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Fourier decay behavior of homogeneous self-similar measures on the complex plane

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Abstract. We prove that the Fourier transform of self-similar measures on the complex plane has fast decay outside of a very sparse set of frequencies, with quantitative estimates, extending the results obtained in the real line, first by R. Kaufman, and later, with quantitative bounds, by the first author and P. Shmerkin. We also derive several applications concerning correlation dimension and Frostman exponent of complex Bernoulli convolutions. Furthermore, we present a generalization for a particular case on $\mathbb{R}^d$, with $d \geq 3$.

1. Introduction

Given μ a finite Borel measure on the complex plane, its Fourier transform is defined as

$$\widehat{\mu}(\xi) := \int_{\mathbb{C}} e^{2\pi i \operatorname{Re}(z\bar{\xi})}\, d\mu(z).$$

The behaviour of $\widehat{\mu}(\xi)$ when $|\xi| \to \infty$ is a fundamental characteristic of the measure μ. Measures for which $|\widehat{\mu}(\xi)| \to 0$ when $|\xi| \to \infty$ are called *Rachjman measures*. By the Riemann–Lebesgue lemma, every absolutely continuous measure is Rachjman, but many singular measures are too. Among the (possibly) singular measures, an important group are the homogeneous self-similar measures (see Section 2), which are the measures that we consider in this paper. In particular, we focus on self-similar measures supported in the complex plane.

We say that $\widehat{\mu}(\xi)$ has *polynomial decay* if there exist $\sigma, C_\sigma > 0$ such that $|\widehat{\mu}(\xi)| \leq C_\sigma |\xi|^{-\sigma/2}$. Furthermore, the *Fourier dimension* of μ is defined as

$$\dim_F(\mu) := 2\sup\big\{\sigma \geq 0 : |\widehat{\mu}(\xi)| \leq C_\sigma |\xi|^{-\sigma} \text{ for some } C_\sigma > 0 \text{ and all } \xi \neq 0\big\},$$

and then one can say that $\widehat{\mu}$ has *polynomial decay* if and only if $\dim_F(\mu) > 0$. For many purposes, the simple convergence of $\widehat{\mu}$ to zero is not enough, and some quantitative decay is needed. For example, if $\dim_F(\mu) > 0$, then μ-almost every number is

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normal to any base, see [1, 10], and if $\dim_F(\mu) > 0$ and μ satisfies a Frotsman-type condition $\mu(B(x,r)) \leq C\, r^s$, then μ satisfies a restriction theorem analogous to the Stein–Tomas theorem for the sphere, see [7, 8].

Regardless of its significance, the Fourier dimension of a measure is particularly difficult to calculate or even give some estimative and, in some cases, there is no decay at all for $\widehat{\mu}$. Furthermore, in some cases, there are not decay at all for $\widehat{\mu}$. For example, consider the complex Bernoulli convolutions μ_λ, being the distribution of the random series $\sum_{n=1}^{+\infty} \pm\lambda^n$, where the signs are chosen independently with probabilities $\{1/2, 1/2\}$ and $\lambda \in \mathbb{D}$, the open unit disk. Also, μ_λ can be defined as the self-similar measure associated to the iterated function system (IFS) $\{\lambda z - 1, \lambda z + 1\}$. For these particular measures, Solomyak and Xu [16] proved that if θ is a complex Pisot number and $1 < |\theta| < \sqrt{2}$, then $|\widehat{\mu}_\lambda(\xi)| \nrightarrow 0$ as $|\xi| \to \infty$ when $\lambda = \frac{1}{\theta}$. This implies that $\dim_F(\mu_\lambda) = 0$ and, in particular, μ_λ is singular for $\lambda = 1/\theta$. Recall that a non-real algebraic integer θ, with $|\theta| > 1$, is called a *complex Pisot number* if all its Galois conjugates, except θ, are less than one in modulus. On the other hand, Shmerkin and Solomyak [14] proved that the Fourier transform of complex Bernoulli convolutions have power decay for all parameter λ, outside of an exceptional set of parameters of zero Hausdorff dimension.

