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Iterated function systems based on the degree of nondensifiability

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Abstract. In the present paper we introduce the concept of iterated function systems (IFS) having at least one ϕ-condensing mapping which belongs to the finite set of self-mappings that define the IFS. It is shown the existence of an invariant for those IFS. Whenever all the self-mappings are ϕ-condensing we prove that the invariant set is compact. We propose some applications of those IFS having ϕ-condensing self-mappings to the superposition operator defined on the Banach space $\mathcal{C}([0,1])$.

1. Introduction

Since the publication of the Hutchinson's work [16], and its subsequent popularization due to Barnsley [4] in the 90s, many authors have extended, in several directions, the classical theory of iterated function systems. For concrete references and results, see [5, 8, 12, 21]. In the following lines, for a better comprehension of the manuscript, we recall some elementary definitions and results related with iterated function systems.

Let (X, d) be a complete metric space and, as usual, for a given non-empty set $B \subset X$, $\bar{B}$ is the closure of B. We recall that for two non-empty and bounded subsets A, B of X, the Hausdorff–Pompeiu semi-metric is given by

$$\delta(A,B) := \inf\Big\{\varepsilon > 0 : A \subset \bigcup_{b\in B} U(b,\varepsilon),\, B \subset \bigcup_{a\in A} U(a,\varepsilon)\Big\},$$

where $U(x, r)$ stands for the closed ball centered at $x \in X$ of radius $r > 0$. If $\mathcal{C}$ is the class of the non-empty, bounded and closed subsets of X, then $(\mathcal{C}, \delta)$ is a metric space [6, Lemma 4], and if $\mathcal{K}$ is the class of the non-empty and compact subsets of X, then $(\mathcal{K}, \delta)$ is a complete (in fact, compact) metric space [27, Theorem 9.2].

To set the terminology, we give the following definition (see [6]).

Definition 1.1. Let $\mathcal{Q}$ be a class of non-empty subsets of a metric space X, $\{f_1, \dots, f_n\}$ a finite family of self-mappings of X and assume that for each $Q \in \mathcal{Q}$,

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$f_1(Q) \cup \ldots \cup f_n(Q) \in \mathcal{Q}$. The mapping $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ defined by

$$\mathcal{F}(Q) := \bigcup_{i=1}^{n} f_i(Q) \quad \text{for all } Q \in \mathcal{Q}$$

is called the *invariance operator* associated to $\mathcal{Q}$. We say that a set $H \in \mathcal{Q}$ is:

(i) $\mathcal{F}$*-invariant* if it is a fixed point of $\mathcal{F}$, i.e., $\mathcal{F}(H) = H$.

(ii) $\mathcal{F}$*-subinvariant* if $H \subset \mathcal{F}(H)$.

(iii) $\mathcal{F}$*-fractal* if it is $\mathcal{F}$-invariant and compact.

Note that the invariance operator $\mathcal{F}$ is well defined, that is, $\mathcal{F}(Q) \in \mathcal{Q}$ because we are assuming that $f_1(Q) \cup \cdots \cup f_n(Q) \in \mathcal{Q}$ whenever $Q \in \mathcal{Q}$. To simplify the notation, for a metric space X, we denote by $\mathcal{F} = \{f_1, \ldots, f_n\}$ the invariance operator associated with the self-mappings $f_1, \ldots, f_n$ of X and, unless it is specified otherwise, we assume that $\mathcal{F}$ is defined on the class of all non-empty subsets of X, i.e., $\mathcal{Q}$ is the class of the non-empty subsets of X, unless it is specified otherwise.

We recall that a mapping $f : X \to X$, (X, d) being a metric space, is said to be r*-contractive*, for some $r \in [0, 1)$ (often called the *contractiveness ratio of* f), if the inequality $d(f(x), f(y)) \leq rd(x, y)$ holds for all $x, y \in X$. From this concept, we give the following definition (see [4, 27]).

Definition 1.2. Let X be a complete metric space and $\mathcal{K}$ be the class of non-empty and compact subsets of X. Assume that $\mathcal{F} := \{f_1, \ldots, f_n\}$ consists of r_i-contractions self-mappings of X for $i = 1, \ldots, n$. Assuming that $\mathcal{F}$ is defined on $\mathcal{K}$, $\mathcal{F}$ is called an *iterated function system* (IFS).

Note that, in the above definition, we give the classical definition of IFS. However, there are other more general definitions. For instance, in [5], the concept of IFS is defined from self-mappings of X. In [21], generalized IFS are defined, where the (possibly infinite) set of mappings f_i are affine contractions.

Next, we state the following well-known result (see, for instance, [4, 16]).

