing $\lambda_1, \ldots, \lambda_n$. On the other hand, we note the inequality

$$\varepsilon^{l-c_{39}} \leq |g|^{\frac{i}{n}(p-\underline{p}-\eta/2)-\frac{\eta}{2n}} \leq |g|^{\frac{i}{n}(p-\underline{p}-\eta)}$$

by using the number $c_{39} = (\eta/2)/(\min I - \underline{p} - \eta/2)$. We get the estimate

$$\begin{aligned} |H^{(i)}(\alpha\varepsilon) - H^{(i)}(0)| &\leq \sum_{l=1}^{n-i} \frac{(l+i)!}{i!}|g_{l+i}|(\alpha\varepsilon)^l \\ &\leq c_2\sum_{l=1}^{n-i}\frac{(l+i)!}{i!}|g|^{t_{l+i}}|g|^{\frac{i}{n}(p-\underline{p}-\eta)}\varepsilon^{c_{39}} \\ &\leq c_{38}(i)|g|^{1-\frac{i}{n}(p-\underline{p}-\eta)}\varepsilon^{c_{39}}. \end{aligned}$$

Thus,

$$\begin{aligned} &|B_i(\varepsilon) - B_i(0)| \\ &\quad= |H^{(i)}(\alpha\varepsilon) - H^{(i)}(0)||H^{(i)}(\alpha\varepsilon)^{\lambda_i-1} + H^{(i)}(\alpha\varepsilon)^{\lambda_i-2}H^{(i)}(0) + \cdots + H^{(i)}(0)^{\lambda_i-1}| \\ &\quad\leq c_{38}(i)^{\lambda_i}|g|^{\lambda_i-\frac{i\lambda_i}{n}(p-\underline{p}-\eta)}\varepsilon^{c_{39}}. \end{aligned}$$

Moreover,

$$\begin{aligned} |A(\varepsilon)| &= |(|g| + \operatorname{sign}(g)x(\alpha\varepsilon))^{p-j}| \\ &\leq \begin{cases} (|g| + |x(\alpha\varepsilon)|)^{p-j} & \text{if } p-j \geq 0, \\ (|g| - |x(\alpha\varepsilon)|)^{p-j} & \text{if } p-j < 0 \end{cases} \\ &\leq c_{40}|g|^{p-j} \end{aligned}$$

with $c_{40} = \max\{(3/2)^{\max I - 1}, (1/2)^{\min I - n}\}$. Consequently, we obtain

$$\begin{aligned}
&|A(\varepsilon)B_1(\varepsilon)\cdots B_{i-1}(\varepsilon)(B_i(\varepsilon)-B_i(0))B_{i+1}(0)\cdots B_n(0)| \\
&\quad \le c_{40}c_{38}(1)^{\lambda_1}\cdots c_{38}(n)^{\lambda_n}\varepsilon^{c_{39}}|g|^{p-j+\lambda_1+\cdots+\lambda_n-\frac{\lambda_1+2\lambda_2+\cdots+n\lambda_n}{n}(p-\underline{p}-\eta)} \\
&\quad = c_{41}(\varepsilon)|g|^{\underline{p}+\eta} \qquad (3.12)
\end{aligned}$$

by putting $c_{41}(\varepsilon) = c_{40}c_{38}(1)^{\lambda_1}\cdots c_{38}(n)^{\lambda_n}\varepsilon^{c_{39}}$. By Claim 4 and inequalities (3.11) and (3.12) the function $\tilde{g}_{n,p}(\varepsilon) = Z(\varepsilon) + (F^{(n)}(\alpha\varepsilon) - F^{(n)}(0))/n!$ fulfills the assertion of the claim.

Claim 6. *When $n \ge 1$, the main assertion (3.7) holds by putting $\varepsilon_1 > 0$ so that $c_{32}(\varepsilon) < 1/2$ for $\varepsilon < \varepsilon_1$.*

Indeed, by virtue of Claim 2 and Claim 5, this assertion is given with $c_{20}(\varepsilon) = c_{31}(\varepsilon) + c_{35}(\varepsilon)$. Hence, the proof is complete. ■

Denote by $\mathcal{L}(\mathcal{X})$ the set of all bounded linear operators acting on a normed linear space $\mathcal{X}$.

Lemma 3.6. *Assume that the incidence matrix of E^∞ is finitely irreducible and the conditions (g.1)–(g.5) and (ψ.1)–(ψ.4) with a fixed nonnegative integer n are satisfied. Then for any nonempty compact subset $I \subset (p(n), \infty)$, there exist operators $\mathcal{L}_{1,p}, \ldots, \mathcal{L}_{n,p} \in \mathcal{L}(F_{\theta,b}(E^\infty))$ and $\tilde{\mathcal{L}}_{n,p}(\varepsilon,\cdot) \in \mathcal{L}(C_b(E^\infty))$ ($p \in I$) such that*

$$\mathcal{L}_{p\log|g(\varepsilon,\cdot)|+\log\psi(\varepsilon,\cdot)} = \mathcal{L}_{p\log|g|+\log\psi} + \mathcal{L}_{1,p}\varepsilon + \cdots + \mathcal{L}_{n,p}\varepsilon^n + \tilde{\mathcal{L}}_{n,p}(\varepsilon,\cdot)\varepsilon^n, \quad (3.13)$$

$$\sup_{p\in I} \|\tilde{\mathcal{L}}_{n,p}(\varepsilon,\cdot)\|_\infty \to 0 \quad (3.14)$$

and $\sup_{p\in I}\|\mathcal{L}_{k,p}\|_\theta < \infty$, where $\mathcal{L}_{k,p}f(\omega) = \sum_{e\in E:t(e)=i(\omega_0)} \zeta_{k,p}(e\cdot\omega)f(e\cdot\omega)$ and $\zeta_{k,p} = \sum_{i=0}^k g_{i,p}\psi_{k-i}$.

Proof. By virtue of the expansions (3.3) and (1.2), we get the expansion (3.13) and convergence (3.14). It remains to check $\sup_{p\in I}\|\mathcal{L}_{k,p}\|_\theta < \infty$ for $k = 1, 2, \ldots, n$. First, we show $\sup_{p\in I}\|\mathcal{L}_{k,p}\|_\infty < \infty$. Let $\eta > 0$ be given so that $p(n) + \eta < \min I$. From (3.5) and condition (ψ.3) in addition to Remark 3.4, we get

$$\begin{aligned}
\|\mathcal{L}_{k,p}f\|_\infty &\le c_{42}\|\mathcal{L}_{(p-\frac{k}{n}p(n)+\frac{k}{n}\underline{p})\log|g|+\log\psi}1\|_\infty\|f\|_\infty \\
&\le c_{42}\|\mathcal{L}_{(\underline{p}+\eta)\log|g|+\log\psi}1\|_\infty\|f\|_\infty
\end{aligned}$$

for any $p \in I$ by putting $c_{42} = (n+1)c_{18}c_5$. Furthermore, $\|\mathcal{L}_{(\underline{p}+\eta)\log|g|+\log\psi}1\|_\infty < \infty$ by $P((\underline{p}+\eta)\log|g| + \log\psi) < +\infty$ (see Proposition A.1). Therefore, we have

$\|\mathcal{L}_{k,p}\|_\infty < \infty$ uniformly in $p \in I$. Next, we check the boundedness of $[\mathcal{L}_{k,p} f]_\theta$ for $f \in F_{\theta,b}(E^\infty)$. It follows from (3.5) and (3.6) that for $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$

$$\begin{aligned}
&|\mathcal{L}_{k,p} f(\omega) - \mathcal{L}_{k,p} f(\upsilon)| \\
&\le \sum_{\substack{e\in E:\\ t(e)=i(\omega_0)}} \big(|\zeta_{k,p}(e\cdot\omega) - \zeta_{k,p}(e\cdot\upsilon)||f(e\cdot\omega)| + |\zeta_{k,p}(e\cdot\upsilon)||f(e\cdot\omega) - f(e\cdot\upsilon)|\big) \\
&\le c_{43} \sum_{\substack{e\in E:\\ t(e)=i(\omega_0)}} |g(e\cdot\omega)|^{p-\frac{k}{n}p(n)+\frac{k}{n}\underline{p}}\psi(e\cdot\omega)\|f\|_\infty d_\theta(e\cdot\omega, e\cdot\upsilon) \\
&\quad + c_{42} \sum_{\substack{e\in E:\\ t(e)=i(\omega_0)}} |g(e\cdot\upsilon)|^{p-\frac{k}{n}p(n)+\frac{k}{n}\underline{p}}\psi(e\cdot\upsilon)[f]_\theta d_\theta(e\cdot\omega, e\cdot\upsilon) \\
&\le \theta(c_{43}+c_{42})\|\mathcal{L}_{(\underline{p}+\eta)\log|g|+\log\psi}1\|_\infty \|f\|_\theta d_\theta(\omega,\upsilon)
\end{aligned}$$

with $c_{43} = (n+1)(c_{18}c_6 + c_{19}(1+c_6))$. Thus,

$$[\mathcal{L}_{k,p} f]_\theta \le \theta(c_{43}+c_{42})\|\mathcal{L}_{(\underline{p}+\eta)\log|g|+\log\psi}1\|_\infty \|f\|_\theta$$

uniformly in $p \in I$. Hence, we obtain the boundedness of $\sup_{p\in I}\|\mathcal{L}_{k,p}\|_\theta$. ■

Lemma 3.7. *Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph. Let $\varphi \in F_\theta(E^\infty, \mathbb{R})$ satisfy (g.2) and (g.3). Then for any number $\eta > 0$ and integer $k \ge 1$, the d_θ-Lipschitz norm of the function $\omega \mapsto |g(\omega)|^\eta(\log|g(\omega)|)^k$ is bounded by a constant $c_{44} = c_{44}(\eta, k)$.*

Proof. From the upper bound of $x \mapsto -x\log x$ is e^{-1}, the norm $\||g|^\eta(\log|g|)^k\|_\infty$ is bounded by $k^k/(e^k\eta^k)$. On the other hand, let $\alpha = \eta/k$. We have for $\omega, \upsilon \in E^\infty$ with $\omega_0 = \upsilon_0$

$$\begin{aligned}
&\big||g(\omega)|^\alpha \log|g(\omega)| - |g(\upsilon)|^\alpha \log|g(\upsilon)|\big| \\
&\quad \le \big|g(\omega)|^\alpha\big|\log|g(\omega)| - \log|g(\upsilon)|\big| + |\log(|g(\upsilon)|)|g(\upsilon)|^\alpha|e^{\alpha\log|g(\omega)| - \alpha\log|g(\upsilon)|} - 1| \\
&\quad \le c_{45} d_\theta(\omega,\upsilon)
\end{aligned}$$

with $c_{45} = (c_1 + (\alpha e)^{-1} e^{\alpha c_1 \theta}\alpha c_1)$. Thus, we obtain

$$\big[|g|^\eta(\log|g|)^k\big]_\theta \le k c_{45}\big(k/(e\eta)\big)^{k-1}.$$

Hence, the assertion is fulfilled. ■

The function $(p,\varepsilon) \mapsto \mathcal{L}_{p\log g(\varepsilon,\cdot)+\log\psi(\varepsilon,\cdot)}$ has also an asymptotic expansion in the sense of the following lemma:

Lemma 3.8. *Assume that the incidence matrix of E^∞ is finitely irreducible and conditions (g.1)–(g.5) and (ψ.1)–(ψ.4) with a fixed nonnegative integer n are satisfied. Choose any nonempty compact subset $I \subset (p(n), +\infty)$. Then for any $s, p \in I$, there exist operators $\mathcal{Z}_{v,q,s} \in \mathcal{L}(F_{\theta,b}(E^\infty))$ $(0 \le v, q \le n)$ and $\hat{\mathcal{Z}}_{n,s,p} \in \mathcal{L}(C_b(E^\infty))$ such that the Ruelle operator of $p \log |g(\varepsilon, \cdot)| + \log \psi(\varepsilon, \cdot)$ is expanded as*

$$\mathcal{L}_{p \log |g(\varepsilon,\cdot)| + \log \psi(\varepsilon,\cdot)} = \sum_{v=0}^{n} \sum_{q=0}^{n} \mathcal{Z}_{v,q,s} \varepsilon^v (p-s)^q + (p-s)^{n+1} \hat{\mathcal{Z}}_{n,s,p} + \tilde{\mathcal{L}}_{n,p}(\varepsilon, \cdot)\varepsilon^n, \tag{3.15}$$

and $\sup_{s,p\in I} \|\hat{\mathcal{Z}}_{n,s,p}\|_\infty < +\infty$, where $\mathcal{Z}_{0,0,s}$ equals $\mathcal{L}_{s \log |g| + \log \psi}$. Here $\mathcal{Z}_{v,q,s} f := \mathcal{L}_0(h_{v,q,s} f)$ and $\hat{\mathcal{Z}}_{n,s,p} f := \sum_{v=0}^{n} \mathcal{L}_0(\hat{h}_{v,s,p} f)\varepsilon^v$ for $f \in C_b(E^\infty)$ are given as

$$h_{v,q,s} := \sum_{k=0}^{v} \sum_{l=0}^{k} \sum_{j=0}^{\min(l,q)} \frac{a_{l,j,s}}{(q-j)!} (\log |g|)^{q-j} G^s_{l,k} \psi_{v-k}$$

$$\hat{h}_{v,s,p} := \sum_{k=0}^{v} \sum_{l=0}^{k} \sum_{j=0}^{l} a_{l,j,s} \hat{\Gamma}_{s,p,n-j+1} G^s_{l,k} \psi_{v-k}$$

$$\hat{\Gamma}_{s,p,i}(\omega) := \int_0^1 \frac{(1-u)^{i-1}}{(i-1)!} |g(\omega)|^{u(p-s)} (\log |g(\omega)|)^i \, du,$$

where $\mathcal{L}_0 f$ means $\sum_{e \in E : t(e) = i(\omega_0)} f(e \cdot \omega)$, and $a_{l,j,s}$ are numbers defined in (3.16) below.

Proof. First we show the expansion (3.15). We take $\eta \in (0, \min I - p(n))$ and $p_k = \underline{p} + (n/k)(1 - t_k)$. By Theorem 3.6, we have the expansion of $\mathcal{L}_{p \log |g(\varepsilon,\cdot)| + \log \psi(\varepsilon,\cdot)}$ as $\sum_{v=0}^{n} \mathcal{L}_{v,p} \varepsilon^v + \tilde{\mathcal{L}}_{n,p}(\varepsilon, \cdot)\varepsilon^n$, where $\mathcal{L}_{v,p}$ and $\tilde{\mathcal{L}}_{n,p}$ are defined in Lemma 3.6. Therefore, we will expand the operator $\mathcal{L}_{v,p}$. We remark that the expansion of the map $p \mapsto |g|^p$ is as follows:

$$|g|^p = e^{p \log |g|} = \sum_{q=0}^{i} \frac{(\log |g|)^q}{q!} |g|^s (p-s)^q + (p-s)^{i+1} |g|^s \hat{\Gamma}_{s,p,i+1}.$$

for any $i \ge 0$. In addition to the notation $G^p_{l,k}$ in (3.1), $g_{k,p}$ in (3.4), $\mathcal{L}_{k,p}$, and $\zeta_{k,p}$ in (3.6), we obtain

$$\begin{aligned}
\mathcal{L}_{v,p} f &= \sum_{k=0}^{v} \sum_{l=0}^{k} \mathcal{L}_0 (\psi_{v-k} \binom{p}{l} G^p_{l,k} f) \\
&= \sum_{k=0}^{v} \sum_{l=0}^{k} \sum_{j=0}^{l} \mathcal{L}_0 \Bigg(\psi_{v-k} a_{l,j,s} \Bigg(\sum_{q=0}^{n-j} \frac{(\log |g|)^q}{q!} G^s_{l,k} (p-s)^{q+j} \\
&\qquad + \hat{\Gamma}_{s,p,n-j+1} G^s_{l,k} (p-s)^{n+1} \Bigg) f \Bigg) \\
&= \sum_{q=0}^{n} \mathcal{Z}_{v,q,s} f (p-s)^q + \mathcal{L}_0(\hat{h}_{v,s,p} f)(p-s)^{n+1},
\end{aligned}$$

where we use the expansion $\binom{p}{l} = \sum_{j=0}^{l} a_{l,j,s}(p-s)^j$ by putting

$$a_{l,j,s} = \begin{cases} \sum_{\substack{0\le i_1,\dots,i_{l-j}\le l-1: \\ i_1<\dots<i_{l-j}}} \frac{1}{l!}\prod_{q=1}^{l-j}(s-i_q) & \text{if } l\ge 1 \text{ and } 0\le j<l \\ 1/l! & \text{if } l\ge 1 \text{ and } j=l \\ 0 & \text{if } l<j. \end{cases} \tag{3.16}$$

Thus, the equation (3.15) is valid.