Nevertheless, if a measure μ has zero Fourier dimension, it may happen that $\widehat{\mu}$ has fast decay outside of a very sparse set of frequencies. In fact, for self-similar measures on the real line, Kaufman [5] and Tsujii [17] proved that for any $\varepsilon > 0$, there exists $\delta > 0$ such that the set

$$\left\{\xi \in [-T, T] :\, |\widehat{\mu}(\xi)| \geq T^{-\delta}\right\}$$

can be covered by T^ε intervals of side length 1. Kaufman treated the homogeneous case using a version of the well-known Erdős–Kahane argument whereas the proof of Tsujii for the non-homogeneous case is based on large deviation estimates. Recently, the first author and Shmerkin [9] made the dependence of δ on ε quantitative in the homogeneous case.

Furthermore, if $\dim_F(\mu) = 0$, Kaufman [5] established, for the case of Bernoulli convolutions in the real line with parameter $\lambda \in (0, 1/2)$, that if $F : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of class C^2 with $F'' > 0$, then $\dim_F(F\mu) > 0$, where here and below $F\mu$ denotes the push-forward measure, that is, $F\mu(A) = \mu(F^{-1}(A))$ for all Borel sets $A \subseteq \mathbb{R}$. Later, in [9], the authors extended this result to any non-atomic homogeneous self-similar measure in the real line.

The goal of this paper is to extend the results mentioned above for homogeneous self-similar measures on the real line to the complex plane. More precisely, we prove that the Fourier transform of homogeneous self-similar measures on the complex plane have fast decay outside of a very sparse set of frequencies, extending the results obtained in [9] in dimension one, making the dependence of δ on ε explicit (see Propositions 2.2 and 2.5). Although the approach is similar to the one used in [5, 9], and is based on the Erdős–Kahane argument, there are some new features, mainly because we have to deal with two different cases: when the parameter $\lambda \in \mathbb{D}$, the contraction ratio of the iterated function system (see Section 2), has non-zero real part or not. As an application, we obtain a generalization of the Kaufman's result about the power Fourier decay of non-linear smooth images of homogeneous self-similar measures. More precisely, if F is an analytic function with $F'' \neq 0$ in a neighborhood of the supp μ, then $\widehat{F\mu}$ has polynomial decay, with quantitative estimates (see Theorem 3.1).

Given ν a Borel measure, since convolution (of a measure, of a function, etc.) with ν is a smoothing operation, a natural problem is quantifying the additional degree of smoothness ensured by convolving with ν. In [12], it was proven that uniformly perfect measures on the real line (which include the Ahlfors-regular measures as a proper subset) have the property that convolving with them results in a strict increase of the L^q dimension. A particular case of this was proved before in [9], when ν is a homogeneous self-similar measure on the real line and $q = 2$, but with quantitative estimates. Moreover, non-quantitative results could be deduced before from [13]. In the present work, we prove that convolving with a homogeneous self-similar measure on the complex plane increases the correlation dimension (see Section 4.1 for the definitions) by a quantitative amount (see Theorem 4.1) and that the Frostman exponent of complex Bernoulli convolutions tends to 2 as the modulus of the contraction ratio tends to 1 (see Theorem 4.2).

Besides the real line and the complex plane, the decay properties of the Fourier transform of self-similar (or self-affine) measures were also considered in higher dimensions; see, for example, [6, 11, 15]. Recently, Solomyak in [15] proved that for almost all d-tuplas $(\theta_1, \ldots, \theta_d)$, with $|\theta| > 1$, the Fourier transform of any self-affine measure associated to a homogeneous iterated function system of the form $\{Ax + w_j\}_{j=1}^m$ in $\mathbb{R}^d$, where A^{-1} is a diagonal matrix with entries $(\theta_1, \ldots, \theta_d)$, has power decay at infty. To conclude, we generalized some of our results to higher dimensions in the particular case when A is a contractive similitude diagonalizable over $\mathbb{R}$, that is, $A = \lambda O$, where O is an orthogonal matrix diagonalizable over $\mathbb{R}$ and $\lambda \in (0, 1)$ (see Proposition 5.2).

2. Fourier decay outside of a sparse set of frequencies

Given $p = (p_1, \ldots, p_m)$ a probability vector, i.e., a vector in $\mathbb{R}^m$ with $p_i > 0$ for all $i = 1, \ldots, m$ and $p_1 + \ldots + p_m = 1$, $w = (w_1, \ldots, w_m)$ a vector in $\mathbb{C}^m$, and $\lambda \in \mathbb{D}$, let $\mu_{\lambda,w}^p$ be the self-similar measure corresponding to the IFS $\{f_i\}_{i=1}^m$, with probability vector p and where $f_i = \lambda z + w_i$. That is, $\mu^p_{\lambda,w}$ is the only Borel probability measure satisfying the relation

$$\mu^p_{\lambda,w} = \sum_{i=1}^{m} p_i f_i \mu^p_{\lambda,w}.$$

On the other hand, $\mu^p_{\lambda,w}$ can also be defined as the distribution of the random sum

$$\sum_{n=1}^{+\infty} \lambda^n X_n,$$

where X_n are random variables i.i.d. with $P(X_n = w_i) = p_i$.