Theorem 1.3. *With the notation of Definition 1.2, assume that* $\mathcal{F} := \{f_1, \ldots, f_n\}$ *is an IFS. Then there exists a unique* $\mathcal{F}$*-fractal* $\mathcal{A}^*$*. Moreover, given* $K_0 \in \mathcal{K}$*, the sets defined by* $K_m := \mathcal{F}(K_{m-1})$ *for* $m \geq 1$ *satisfy*

$$\delta(K_m, \mathcal{A}^*) \leq \frac{r^m}{1-r}\delta(K_1, K_0) \quad \text{with } r := \max\{r_1, \ldots, r_n\}.$$

The above result relies on the celebrated Banach Contraction Principle. Indeed, for a given IFS $\mathcal{F} := \{f_1, \ldots, f_n\}$, the operator $\mathcal{F}$ is really an r-contractive self-mapping of the complete metric space $(\mathcal{K}, \delta)$, with $r := \max\{r_i : i = 1, \ldots, n\}$ (each r_i being the contractiveness ratio of f_i). Therefore the existence of a fixed point of $\mathcal{F}$ (as well as the iterative procedure to approximate it) follows from the Banach Contraction Principle.

It is important to stress that, by using the so-called measures of noncompactness (see, for instance, [1, 3]), some interesting approaches to the IFS theory have been developed in [6, 19, 20]. Before describing these results, it is convenient to state two important and well-known fixed point theorems based on the measure of noncompactness. First, we recall two widely studied measures of noncompactness, which will be used in later examples, namely those called of Hausdorff and Kuratowski (see, for instance, [1,3]), denoted by χ and κ, respectively, and defined as

$$\chi(B) := \inf\big\{\varepsilon > 0 : B \text{ can be covered by finitely many balls of radius at most } \varepsilon\big\},$$

and

$$\kappa(B) := \inf\big\{\varepsilon > 0 : B \text{ can be covered by finitely many sets of diameter at most } \varepsilon\big\},$$

for each non-empty and bounded subset B of a metric space (X, d).

Following [3, Definition II.5.1], we give the following definition.

Definition 1.4. Let X be a metric space and $f : X \to X$ a mapping such that $f(B)$ is bounded for each non-empty and bounded subset B of X, and let μ be a measure of noncompactness. We say that:

(i) f is (μ, r)-*contractive*, for some $r \in [0, 1)$, if $\mu(f(B)) \le r\mu(B)$ for all non-empty and bounded sets $B \subset X$.

(ii) f is μ-*condensing* if $\mu(f(B)) < \mu(B)$ for all non-empty and bounded sets $B \subset X$ with $\mu(B) > 0$.

The first, and one of the most remarkable fixed point results based in the measure of noncompactness, is the following (see [1,3]).

Theorem 1.5 (Darbo fixed point theorem). *Let C be a non-empty, bounded, convex and closed subset of a Banach space E. Assume that $f : C \to C$ is a (μ, r)-contractive mapping, for some measure of noncompactness μ and $r \in [0, 1)$. Then f has a fixed point.*

The above result has been extended in many senses and directions (see, for instance, [17, 25] and references therein). The first of them is due to Sadovskiĭ [26].

Theorem 1.6 (Sadovskiĭ fixed point theorem). *Let C be a non-empty, bounded, convex and closed subset of a Banach space E. Assume that $f : C \to C$ is μ-condensing, for some measure of noncompactness μ. Then f has a fixed point.*

Remark 1.7. Apparently, the concept of μ-condensing mapping is only a slight generalization of the (μ, r)-contractive one. However, as it was proved in [3, Chapter II], Theorem 1.6 is a very deep generalization of Theorem 1.5. This is due to the fact that the class of (μ, r)-contractive mappings is residual (in the category sense) in the class of μ-condensing mappings.

As it has been pointed out above, there are some results that connect the IFS with the measures of noncompactness. Here, we focus in the following result proved in [6, Theorem 2].

Theorem 1.8. *Let C be a non-empty, bounded, convex and closed subset of a Banach space E. Assume that $\mathcal{F} := \{f_1, \ldots, f_n\}$ consists of χ-condensing self-mappings of C, χ being the Hausdorff measure of noncompactness. Then there exists an $\mathcal{F}$-fractal.*

In the present paper, we prove a result similar to Theorem 1.8 by using the so-called degree of nondensifiability, exposed in detail in Section 2. As it will be shown in Example 2.4, the degree of nondensifiability is not a measure of noncompactness (here the definition of measure of noncompactness is that of [3]). The proof of the main result, see Theorem 3.4, adopts the ideas of Sadovskiĭ [26] and Bessenyei and Pénzes [6].

To conclude our exposition, in Section 4 we apply our results to prove, under suitable conditions, the existence of an $\mathcal{F}$-invariant set for certain IFS on the Banach space of the continuous functions defined on $[0, 1]$. Furthermore, the existence of an $\mathcal{F}$-fractal follows whenever the IFS is defined by integral operators.

2. The degree of nondensifiability: ϕ-condensing mappings

We start this section by recalling the following concepts introduced in [22].

Definition 2.1. Let B be a non-empty and bounded subset of a metric space (X, d) and $\alpha \geq 0$. A mapping $\gamma : [0, 1] \to X$ is said to be an *α-dense curve in B* if

(i) $\gamma([0, 1]) \subset B$.

(ii) For each $x \in B$, there is $y \in \gamma([0, 1])$ such that $d(x, y) \leq \alpha$.

If for each $\alpha > 0$, there is an α-dense curve in B, then B is said to be *densifiable*.