Next, we will prove $\mathcal{Z}_{v,q,s} \in \mathcal{L}(F_{\theta,b}(E^\infty))$. In the expression of $h_{v,q,s}$, we rewrite $(\log|g|)^{q-j}G^s_{l,k}\psi_{v-k} = ((\log|g|)^{q-j}|g|^{\eta/2})|g|^{-\eta/2}G^s_{l,k}\psi_{v-k}$. Then the function $(\log|g|)^{q-j}|g|^{\eta/2}$ is in $F_{\theta,b}(E^\infty,\mathbb{R})$ from Lemma 3.7. By Proposition 3.1 with $f_0 = (\log|g|)^{q-j}|g|^{\eta/2}$, $h_0=1$, $f_1=|g|^{-\eta/2}$, $h_1=|g|^{-\eta/2}$, $f_2=G^s_{l,k}$, $h_2=|g|^{s+\eta}$, $f_3=\psi_{v-k}$, $h_3=\psi$, $c_{11}(0)=c_{44}(\eta/2,q-j)$, $c_{11}(1)=1$, $c_{11}(2)=c_{15}(k,l)$, $c_{11}(3)=c_5$, $c_{12}(0)=c_{44}(\eta/2,q-j)$, $c_{12}(1)=c_1(1+c_1\theta)^{\eta/2}$, $c_{12}(2)=c_{16}(k,l)$, and $c_{12}(3)=c_6$, we get the estimate

$$\begin{aligned} &|(\log|g|)^{q-j}G^s_{l,k}\psi_{v-k}| \le c_{46}(k,l,j)|g|^{s+\eta/2}\psi \\ &|(\log|g(\omega)|)^{q-j}G^s_{l,k}(\omega)\psi_{v-k}(\omega) - (\log|g(\upsilon)|)^{q-j}G^s_{l,k}(\upsilon)\psi_{v-k}(\upsilon)| \\ &\qquad \le c_{47}|g(\omega)|^{s+\eta/2}d_\theta(\omega,\upsilon) \end{aligned}$$

for constants $c_{46}=c_{46}(k,l,j)=c_{44}(\eta/2,q-j)c_{15}(k,l)c_5$ and $c_{47}=c_{47}(k,l,j)=c_{13}$ with $n=3$. Thus, we obtain

$$\begin{aligned} &\|\mathcal{Z}_{v,q,s}\|_\infty \le c_{48}\|\mathcal{L}_{(\underline{p}+\eta/2)\log|g|+\log|\psi|}1\|_\infty \\ &[\mathcal{Z}_{v,q,s}f]_\theta \le (c_{49}\theta\|f\|_\infty + c_{48}\theta[f]_\theta)\|\mathcal{L}_{(\underline{p}+\eta/2)\log|g|+\log|\psi|}1\|_\infty, \end{aligned}$$

where we define constants c_{48} and c_{49} by

$$c_{48} = \sum_{k=0}^{v}\sum_{l=0}^{k}\sum_{j=0}^{\min(l,q)} \frac{a_{l,j,s}}{(q-j)!}c_{46}(k,l,j),\quad c_{49} = \sum_{k=0}^{v}\sum_{l=0}^{k}\sum_{j=0}^{\min(l,q)} \frac{a_{l,j,s}}{(q-j)!}c_{47}(k,l,j).$$

Hence, $\mathcal{Z}_{v,q,s} \in \mathcal{L}(F_{\theta,b}(E^\infty,\mathbb{R}))$ is guaranteed.

It remains to check the boundedness of $\hat{\mathcal{Z}}_{n,s,p}$. In the expression of $\hat{h}_{v,s,p}$, we have

$$\begin{aligned} &|\hat{\Gamma}_{s,p,n-j+1}(\omega)G^s_{l,k}(\omega)\psi_{v-k}(\omega)| \\ &\quad \le c_{15}(k,l)c_5|g(\omega)|^{s-\frac{k}{n}p(n)+\frac{k}{n}\underline{p}}\psi(\omega)|\hat{\Gamma}_{s,p,n-j+1}(\omega)| \\ &\quad \le c_{15}(k,l)c_5|g(\omega)|^{\underline{p}+\eta}\psi(\omega)\int_0^1 \frac{(1-u)^{n-j}}{(n-j)!}|\log(|g(\omega)|)|^{n-j+1}\,du \\ &\qquad\qquad (\because \text{Remark } 3.4) \\ &\quad \le \frac{c_{15}(k,l)c_5c_{44}(\eta/2,n-j+1)}{(n-j+1)!}\psi(\omega)|g(\omega)|^{\underline{p}+\eta/2} \quad (\because \text{ Lemma } 3.7). \end{aligned}$$

As a result, we get $|\hat{h}_{v,s,p}(\omega)| \le c_{50}\psi(\omega)|g(\omega)|^{\underline{p}+\eta/2}$ for some constant $c_{50}>0$. Thus, $\hat{\mathcal{Z}}_{n,s,p}$ is bounded by $c_{50}\|\mathcal{L}_{(\underline{p}+\eta/2)\log|g|+\log\psi}1\|_\infty < +\infty$ uniformly in $s,p\in I$. ∎

We will describe a remark concerning the coefficient of the solution $s(\varepsilon)$ before the proof of Theorem 1.1.

Remark 3.9. Assume that the incidence matrix of E^∞ is finitely irreducible and conditions (g.1)–(g.5) and (ψ.1)–(ψ.4) with a fixed nonnegative integer n are satisfied. If we put for each $0 \le k \le n$

$$s_k = \frac{-1}{\nu(h \log|g|)} \Bigg(\sum_{i=1}^{k-1} \nu_i(\mathcal{L}_{s(0)\log|g|+\log\psi|}(h\log|g|)) s_{k-i} + \sum_{i=0}^{k-1} \nu_i(\mathcal{N}_{k-i}h) \Bigg), \tag{3.17}$$

$$\tilde{s}_0(\varepsilon) = -\frac{\nu(\varepsilon, \tilde{\mathcal{L}}_{0,s(\varepsilon)}(\varepsilon, h))}{\nu(\varepsilon, \hat{\mathcal{Z}}_{0,s(0),s(\varepsilon)}h)} \tag{3.18}$$

$$\tilde{s}_n(\varepsilon) = \frac{-1}{\nu(h\log|g|)} \Bigg(\sum_{u=1}^{n} \tilde{\nu}_{n-u}(\varepsilon, \mathcal{N}_u h) + \sum_{i=1}^{n-1} \tilde{s}_{n-i}(\varepsilon)\nu_i(\mathcal{Z}_{0,1,s(0)}h) \tag{3.19}$$

$$+ \nu(\varepsilon, \tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h)) + \frac{s(\varepsilon)-s(0)}{\varepsilon} \tilde{\nu}_{n-1}(\varepsilon, \mathcal{Z}_{0,1,s(0)}h) + \nu(\varepsilon, \hat{\mathcal{N}}_{n+1}(\varepsilon, h))\varepsilon \Bigg),$$

then $s(\varepsilon)$ can be expanded as form (1.3). Here ν, ν_k, $\nu(\varepsilon, \cdot)$ and $\tilde{\nu}_{n-u}(\varepsilon, \cdot)$ appear in the asymptotic expansions $\nu(\varepsilon, f) = \nu(f) + \sum_{k=1}^m \nu_k(f)\varepsilon^k + \tilde{\nu}_m(\varepsilon, f)\varepsilon^m$ $(0 \le m \le n-1)$ of the Perron eigenvector $\nu(\varepsilon, \cdot)$ of $\mathcal{L}_{s(\varepsilon)\log|g(\varepsilon,\cdot)|+\log\psi}$ given by Corollary B.2. We define operators $\mathcal{N}_u, \hat{\mathcal{N}}_{n+1}(\varepsilon, \cdot) \in \mathcal{L}(C_b(E^\infty))$ by

$$\mathcal{N}_u = \sum_{\substack{0 \le v \le u, \\ 0 \le q \le u-v: \\ (v,q) \ne (0,1)}} s_{q,u-v} \mathcal{Z}_{v,q,s(0)} \quad (u = 1, \dots, n), \tag{3.20}$$

$$\hat{\mathcal{N}}_{n+1}(\varepsilon, f) = \sum_{\substack{0 \le v, \\ q \le n: \\ (v,q) \ne (0,1)}} \tilde{s}_{q,n-v}(\varepsilon) \mathcal{Z}_{v,q,s(0)} f + \hat{\mathcal{Z}}_{n,s(0),s(\varepsilon)} f \left(\frac{s(\varepsilon)-s(0)}{\varepsilon} \right)^{n+1}, \tag{3.21}$$

where $s_{q,i}$ and $\tilde{s}_{q,i}(\varepsilon)$ are the coefficient and the remainder of the expansion $(s(\varepsilon)-s(0))^k = \sum_{i=0}^{j} s_{q,i}\varepsilon^i + \tilde{s}_{q,j}(\varepsilon)\varepsilon^j$, respectively (see (3.24) for detail).

Proof of Theorem 1.1. Put $\Phi(\varepsilon, s, \cdot) = s \log|g(\varepsilon, \cdot)| + \log\psi(\varepsilon, \cdot)$ and $\Phi(s, \cdot) = s\log|g| + \log\psi$ for convenience. For $s \in I$ and $\varepsilon > 0$, let $(\lambda_s(\varepsilon), h_s(\varepsilon, \cdot), \nu_s(\varepsilon, \cdot))$ be the triplet of the operator $\mathcal{L}_{\Phi(\varepsilon,s,\cdot)}$ and (λ_s, h_s, ν_s) the triplet of $\mathcal{L}_{\Phi(s,\cdot)}$ given by Theorem A.2. We may write $\nu(\varepsilon, \cdot) := \nu_{s(\varepsilon)}(\varepsilon, \cdot)$, $\nu := \nu_{s(0)}$ and $h := h_{s(0)}$.

Claim 1. *The solution $s = s(\varepsilon)$ of the equation $P(s\log|g(\varepsilon,\cdot)| + \log\psi(\varepsilon,\cdot)) = p_0$ exists in I for any small $\varepsilon > 0$, and converges to $s(0)$ as $\varepsilon \to 0$.*

Indeed, it follows from (3.15) with $n = 0$, $a_{0,0,s} = 1$, $G^s_{0,0} = |g|^s$ and $\psi_0 = \psi$ that

$$\mathcal{L}_{\Phi(\varepsilon,s,\cdot)} h = \mathcal{L}_{\Phi(s(0),\cdot)} h + (s - s(0)) \hat{\mathcal{Z}}_{0,s(0),s} h + \tilde{\mathcal{L}}_{0,s}(\varepsilon, h),$$
$$\hat{\mathcal{Z}}_{0,s(0),s} h = \mathcal{L}_0\Big(\int_0^1 |g|^{u(s-s(0))+s(0)}\, du \log |g| \psi h\Big).$$

By using the equations $\mathcal{L}^*_{\Phi(\varepsilon,s,\cdot)} \nu_s(\varepsilon,\cdot) = \lambda_s(\varepsilon)\nu_s(\varepsilon,\cdot)$ and $\mathcal{L}_{\Phi(s(0),\cdot)} h = \lambda_{s(0)} h$, we obtain

$$\nu_s(\varepsilon, h)(\lambda_s(\varepsilon) - \lambda_{s(0)}) = \nu_s(\varepsilon, \hat{\mathcal{Z}}_{0,s(0),s} h)(s - s(0)) + \nu_s(\varepsilon, \tilde{\mathcal{L}}_{0,s}(\varepsilon, h)). \quad (3.22)$$

Let us now choose any small $\eta > 0$ so that $s(0) + \eta, s(0) - \eta \in I$. For any $s \in [s(0) - \eta, s(0) + \eta]$, we have the estimate

$$\begin{aligned} -\nu_s(\varepsilon, \hat{\mathcal{Z}}_{0,s(0),s} h_{s(0)}) &\ge \begin{cases} \nu_s(\varepsilon, \mathcal{L}_0(|g|^s \psi(-\log|g|) h_{s(0)})) & \text{if } s \ge s(0), \\ \nu_s(\varepsilon, \mathcal{L}_0(|g|^{s(0)} \psi(-\log|g|) h_{s(0)})) & \text{if } s < s(0) \end{cases} \\ &\ge \begin{cases} -\log\|g\|_\infty \nu_s\big(\varepsilon, \mathcal{L}_{\Phi(\max I,\cdot)} h_{\max I} \frac{h_{s(0)}}{h_{\max I}}\big) & \text{if } s \ge s(0), \\ -\log\|g\|_\infty \lambda_{s(0)} \nu_s(\varepsilon, h_{s(0)}) & \text{if } s < s(0) \end{cases} \\ &\qquad\qquad (\because s \le \max I) \\ &\ge -c_{51} \log\|g\|_\infty > 0 \end{aligned} \quad (3.23)$$

with $c_{51} = \min(\lambda_{\max I} \inf_\omega h_{\max I}(\omega)/\|h_{\max I}\|_\infty, \lambda_{s(0)}) \inf_\omega h_{s(0)}(\omega)$. Now fix $s \in [s(0) - \eta, s(0) + \eta]$ with $s \ne s(0)$. For any small $\varepsilon > 0$ with

$$\frac{\|\tilde{\mathcal{L}}_{0,s}(\varepsilon, h_{s(0)})\|_\infty}{|s - s(0)|} \le -\frac{c_{51}}{2} \log(\|g\|_\infty),$$

equation (3.22) yields

$$\begin{aligned} \nu_s(\varepsilon, h_{s(0)}) \frac{\lambda_s(\varepsilon) - \lambda_{s(0)}}{s - s(0)} &= \nu_s(\varepsilon, \hat{\mathcal{Z}}_{0,s(0),s(\varepsilon)} h_{s(0)}) + \frac{\nu_s(\varepsilon, \tilde{\mathcal{L}}_{0,s(\varepsilon)}(\varepsilon, h_{s(0)}))}{s - s(0)} \\ &\le \frac{c_{51}}{2} \log\|g\|_\infty < 0. \end{aligned}$$

In addition to $\nu_s(\varepsilon, h_{s(0)}) > 0$, this implies that $\lambda_s(\varepsilon) < \lambda_{s(0)}$ if $s > s(0)$ and $\lambda_s(\varepsilon) > \lambda_{s(0)}$ if $s < s(0)$ for any small $\varepsilon > 0$. Since $\log \lambda_s(\varepsilon) = P(\Phi(\varepsilon, s, \cdot))$ and $\log \lambda_{s(0)} = P(\Phi(s(0), \cdot)) = p_0$ are satisfied, we obtain

$$P(\Phi(\varepsilon, s(0) + \eta, \cdot)) < p_0 < P(\Phi(\varepsilon, s(0) - \eta, \cdot))$$

for a fix number $\eta > 0$ with $s(0) + \eta, s(0) - \eta \in I$ and for any small $\varepsilon > 0$. It follows from this inequality and the strictly monotone decreasing of the map $s \mapsto P(\Phi(\varepsilon, s, \cdot))$ that there exists a unique $s(\varepsilon) \in [s(0) - \eta, s(0) + \eta]$ so that $P(\Phi(\varepsilon, s(\varepsilon), \cdot)) = p_0$. By arbitrarily choosing $\eta > 0$, $s(\varepsilon)$ converges to $s(0)$ as $\varepsilon \to 0$. In particular, $\tilde{s}_0(\varepsilon) = s(\varepsilon) - s(0)$ estimates as (3.18) from (3.22).

Claim 2. *$s(\varepsilon) = s(0) + O(\varepsilon)$ as $\varepsilon \to 0$ in the case $n \geq 1$.*

Indeed, since the form $\tilde{\mathcal{L}}_{0,s(\varepsilon)}(\varepsilon,\cdot) = \mathcal{L}_{1,s(\varepsilon)}\varepsilon + \tilde{\mathcal{L}}_{1,s(\varepsilon)}(\varepsilon,\cdot)\varepsilon$ is satisfied from (3.13), (3.18) implies

$$\frac{s(\varepsilon)-s(0)}{\varepsilon} = \frac{-\nu(\varepsilon, \mathcal{L}_{1,s(\varepsilon)} + \tilde{\mathcal{L}}_{1,s(\varepsilon)}(\varepsilon,\cdot))}{\nu(\varepsilon, \hat{\mathcal{Z}}_{0,s(0),s(\varepsilon)}h)}.$$

By virtue of Theorem B.1 by regarding $(\mathcal{X}_0, \|\cdot\|_0) = (C_b(E^\infty), \|\cdot\|_\infty)$ and $(\mathcal{X}_1, \|\cdot\|_1) = (F_{\theta,b}(E^\infty), \|\cdot\|_\theta)$, $\mathcal{L}(\varepsilon,\cdot) = \mathcal{L}_{\Phi(\varepsilon,s(\varepsilon),\cdot)}$ and $\mathcal{L} = \mathcal{L}_{\Phi(s(0),\cdot)}$, the measure $\nu(\varepsilon,\cdot)$ converges to ν weakly. In fact, conditions (L.1) and (L.2) are satisfied by Theorem A.2, condition (L.3) is yielded by $\|\nu(\varepsilon,\cdot)\| = 1$ and $0 < \inf h \leq \sup h < \infty$, and condition (L.4) is fulfilled from $\mathcal{L}_{\Phi(\varepsilon,s(\varepsilon),\cdot)} \to \mathcal{L}_{\Phi(s(0),\cdot)}$ in $C_b(E^\infty)$. Moreover, the operator $\mathcal{L}_{1,s(\varepsilon)}$ is bounded by Lemma 3.6 for any small $\varepsilon > 0$. In addition to (3.23) and (3.14), it follows from the boundedness of $\mathcal{L}_{1,s(\varepsilon)}$ and convergence of $\nu(\varepsilon,\cdot)$ that $(s(\varepsilon)-s(0))/\varepsilon$ is bounded.