From the above definition, it is clear that α-dense curves are a generalization of the so-called space-filling curves, see [27]. Let us note that for a non-empty and bounded subset B of a metric space X, there is always an α-dense curve in B for each $\alpha \geq \mathrm{Diam}(B)$, the diameter of B. Indeed, given $x_0 \in B$, the mapping $\gamma(t) := x_0$ for all $t \in [0, 1]$ is trivially an α-dense curve. For a detailed exposition of the above concepts, see [9, 15, 22–24] and references therein.

Now, we can define the following (see [13, 23]).

Definition 2.2. For a given non-empty and bounded subset B of a metric space X, its *degree of nondensifiability* is defined as

$$\phi(B) := \inf\left\{\alpha \geq 0 : \Gamma_{B,\alpha} \neq \emptyset\right\},$$

where $\Gamma_{B,\alpha}$ denotes the class of the α-dense curves in B.

Note that the degree of nondensifiability ϕ is well defined. Indeed, we have that $\phi(B) \in [0, \mathrm{Diam}(B)]$ for each non-empty and bounded subset B of the metric space X. Some basic properties of the degree of nondensifiability are stated in the following result (see [13]).

Proposition 2.3. Let X be a metric space. The degree of nondensifiability ϕ satisfies:

(M-1) *Regularity on arc-connected sets.* $\phi(B) = 0$ if, and only if, B is precompact, for each non-empty, arc-connected and bounded subset B of X.

(M-2) *Invariance under closure.* $\phi(B) = \phi(\bar{B})$ for each non-empty and bounded subset B of X.

(M-3) *Semi-additivity on arc-connected sets.* If $B_1, \ldots, B_n$ are non-empty, arc-connected and bounded subset of X such that $B_1 \cap \ldots \cap B_n \neq \emptyset$, then

$$\phi(B_1 \cup \cdots \cup B_n) \leq \max\{\phi(B_1), \ldots, \phi(B_n)\}.$$

(B-1) *Semi-homogeneity.* $\phi(cB) = |c|\phi(B)$, for each $c \in \mathbb{R}$ and each non-empty and bounded subset B of E.

(B-2) *Invariance under translations.* $\phi(x_0 + B) = \phi(B)$, for each $x_0 \in E$ and each non-empty and bounded subset B of E.

(B-3) $\phi(\mathrm{Conv}(B)) \leq \phi(B)$ for each non-empty and bounded subset B of E, where $\mathrm{Conv}(B)$ stands for the convex hull of B.

(B-4) $\phi(\mathrm{Conv}(B_1 \cup \cdots \cup B_n)) \leq \max\{\phi(\mathrm{Conv}(B_1)), \ldots, \phi(\mathrm{Conv}(B_n))\}$, for each non-empty and bounded subsets $B_1, \ldots, B_n$ of E.

As we have pointed out in Section 1, the degree of nondensifiability ϕ is not a measure of noncompactness, at least in the sense given in [3]. This fact is evidenced in the following example.

Example 2.4. Let L^1 be the Banach space of the Lebesgue integrable functions $f : [0, 1] \to \mathbb{R}$ endowed with its usual norm. Consider the set

$$C := \left\{f \in L^1 : \int_0^1 f(x)dx = 1, f \geq 0 \text{ almost everywhere}\right\}.$$

In [13], it was proved that $\phi(C) = 2$ and $\phi(U_{L^1}) = 1$, where U_{L^1} denoted the closed unit ball of L^1. By recalling that a measure of noncompactness μ satisfies the inequality $\mu(A) \leq \mu(B)$ for all non-empty and bounded subsets A, B of L^1 with $A \subset B$, and noticing that $C \subset U_{L^1}$, it follows that, in general, this monotony property does not hold for the degree of nondensifiability ϕ.

In view of Definition 1.4 (see also [14]), we give the following definition.

Definition 2.5. Let X be a metric space and $f : X \to X$ a continuous mapping such that $f(B)$ is bounded for each non-empty and bounded subset B of X, and let ϕ be the degree of nondensifiability. We say that:

(i) f is (ϕ, r)-*contractive* if $\phi(f(B)) \leq r\phi(B)$ for some $r \in [0, 1)$, for all non-empty, arc-connected and bounded subsets B of X.

(ii) f is ϕ-*condensing* if $\phi(f(B)) < \phi(B)$ for all non-empty, arc-connected and bounded subsets B of X with $\phi(B) > 0$.

Note from the definition above that the class of (ϕ, r)-contractive mappings is contained in that of the ϕ-condensing mappings.

Proposition 2.6. Let X be a metric space, ϕ the degree of nondensifiability and let $f : X \to X$ be an r-contractive mapping, for some $r \in [0, 1)$. Then f is (ϕ, r)-contractive.

Proof. Let B be a non-empty, arc-connected and bounded subset of X. Consider $\phi(B)$, by Definition 2.2 for any $\alpha \geq \phi(B)$ there exists $\gamma : [0, 1] \to X$ an α-dense curve in B. Then $f \circ \gamma$ is $r\alpha$-dense in $f(B)$ (see also [15, Proposition 3.1]). Therefore $\phi(f(B)) \leq r\alpha$. Since the above inequality is true for all $\alpha \geq \phi(B)$, it implies $\phi(f(B)) \leq r\phi(B)$. ∎

From the above result, we deduce that the class of r-contractive mappings is contained in the class of (ϕ, r)-contractive mappings. In the next example, we show that such inclusion is strict.

Example 2.7. Let $(\ell_2, \|\cdot\|)$ be the Banach space of the real summable square sequences endowed with its usual norm $\|x\| := (\sum_{n\geq 1} |x_n|^2)^{1/2}$ for each $x := (x_1, x_2, \ldots, x_n, \ldots) \in \ell_2$, and let U_{ℓ_2} be its closed unit ball. Given $0 < \beta < 1/2$, we define $f : U_{\ell_2} \to U_{\ell_2}$ as

$$f(x) := \left(\sqrt{1 - \|x\|}, \beta x_1, \beta x_2, \ldots, \beta x_n, \ldots\right) \quad \text{for all } x \in U_{\ell_2}.$$

Then if $\|x\| = 1$, we have $\|f(x) - f(\theta)\| = \sqrt{1 + \beta^2} > 1 = \|x - \theta\|$, θ being the null vector of ℓ_2. Therefore f is not r-contractive for any $0 \leq r < 1$.

However, in [15, Proposition 3.1] it was proved that $\phi(f(B)) \leq 2\beta\phi(B)$ for each arc-connected $B \subset U_{\ell_2}$, where ϕ is the degree of nondensifiability. Consequently, f is $(\phi, 2\beta)$-contractive.

In Theorem 1.8, the Sadovskiĭ fixed point theorem plays a crucial role. In the next result, we introduce a Sadovskiĭ-type fixed point theorem based on the degree of nondensifiability.

Theorem 2.8. *Let C be a non-empty, convex, closed and bounded subset of a Banach space X, and let ϕ be the degree of nondensifiability. Then each ϕ-condensing mapping $\phi : C \to C$ has a fixed point.*

Proof. See the proof of [14, Theorem 3.1] or, in a more general context, the proof of Theorem 3.4 in Section 3. ∎

An immediate consequence of the above result is the following (see also [10, Theorem 3.2] and [14, Theorem 3.1]).

Corollary 2.9. Let C a non-empty, bounded, closed and arc-connected subset of a Banach space E and let $f : C \to C$ be a mapping. Assume that there is a monotone increasing function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(s) < s$ for all $s > 0$ and $\psi(0) = 0$, such that

$$\phi(f(B)) \leq \psi(\phi(B))$$

for all non-empty, closed and arc-connected $B \subset C$, where ϕ denotes the degree of nondensifiability. Then f has a fixed point.

We present two examples where the Darbo and Sadovskiĭ fixed point theorems cannot be applied. However, the degree of nondensifiability fixed point theory ensures the existence of a fixed point.

Example 2.10. Let $\mathcal{C}([0, 1])$ be the Banach space of the continuous functions defined on $[0, 1]$, endowed with its usual supremum norm. Consider the mapping

$$f : \mathcal{C}([0, 1]) \to \mathcal{C}([0, 1])$$

given by, for $x(t) \in \mathcal{C}([0, 1])$,

$$f(x(t)) := \begin{cases} \frac{1}{2}x(2t) + \frac{1}{2}x(0), & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{2}x(2t-1) + \frac{1}{2}x(1), & \frac{1}{2} < t \leq 1, \end{cases}$$

and define the closed and convex set

$$C := \{x(t) \in \mathcal{C}([0, 1]) : 0 = x(0) \leq x(t) \leq x(1) = 1, t \in [0, 1]\}.$$

Then $\chi(C) = \chi(f(C)) = \frac{1}{2}$ (see [3, Example X.2]) and therefore the contractiveness condition of the Darbo fixed point theorem does not hold in this case. However, in view of Proposition 2.6 (see also [11, Example 3.4]), we have $\phi(f(B)) \leq \phi(B)/2$, ϕ being the degree of nondensifiability, for each non-empty, arc-connected and bounded

$B \subset C$. Therefore f is $(\phi, \frac{1}{2})$-contractive. Consequently, by applying Corollary 2.9, f has a fixed point.

Example 2.11. Let ℓ_2 be as in Example 2.7 and consider

$$f(x) := \sum_{n\geq 1} \max\left\{0, 1 - \frac{\|x - e_n\|}{1-\beta}\right\}\xi_n \quad \text{for all } x \in U_{\ell_2},$$

where $(e_n)_{n\geq 1}$ is the standard basis of ℓ_2, β is a real number with $2^{-1/2} < \beta < 1$ and $(\xi_n)_{n\geq 1}$ is a dense sequence in βU_{ℓ_2}, U_{ℓ_2} being the closed unit ball of ℓ_2. As shown in [7, Remark 3.10], $f(U_{\ell_2}) \subset \beta U_{\ell_2}$ and

$$\kappa(B) = \sqrt{2} < 2\beta = \kappa(f(B)),$$

where $B = \{e_n : n \geq 1\}$. Therefore f is not κ-condensing.