Claim 3. *The asymptotic expansion (1.6) of $s(\varepsilon)$ is satisfied in the case $n \geq 1$.*

Let $n \geq 1$. We show that if $s(\varepsilon)$ has the form $(n-1)$-order asymptotic expansion $s(\varepsilon) = s_0 + s_1\varepsilon + \cdots + s_{n-1}\varepsilon^{n-1} + o(\varepsilon^{n-1})$ with $s_0 = s(0)$, then so is for n. To check the n-order asymptotic behaviour of $s(\varepsilon)$, we need the expansion of $(s(\varepsilon)-s(0))^k$ for $k \geq 0$:

$$(s(\varepsilon)-s(0))^k = \begin{cases} s_{1,0}+s_{1,1}\varepsilon+\ldots+s_{1,n-1}\varepsilon^{n-1}+\tilde{s}_{n-1}(\varepsilon)\varepsilon^{n-1} & \text{if } k=1, \\ s_{k,0}+s_{k,1}\varepsilon+\ldots+s_{k,n-1}\varepsilon^{n-1}+s_{k,n}\varepsilon^n+\tilde{s}_{k,n}(\varepsilon)\varepsilon^n & \text{if } k \geq 2 \end{cases}$$

with

$$s_{k,i} = \begin{cases} 0 & \text{if } k \geq 1 \text{ and } 0 \leq i \leq k-1, \\ s_i & \text{if } k = 1 \text{ and } 1 \leq i \leq n-1, \\ \displaystyle\sum_{\substack{j_1,\ldots,j_{i-1}\geq 0: \\ j_1+\cdots+j_{i-1}=k \\ j_1+2j_2+\cdots+(i-1)j_{i-1}=i}} \frac{s_1^{j_1}\cdots s_{i-1}^{j_{i-1}}}{j_1!\cdots j_{i-1}!} & \text{if } k \geq 2 \text{ and } k \leq i \leq n \end{cases} \tag{3.24}$$

with $\tilde{s}_{k,n}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that $s_{1,i} = s_i$ holds for $1 \leq i \leq n-1$. Thus, the expansion (3.15) implies

$$\mathcal{L}_{\Phi(\varepsilon,s(\varepsilon),\cdot)}f = \sum_{v=0}^{n}\sum_{q=0}^{n}(s(\varepsilon)-s(0))^q \mathcal{Z}_{v,q,s(0)}f\varepsilon^v$$
$$+ (s(\varepsilon)-s(0))^{n+1}\hat{\mathcal{Z}}_{n,s(0),s(\varepsilon)}f + \tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon,f)\varepsilon^n$$

$$= (s(\varepsilon) - s(0))\mathcal{Z}_{0,1,s(0)} f + \sum_{\substack{0 \le v,q \le n: \\ (v,q) \ne (0,1)}} (s(\varepsilon) - s(0))^q \mathcal{Z}_{v,q,s(0)} f \varepsilon^v$$
$$+ (s(\varepsilon) - s(0))^{n+1} \widehat{\mathcal{Z}}_{n,s(0),s(\varepsilon)} f + \tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, f)\varepsilon^n$$
$$= \mathcal{L}_{\Phi(s(0),\cdot)} f + \mathcal{Z}_{0,1,s(0)} f(s(\varepsilon) - s(0))$$
$$+ \sum_{u=1}^{n} \mathcal{N}_u f \varepsilon^u + \widehat{\mathcal{N}}_{n+1}(\varepsilon, f)\varepsilon^{n+1} + \tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, f)\varepsilon^n \tag{3.25}$$

for $f \in C_b(E^\infty)$ using (3.20) and (3.21). By the definition of $\widehat{\mathcal{N}}_{n+1}(\varepsilon, \cdot)$, this operator is bounded uniformly in any small $\varepsilon > 0$. Since $s(\varepsilon)$ has $(n-1)$-order asymptotic behaviour, we see by (3.25) that $\mathcal{L}_{\Phi(\varepsilon,s(\varepsilon),\cdot)}$ has at least $(n-1)$-order asymptotic expansion in $\mathcal{L}(C_b(E^\infty))$. Thus, it follows from Corollary B.2 that $\nu(\varepsilon, \cdot)$ has the form $\nu(\varepsilon, \cdot) = \nu + \sum_{k=1}^{n-1} \nu_k \varepsilon^k + \tilde{\nu}_{n-1}(\varepsilon, \cdot)\varepsilon^{n-1}$ and $|\tilde{\nu}_{n-1}(\varepsilon, f)| \to 0$ for each $f \in F_{\theta,b}(E^\infty)$. We obtain

$$0 = \nu(\varepsilon, (e^{p_0} - e^{p_0})h) = \nu(\varepsilon, (\mathcal{L}_{\Phi(\varepsilon,s(\varepsilon),\cdot)} - \mathcal{L}_{\Phi(s(0),\cdot)})h)$$
$$= (s(\varepsilon) - s(0))\nu(\varepsilon, \mathcal{Z}_{0,1,s(0)} h)$$
$$+ \sum_{u=1}^{n} \nu(\varepsilon, \mathcal{N}_u h)\varepsilon^u + \nu(\varepsilon, \widehat{\mathcal{N}}_{n+1}(\varepsilon, h))\varepsilon^{n+1} + \nu(\varepsilon, \tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h))\varepsilon^n.$$

Consequently, we get the form $\tilde{s}_{n-1}(\varepsilon) = s_n \varepsilon + \tilde{s}_n(\varepsilon)\varepsilon$ by putting (3.17) with $k = n$ and (3.19) and $\tilde{s}_n(\varepsilon)$ vanishes. Thus, this claim is satisfied.

Claim 4. *The estimate (1.7) of the remainder $\tilde{s}_n(\varepsilon)$ is valid.*

First assume $n = 0$. Recall the form (3.18) of $\tilde{s}_0(\varepsilon)$. Since $s(\varepsilon)$ converges to $s(0)$, $\mathcal{L}_{s(\varepsilon)\log|g(\varepsilon,\cdot)|+\log\psi(\varepsilon,\cdot)}$ converges to $\mathcal{L}_{s(0)\log|g|+\log\psi}$ in $C_b(E^\infty, \mathbb{C})$ as $\varepsilon \to 0$ by Lemma 3.8. Therefore, the measure $\nu(\varepsilon, \cdot)$ converges to ν weakly from Corollary B.2. Moreover, it follows from the Mean Value Theorem that $\widehat{\Gamma}_{s(0),s(\varepsilon),1}(\varepsilon, \omega) = \int_0^1 |g(\omega)|^{u(s(\varepsilon)-s(0))} \log|g(\omega)|\, du$ converges to $\log|g(\omega)|$ uniformly in $\omega \in E^\infty$. Thus, $\nu(\varepsilon, \widehat{\mathcal{Z}}_{0,s(0),s(\varepsilon)} h) \to \nu(h \log|g|)$. The assertion is valid in the case $n = 0$.

Next, assume $n \ge 1$. Since $s(\varepsilon)$ has an n-order asymptotic expansion, so does $\mathcal{L}_{s(\varepsilon)\log|g(\varepsilon,\cdot)|+\log\psi(\varepsilon,\cdot)}$ in $\mathcal{L}(C_b(E^\infty))$ from the expansion (3.25). Thus, Corollary B.2 says that $\nu(\varepsilon, \cdot)$ has an n-order asymptotic expansion. In the expression (3.19) of $\tilde{s}_n(\varepsilon)$, we obtain the form $\tilde{s}_{n-1}(\varepsilon) = (s_n + \tilde{s}_n(\varepsilon))\varepsilon$ and $\tilde{\nu}_{n-1}(\varepsilon, f) = (\nu_n(f) + \tilde{\nu}_n(\varepsilon, f))\varepsilon$. Hence, the proof is complete. ∎

3.2. Proof of Proposition 1.3

Proof. Choose any compact neighborhood I of s_0 so that

$$I \subset (p(n), \underline{p} + (n+1)(1 - t_0)) \setminus \mathbb{Z}.$$

Put $x(\varepsilon, \omega) = |g(\varepsilon, \omega)| - |g(\omega)|$. We begin with the estimate of $x(\varepsilon, \omega)$.

Claim 1. *There exists $c_{52} > 0$ such that for any $\omega \in E^\infty$, $c_{52}|g(\omega)|^{t_0}\varepsilon \le x(\varepsilon,\omega)$ for any small $\varepsilon > 0$.*

Indeed, let $s_\omega = \mathrm{sign}(g(\omega))$. It follows from (g.9) that for any $2 \le k \le n$, $|g_k(\omega)| \le c_2|g(\omega)|^{t_k} \le c_2|g(\omega)|^{t_0}$. Then we have

$$\frac{\mathrm{sign}(g(\omega))g(\varepsilon,\omega) - |g(\omega)|}{\varepsilon|g(\omega)|^{t_0}} = \sum_{k=1}^{n} \frac{\mathrm{sign}(g(\omega))g_k(\omega)}{|g(\omega)|^{t_0}}\varepsilon^{k-1}$$
$$\ge c_8 - \sum_{k=2}^{n} c_2\varepsilon^{k-1} > \frac{c_8}{2} > 0$$

for any small $\varepsilon > 0$. This implies that the signature of $\mathrm{sign}(g(\omega))g(\varepsilon,\omega)$ is plus for any small $\varepsilon > 0$, and therefore, the signature of $g(\varepsilon,\omega)$ equals the signature of $g(\omega)$. This also yields $x(\varepsilon,\omega) = \mathrm{sign}(g(\omega))g(\varepsilon,\omega) - |g(\omega)| > 0$, and thus, the assertion is valid by putting $c_{52} = c_8/2$.

Claim 2. $\|\mathcal{L}_{(\min I - n(1-t_1))\log|g|}1\|_\infty < \infty$ *and* $\|\mathcal{L}_{(\max I-(n+1)(1-t_0))\log|g|}1\|_\infty = \infty$.

Since $\min I - (1-t_1)n$ is greater than $\underline{p}$, $P((\min I - (1-t_1)n)\log|g|)$ is finite and so is $\|\mathcal{L}_{(\min I-(1-t_1)n)\log|g|}1\|_\infty$. On the other hand, from $\max I - (1-t_0)(n+1)$ is less than $\underline{p}$, $P((\max I - (1-t_1)n)\log|g|)$ is infinite and it yields

$$\|\mathcal{L}_{(\max I-(1-t_1)n)\log|g|}1\|_\infty = \infty.$$

Therefore, this claim is valid.

We let $E(\varepsilon) = \{e \in E : \inf_{\omega\in[e]}|g(\omega)| \ge 2c_2\varepsilon\}$. Then we see that $E(\varepsilon)$ is an including finite set and $\lim_{\varepsilon\to 0} E(\varepsilon) = E$. We will use the fact that for any $e \in E(\varepsilon)$, $\omega \in [e]$ and $0 < \varepsilon < 1/2$

$$|x(\varepsilon,\omega)| \le c_2 \sum_{k=1}^{n} |g(\omega)|^{t_k}\varepsilon^k \le 2c_2\varepsilon \le |g(\omega)|. \tag{3.26}$$

Claim 3. *There exists $\tilde{b} \in E$ such that*

$$\inf_{\omega\in[\tilde{b}]} \sum_{e\in E(\varepsilon):t(e)=i(\tilde{b})} |g(e\cdot\omega)|^{\max I-(n+1)(1-t_0)} \to \infty$$

as $\varepsilon \to 0$.

Choose any large number $M > 0$. Since an incidence matrix is finitely irreducible, there exists a finite subset $\{b_1,\dots,b_N\}$ of E such that for any $e \in E$, $t(e) = i(b_k)$ for some k. Namely, when we put $E_k = \{e \in E : t(e) = i(b_k)\}$ for each $k = 1,2,\dots,N$, then $E = \bigcup_{k=1}^N E_k$ is satisfied. From $\|\mathcal{L}_{(\max I-(1-t_0)(n+1))\log|g|}1\|_\infty = +\infty$, there is $\omega \in E^\infty$ so that $\mathcal{L}_{(\max I-(1-t_0)(n+1))\log|g|}1(\omega) > M$. Moreover, there exists $\varepsilon_0 > 0$

such that for any $0 < \varepsilon < \varepsilon_0$, $\sum_{e\in E(\varepsilon):e\cdot\omega\in E^\infty} |g(e\cdot\omega)|^{\max I-(1-t_0)(n+1)} > M$. We notice that for any $\omega^k \in [b_k]$ $(k = 1, 2, \dots, N)$

$$\begin{aligned}
&\sum_{e\in E(\varepsilon):t(e)=i(\omega_0)} |g(e\cdot\omega)|^{\max I-(1-t_0)(n+1)} \\
&\quad \le \sum_{k=1}^{N} \sum_{e\in E_k\cap E(\varepsilon):t(e)=i(\omega_0)} |g(e\cdot\omega)|^{\max I-(1-t_0)(n+1)} \\
&\quad \le \sum_{k=1}^{N} \sum_{e\in E_k\cap E(\varepsilon):t(e)=i(b_k)} |g(e\cdot\omega^k)|^{\max I-(1-t_0)(n+1)}(1+c_{14}\theta)
\end{aligned}$$

by Lemma 3.2. Thus,

$$M \le \sum_{k=1}^{N} \inf_{\upsilon\in[b_k]} \sum_{e\in E(\varepsilon):t(e)=i(b_k)} |g(e\cdot\upsilon)|^{\max I-(1-t_0)(n+1)}(1+c_{14}\theta).$$

Since M is an arbitrary large number, the right hand side tends to $+\infty$ as $\varepsilon \to 0$. Hence, the assertion of the claim is valid for some $\tilde{b} \in \{b_k\}_{k=1}^N$.

Claim 4. *There exist a sign $\tilde{s} \in \{+1, -1\}$ and a constant $c_{53} > 0$ such that $\tilde{s}\tilde{\mathcal{L}}_{n,s}(\varepsilon, h)(\omega) \ge -c_{53}$ uniformly in $s \in I$ and $\omega \in E^\infty$, and*

$$\inf_{s\in I} \inf_{\omega\in[\tilde{b}]} \tilde{s}\tilde{\mathcal{L}}_{n,s}(\varepsilon, h)(\omega)/\varepsilon \to +\infty.$$

Let $\omega \in E^\infty$. From the Taylor expansion of the function $F: x \mapsto (|g(\omega)| + x)^s$, we obtain the form

$$\begin{aligned}
|g(\varepsilon,\omega)|^s &= \sum_{l=0}^{n} \binom{s}{l} |g(\omega)|^{s-l} x(\varepsilon,\omega)^l \\
&\qquad + \binom{s}{n+1} (|g(\omega)| + \alpha x(\varepsilon,\omega))^{s-n-1} x(\varepsilon,\omega)^{n+1} \\
&= g_{0,s} + g_{1,s}\varepsilon + \cdots + g_{n,s}\varepsilon^n + \tilde{g}_{n,s}(\varepsilon,\cdot)\varepsilon^n
\end{aligned}$$

with

$$\begin{aligned}
\tilde{g}_{n,s}(\varepsilon,\omega) &= \sum_{l=0}^{n} \binom{s}{l} |g(\omega)|^{s-l} \operatorname{sign}(g(\omega))^l \\
&\qquad \times \sum_{j=n+1}^{nl} \sum_{\substack{j_1,\dots,j_n\ge 0:\\ j_1+\cdots+j_n=l\\ j_1+2j_2+\cdots+nj_n=j}} \frac{l!}{j_1!\cdots j_n!} g_1(\omega)^{j_1}\cdots g_n(\omega)^{j_n} \varepsilon^{j-n} \\
&\qquad + \binom{s}{n+1} (|g(\omega)| + \alpha x(\varepsilon,\omega))^{s-n-1} \left(\frac{x(\varepsilon,\omega)}{\varepsilon}\right)^{n+1} \varepsilon \\
&= I_1(\varepsilon,\omega) + I_2(\varepsilon,\omega)
\end{aligned}$$

for some $\alpha = \alpha(s, \varepsilon, \omega) \in [0, 1]$. By $t_1 \le t_0 \le t_k$ for $k \ge 2$ and by the estimate $|g|^{s-l}|g_1|^{j_1} \cdots |g_n|^{j_n} \le c_2^l |g|^{s-l+lt_1} \le \max(c_2, 1)^n |g|^{s-(1-t_1)n}$, we get the inequality

$$|I_1(\varepsilon, \omega)| \le c_{54}|g(\omega)|^{s-(1-t_1)n}\varepsilon \tag{3.27}$$

for any $\omega \in E^\infty$ for some constant $c_{54} > 0$. To estimate $I_2(\varepsilon, \cdot)$, we note that since the integers $1, 2, \ldots, n$ are not in I, the signature of $\binom{s}{n+1}$ for $s \in I$ does not depend on the choice of $s \in I$. Then we put $\tilde{s} = \mathrm{sign}(\binom{s}{n+1})$ for a $s \in I$. For the same reason as above, $\inf_{s\in I} |\binom{s}{n+1}| =: c_{55}$ is positive. Notice also that for $e \in E(\varepsilon)$

$$\begin{aligned}
&(|g(e\cdot\omega)| + \alpha x(\varepsilon, e\cdot\omega))^{s-n-1} \\
&\quad \ge \begin{cases} |g(e\cdot\omega)|^{s-n-1} & \text{if } s-n-1 \ge 0, \\ (|g(e\cdot\omega)| + x(\varepsilon, e\cdot\omega))^{s-n-1} & \text{if } s-n-1 < 0 \end{cases} \\
&\quad \ge \begin{cases} |g(e\cdot\omega)|^{s-n-1} & \text{if } s-n-1 \ge 0, \\ 2^{s-n-1}|g(e\cdot\omega)|^{s-n-1} & \text{if } s-n-1 < 0 \end{cases} \\
&\quad \ge 2^{-|s-n-1|}|g(e\cdot\omega)|^{s-n-1}
\end{aligned}$$

by (3.26). Thus, we obtain that for any $s \in I$ and $\omega \in E^\infty$,

$$\begin{aligned}
\frac{\tilde{s} I_2(\varepsilon, e\cdot\omega)}{\varepsilon} &= \left|\binom{s}{n+1}\right| (|g(e\cdot\omega)| + \alpha x(\varepsilon, e\cdot\omega))^{s-n-1} \left(\frac{x(\varepsilon, e\cdot\omega)}{\varepsilon}\right)^{n+1} \\
&\ge \begin{cases} c_{56}|g(e\cdot\omega)|^{\max I-(1-t_0)(n+1)} & \text{if } \omega \in [e] \text{ for some } e \in E(\varepsilon), \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{3.28}$$

with $c_{56} = c_{55}2^{-\sup_{s\in I}|s-n-1|}c_{52}^{n+1} > 0$. Consequently, (3.27) and (3.28) imply that for any $\omega \in E^\infty$

$$\begin{aligned}
\frac{\tilde{s}\tilde{\mathcal{L}}_{n,s}(\varepsilon, h)(\omega)}{\varepsilon} &= \sum_{e\in E: t(e)=i(\omega_0)} (\tilde{s} I_1(\varepsilon, e\cdot\omega) + \tilde{s} I_2(\varepsilon, e\cdot\omega))h(e\cdot\omega)\frac{1}{\varepsilon} \\
&\ge \begin{cases} -c_{53} + c_{56}(\inf_\upsilon h(\upsilon)) \displaystyle\sum_{\substack{e\in E(\varepsilon):\\ t(e)=i(\omega_0)}} |g(e\cdot\omega)|^{\max I-(n+1)(1-t_0)} & \text{if } \omega \in [\tilde{b}], \\ -c_{53} & \text{otherwise} \end{cases}
\end{aligned}$$

by putting $c_{53} = c_{54}\|\mathcal{L}_{(\min I-(1-t_1)n)\log|g|}1\|_\infty \|h\|_\infty$. Thus, the assertion is valid.

Claim 5. *The assertion of this corollary is valid.*

Recall the form $\tilde{s}_n(\varepsilon)$ $(n \ge 1)$ of (1.7) in Theorem 1.1. Then we have

$$\begin{aligned}
\frac{\tilde{s}\tilde{s}_n(\varepsilon)}{\varepsilon} &= \frac{1}{-\nu(h\log|g|)}\nu\left(\varepsilon, \frac{\tilde{s}\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h)}{\varepsilon}\right) + O(1) \\
&\ge -\frac{-c_{53}}{-\nu(h\log|g|)} + \frac{\nu(\varepsilon, [\tilde{b}])}{-\nu(h\log|g|)} \frac{\inf_{s\in I}\inf_{\omega\in[\tilde{b}]} \tilde{s}\tilde{\mathcal{L}}_{n,s}(\varepsilon, h)(\omega)}{\varepsilon} + O(1) \\
&\to +\infty
\end{aligned}$$

as $\varepsilon \to 0$ by using the above claim in addition to the fact $\nu(\varepsilon, [\tilde{b}]) \to \nu([\tilde{b}]) > 0$. Hence, the assertion $|\tilde{s}_n(\varepsilon)|/\varepsilon \to +\infty$ is guarantied. ■

3.3. Proof of Theorem 2.2

For the sake of convenience, we write the composite map $T_{\omega_0}(\varepsilon,\cdot)T_{\omega_1}(\varepsilon,\cdot)\cdots T_{\omega_n}(\varepsilon,\cdot)$ as $T_{\omega_0\omega_1\cdots\omega_n}(\varepsilon,\cdot)$. Similarly, $T_{\omega_0\omega_1\cdots\omega_n}$ means $T_{\omega_0}T_{\omega_1}\cdots T_{\omega_n}$. Assume that condition $(G.1)_n$ is satisfied. We take open and relative compact subsets (U_v) of $\mathbb{R}^D$ and numbers $r \in (0,1)$ and $r_0 > 0$ such that $U_v = \bigcup_{x\in J_v} B(x, r_0)$, $J_v \subset U_v \subset \overline{U_v} \subset O_v$ for any $v \in V$, and $\sup_{e\in E}\sup_{x\in U_{t(e)}} \|T_e'(x)\| \le r$, where $B(x, r_0)$ is the open ball with center x and radius r_0. We begin with the following fact:

Lemma 3.10. *For any $n \ge 1$, if conditions $(G.1)_n$ and $(G.2)_n$ are satisfied, then so are conditions $(G.1)_{n-1}$ and $(G.2)_{n-1}$.*

Proof. Assume that conditions $(G.1)_n$ and $(G.2)_n$ are satisfied. It suffices to prove that conditions (iii) and (iv) in $(G.2)_{n-1}$ are fulfilled for $n \ge 2$. Since $\tilde{T}_{e,n-1}(\varepsilon,\cdot)$ has the form $T_{e,n}\varepsilon + \tilde{T}_{e,n}(\varepsilon, x)\varepsilon$, convergence

$$\sup_{e\in E}\sup_{x\in J_{t(e)}} \frac{\left\|\frac{\partial}{\partial x}\tilde{T}_{e,n-1}(\varepsilon, x)\right\|}{\|T_e'(x)\|^{\tilde{t}_1}} \to 0$$

is yielded by putting $\tilde{t}_1 = \min(t(n,1), \tilde{t}_0)$. Therefore, (iii) is valid for $(G.2)_{n-1}$. To check (iv) in $(G.1)_{n-1}$, we note the forms

$$\begin{aligned} p(n-1) &= \max\Big\{\underline{p} - \frac{n-1}{1}(1-t_1), \ldots, \underline{p} - \frac{n-1}{n-1}(1-t_{n-1}), \\ &\qquad \underline{p}/t_1, \ldots, \underline{p}/t_{n-1}, \underline{p} + 1 - \tilde{s}, \underline{p}/\tilde{s}\Big\} \\ \tilde{s} &= \min\Big\{t_{n-1}, \tilde{t}_1, \frac{\tilde{t}_1}{D} + \frac{D-1}{D}t(1,1), \ldots, \frac{\tilde{t}_1}{D} + \frac{D-1}{D}t(n-1,1)\Big\}. \end{aligned}$$

By the definition of t_k in (2.1), the inequality $t_{n-1} \ge t_n$ holds. By the same reasoning, we have $t(n,1), \ldots, t(1,1) \ge t_n$ and therefore $\tilde{s} \ge \tilde{t}$, where $\tilde{t}$ is defined in (2.2). Thus, we see $p(n-1) \le p(n)$. Hence, $\dim_H K/D > p(n)$ implies $\dim_H K/D > p(n-1)$. ■

Lemma 3.11. *If condition $(G.1)_n$ (ii) holds, then there exists a constant $c_{57} > 0$ such that for any $e \in E$, $x \in J_{t(e)}$, $y \in O_{t(e)}$ with $|x-y| < r_0$, $0 \le l \le n$, and $0 \le k \le 1 + n - l$*

$$\left\|T_{e,l}^{(k)}(x) - T_{e,l}^{(k)}(y)\right\| \le c_{57}\left\|T_e'(x)\right\|^{t(l,k)}|x-y|^{\beta}, \quad \left\|T_{e,l}^{(k)}(y)\right\| \le c_{57}\left\|T_e'(x)\right\|^{t(l,k)}.$$

Proof. We have

$$\begin{aligned}
&\left|\frac{T_{e,l}^{(k)}(x)}{\|T_e'(x)\|^{t(l,k)}}-\frac{T_{e,l}^{(k)}(y)}{\|T_e'(x)\|^{t(l,k)}}\right| \\
&\quad\le\left|\frac{T_{e,l}^{(k)}(x)}{\|T_e'(x)\|^{t(l,k)}}-\frac{T_{e,l}^{(k)}(y)}{\|T_e'(y)\|^{t(l,k)}}\right|+\left|\frac{T_{e,l}^{(k)}(y)}{\|T_e'(y)\|^{t(l,k)}}-\frac{T_{e,l}^{(k)}(y)}{\|T_e'(x)\|^{t(l,k)}}\right| \\
&\quad\le c_{58}|x-y|^{\beta}+c_{59}\|T_e'(y)\|^{t(l,k)}\frac{\left|\|T_e'(x)\|^{t(l,k)}-\|T_e'(y)\|^{t(l,k)}\right|}{\|T_e'(x)\|^{t(l,k)}\|T_e'(y)\|^{t(l,k)}} \\
&\quad\le (c_{58}+c_{59}(1+c_9r_0^{\beta}))|x-y|^{\beta}
\end{aligned}$$

for some $c_{58}, c_{59} > 0$ by using condition $(G.2)_n$ (ii) and condition (v) in GDMS. Therefore, the former assertion is fulfilled. The latter assertion follows from the above inequality. ■

Let $\pi(\varepsilon,\cdot)$ be the coding map of $K(\varepsilon)$ for $\varepsilon>0$.

Lemma 3.12. *Assume that conditions*$(G.1)_n$ *and* $(G.2)_n$ *are satisfied. Choose any* $r_1\in(r,1)$. *Then there exist functions* $\pi_1,\pi_2,\dots,\pi_n\in F_{r_1,b}(E^{\infty},\mathbb{R}^D)$ *and* $\tilde{\pi}(\varepsilon,\cdot)\in C_b(E^{\infty},\mathbb{R}^D)$ *such that* $\pi(\varepsilon,\cdot)=\pi+\pi_1\varepsilon+\cdots+\pi_n\varepsilon^n+\tilde{\pi}_n(\varepsilon,\cdot)\varepsilon^n$ *and* $\|\tilde{\pi}_n(\varepsilon,\cdot)\|_{\infty}:=\sup_{\omega\in E^{\infty}}|\tilde{\pi}_n(\varepsilon,\omega)|\to 0$ *as* $\varepsilon\to 0$.

Proof. This assertion mostly follows from the proof of [25, Lemma 3.1]. When we use this proof, we need the boundedness of $|T_{e,k}^{(i)}(y)|$ uniformly in $e\in E$ and $y\in U_{t(e)}$ for each k and i. This fact is satisfied by condition $(G.2)_n$ (ii) in particular. Therefore, the proof of [25, Lemma 3.1] implies

$$\pi_j(\omega)=\sum_{k=0}^{\infty}T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(R_j(\pi\sigma^k\omega)),$$

$$\tilde{\pi}_n(\varepsilon,\omega)=\sum_{k=0}^{\infty}T'_{\omega_0\cdots\omega_{k-1}}(\pi\sigma^k\omega)(\tilde{R}_n(\varepsilon,\pi\sigma^k\omega)),$$

where R_j and $\tilde{R}_n(\varepsilon,\cdot)$ are defined inductively

$$R_j(\omega)=T_{\omega_0,j}(\pi\sigma\omega)+\sum_{\substack{0\le l\le j-1,\\ 1\le k\le j-l:\\ (l,k)\ne(0,1)}}\ \sum_{\substack{i_1,\dots,i_k\ge 1:\\ i_1+\cdots+i_k=j-l}}\frac{T_{\omega_0,l}^{(k)}(\pi\sigma\omega)(\pi_{i_1}(\sigma\omega),\dots,\pi_{i_k}(\sigma\omega))}{k!}$$

$$\tilde{R}_n(\varepsilon,\omega)=\sum_{\substack{0\le l\le n-1,\\ 1\le k\le n-l:\\ (l,k)\ne(0,1)}}\ \sum_{i=n-l+1}^{kn}\ \sum_{\substack{1\le i_1,\dots,i_k\le n-1:\\ i_1+\cdots+i_k=i}}\frac{T_{\omega_0,l}^{(k)}(x)(\pi_{i_1}(\sigma\omega),\dots,\pi_{i_k}(\sigma\omega))}{k!}\varepsilon^{i-n+l}$$

$$+ \sum_{\substack{0\le l\le n-1,\\ 1\le k\le n-l:\\ (l,k)\ne(0,1)}} \sum_{i=1}^{k} \frac{T^{(k)}_{\omega_0,l}(x)}{k!}(\underbrace{z(\varepsilon),\dots,z(\varepsilon)}_{i-1},\underbrace{\tilde{\pi}_{n-1}(\varepsilon,\sigma\omega)}_{i\text{-th}},\underbrace{x(\varepsilon)-x,\dots,x(\varepsilon)-x}_{k-i})\varepsilon^{l-1}$$

$$+ \sum_{l=0}^{n} L(n-l,T_{\omega_0,l},x(\varepsilon),x)\left(\frac{x(\varepsilon)-x}{\varepsilon}\right)^{n-l} + \tilde{T}_{\omega_0,n}(\varepsilon,x(\varepsilon)),$$

where $z(\varepsilon)=\sum_{k=1}^{n-1}\pi_k(\sigma\omega)\varepsilon^k$, $x(\varepsilon)=\pi(\varepsilon,\sigma\omega)$, $x=\pi\sigma\omega$, and

$$L(n-l,T_{\omega_0,l},x(\varepsilon),x)$$
$$=\int_0^1 \frac{(1-t)^{n-l-1}}{(n-l-1)!}(T^{(n-l)}_{\omega_0,l}(x+t(x(\varepsilon)-x))-T^{(n-l)}_{\omega_0,l}(x))\,dt.$$

The facts $\pi_k\in F_{r_1,b}(E^\infty,\mathbb{R}^D)$ and $\|\tilde{\pi}_n(\varepsilon,\cdot)\|_\infty\to 0$ follow from [25] again. ■

We give the asymptotic expansion of the function $\omega\mapsto \det\frac{\partial}{\partial x}T_{\omega_0}(\varepsilon,\pi(\varepsilon,\sigma\omega))$. Let us put

$$u(k,i)=\min\{t(i_1,j_1+1)+\cdots+t(i_D,j_D+1):$$
$$0\le i_1,\dots,i_D\le k, i_1+\cdots+i_D=k$$
$$0\le j_1,\dots,j_D\le i, j_1+\cdots+j_D=i\}$$

for each $k=1,\dots,n$. Then we see the equation

$$t_k=\frac{1}{D}\min(\{u(k,0)\}\cup\{u(l,i): l=0,\dots,k-1, i=1,\dots,k-l\}).$$

Lemma 3.13. *Assume that conditions* (G.1)$_n$ *and* (G.2)$_n$ *are satisfied. Then the functions* $g(\varepsilon,\omega):=\det\frac{\partial}{\partial x}T_{\omega_0}(\varepsilon,\pi(\varepsilon,\sigma\omega))$ *and* $g(\omega):=\det T'_{\omega_0}(\pi\sigma\omega)$ *satisfy conditions* (g.1)–(g.5).