Given any $\sqrt{2}/2\beta < \varepsilon < 1$, define $f_\varepsilon(x) := \varepsilon f(x)$ for all $x \in U_{\ell_2}$. Then $f_\varepsilon(U_{\ell_2}) \subset \varepsilon\beta U_{\ell_2}$ and f_ε is not κ-condensing.

Now, we claim that f_ε is ϕ-condensing, ϕ being the degree of nondensifiability. Indeed, let B be a non-empty, closed, arc-connected and not precompact subset of U_{ℓ_2}. By the invariance under translations of ϕ (see Proposition 2.3), there is no loss of generality if we assume $0 \in B$. As $f_\varepsilon(0) = 0$, the constant mapping $\gamma(t) = 0$ for all $t \in [0, 1]$ satisfies the condition $\gamma([0, 1]) \subset f_\varepsilon(B)$ and therefore it is an $\varepsilon\beta$-dense curve in $f_\varepsilon(B)$ as $f_\varepsilon(B) \subset \varepsilon\beta U_{\ell_2}$. Then, we have

$$\phi(f_\varepsilon(B)) \leq \varepsilon\beta. \tag{2.1}$$

Again, from [7, Remark 3.10], $\chi(B) \geq \beta$ and then, noticing (2.1) and the properties of ϕ (see [13]), we get

$$\phi(f_\varepsilon(B)) \leq \varepsilon\beta < \beta \leq \chi(B) \leq \phi(B),$$

and this proves that f_ε is a ϕ-condensing mapping. Consequently, by applying Theorem 2.8, f has a fixed point.

3. The main result

As we have pointed out in Section 2, a key result to prove Theorem 1.8 is the Sadovskiĭ fixed point theorem. However, there is another crucial result in the proof of Theorem 1.8, namely the following lemma (see [6, Lemma 1]).

Lemma 3.1. *Assume that $\mathcal{F} := \{f_1, \ldots, f_n\}$ consists of self-mappings of a metric space X and $H_1 \subset X$ is a $\mathcal{F}$-subinvariant set. Then, by putting*

$$H_{m+1} := \mathcal{F}(H_m) \quad \textit{and} \quad H := \bigcup_{m\geq 1} H_m,$$

we have that H is an $\mathcal{F}$-invariant set.

Remark 3.2. The sequence of sets $(H_m)_{m\geq 1}$ in the above lemma is often known as the Kantorovich iteration, and the set H is its limit (see [6] and references therein). These sets were used by Kantorovitch [18] in 1939 to obtain order-theoretic fixed point results.

We will need also the following lemma (see [6, Lemma 3]).

Lemma 3.3. *If $\mathcal{F}$ is a finite family of continuous self-mappings of a metric space and H is a relatively compact $\mathcal{F}$-invariant set, then $\overline{H}$ is $\mathcal{F}$-invariant.*

Now, we can state and prove our main result.

Theorem 3.4. *Let C be a non-empty, closed and convex subset of a Banach space E, let ϕ be the degree of nondensifiability and let $\mathcal{F} := \{f_1, \dots, f_n\}$ be a family of self-mappings of C.*

(1) *If $\mathcal{F}$ contains a ϕ-condensing mapping, then there is a non-empty $\mathcal{F}$-invariant set.*

(2) *If $\mathcal{F}$ consists of ϕ-condensing mappings, then there exists an $\mathcal{F}$-fractal.*

Proof. Let us prove (1). Assuming that f_1 is ϕ-condensing, in view of Theorem 2.8, f_1 has a fixed point $x^* \in C$. By defining $H_1 := \{x^*\}$ and $H_{m+1} := \mathcal{F}(H_m)$ for each $m \geq 1$, by Lemma 3.1, the set $\mathcal{A}^* := \cup_{m\geq 1} H_m$ is $\mathcal{F}$-invariant.

To prove (2), assume that each f_i, for $i = 1, \dots, n$, is a ϕ-condensing mapping. For a given $x_0 \in C$, we define

$$\Omega := \big\{B \subset C : B = \overline{\text{Conv}}(B), x_0 \in B, \mathcal{F}(B) \subset B\big\}.$$

Note that Ω is non-empty because $C \in \Omega$. Define the sets

$$\mathcal{H} := \bigcap_{B\in\Omega} B \quad \text{and} \quad \mathcal{G} := \overline{\text{Conv}}(\mathcal{F}(\mathcal{H}) \cup \{x_0\}).$$

Since $x_0 \in \mathcal{H}$, $\mathcal{H} \neq \emptyset$. We claim that $\mathcal{H} = \mathcal{G}$.