Proof. For each $e\in E$, $x\in O_{t(e)}$, $0\le k\le n$, and $\varepsilon>0$, we write $T_e(\varepsilon,x)=(t_{e,1}(\varepsilon,x),\dots,t_{e,D}(\varepsilon,x))$, $T_{e,k}(x)=(t_{e,k,1}(x),\dots,t_{e,k,D}(x))$ and $\tilde{T}_{e,n}(\varepsilon,x)=(\tilde{t}_{e,n,1}(\varepsilon,x),\dots,\tilde{t}_{e,n,D}(\varepsilon,x))$, where $T_{e,0}=T_e$. Note the form

$$\det\frac{\partial}{\partial x}T_e(\varepsilon,x)=\sum_\eta \operatorname{sgn}(\eta)\frac{\partial t_{e,1}(\varepsilon,x)}{\partial x_{\eta(1)}}\cdots\frac{\partial t_{e,D}(\varepsilon,x)}{\partial x_{\eta(D)}}$$

for $x=(x_1,x_2,\dots,x_D)\in J_{t(e)}$, where η is taken over all permutations on $\{1,2,\dots,D\}$ and $\operatorname{sgn}(\eta)$ denotes the sign of η. We also recall the form

$$\det\frac{\partial}{\partial x}T_e(\varepsilon,x)=\det T'_e(x)+\kappa_{e,1}(x)\varepsilon+\cdots+\kappa_{e,n}(x)\varepsilon^n+\tilde{\kappa}_{e,n}(\varepsilon,x)\varepsilon^n,$$

where we let

$$\kappa_{e,k}(x)=\sum_\eta \operatorname{sgn}(\eta)\sum_{\substack{0\le i_1,\dots,i_D\le k:\\ i_1+\cdots+i_D=k}}\prod_{p=1}^{D}\frac{\partial t_{e,i_p,p}(x)}{\partial x_{\eta(p)}}$$

$$\tilde{\kappa}_{e,n}(\varepsilon,x) = \sum_{\eta} \operatorname{sgn}(\eta) \sum_{i=n+1}^{Dn} \sum_{\substack{0\le i_1,\dots,i_D\le n:\\ i_1+\cdots+i_D=i}} \prod_{p=1}^{D} \frac{\partial t_{e,i_p,p}(x)}{\partial x_{\eta(p)}} \varepsilon^{i-n}$$
$$+ \sum_{\eta} \operatorname{sgn}(\eta) \sum_{j=1}^{D} \left\{ \left(\prod_{q=1}^{j-1} \sum_{l=0}^{n} \frac{\partial t_{e,l,q}(x)}{\partial x_{\eta(q)}} \varepsilon^{l} \right) \frac{\partial \tilde{t}_{e,n,j}(\varepsilon,x)}{\partial x_{\eta(j)}} \left(\prod_{p=j+1}^{D} \frac{\partial t_{e,p}(\varepsilon,x)}{\partial x_{\eta(p)}} \right) \right\} \tag{3.29}$$

for each $e \in E$, $x \in O_{t(e)}$, $k = 1,2,\dots,n$ and $\varepsilon > 0$ (see [25, Lemma 3.2]). Note that $\kappa_{e,k}$ is of class $C^{n-k+\beta}$ and has the form

$$\kappa_{e,k}^{(i)}(x) = \sum_{\substack{j_1,\dots,j_D\ge 0:\\ j_1+\cdots+j_D=i}} \sum_{\eta} \operatorname{sgn}(\eta) \sum_{\substack{0\le i_1,\dots,i_D\le n:\\ i_1+\cdots+i_D=k}} \prod_{p=1}^{D} \left(\frac{\partial t_{e,i_p,p}}{\partial x_{\eta(p)}} \right)^{(j_p)} (x)$$

for each $i = 0,1,\dots,n-k$ and $x \in O_{t(e)}$. For $1 \le j \le D$, we let $z(0) \in \mathbb{R}^D$ as $z(0)_j = 1$ and $z(0)_{j_0} = 0$ for $j_0 \ne j$. By using Lemma 3.11, we have

$$\begin{aligned}
&\left\| \left(\frac{\partial t_{e,i,q}}{\partial x_j} \right)^{(p)} (y) \right\| \\
&\quad = \sup_{\substack{z(1),\dots,z_p\in\mathbb{R}^D:\\ |(z(0),z(1),\dots,z(p))|\le 1}} \left| \sum_{1\le i_0\cdots i_p\le D} \frac{\partial^{p+1} t_{e,i,q}(y)}{\partial x_{i_0}\cdots x_{i_p}} (z(0),z(1),\dots,z(p)) \right| \\
&\quad \le \|t_{e,i,q}^{(p+1)}(y)\| \le \|T_{e,i}^{(p+1)}(y)\| \le c_{57} \|T'_e(x)\|^{t(i,p+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \left(\frac{\partial t_{e,i,q}}{\partial x_j} \right)^{(p)} (x) - \left(\frac{\partial t_{e,i,q}}{\partial x_j} \right)^{(p)} (y) \right\| \\
&= \sup_{\substack{z(1),\dots,z_p\in\mathbb{R}^D:\\ |(z(0),\dots,z(p))|\le 1}} \left| \sum_{1\le i_1,\dots,i_p\le D} \left\{ \frac{\partial^{p+1} t_{e,i,q}(x)}{\partial x_{i_0}\cdots x_{i_p}} (z(0),\dots,z(p)) \right.\right. \\
&\qquad\qquad \left.\left. - \frac{\partial^{p+1} t_{e,i,q}(y)}{\partial x_{i_0}\cdots x_{i_p}} (z(0),\dots,z(p)) \right\} \right| \\
&\le \|t_{e,i,q}^{(p+1)}(x) - t_{e,i,q}^{(p+1)}(y)\| \\
&\le \|T_{e,i}^{(p+1)}(x) - T_{e,i}^{(p+1)}(y)\| \le \max\{c_9, c_{57}\} \|T'_e(x)\|^{t(i,p+1)} |x-y|^{\beta}
\end{aligned}$$

for $x \in J_{t(e)}$ and $y \in U_{t(e)}$ with $|x-y| < r_0$, and for $0 \le p \le n-k$. Therefore

$$|\kappa_{e,k}^{(i)}(y)| \le \sum_{\substack{j_1,\dots,j_D\ge 0:\\ j_1+\cdots+j_D=i}} \sum_{\eta} \sum_{\substack{0\le i_1,\dots,i_D\le n:\\ i_1+\cdots+i_D=k}} \prod_{p=1}^{D} \left\| \left(\frac{\partial t_{e,i_p,p}}{\partial x_{\eta(p)}} \right)^{(j_p)} (y) \right\|$$

$$\leq \sum_{\substack{j_1,\dots,j_D\geq 0:\\ j_1+\cdots+j_D=i}} \sum_{\eta} \sum_{\substack{0\leq i_1,\dots,i_D\leq k:\\ i_1+\cdots+i_D=k}} (c_{57})^D \|T_e'(x)\|^{t(i_1,j_1+1)+\cdots+t(i_D,j_D+1)}$$

$$\leq c_{60}\|T_e'(x)\|^{u(k,i)}$$

for some constant c_{60}. Moreover, we have

$$\|\kappa_{e,k}^{(i)}(x)-\kappa_{e,k}^{(i)}(y)\| \leq c_{61}\|T_e'(x)\|^{u(k,i)}|x-y|^{\beta}$$

by using Proposition 3.1 for each $e \in E$, $x \in J_{t(e)}$, $y \in U_{t(e)}$ with $|x-y| < r_0$ and $i = 0, 1, \dots, n-k$ for some constant $c_{61} > 0$. On the other hand, the form (3.29) implies

$$\|\tilde{\kappa}_{e,n}(\varepsilon,x)\| \leq D!(n+1)^D c_{57}^D \varepsilon \|T_e'(x)\|^{\min_{n+1\leq i\leq Dn} u(i,0)} + D!D(2c_{57})^{D-1}c_{10}(\varepsilon)$$
$$\times \sum_{j=1}^{D} \|T_e'(x)\|^{(j-1)\min\{t(1,1),\dots,t(n,1)\}+\tilde{t}+(D-j)\min\{t(1,1),\dots,t(n,1),\tilde{t}\}}$$
$$\leq c_{3.3}(\varepsilon)\|T_e'(x)\|^{\tilde{u}}$$

with $\tilde{u} = \min\{u(n+1,0),\dots,u(Dn,0),\tilde{t}+(D-1)\min\{t(1,1),\dots,t(n,1),\tilde{t}\}\}$ for any $e \in E$, $x \in J_{t(e)}$ and a small $\varepsilon > 0$ by putting

$$c_{62}(\varepsilon) = D!\max((n+1)^D c_{57}^D \varepsilon, D(2c_{57})^{D-1}c_{10}(\varepsilon))$$

with $c_{3.3}(\varepsilon) \to 0$. Consequently, by the asymptotic expansion of the composite functions (see [25, Proposition 2.3]), we obtain $g(\varepsilon,\omega) = g(\omega) + \sum_{k=1}^{n} g_k(\omega)\varepsilon^k + \tilde{g}_n(\varepsilon,\omega)\varepsilon^n$ with

$$g_j(\omega) = \kappa_{\omega_0,j}(\pi\sigma\omega) + \sum_{l=0}^{j-1}\sum_{k=1}^{j-l} \sum_{\substack{i_1,\dots,i_k\geq 1:\\ i_1+\cdots+i_k=j-l}} \frac{\kappa_{\omega_0,l}^{(k)}(x)(\pi_{i_1}(\sigma\omega),\dots,\pi_{i_k}(\sigma\omega))}{k!}$$

$$\tilde{g}_n(\varepsilon,\omega) = \sum_{l=0}^{n-1}\sum_{k=1}^{n-l}\sum_{i=n-l+1}^{kn} \sum_{\substack{1\leq i_1,\dots,i_k\leq n:\\ i_1+\cdots+i_k=i}} \frac{\kappa_{\omega_0,l}^{(k)}(x)(\pi_{i_1}(\sigma\omega),\dots,\pi_{i_k}(\sigma\omega))}{k!}\varepsilon^{i-n+l}$$
$$+\sum_{l=0}^{n}\sum_{k=1}^{n-l}\sum_{i=1}^{k} \kappa_{\omega_0,l}^{(k)}(x)(\underbrace{z(\varepsilon),\dots,z(\varepsilon)}_{i-1},\underbrace{\tilde{\pi}_n(\varepsilon,\sigma\omega)}_{i\text{-th}},\underbrace{x(\varepsilon)-x,\dots,x(\varepsilon)-x}_{k-i})\frac{\varepsilon^l}{k!}$$
$$+\sum_{l=0}^{n-1}\int_0^1 \frac{(1-t)^{n-l-1}}{(n-l-1)!}(\kappa_{\omega_0,l}^{(n-l)}(x(\varepsilon)+t(x-x(\varepsilon)))-\kappa_{\omega_0,l}^{(n-l)}(x))\,dt\left(\frac{x(\varepsilon)-x}{\varepsilon}\right)^{n-l}$$
$$+\kappa_{\omega_0,n}(x(\varepsilon))-\kappa_{\omega_0,n}(x)+\tilde{\kappa}_{\omega_0,n}(\varepsilon,x(\varepsilon)),$$

where $z(\varepsilon)=\sum_{k=1}^{n}\pi_k(\sigma\omega)\varepsilon^k$, $x(\varepsilon)=\pi(\varepsilon,\sigma\omega)$ and $x=\pi\sigma\omega$. Then we see

$$\begin{aligned}
|g_j(\omega)| &\le |\kappa_{\omega_0,j}(\pi\sigma\omega)| + \sum_{l=0}^{j-1}\sum_{k=1}^{j-l}\sum_{\substack{i_1,\ldots,i_k\ge 1:\\ i_1+\cdots+i_k=j-l}} \frac{\|\kappa^{(k)}_{\omega_0,l}(\pi\sigma\omega)\|\|\pi_{i_1}\|_\infty\cdots\|\pi_{i_k}\|_\infty}{k!}\\
&\le c_{60}\Bigg(\|T'_{\omega_0}(\pi\sigma\omega)\|^{u(j,0)}\\
&\quad + \sum_{l=0}^{j-1}\sum_{k=1}^{j-l}\sum_{\substack{i_1,\ldots,i_k\ge 1:\\ i_1+\cdots+i_k=j-l}} \|T'_{\omega_0}(\pi\sigma\omega)\|^{u(l,k)}\frac{\|\pi_{i_1}\|_\infty\cdots\|\pi_{i_k}\|_\infty}{k!}\Bigg)\\
&\le c_{63}\|T'_{\omega_0}(\pi\sigma\omega)\|^{\min(\{u(j,0)\}\cup\{u(l,k):l=0,\ldots,j-1,k=1,\ldots,j-l\})}\\
&= c_{63}\|T'_{\omega_0}(\pi\sigma\omega)\|^{Dt_j}
\end{aligned}$$

with constant $c_{63}>0$, where t_j is defined by (2.1). Moreover, for $\omega,\upsilon\in E^\infty$ with $\omega_0=\upsilon_0$,

$$\begin{aligned}
|g_j(\omega)-g_j(\upsilon)| &\le |\kappa_{\omega_0,j}(\pi\sigma\omega)-\kappa_{\omega_0,j}(\pi\sigma\upsilon)|\\
&\quad+\sum_{l=0}^{j-1}\sum_{k=1}^{j-l}\sum_{\substack{i_1,\ldots,i_k\ge 1:\\ i_1+\cdots+i_k=j-l}}\frac{1}{k!}\\
&\qquad\times\Bigg\{\|\kappa^{(k)}_{\omega_0,l}(\pi\sigma\omega)-\kappa^{(k)}_{\omega_0,l}(\pi\sigma\upsilon)\|\|\pi_{i_1}\|_\infty\cdots\|\pi_{i_k}\|_\infty+\\
&\qquad+\sum_{q=1}^{k}\|\kappa^{(k)}_{\omega_0,l}(\pi\sigma\omega)\|_\infty\||\pi_{i_q}(\sigma\omega)-\pi_{i_q}(\sigma\upsilon)|\prod_{\substack{1\le u\le k:\\ k\ne q}}\|\pi_{i_u}\|_\infty\Bigg\}\\
&= c_{64}|g(\omega)|^{t_j}d_\theta(\omega,\upsilon).
\end{aligned}$$

by putting $r_1\in(r^\beta,\theta)$ for some constant $c_{64}>0$. On the other hand, it follows from the definition of the remainder $\tilde g_n(\varepsilon,\omega)$ addition to condition (v) in GDMS's definition that

$$\begin{aligned}
&|\tilde g_n(\varepsilon,\omega)|\\
&\le c_{60}\sum_{l=0}^{n}\sum_{k=1}^{n-l}\left(\varepsilon\frac{(\sum_{j=1}^{n}\|\pi_j\|_\infty)^k}{k!}+\|\tilde\pi_n(\varepsilon,\cdot)\|_\infty\sum_{i=1}^{k}\|z(\varepsilon)\|^{i-1}\|x(\varepsilon)-x\|^{k-i}\right)\\
&\quad\times\|T'_{\omega_0}(\pi\sigma\omega)\|^{u(l,k)}\\
&\quad+c_{60}\sum_{l=0}^{n-1}\frac{n-l}{(n-l+1)!}\left\|\frac{\pi(\varepsilon,\cdot)-\pi}{\varepsilon}\right\|_\infty^{n-l}\|\pi(\varepsilon,\cdot)-\pi\|_\infty\|T'_{\omega_0}(\pi\sigma\omega)\|^{u(l,n-l)}\\
&\quad+c_{61}\|\pi(\varepsilon,\cdot)-\pi\|^\beta\|T'_{\omega_0}(\pi\sigma\omega)\|^{u(n,0)}+c_{3.3}(\varepsilon)\|T'_e(x(\varepsilon))\|^{\tilde u}\\
&\le c_{65}(\varepsilon)\|T'_{\omega_0}(\pi\sigma\omega)\|^{\min\{Dt_n,\tilde u\}}=c_{65}(\varepsilon)|g(\omega)|^{\tilde t}
\end{aligned}$$

for any $\omega \in E^\infty$ for some number $c_{65}(\varepsilon)$ with $\lim_{\varepsilon\to 0} c_{65}(\varepsilon) = 0$, where the last inequality uses the fact $\|T_e'(x(\varepsilon))\|^{\tilde{u}} \le c_{3.3}(\varepsilon)(1 + c_9\|\pi(\varepsilon,\cdot) - \pi\|_\infty)^{\tilde{u}}\|T_{\omega_0}'(\pi\sigma\omega)\|^{\tilde{u}}$. Hence, the proof is complete. ■

3.4. Proof of Theorem 2.4

Proof. We notice $T_e(\varepsilon, X_v) = T_e(\overline{B(1/2 + a(e)\varepsilon, 1/2)})$. The function $T_e(\varepsilon, z)$ has the expansion $T_e(\varepsilon, z) = T_e(z) + \sum_{k=1}^n T_{e,k}(z)\varepsilon^k + \tilde{T}_n(\varepsilon, z)\varepsilon^n$ with

$$T_{e,k}(z) = (-a(e))^k(e+z)^{-k-1}$$

and

$$\tilde{T}_n(\varepsilon, z) = \varepsilon(-1)^{n+1}a(e)^{n+1}/((e+z)^{n+1}(e+z+a(e)\varepsilon))$$

as in (2.3). In addition to the fact $|e+z| \ge 3/4$ for all $e \in E$ and $z \in O_v$, it is not hard to check that condition (G.2)$_n$ is satisfied with $t(l,k) = \tilde{t} = 1$ for all l, k. Hence, we obtain the assertion. ■

3.5. Proof of Theorem 2.5

Proof. (1) When $a \ge 5$, the number $p(n)$ becomes zero. Therefore $\dim_H K > p(n)$ is satisfied whenever $n \ge 0$. Thus, Theorem 2.2 implies $s(\varepsilon)$ has asymptotic expansion with any order $n \ge 0$. The coefficients of $s(\varepsilon)$ are calculated as follows. Recall the form (3.17) of s_k. We see $h \equiv 1$ and $\nu([e]) = 1/2^e$ for $e \in E$. We obtain

$$\begin{aligned}\mathcal{N}_u h &= \sum_{\substack{0\le v\le u\\ 0\le q\le u-v:\\ (v,q)\ne(0,1)}} s_{q,u-v}\mathcal{Z}_{v,q,s(0)}\\ &= \sum_{\substack{0\le v\le u\\ 0\le q\le u-v:\\ (v,q)\ne(0,1)}} \sum_{j=0}^{\min(v,q)} s_{q,u-v}\frac{a_{v,j,s(0)}}{(q-j)!}(-1)^{q-j}(\log 5)^{q-j}\sum_{e=1}^{\infty} e^{q-j}\left(\frac{5^v}{2a^v}\right)^e.\end{aligned}$$

We notice that this is a constant function. Note the equations $\nu(h\log|g|) = -2\log 5$ and $\mathcal{Z}_{0,1,s(0)}h(\omega) = -2\log 5$. By using $\nu(\varepsilon, 1) \equiv 1$, we obtain $\nu_i(\mathcal{N}_{k-i}h) = \nu_i(\mathcal{Z}_{0,1,s(0)}h) = 0$ for $1 \le i \le k-1$. Thus, we get the equation

$$s_k = \frac{-1}{\nu(h\log|g|)}\left(\sum_{i=1}^{k-1}\nu_i(\mathcal{Z}_{0,1,s(0)}h)s_{k-i} + \sum_{i=0}^{k-1}\nu_i(\mathcal{N}_{k-i}h)\right) = \frac{1}{2\log 5}\nu(\mathcal{N}_k h).$$

This yields the form (2.5) of s_k.

(2) Assume $1 < a < 5$. Put $a_0 = \log 5/\log(5/a)$. We show some claims below:

Claim 1. *For each $n \geq 0$, if $\varepsilon = o(\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h))$ then we have*

$$\tilde{s}_n(\varepsilon) \asymp \tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h).$$

Indeed, note that the Ruelle operator of $\varphi(\varepsilon, \omega) = \log \|\frac{\partial}{\partial x} T_{\omega_0}(\varepsilon, \pi(\varepsilon, \sigma\omega))\|$ has the form $\mathcal{L}_{s(\varepsilon)\varphi(\varepsilon,\cdot)} f(\omega) = \sum_{e=1}^{\infty} (1/5^e + \varepsilon/a^e)^{s(\varepsilon)}$. Here, $\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, \cdot)$ is given as follows: By applying Taylor theorem to the function $x \mapsto (1/5^e + x)^{s(\varepsilon)}$, we get the expansion

$$\left(\frac{1}{5^e} + \frac{\varepsilon}{a^e}\right)^{s(\varepsilon)} = \sum_{k=0}^{n} \binom{s(\varepsilon)}{k} \left(\frac{1}{5^e}\right)^{s(\varepsilon)-k} \left(\frac{\varepsilon}{a^e}\right)^k + \binom{s(\varepsilon)}{n+1} \left(\frac{1}{5^e} + \alpha\frac{\varepsilon}{a^e}\right)^{s(\varepsilon)-n-1} \left(\frac{\varepsilon}{a^e}\right)^{n+1} \tag{3.30}$$

for each $e \in E$ and $\varepsilon > 0$ for some $\alpha = \alpha(e, n+1, \varepsilon, a) \in [0, 1]$. Therefore,

$$\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, f)(\omega) = \binom{s(\varepsilon)}{n+1} \sum_{e \in E} \left(\frac{1}{5^e} + \alpha\frac{\varepsilon}{a^e}\right)^{s(\varepsilon)-n-1} \left(\frac{\varepsilon}{a^e}\right)^{n+1} \frac{1}{\varepsilon^n} f(e \cdot \omega).$$

This implies that $\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h)$ is a constant function. We obtain the claim by (1.7) in Theorem 1.1.

Claim 2. *The inequality $n < a_0 s(0) \leq n+1$ is satisfied, where $s(0) = \log 2/\log 5$.*

Indeed, the assumption $a \leq 5/2^{1/(n+1)}$ implies

$$\frac{\log a}{\log 5} \leq 1 - \frac{1}{n+1} s(0), \qquad \frac{\log 5}{\log(5/a)} \leq \frac{n+1}{s(0)}$$

and, therefore, $s(0)a_0 \leq n+1$. By a similar argument as the one above, if $n \geq 1$, then $5/2^{1/n} < a$ implies $s(0)a_0 > n$. Thus, the claim is valid for any $n \geq 0$.

Given $R_e(\varepsilon) = (1/5^e + \alpha\varepsilon/a^e)^{s(\varepsilon)-n-1} (\varepsilon/a^e)^{n+1}$. Then we have

$$\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, 1)(\omega)\varepsilon^n = \binom{s(\varepsilon)}{n+1} \sum_{e \in E} R_e(\varepsilon).$$

Now we will prove that $\sum_{e \in E} R_e(\varepsilon) \asymp \varepsilon^{a_0 s(\varepsilon)}$. Note that

$$\frac{1}{5^e} > \frac{\varepsilon}{a^e} \iff e < \frac{\log \varepsilon}{\log(a/5)} =: a_1(\varepsilon) \iff e < \lceil a_1(\varepsilon) \rceil =: a_2(\varepsilon),$$

where the notation $\lceil\ \rceil$ means round up to the nearest integer. Recall the notation $M(n+1, s(\varepsilon))$ in Proposition C.1 replacing $n := n+1$, $a := 1/5^e$, $x := \varepsilon/a^e$ and $s := s(\varepsilon)$. Since $\varepsilon \mapsto M(n+1, s(\varepsilon))$ and $\varepsilon \mapsto L(n+1, s(\varepsilon))$ are continuous, there exists $\varepsilon_0 > 0$ such that $M(n+1, s(\varepsilon)) \geq M(n+1, s(0))/2$ and $L(n+1, s(\varepsilon)) \geq$

$L(n+1,s(0))/2$ for any $0<\varepsilon<\varepsilon_0$. Put $a_3=\lceil\frac{\log(M(n+1,s(0))/2)}{\log(a/5)}\rceil$. We decompose $\sum_e R_e(\varepsilon)$ into

$$\sum_e R_e(\varepsilon)=\sum_{e=1}^{a_2(\varepsilon)-1}R_e(\varepsilon)+\sum_{e=a_2(\varepsilon)}^{a_2(\varepsilon)+a_3-1}R_e(\varepsilon)+\sum_{e=a_2(\varepsilon)+a_3}^{\infty}R_e(\varepsilon)$$
$$=I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon).$$

Claim 3. *Let* $1<a<5$ *and* $n\geq 0$ *be the largest integer satisfying* $a\leq 5/2^{1/(n+1)}$. *Then* $\limsup_{\varepsilon\to 0}(I_1(\varepsilon)+I_2(\varepsilon))/(-\varepsilon^{n+1}\log\varepsilon)<+\infty$ *if* $a=5/2^{1/(n+1)}$ *for some* $n\geq 0$, *and* $\limsup_{\varepsilon\to 0}(I_1(\varepsilon)+I_2(\varepsilon))/\varepsilon^{a_0 s(0)}<+\infty$ *otherwise.*

Indeed, we have

$$I_1(\varepsilon)+I_2(\varepsilon)\leq\sum_{e=1}^{a_2(\varepsilon)+a_3-1}\left(\frac{1}{5^e}\right)^{s(0)-n-1}\left(\frac{\varepsilon}{a^e}\right)^{n+1}\qquad(\because s(\varepsilon)\geq s(0))$$
$$\leq\begin{cases}\varepsilon^{n+1}\left(\frac{-\log\varepsilon}{-\log(a/5)}+a_3\right) & \text{if } a=5/2^{1/(n+1)},\\ \varepsilon^{n+1}\dfrac{\left(\frac{5^{n+1}}{2a^{n+1}}\right)^{a_2(\varepsilon)+a_3}-\frac{5^{n+1}}{2a^{n+1}}}{\frac{5^{n+1}}{2a^{n+1}}-1} & \text{if } a<5/2^{1/(n+1)}.\end{cases}$$

To show the assertion, it is sufficient to check that $\varepsilon^{n+1}(5^{n+1}/(2a^{n+1}))^{a_2(\varepsilon)}\leq c\varepsilon^{a_0 s(0)}$ for some $c>0$. This is implied by the facts $a_2(\varepsilon)\leq a_1(\varepsilon)-1$, $(1/2)^{a_1(\varepsilon)}=\varepsilon^{a_0 s(0)}$, and $(5/a)^{(n+1)a_1(\varepsilon)}=\varepsilon^{-n-1}$, and by putting $c=2a^{n+1}/5^{n+1}$. This yields the assertion of the claim.

Claim 4. *Let* $1<a<5$ *and* $n\geq 0$ *be the largest integer satisfying* $a\leq 5/2^{1/(n+1)}$. *Then* $\limsup_{\varepsilon\to 0} I_3(\varepsilon)/\varepsilon^{a_0 s(0)}<+\infty$.

To show this, we will apply (3.30) to the equation (C.1) in Appendix C taking $a:=1/5^e$, $x:=\varepsilon/a^e$ and $s:=s(\varepsilon)$. We obtain the inequality $(a/5)^{a_3}\leq M(n+1,s(0))/2\leq M(n+1,s(\varepsilon))$ and $(a/5)^{a_2(\varepsilon)}/\varepsilon\leq 1$. Therefore, $(a/5)^{a_2(\varepsilon)+a_3}/\varepsilon\leq M(n+1,s(\varepsilon))$ is satisfied. Moreover, it follows from Proposition C.1 that

$$e\geq a_2(\varepsilon)+a_3\Rightarrow\frac{1}{\varepsilon}\left(\frac{a}{5}\right)^e\leq M(n+1,s(\varepsilon))$$
$$\Rightarrow\begin{cases}\alpha\geq L(1,s(\varepsilon)) & \text{if } n=0,\\ \alpha\geq L(n+1,s(\varepsilon))\left(\frac{1}{\varepsilon}\left(\frac{a}{5}\right)^e\right)^{\frac{n-s(\varepsilon)}{n+1-s(\varepsilon)}} & \text{if } n\geq 1.\end{cases}$$

Note also that $L(n+1, s(\varepsilon)) \geq L(n+1, s(0))/2 =: L(n+1) > 0$ for $0 < \varepsilon < \varepsilon_0$. In the case when $n = 0$, we have

$$\begin{aligned} I_3(\varepsilon) &\leq \sum_{e=a_2(\varepsilon)+a_3}^{\infty} \left(\frac{1}{5^e} + L(1)\frac{\varepsilon}{a^e}\right)^{s(\varepsilon)-1} \frac{\varepsilon}{a^e} \\ &\leq \frac{1}{L(1)} \sum_{e=a_2(\varepsilon)+a_3}^{\infty} \left((1+L(1))\frac{\varepsilon}{a^e}\right)^{s(0)} \quad \left(\because \frac{1}{5^e} \leq \frac{\varepsilon}{a^e} \text{ and } s(\varepsilon) > s(0)\right) \\ &\leq \frac{(1+L(1))^{s(0)}}{L(1)(1-1/a)} \left(\frac{1}{a^{s(0)}}\right)^{a_3} \varepsilon^{a_0 s(0)} \\ &\qquad\qquad (\because a^{a_2(\varepsilon)} \geq a^{a_1(\varepsilon)} = \varepsilon^{\log_{a/5} a} = \varepsilon^{-a_0+1}). \end{aligned}$$

Thus, the assertion of the claim holds in the case $n = 0$.

In the case when $n \geq 1$, we obtain

$$\begin{aligned} I_3(\varepsilon) &\leq \sum_{e=a_2(\varepsilon)+a_3}^{\infty} \left(\frac{1}{5^e} + L(n+1)\left(\frac{1}{\varepsilon}\left(\frac{a}{5}\right)^e\right)^{\frac{n-s(\varepsilon)}{n+1-s(\varepsilon)}} \frac{\varepsilon}{a^e}\right)^{s(\varepsilon)-n-1} \left(\frac{\varepsilon}{a^e}\right)^{n+1} \\ &\leq L(n+1)^{s(\varepsilon)-n-1} \sum_{e=a_2(\varepsilon)+a_3}^{\infty} \left(\frac{\varepsilon}{a^e}\right)^{s(\varepsilon)} \left(\frac{1}{\varepsilon}\left(\frac{a}{5}\right)^e\right)^{-n+s(\varepsilon)} \\ &\qquad\qquad (\because s(\varepsilon) - n - 1 < 0) \\ &\leq L(n+1)^{s(0)-n-1} \varepsilon^n \frac{\left(\frac{1}{2}\left(\frac{5}{a}\right)^n\right)^{a_2(\varepsilon)+a_3}}{1 - \frac{1}{2}\left(\frac{5}{a}\right)^n} \qquad (\because s(\varepsilon) > s(0) \text{ and } 5^{s(0)} = 2), \end{aligned}$$

where in the last expression, we remark that $(5/a)^n/2 < 1$ by the definition of n. In the last expression, we notice the estimate $((5/a)^n/2)^{a_2(\varepsilon)} \leq ((5/a)^n/2)^{a_1(\varepsilon)} = \varepsilon^{a_0 s(0)}\varepsilon^{-n}$. Thus, we obtain the assertion of Claim 4.

Claim 5. *Let $1 < a < 5$ and $n \geq 0$ be the largest integer satisfying $a \leq 5/2^{1/(n+1)}$. Then $\liminf_{\varepsilon\to 0} I_1(\varepsilon)/(-\varepsilon^{n+1}\log\varepsilon) > 0$ if $a = 5/2^{1/(n+1)}$, and $\liminf_{\varepsilon\to 0} I_1(\varepsilon)/\varepsilon^{a_0 s(0)} > 0$ otherwise.*

By virtue of Claims 1–4, $s(\varepsilon) = s(0) + t(\varepsilon)\varepsilon^{a_1}$ and $t(\varepsilon) = O(1)$ are satisfied with $a_1 := a_0 s(0) - \eta$ for any small $\eta > 0$. Then we have that for each $n \geq 0$,

$$\begin{aligned} I_1(\varepsilon) &\geq \sum_{e=1}^{a_2(\varepsilon)-1} \left(\frac{1}{5^e} + \frac{\varepsilon}{a^e}\right)^{s(\varepsilon)-n-1} \left(\frac{\varepsilon}{a^e}\right)^{n+1} \\ &\geq \sum_{e=1}^{a_2(\varepsilon)-1} \frac{\left(\frac{1}{5^e}\right)^{s(0)}}{\left(\frac{2}{\varepsilon}\left(\frac{a}{5}\right)^e\right)^{n+1}} \left(\frac{\varepsilon}{a^e}\right)^{t(\varepsilon)\varepsilon^{a_1}} \qquad (\because t(\varepsilon) > 0) \end{aligned}$$

$$\geq \left(\frac{\varepsilon}{a^{a_1(\varepsilon)}}\right)^{t(\varepsilon)\varepsilon^{a_1}} \frac{\varepsilon^{n+1}}{2^{n+1}} \begin{cases} a_1(\varepsilon) - 1 & \text{if } a = 5/2^{1/(n+1)}, \\ \frac{\left(\frac{5^{n+1}}{2a^{n+1}}\right)^{a_1(\varepsilon)} - \frac{5^{n+1}}{2a^{n+1}}}{\frac{5^{n+1}}{2a^{n+1}} - 1} & \text{if } a < 5/2^{1/(n+1)} \end{cases}$$

$$(\because a_2(\varepsilon) \geq a_1(\varepsilon)).$$

Here we notice that $a_2(\varepsilon) \geq a_1(\varepsilon) = \log \varepsilon / \log(a/5)$ and $\left(5^{n+1}/(2a^n)\right)^{a_1(\varepsilon)} = \varepsilon^{a_0 s(0)} \varepsilon^{-n-1}$. Note that

$$\begin{aligned} \left(\frac{\varepsilon}{a^{a_1(\varepsilon)}}\right)^{t(\varepsilon)\varepsilon^{a_1}} &= \exp(t(\varepsilon)\varepsilon^{a_1} \log \varepsilon) \exp(-t(\varepsilon) \frac{\log a}{\log(a/5)} \varepsilon^{a_1} \log \varepsilon) \\ &\to \exp(0) \exp(0) = 1 \end{aligned}$$

as $\varepsilon \to 0$. Thus, the assertion of Claim 5 is yielded.