For any $D \in \mathcal{H}$, we have

$$\mathcal{F}(\mathcal{H}) = \bigcup_{i=1}^{n} f_i\Big(\bigcap_{B\in\Omega} B\Big) \subset \bigcup_{i=1}^{n} f_i(D) = \mathcal{F}(D) \subset D,$$

and from the arbitrariness of D, we infer that $\mathcal{F}(\mathcal{H}) \subset \mathcal{H}$. As $x_0 \in \mathcal{H}$, $\mathcal{F}(\mathcal{H}) \subset \mathcal{H}$ and $\mathcal{H}$ is closed and convex, we have $\mathcal{G} \subset \mathcal{H}$. From this inclusion, we deduce that

$$\mathcal{F}(\mathcal{G}) \subset \mathcal{F}(\mathcal{H}) \subset \overline{\text{Conv}}(\mathcal{F}(\mathcal{H}) \cup \{x_0\}) = \mathcal{G}.$$

This proves that $\mathcal{F}(\mathcal{G}) \in \Omega$ and, in particular, $\mathcal{H} \subset \mathcal{G}$. So $\mathcal{H} = \mathcal{G}$ as claimed.

On the other hand, assume that $\mathcal{H}$ is not precompact. By Proposition 2.3, since each f_i is a ϕ-condensing mapping, we have

$$\begin{aligned}\phi(\mathcal{H}) = \phi(\mathcal{G}) &\leq \phi\big(\mathrm{Conv}(\mathcal{F}(\mathcal{H}) \cup \{x_0\})\big) \leq \phi\big(\mathrm{Conv}(\mathcal{F}(\mathcal{H}))\big) \\ &= \phi\Big(\mathrm{Conv}\Big(\bigcup_{i=1}^{n} f_i(\mathcal{H})\Big)\Big) \leq \max\big\{\phi(f_i(\mathcal{H})) : i = 1, \dots, n\big\} < \phi(\mathcal{H}),\end{aligned}$$

which is contradictory. Therefore $\mathcal{H}$ is precompact.

Now, as $f_1(\mathcal{H}) \subset \mathcal{F}(\mathcal{H}) \subset \mathcal{H}$, according to Theorem 2.8, we can take some $x_1^* \in \mathcal{H}$ which is a fixed point of f_1. The set $\{x_1^*\}$ is $\mathcal{F}$-subinvariant. Then, by applying Lemma 3.1 taking the metric space X as $\mathcal{H}$ (note that, by the definition of $\mathcal{H}$, $f_i(\mathcal{H}) \subset \mathcal{H}$ for each $i = 1, \dots, n$), there exists $\mathcal{H}^*$ in $\mathcal{H}$ that is $\mathcal{F}$-invariant. Since a subset of a precompact set is precompact, so is $\mathcal{H}^*$. Noticing that in a Banach space the relative compactness coincides with the precompactness, $\mathcal{H}^*$ is relatively compact. Now, by applying Lemma 3.3, $\bar{\mathcal{H}}^*$ is $\mathcal{F}$-invariant, i.e., $\bar{\mathcal{H}}^*$ is an $\mathcal{F}$-fractal. ∎

In the above result, if some f_i is not a ϕ-condensing mapping, the $\mathcal{F}$-invariant set is not necessarily compact, that is, such a set might not be an $\mathcal{F}$-fractal. This fact will be evidenced in Example 4.2.

Note that, unlike Theorem 1.8, neither Theorem 1.3 nor Theorem 3.4 provide an iterative procedure to approximate (with a prescribed and arbitrary small error) an $\mathcal{F}$-invariant set. However, if we know a fixed point of f_1, we can obtain, recursively, the sets H_m of Lemma 3.1.

It is clear that each set H_m of Lemma 3.1 is a set of n^{m-1} points. But we can obtain easily $1 \leq k < n^{m-1}$ points in H_m with the scheme given in Algorithm 3.1. The notation used is the following: given two positive integers $m > 1$ and $1 \leq k < n^{m-1}$, $\widehat{H}_{m,k}$ denotes a subset of H_m with k elements and σ_{m-1} is a set of indices of length $m-1$ obtained from the set $\{1, \dots, n\}$. For instance, for $n = 2$ and $m = 3$, σ_{m-1} can be any of the sets $\{1, 1\}, \{1, 2\}, \{2, 1\}, \{2, 2\}$. For $\sigma_{m-1} := \{i_1, \dots, i_{m-1}\}$, with $1 \leq i_j \leq n$ for each $j = 1, \dots, m-1$, we define $f_{\sigma_{m-1}} := f_{i_1} \circ \cdots \circ f_{i_{m-1}}$.

Remark 3.5. Note that Algorithm 3.1 computes some points of the set H_m from Lemma 3.1, but it is not a chaos game (see, for instance, [4, 5]) in the strict sense.

On the other hand, if C is a non-empty, bounded, convex and closed subset of a Banach space E, ϕ is the degree of nondensifiability and $f_i : C \to C$, $i = 1, \dots, n$, are compact mappings, i.e., f_i maps bounded sets into precompact ones (see, for instance [3, Definition I.2.5]), then $\mathcal{F} = \{f_1, \dots, f_n\}$ is a IFS as in (2) of Theorem 3.4. Indeed, from (M-1) of Proposition 2.3 each f_i is a ϕ-condensing mapping, for $i = 1, \dots, n$. Hence, by Theorem 3.4, there is an $\mathcal{F}$-fractal.