By Claims 3–5 and the fact $\sum_{e \in E} R_e(\varepsilon) \geq I_1(\varepsilon)$, we obtain $\sum_{e \in E} R_e(\varepsilon) \asymp -\varepsilon^{n+1} \log \varepsilon$ if $a = 5/2^{1/(n+1)}$, and $\asymp \varepsilon^{a_0 s(0)}$ otherwise. Thus, so is $\tilde{\mathcal{L}}_{n,s(\varepsilon)}(\varepsilon, h)$. Hence, the proof follows from Claim 1. ∎

3.6. Proof of Proposition 2.6

Proof. We take t_1 and t_2 so that

$$t_1 = \sup\left\{t \in (0,1] \colon \sup_{\omega \in E^\infty} \frac{|g_1(\omega)|}{|g(\omega)|^t} = \sup_{e \in E} \left(\frac{5^t}{4}\right)^e < +\infty\right\} \iff t_1 = \frac{\log 4}{\log 5},$$
$$t_2 = \sup\left\{t \in (0,1] \colon \sup_{\omega \in E^\infty} \frac{|g_1(\omega)|}{|g(\omega)|^t} = \sup_{e \in E} \left(\frac{5^t}{3}\right)^e < +\infty\right\} \iff t_2 = \frac{\log 3}{\log 5}.$$

In view of Theorem 1.1, the number $p(n)$ is given by

$$p(n) = \max\left(n(1-t_1), \frac{n(1-t_2)}{2}\right)$$

by $\underline{p} = 0$ and $\tilde{t} = 0$. Therefore,

$$p(n) = n \max\left(\frac{1 - \log 4}{\log 5}, \frac{(1 - \log 3/\log 5)}{2}\right) = \frac{n(1 - \log 3/\log 5)}{2}$$

by $1 - \log 4/\log 5 = 0.1386\ldots$ and $(1 - \log 3/\log 5)/2 = 0.1586\ldots$. By virtue of Corollary 1.2, when $s(0) > p(n)$, the dimension $\dim_H K(\varepsilon)$ has an asymptotic expansion with order n at $\varepsilon = 0$. We see

$$\begin{aligned} s(0) = \frac{\log 2}{\log 5} > n \frac{1 - \log 3/\log 5}{2} &\iff n < \frac{\log 2/\log 5}{(1 - \log 3/\log 5)/2} = 2.713\ldots \\ &\iff n \leq 2. \end{aligned}$$

Hence, $\dim_H K(\varepsilon)$ has at least the 2-order asymptotic expansion at $\varepsilon = 0$. ∎

3.7. Proof of Theorem 2.7

Proof. We will check the conditions $(G.1)_n$ and $(G.2)_n$ in Theorem 2.2. Let $\varphi(\omega) = \log \|T'_{\omega_0}(\pi\sigma\omega)\|$ for $\omega = w_0 w_1 \cdots \in E^\infty$ and recall the number $\underline{p}$ defined by (1.4) with $g(\omega) = (1/D) \det |T'_{\omega_0}(\pi\sigma\omega)|$ and $\psi \equiv 1$. The condition $(G.1)_n$ is yielded by condition (K.1). We will check $(G.2)_n$ (i). Since any inversion map is of C^∞ and since $r(\varepsilon, v)$ and $a(\varepsilon, v)$ have the n-order expansions (2.7) and (2.8), it is not hard to see that the map $T_e(\varepsilon, \cdot)$ for $e \in E_0$ has the n-order asymptotic expansion $T_e(\varepsilon, \cdot) = T_e + \sum_{k=1}^n T_{e,k}\varepsilon^k + \tilde{T}_{e,n}(\varepsilon, \cdot)\varepsilon^n$ for some $C^\infty(O_{t(e)})$ maps $T_{e,k}$ and $\tilde{T}_{e,n}(\varepsilon, \cdot)$ with convergence $\sup_{x \in O_{t(e)}} |\tilde{T}_{e,n}(\varepsilon, x)| \to 0$ and $\sup_{x \in O_{t(e)}} \|\frac{\partial}{\partial x}\tilde{T}_{e,n}(\varepsilon, x)\| \to 0$ as $\varepsilon \to 0$. Since $T_w(\varepsilon, \cdot)$ is a composite of a finite number of such functions, we also get an n-order asymptotic expansion of $T_w(\varepsilon, \cdot)$ for each $w \in E$ and therefore $(G.2)_n$ (i) is fulfilled. By virtue of condition (K.2), the maps $T_w(\varepsilon, \cdot)$ that change by ε are of a finite number. Thus, conditions $(G.2)_n$ (ii)–(iv) are valid by putting $t(l, k) = \tilde{t} = 1$. Hence, the assertion is obtained from Theorem 2.2. ∎

A. Thermodynamic formalism and Ruelle operators

In this section, we will present useful results for the proof of the main theorem. We will recall the notion of thermodynamic formalism and some facts of Ruelle transfer operators which were mainly introduced by [17].

We use the notation defined in Section 1. The incidence matrix A of the graph G is called *finitely primitive* if there exist an integer $n \geq 1$ and a finite subset F of E^n such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. Note that A is finitely primitive if and only if (E^∞, σ) is topologically mixing and A has the BIP property (see [23]). Then it is stronger than finitely irreducible. Here the matrix A has the *big images and pre-images property (BIP property)* if there is a finite subset $\{e_1, \ldots, e_N\}$ of E such that for any $e \in E$, there exist $1 \leq i, j \leq N$ such that $e_i e$ and ee_j are paths on the graph G. A function $\psi \colon E^\infty \to \mathbb{R}$ is *acceptable* if there exists a constant $c_{66} \geq 1$ such that for any $e \in E$ and $\omega, \upsilon \in [e]$, $e^{\psi(\omega) - \psi(\upsilon)} \leq c_{66}$ (see [17] for the terminology). For a real-valued function ψ on E^∞, the topological pressure $P(\psi)$ of ψ is formally given by

$$P(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w \in E^n : [w] \neq \emptyset} \exp\left(\sup_{\omega \in [w]} \sum_{k=0}^{n-1} \psi(\sigma^k \omega) \right). \tag{A.1}$$

If ψ is acceptable, then $P(\psi)$ exists in $[-\infty, +\infty]$ (see [17]). We mainly consider the pressure function $t \mapsto P(t\psi) \in [-\infty, +\infty]$ with $\sup_{\omega \in E^\infty} \psi(\omega) < 0$. In this case, it is a basic fact that the pressure function is strictly monotone decreasing and convex (being a limit of convex functions), and thus continuous. In particular, $\lim_{t\to+\infty} P(t\psi) = -\infty$ holds.

For a real-valued function ψ on E^∞, the Ruelle operator $\mathcal{L}_\psi$ associated to ψ is defined by

$$\mathcal{L}_\psi f(\omega) = \sum_{e\in E:t(e)=i(\omega_0)} e^{\psi(e\cdot\omega)} f(e\cdot\omega)$$

if this series converges in $\mathbb{C}$ for a complex-valued function f on E^∞ and for $\omega \in E^\infty$. Here $e\cdot\omega$ is the concatenation of e and ω, i.e., $e\cdot\omega = e\omega_0\omega_1\cdots$. It is known that if the incidence matrix is finitely irreducible and ψ is in $F_\theta(E^\infty, \mathbb{R})$ with finite topological pressure, then $\mathcal{L}_\psi$ becomes a bounded linear operator acting on both the Banach spaces $F_{\theta,b}(E^\infty)$ and $C_b(E^\infty)$. We begin with the following proposition.

Proposition A.1 ([17, Proposition 2.1.9]). *Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph such that the incidence matrix of E^∞ is finitely irreducible. Take an acceptable potential function $\psi\colon F_\theta(E^\infty) \to \mathbb{R}$. Then $P(\psi) < \infty$ if and only if $\|\mathcal{L}_\psi 1\|_\infty < \infty$ if and only if $Z := \sum_{e\in E} \exp(\sup_{\omega\in[e]} \psi(\omega)) < \infty$.*

A Borel probability measure μ on E^∞ is said to be a *Gibbs measure* of the potential ψ if there exist constants $c \geq 1$ and $P \in \mathbb{R}$ such that for any $\omega \in E^\infty$ and $n \geq 1$

$$c^{-1} \leq \frac{\mu\big(\{\upsilon \in E^\infty\colon \upsilon_i = \omega_i, 0 \leq i < n\}\big)}{\exp(-nP + \sum_{k=0}^{n-1} \psi(\sigma^k\omega))} \leq c.$$

Recall the notation $\mathcal{L}(\mathcal{X})$ which is the set of all bounded linear operators acting on a normed linear space $\mathcal{X}$. The following is a version of Ruelle–Perron–Frobenius Theorem:

Theorem A.2 ([1, 2, 17, 21, 22]). *Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph such that the incidence matrix of E^∞ is finitely irreducible. Assume that $\psi \in F_\theta(E^\infty, \mathbb{R})$ with $P(\psi) < \infty$. Then there exists a unique triplet $(\lambda, h, \nu) \in \mathbb{R} \times F_{\theta,b}(E^\infty) \times C_b(E^\infty)^*$ such that the following are satisfied:*

(1) *The number λ is positive and a simple maximal eigenvalue of the operator $\mathcal{L}_\psi \in \mathcal{L}(F_{\theta,b}(E^\infty))$ and is equal to* $\exp(P(\psi))$.

(2) *The operator $\mathcal{L}_\psi \in \mathcal{L}(F_{\theta,b}(E^\infty))$ has the decomposition $\mathcal{L}_\psi = \lambda\mathcal{P} + \mathcal{R}$ with $\mathcal{P}\mathcal{R} = \mathcal{R}\mathcal{P} = O$. Here the operator $\mathcal{P}$ is a projection onto the one-dimensional eigenspace of the eigenvalue λ. Moreover, this has the form $\mathcal{P}f = \int_{E^\infty} f h\, d\nu$ for $f \in C_b(E^\infty)$, where $h \in F_{\theta,b}(E^\infty, \mathbb{R})$ is the corresponding eigenfunction of λ, and ν is the corresponding eigenvector of λ of the dual $\mathcal{L}_\psi^*$ with $\nu(h) = 1$. Here h is bounded uniformly away from zero and infinity, and ν is a Borel probability measure on E^∞. In particular, $h\nu$ is the σ-invariant Gibbs measure of ψ.*

(3) *The spectrum of $\mathcal{R} \in \mathcal{L}(F_{\theta,b}(E^\infty))$ is contained in $\{z \in \mathbb{C}\colon |z - \lambda| \geq \rho\}$ for some $\rho > 0$.*

B. Asymptotic perturbation of eigenvectors of bounded linear operators

In this section, we study asymptotic behaviour of the eigenvalues and eigenvectors of perturbed bounded linear operators under an abstract setting.

Put $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $(\mathcal{X}_0, \|\cdot\|_0)$ be a normed linear space over $\mathbb{K}$ and $(\mathcal{X}_1, \|\cdot\|_1)$ a Banach space over $\mathbb{K}$ such that $\mathcal{X}_1 \subset \mathcal{X}_0$ and $\|f\|_0 \le \|f\|_1$ for any $f \in \mathcal{X}_1$. We write $\mathcal{X}^*$ as the dual space of $\mathcal{X}$ and $\mathcal{L}^* \in \mathcal{L}(\mathcal{X}^*)$ as the dual operator of $\mathcal{L} \in \mathcal{L}(\mathcal{X})$.

Let $\mathcal{L} \in \mathcal{L}(\mathcal{X}_0) \cap \mathcal{L}(\mathcal{X}_1)$ and $\mathcal{L}(\varepsilon, \cdot) \in \mathcal{L}(\mathcal{X}_0)$. Take $(\lambda, \nu), (\lambda(\varepsilon), \nu(\varepsilon, \cdot)) \in \mathbb{K} \times \mathcal{X}_0^*$ so that $\mathcal{L}^*\nu = \lambda\nu$ and $\mathcal{L}(\varepsilon, \cdot)^*\nu(\varepsilon, \cdot) = \lambda(\varepsilon)\nu(\varepsilon, \cdot)$. We assume the following conditions:

(L.1) There exists $h \in \mathcal{X}_1$ such that $\mathcal{L}h = \lambda h$ and $\nu(h) = 1$.

(L.2) The operator $\mathcal{L}$ has the decomposition $\mathcal{L} = \lambda\mathcal{P} + \mathcal{R}$ satisfying that (i) $\mathcal{P}$ is in $\mathcal{L}(\mathcal{X}_0) \cap \mathcal{L}(\mathcal{X}_1)$ and forms $\mathcal{P}f = \nu(f)h$, (ii) $\mathcal{P}\mathcal{R} = \mathcal{R}\mathcal{P} = O$, and (iii) λ is in the resolvent set of the operator $\mathcal{R} \in \mathcal{L}(\mathcal{X}_1)$.

(L.3) $\limsup_{\varepsilon \to 0} \|\nu(\varepsilon, \cdot)\|_0^*/\nu(\varepsilon, h) < \infty$, where

$$\|\nu(\varepsilon, \cdot)\|_0^* := \sup_{f \in \mathcal{X}_0 : \|f\|_0 \le 1} |\nu(\varepsilon, f)|.$$

(L.4) There exist operators $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathcal{L}(\mathcal{X}_0) \cap \mathcal{L}(\mathcal{X}_1)$ and $\tilde{\mathcal{L}}_n(\varepsilon, \cdot) \in \mathcal{L}(\mathcal{X}_0)$ such that $\mathcal{L}(\varepsilon, \cdot) = \mathcal{L} + \mathcal{L}_1\varepsilon + \cdots + \mathcal{L}_n\varepsilon^n + \tilde{\mathcal{L}}_n(\varepsilon, \cdot)\varepsilon^n$ and $\|\tilde{\mathcal{L}}_n(\varepsilon, f)\|_0 \to 0$ as $\varepsilon \to 0$ for each $f \in \mathcal{X}_1$.

Let $\mathcal{S} \in \mathcal{L}(\mathcal{X}_1)$ be $\mathcal{S} = (\mathcal{R} - \lambda\mathcal{I})^{-1}(\mathcal{I} - \mathcal{P})$, where $\mathcal{I}$ is the identity operator on $\mathcal{X}_1$. Numbers $\lambda_k \in \mathbb{K}$ and linear functionals $\kappa_k \colon \mathcal{X}_1 \to \mathbb{K}$ $(1 \le k \le n)$ are defined by

$$\lambda_k = \sum_{j=1}^{k} \kappa_{k-j}(\mathcal{L}_j h), \tag{B.1}$$

$$\kappa_k(f) = \sum_{j=1}^{k} \kappa_{k-j}((\lambda_j\mathcal{I} - \mathcal{L}_j)\mathcal{S}f) \quad \text{for each } f \in \mathcal{X}_1 \tag{B.2}$$

inductively with $\kappa_0 = \nu$ and $\mathcal{L}_0 = \mathcal{L}$. We have the following:

Theorem B.1. *Assume that the conditions (L.1)–(L.4) are satisfied for a fixed integer $n \ge 0$. We define $\kappa(\varepsilon, \cdot) \in \mathcal{X}_0^*$ by $\kappa(\varepsilon, f) = \nu(\varepsilon, f)/\nu(\varepsilon, h)$ for $f \in \mathcal{X}_0$. Then*

(1) $\lambda(\varepsilon) = \lambda + \lambda_1\varepsilon + \cdots + \lambda_n\varepsilon^n + o(\varepsilon^n)$ *in* $\mathbb{K}$;

(2) $\kappa(\varepsilon, f) = \nu(f) + \kappa_1(f)\varepsilon + \cdots + \kappa_n(f)\varepsilon^n + o(\varepsilon^n)$ *in* $\mathbb{K}$ *for each* $f \in \mathcal{X}_1$.