Algorithm 3.1: Computation of the set $\widehat{H}_{m,k}$

Data: The numbers $k \geq 2$, $m > 1$, the mappings $f_i's$ and a fixed point $x^* \in C$ of f_1.

Result: The set $\widehat{H}_{m,k} \subset H_m$.

`/* Initialization */`

$\widehat{H}_{m,k} \leftarrow \emptyset$, Ind $\leftarrow \emptyset$, $j \leftarrow 1$;

`/* The loop */`

for $j \leftarrow 1$ **to** k **do**

  Randomly select σ_{m-1} such that $\sigma_{m-1} \notin$ Ind;

  Ind $\leftarrow$ Ind $\cup \{\sigma_{m-1}\}$;

  $\widehat{H}_{m,k} \leftarrow \widehat{H}_{m,k} \cup \{f_{\sigma_{m-1}}(x^*)\}$;

`/* Return the set` $\widehat{H}_{m,k}$ `*/`

return $\widehat{H}_{m,k}$;

We formalize the above comments in the following consequence of Theorem 3.4, which is a Schauder–Tychonoff-type fixed point result.

Corollary 3.6. Let ϕ the degree of nondensifiability, let C be a non-empty, closed and convex subset of a Banach space E and let $\mathcal{F} := \{f_1, \ldots, f_2\}$ be a family of self-mappings of C such that each f_i is compact for $i = 1, \ldots, n$. Then there is an $\mathcal{F}$-fractal.

4. Application: IFS defined from the superposition operator

Through this section, $E := \mathcal{C}([0,1])$ will be the Banach space of the continuous functions defined on $[0,1]$, as in Example 2.10, endowed with the supremum norm $\|\cdot\|_\infty$. Although usually the examples of IFS are defined on subsets of $\mathbb{R}^n$, there are interesting results for IFS defined in the context of infinite dimensional Banach spaces (see [12] and references therein). In this section, we study certain IFS defined on non-empty, closed and convex subsets of E and from the so-called superposition operator (sometimes called Nemytskij operator).

For a given (not necessarily continuous) $f : [0,1] \times E \to \mathbb{R}$, we recall that the superposition operator $F : E \to E$ associated with f is defined by

$$F(x)(t) := f(t, x(t)) \quad \text{for all } t \in [0,1] \text{ and } x \in E.$$

This application F plays a crucial role in certain functional equations, see [2, 28] and references therein. According to [2], the operator F is continuous if, and only if, f is continuous. Also, to simplify the notation, we define

$$\Psi := \big\{\psi : [0, +\infty) \longrightarrow [0, +\infty) \ : \ \psi \text{ is monotone increasing with } \psi(s) < s \text{ for all } s > 0 \text{ and } \psi(0) = 0\big\},$$

that is, Ψ is the class of functions satisfying the conditions of Corollary 2.9.

Now, for a given mapping $f : [0, 1] \times E \to \mathbb{R}$, consider the corresponding superposition operator F, a non-empty, bounded, convex and closed subset C of E and the inequality

$$\phi(F(B)) \leq \psi(\phi(B)) \tag{4.1}$$

for some $\psi \in \Psi$ and all non-empty and arc-connected subsets B of C.

Let us note that (4.1) is satisfied for a large class of mappings:

(I) If f is compact, then (4.1) is trivially fulfilled.

(II) If there is $\psi \in \Psi$ such that for each $t \in [0, 1]$, the inequality

$$|f(t, r) - f(t, s)| \leq \psi(|r - s|)$$

is satisfied for all $r, s \in \mathbb{R}$, then it is immediate to prove $\|F(x) - F(y)\|_\infty \leq \|x - y\|_\infty$ for all $x, y \in C$. Consequently (see the proof of Proposition 2.6), the condition (4.1) holds.

Next, let $f_i : [0, 1] \times E \to \mathbb{R}$ given mappings and F_i the corresponding superposition operators, for $i = 1, \ldots, n$. Also, let the conditions:

(C1) There is a non-empty, bounded, convex and closed subset C of E such that $f_i(\cdot, x(\cdot))$ is continuous for all $x \in C$ and $F_i(C) \subset C$, for each $i = 1, \ldots, n$.

(C2) F_1 satisfies (4.1).

The main result of this section is the following.

Proposition 4.1. Assume conditions (C1) and (C2) and let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of self-mappings of C, the non-empty, bounded, convex and closed subset C of E given in (C1). Then there exists an $\mathcal{F}$-invariant set.