Corollary B.2. *In addition to the conditions (L.1)–(L.4), we assume*

$$\liminf_{\varepsilon\to 0} |\nu(\varepsilon, h)| > 0,$$

and there exists $1_{\mathcal{X}} \in \mathcal{X}_1$ *such that* $\nu(\varepsilon, 1_{\mathcal{X}}) = 1$ *for any* $\varepsilon > 0$. *Then the eigenvector of* $\lambda(\varepsilon)$ *has the expansion*

$$\nu(\varepsilon, f) = \frac{\nu(f)}{\nu(1_{\mathcal{X}})} + \nu_1(f)\varepsilon + \cdots + \nu_n(f)\varepsilon^n + o(\varepsilon^n) \text{ in } \mathbb{K}$$

for each $f \in \mathcal{X}_1$, *where we put* $\nu_k(f) = \sum_{0\le i,j\le k: i+j=k} b_j \kappa_i(f)$, $b_0 = 1/\nu(1_{\mathcal{X}})$ *and*

$$b_j = \sum_{l=1}^{j} \frac{1}{\nu(1_{\mathcal{X}})^{l+1}} \sum_{\substack{i_1,\dots,i_j\ge 0:\\ i_1+\dots+i_j=l\\ i_1+2i_2+\dots+j\cdot i_j=j}} \frac{(-1)^l l!}{i_1!\cdots i_j!} \kappa_1(1_{\mathcal{X}})^{i_1}\cdots\kappa_l(1_{\mathcal{X}})^{i_j}$$

for $1 \le j \le n$.

Proof. Put $f = 1_{\mathcal{X}}$ in Theorem B.1(2). Then we have $\nu(\varepsilon, h)^{-1} = \nu(1_{\mathcal{X}}) + \sum_{k=1}^{n} \kappa_k(1_{\mathcal{X}})\varepsilon^k + o(\varepsilon^n)$ and $\nu(1_{\mathcal{X}}) \ne 0$. The assertion follows from the asymptotic expansion of $\nu(\varepsilon, h)$ and the form $\nu(\varepsilon, f) = \kappa(\varepsilon, f)\nu(\varepsilon, h)$. ∎

Remark B.3. (1) The results are generalizations of the results of [24] which gave the asymptotic behaviour of the maximal eigenvalue of the Ruelle operators with finite state and the corresponding eigenprojection.

(2) When the remainder $\tilde{\mathcal{L}}_n(\varepsilon,\cdot)$ satisfies $\|\tilde{\mathcal{L}}_n(\varepsilon,\cdot)\|_1 \to 0$, the above results are implied by the general asymptotic perturbation theory [12]. If $\mathcal{L}_n(\varepsilon,\cdot)$ is a Ruelle operator with finite state and fulfills $\|\tilde{\mathcal{L}}_n(\varepsilon,\cdot)\|_0 \to 0$ with $\|\cdot\|_0 := \|\cdot\|_\infty$, then similar assertions follow from [24]. Keller and Liverani in [13] considered convergence of eigenvalue and eigenprojection in an abstract setting under a uniform Lasota–Yorke type inequality such as $\|\mathcal{L}(\varepsilon, f)^n\|_1 \le c\alpha^n \|f\|_1 + cM^n\|f\|_0$ for any $\varepsilon > 0$ and $f \in \mathcal{X}_1$ for some constant $c > 0$, $0 < M \le \sup_{\varepsilon>0} \|\mathcal{L}(\varepsilon,\cdot)\|_0$ and $0 < \alpha < M$. Under such an inequality, Gouëzel and Liverani in [8, Section 8] studied the asymptotic perturbation of bounded linear operators. We emphasize that our assertion does not need a uniform Lasota–Yorke type inequality.

Proof of Theorem B.1. We start with the equation $(\mathcal{L} - \lambda\mathcal{I})\mathcal{S} = \mathcal{I} - \mathcal{P}$ on $\mathcal{X}_1$. By the definition of the operator $\mathcal{P}$, this is a projection, i.e., $\mathcal{P}^2 = \mathcal{P}$. The equation follows from $(\mathcal{I} - \mathcal{P})(\mathcal{R} - \lambda\mathcal{I}) = \mathcal{R} - \lambda\mathcal{I} + \lambda\mathcal{P} = \mathcal{L} - \lambda\mathcal{I}$.

We first prove assertions (1) and (2) in the case when $n = 0$. Consider the equation

$$(\lambda(\varepsilon) - \lambda)\nu(\varepsilon, h) = \nu(\varepsilon, (\mathcal{L}(\varepsilon,\cdot) - \mathcal{L})h) \tag{B.3}$$

by using $\mathcal{L}(\varepsilon,\cdot)^*\nu(\varepsilon,\cdot)=\lambda(\varepsilon)\nu(\varepsilon,\cdot)$ and $\mathcal{L}h=\lambda h$. This yields the inequality $|\lambda(\varepsilon)-\lambda|\le\|\kappa(\varepsilon,\cdot)\|_0^*\|\tilde{\mathcal{L}}_0(\varepsilon,h)\|_0\to 0$ with conditions (L.3) and (L.4). Therefore, we have $\lambda(\varepsilon)\to\lambda$. On the other hand, we obtain that for each $f\in\mathcal{X}_1$

$$\begin{aligned}|\kappa(\varepsilon,(\mathcal{I}-\mathcal{P})f)|&=|\kappa(\varepsilon,(\mathcal{L}-\lambda\mathcal{I})\mathcal{S}f)|\\&=|\kappa(\varepsilon,(\mathcal{L}-\mathcal{L}(\varepsilon,\cdot)+(\lambda(\varepsilon)-\lambda)\mathcal{I})\mathcal{S}f)|\\&\le\|\kappa(\varepsilon,\cdot)\|_0^*(\|(\mathcal{L}(\varepsilon,\cdot)-\mathcal{L})\mathcal{S}f\|_0+|\lambda(\varepsilon)-\lambda|\|\mathcal{S}f\|_0)\to 0\end{aligned}$$

as $\varepsilon\to 0$. This and the fact $\kappa(\varepsilon,h)\equiv 1$ imply $\kappa(\varepsilon,f)\to\nu(f)$ for $f\in\mathcal{X}_1$.

Assume $n\ge 1$. To show the assertions (1), (2), we assume that the assertions (1), (2), (B.1) and (B.2) are valid for each $n'=0,1,\dots,n-1$. We will check the case $n'=n$. By (B.1), for each $n'=1,2,\dots,n-1$, and equation (B.3), we have the following:

$$\begin{aligned}&\frac{\lambda(\varepsilon)-\lambda-\lambda_1\varepsilon-\cdots-\lambda_{n-1}\varepsilon^{n-1}}{\varepsilon^n}\\&=\kappa\Big(\varepsilon,\frac{\lambda(\varepsilon)-\lambda-\lambda_1\varepsilon-\cdots-\lambda_{n-1}\varepsilon^{n-1}}{\varepsilon^n}h\Big)\\&=\kappa(\varepsilon,\frac{\mathcal{L}(\varepsilon,\cdot)-\mathcal{L}-\sum_{l=1}^{n-1}\mathcal{L}_l\varepsilon^l}{\varepsilon^n}h)+\sum_{l=1}^{n-1}\frac{\kappa(\varepsilon,\mathcal{L}_lh)\varepsilon^l-\sum_{j=1}^{l}\kappa_{l-j}(\mathcal{L}_jh)\varepsilon^j}{\varepsilon^n}\\&=\kappa(\varepsilon,\frac{\mathcal{L}(\varepsilon,\cdot)-\mathcal{L}-\sum_{l=1}^{n-1}\mathcal{L}_l\varepsilon^l}{\varepsilon^n}h)+\sum_{l=1}^{n-1}\frac{\kappa(\varepsilon,\mathcal{L}_lh)-\sum_{j=0}^{n-l-1}\kappa_j(\mathcal{L}_lh)\varepsilon^j}{\varepsilon^{n-l}}\\&\to\nu(\mathcal{L}_nh)+\sum_{l=1}^{n-1}\kappa_{n-l}(\mathcal{L}_lh)=\sum_{l=1}^{n}\kappa_{n-l}(\mathcal{L}_lh)=:\lambda_n.\end{aligned}$$

Thus, (1) and (B.1) are valid for n. Finally, we check (2) and (B.2). We obtain

$$\begin{aligned}&\frac{\kappa(\varepsilon,f)-\sum_{l=0}^{n-1}\kappa_l(f)\varepsilon^l}{\varepsilon^n}=\kappa\Big(\varepsilon,\frac{\mathcal{I}-\mathcal{P}}{\varepsilon^n}f\Big)-\sum_{l=1}^{n-1}\frac{\kappa_l(f)\varepsilon^l}{\varepsilon^n}\\&=\kappa\Big(\varepsilon,\frac{\mathcal{L}-\mathcal{L}(\varepsilon,\cdot)+(\lambda(\varepsilon)-\lambda)\mathcal{I}}{\varepsilon^n}\mathcal{S}f\Big)-\sum_{l=1}^{n-1}\frac{\sum_{i=1}^{l}\kappa_{l-i}((\lambda_i\mathcal{I}-\mathcal{L}_i)\mathcal{S}f)}{\varepsilon^{n-l}}\\&=-\kappa\Big(\varepsilon,\frac{\mathcal{L}(\varepsilon,\cdot)-\mathcal{L}-\sum_{l=1}^{n-1}\mathcal{L}_l\varepsilon^l}{\varepsilon^n}\mathcal{S}f\Big)+\kappa(\varepsilon,\frac{\lambda(\varepsilon)-\sum_{l=0}^{n-1}\lambda_l\varepsilon^l}{\varepsilon^n}\mathcal{S}f)\\&\qquad+\kappa\Big(\varepsilon,\frac{\sum_{l=1}^{n-1}(\lambda_l\mathcal{I}-\mathcal{L}_l)\varepsilon^l}{\varepsilon^n}\mathcal{S}f\Big)-\sum_{l=1}^{n-1}\frac{\sum_{i=1}^{l}\kappa_{l-i}((\lambda_i\mathcal{I}-\mathcal{L}_i)\mathcal{S}f)}{\varepsilon^{n-l}}\end{aligned}$$

$$= -\kappa\left(\varepsilon, \frac{\mathcal{L}(\varepsilon,\cdot) - \mathcal{L} - \sum_{l=1}^{n-1} \mathcal{L}_l \varepsilon^l}{\varepsilon^n} \mathcal{S} f\right) + \frac{\lambda(\varepsilon) - \lambda - \sum_{l=1}^{n-1} \lambda_l \varepsilon^l}{\varepsilon^n} \kappa(\varepsilon, \mathcal{S} f)$$
$$+ \sum_{l=1}^{n-1} \frac{\kappa(\varepsilon,\cdot) - \nu(\cdot) - \sum_{j=1}^{n-l-1} \kappa_j(\cdot)\varepsilon^j}{\varepsilon^{n-l}} ((\lambda_l \mathcal{I} - \mathcal{L}_l)\mathcal{S} f)$$
$$\to -\nu(\mathcal{L}_n \mathcal{S} f) + \lambda_n \nu(\mathcal{S} f) + \sum_{l=1}^{n-1} \kappa_{n-l}\big((\lambda_l \mathcal{I} - \mathcal{L}_l)\mathcal{S} f\big)$$
$$= \sum_{l=1}^{n} \kappa_{n-l}\big((\lambda_l \mathcal{I} - \mathcal{L}_l)\mathcal{S} f\big) =: \kappa_n(f).$$

Hence, (2) and (B.2) are fulfilled for n. ∎

C. Estimate of an intermediate point of the binomial expansion

The Taylor expansion implies that for any $n \geq 1$, $a > 0$ and $s \in (0, 1]$, the map $x \mapsto (a + x)^s$ has the form

$$(a + x)^s = a^s + \sum_{k=1}^{n-1} \binom{s}{k} a^{s-k} x^k + \binom{s}{n} (a + x\alpha)^{s-n} x^n \tag{C.1}$$

for some constant $\alpha = \alpha(n, a, s, x) \in [0, 1]$, where $\binom{s}{k}$ is the binomial coefficient $s(s-1)\dots(s-k+1)/k!$. In this section, we will estimate the lower bound of the intermediate point α which plays an important role in giving the asymptotic expansion of $\exp(t\varphi(\varepsilon,\cdot))$ (see the Proof of Theorem 2.5). Note that the estimate of α was studied by [9, 10].

Proposition C.1. *Assume that the map $x \mapsto (a + x)^s$ has the expansion (C.1). Then there exist two positive continuous functions $(0, 1) \ni s \mapsto L(n, s)$, $(0, 1) \ni s \mapsto M(n, s) \in (0, 1)$ such that for any $a, x > 0$ with $0 < a/x \leq M(n, s)$*

(1) *if $n = 1$, then $\alpha(n, a, s, x) \geq L(n, s)$;*

(2) *if $n \geq 2$, then $\alpha(n, a, s, x) \geq (a/x)^{\frac{n-1-s}{n-s}} L(n, s)$.*

Proof. (1) Assume that $n = 1$. We have $(1 + a/x)^s - (a/x)^s = s(\alpha + a/x)^{s-1}$. If x, a satisfy $a/x \leq s^{1/(1-s)} 2^{-s/(1-s)-1}$, then

$$\alpha = \frac{s^{1/(1-s)}}{\left(\left(\frac{a}{x} + 1\right)^s - \left(\frac{a}{x}\right)^s\right)^{1/(1-s)}} - \frac{a}{x} \geq \frac{s^{1/(1-s)}}{2^{s/(1-s)}} - \frac{1}{2}\frac{s^{1/(1-s)}}{2^{s/(1-s)}}$$
$$= \frac{s^{1/(1-s)}}{2 \cdot 2^{s/(1-s)}} = \frac{s^{1/(1-s)}}{2^{1/(1-s)}}.$$

Thus, we obtain the assertion by putting $L(1,s)=s^{1/(1-s)}2^{-1/(1-s)}$ and $M(1,s)=s^{1/(1-s)}2^{-s/(1-s)-1}$.

(2) Assume that $n\geq 2$. We will solve the equation (C.1) for α. This equation implies

$$\left(\frac{a}{x}+1\right)^{s}=\sum_{k=0}^{n-1}\binom{s}{k}\left(\frac{a}{x}\right)^{s-k}+\binom{s}{n}\left(\frac{a}{x}+\alpha\right)^{s-n}.$$

Noting the fact sign $\binom{s}{n}=(-1)^{n-1}$, we have

$$\begin{aligned}\left|\binom{s}{n}\right|\left(\frac{a}{x}+\alpha\right)^{s-n}&=(-1)^{n-1}\left(\frac{a}{x}+1\right)^{s}+(-1)^{n}\sum_{k=0}^{n-1}\binom{s}{k}\left(\frac{a}{x}\right)^{s-k}\\&=\left(\frac{a}{x}\right)^{s-n+1}\left(b(s,a,x)+\left|\binom{s}{n-1}\right|\right)\end{aligned}$$

by putting

$$b(s,a,x)=(-1)^{n-1}\left(\frac{a}{x}\right)^{-s+n-1}\left(1+\frac{a}{x}\right)^{s}+(-1)^{n}\sum_{k=0}^{n-2}\binom{s}{k}\left(\frac{a}{x}\right)^{n-1-k}.$$

Thus,

$$\alpha=\left(\frac{a}{x}\right)^{\frac{n-s-1}{n-s}}\left\{\left(\frac{\left|\binom{s}{n}\right|}{\left|\binom{s}{n-1}\right|+b(s,a,x)}\right)^{1/(n-s)}-\left(\frac{a}{x}\right)^{\frac{1}{n-s}}\right\}.$$

When $a\leq x$, we see

$$|b(s,a,x)|\leq\left(\frac{a}{x}\right)^{-s+n-1}2^{s}+\sum_{k=0}^{n-2}\left|\binom{s}{k}\right|\left(\frac{a}{x}\right)^{n-1-k}\leq\left(\frac{a}{x}\right)^{1-s}c_{67}$$

with $c_{67}=c_{67}(s,a,x)=2^{s}+\sum_{k=0}^{n-2}\left|\binom{s}{k}\right|$. Consequently, for any $x,a>0$ satisfying that $a/x\leq M(n,s):=\min(1,(1/c_{67})^{1/(1-s)},(1/2)^{n-s}\left|\binom{s}{n}\right|/(\left|\binom{s}{n-1}\right|+1))$, we obtain that $|b(s,a,x)|\leq 1$ and

$$\alpha\geq\left(\frac{a}{x}\right)^{\frac{n-s-1}{n-x}}\frac{1}{2}\left(\frac{\left|\binom{s}{n}\right|}{\left|\binom{s}{n-1}\right|+1}\right)^{1/(n-s)}.$$

Hence, the assertion is fulfilled by putting $L(n,s)=\frac{1}{2}\left(\left|\binom{s}{n}\right|/(\left|\binom{s}{n-1}\right|+1)\right)^{1/(n-s)}$. ∎

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