Proof. Let ϕ be the degree of nondensifiability. By condition (C1), $F_i(C) \subset C$ for each $i = 1, \ldots, n$. Also, noticing condition (C2) and Corollary 2.9, F_1 is ϕ-condensing, and the result follows from Theorem 3.4. ■

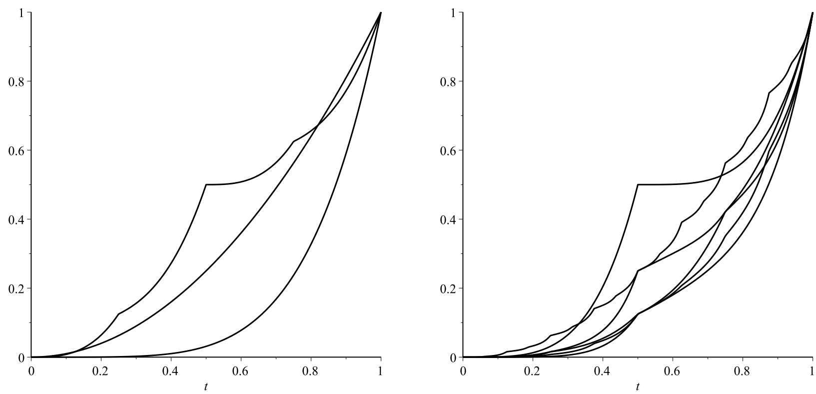
At this point, we show an example to illustrate the above result.

Example 4.2. Let C and f_1 be as in Example 2.10 and $f_2(t, x(t)) := tx(t)$ for all $x \in E$ and $t \in [0, 1]$. Note that f_2 has no fixed point in C, and therefore F_2 does not satisfy the inequality (4.1) for any $\psi \in \Psi$ (otherwise, by Corollary 2.9, F_2 has a fixed point, which is not possible). Clearly, f_2 is continuous and $F_2(C) \subset C$, that is to say, F_2 obeys condition (C1).

As we have shown in Example 2.10, F_1 satisfies conditions (C1) and (C2) for $\psi(r) := r/2$. Then, by Proposition 4.1, there is some $\mathcal{A}^* \subset C$ such that

$$\mathcal{A}^* = f_1(\mathcal{A}^*) \cup f_2(\mathcal{A}^*).$$

In Figure 4.2, we show some sets of type $\widehat{H}_{m,k}$ computed with Algorithm 3.1. Let us note that the set $\mathcal{A}^*$ is not compact. Indeed, otherwise given a sequence $(x_n(t))_{n\geq 1} \subset \mathcal{A}^*$, there exists a subsequence, for simplicity noted in the same way, of $(x_n(t))_{n\geq 1}$ such that $x_n \to x^*$, for some $x^* \in \mathcal{A}^*$, where the convergence means in supremum norm. Then, by the continuity of f_2, we find that $f_2(t, x_n(t)) \to f_2(t, x^*(t)) = tx^*(t)$ for all $t \in [0, 1]$. This means that f_2 has a fixed point, which is not possible.



(a) A set $\widehat{H}_{4,3} \subset H_4$. (b) A set $\widehat{H}_{8,6} \subset H_8$.

Figure 1. Some functions belonging to the set H_m, obtained with Algorithm 3.1, for the indicated values of m.

On the other hand, if each F_i is compact (in particular, the above condition (C2) follows trivially) and condition (C1) holds, then there exists some $\mathcal{F}$-invariant set. In fact, by Corollary 3.6, there exists an $\mathcal{F}$-fractal set, with $\mathcal{F} = \{F_1, \ldots, F_n\}$ and C as in Proposition 4.1. In the following lines, we show how we can derive (as consequence of Proposition 4.1) the existence of an $\mathcal{F}$-fractal for certain IFS defined from integral operators.

It is a well-known fact (see, for instance, [3, Example I.3]) that, in the Banach space E, integral operators with sufficiently regular kernels provide one of the most important examples of compact operators. In particular, given the continuous mappings $h_i : [0, 1] \longrightarrow [0, 1]$ and $K_i : [0, 1]^2 \times E \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$, the mappings

$f_i : [0,1] \times E \longrightarrow \mathbb{R}$ defined as

$$f_i(t, x(t)) := g_i(t) + \int_0^t K_i(t, s, x(s))\, ds \quad \text{for all } t \in [0,1], x \in E \tag{4.2}$$

are compact. As above, we denote by F_i the corresponding superposition operator associated with mappings f_i for $i = 1, \dots, n$. The continuity of F_i follows from the continuity of f_i, for each $i = 1, \dots, n$. Consider the following condition:

(D1) There is $R > 0$ such that

$$|f(t, x(t))| \leq R \quad \text{for all } t \in [0,1] \text{ whenever } \|x\|_\infty \leq R.$$

Now, we have the following corollary.

Corollary 4.3. Assume condition (D1) holds for $f_1, \dots, f_n$ defined in (4.2) and let $\mathcal{F} = \{F_1, \dots, F_n\}$ be the family of superposition operators associated with $f_1, \dots, f_n$ defined in RU_E, U_E being the closed unit ball of E and $R > 0$ the number given by condition (D1). Then, there exists an $\mathcal{F}$-fractal.

Proof. It is clear that the mappings f_i are continuous, and consequently the F_i too, for each $i = 1, \dots, n$. Also, as each f_i satisfies condition (D1), $F_i(RU_E) \subset RU_E$ for $i = 1, \dots, n$. Then, by Proposition 4.1, there is a $\mathcal{F}$-invariant set. But, as each F_i is compact (because of f_i is compact), such $\mathcal{F}$-invariant set is, according to Lemma 3.3, an $\mathcal{F}$-fractal. ∎

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