Throughout the paper, given $z \in \mathbb{C}$, we denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ its real and imaginary part, respectively.

By the definition of $\mu^p_{\lambda,w}$ as a self-similar measure, we can express its Fourier transform as follows:

$$\begin{aligned}\widehat{\mu}^p_{\lambda,w}(\xi) &= \int_{\mathbb{C}} e^{2\pi i \mathrm{Re}(z\cdot\overline{\xi})} d\mu^p_{\lambda,w}(z) \\ &= \prod_{n=0}^{\infty} \sum_{j=1}^{m} p_j \exp(2\pi i \mathrm{Re}(\lambda^n w_j \overline{\xi})) = \prod_{n=0}^{\infty} \Phi(\lambda^n \overline{\xi}),\end{aligned}$$

where $\Phi(u) := \sum_{j=1}^{m} p_j \exp(2\pi i \mathrm{Re}(w_j u))$.

Observe that since replacing w_i by $(w_i - w_1)/(w_2 - w_1)$ has the effect of applying a linear map to the measures in question, we may always assume without loss of generality that $w_1 = 0$ and $w_2 = 1$.

Lemma 2.1. *The following holds for all $z \in \mathbb{C}$ and $c \in (0, 1)$: If $\|\mathrm{Re}(z)\| > c/2$, then $|\Phi(z)| < 1 - \eta(c, p)$, where*

$$\eta(c, p) = p_1 + p_2 - \sqrt{p_1^2 + 2p_1 p_2 \cos(\pi c) + p_2^2},$$

and $\|x\|$ denotes the distance of a real number x to the closest integer.

Proof. By definition of Φ, we have

$$\begin{aligned}|\Phi(z)| &\le |p_1 + p_2 e^{2\pi i \mathrm{Re}(z)}| + (1 - p_1 - p_2) \\ &= |p_1 + p_2 \cos(2\pi \mathrm{Re}(z)) + p_2 i \sin(2\pi \mathrm{Re}(z))| + (1 - p_1 - p_2) \\ &= \sqrt{p_1^2 + p_2^2 + 2p_1 p_2 \cos(2\pi \mathrm{Re}(z))} + (1 - p_1 - p_2).\end{aligned}$$

Now, using that $\|\mathrm{Re}(z)\| > c/2$, we obtain that $\cos(2\pi \mathrm{Re}(z)) < \cos(\pi c)$ and then $|\Phi(z)| \le 1 - \eta(c, p)$. ∎

The next proposition is the main result of this section. We prove that outside of a very sparse set of frequencies, the Fourier transform of $\mu_{\lambda,w}^p$ has fast decay, with quantitative bounds. First, we consider the case when the complex parameter $\lambda \in \mathbb{D}$ is outside of the real line.

Proposition 2.2. *Given $\lambda \in \mathbb{D} \setminus \mathbb{R}$ and a probability vector $p = (p_1, \dots, p_m)$, there is a constant $C = C_\lambda > 0$ such that for each $\varepsilon > 0$ small enough (depending continuously on λ), the following holds for all T large enough: The set of frequencies $|\xi| \le T$ such that $|\widehat{\mu}_{\lambda,w}^p(\xi)| \ge T^{-\varepsilon}$ can be covered by $C_\lambda T^\delta$ squares of side-length 1, where*

$$\delta = \frac{\log\left(\lceil 1 + \frac{3}{|\lambda|^2} \rceil\right)\widetilde{\varepsilon} + h(\widetilde{\varepsilon})}{\log\left(\frac{1}{|\lambda|}\right)}, \qquad \widetilde{\varepsilon} = \frac{\log(|\lambda|)}{\log\left(1 - \eta\left(\frac{|\lambda|^2}{|\lambda|^2+3}, p\right)\right)}\, \varepsilon, \tag{2.1}$$

and $h(\widetilde{\varepsilon}) = -\widetilde{\varepsilon}\log(\widetilde{\varepsilon}) - (1-\widetilde{\varepsilon})\log(1-\widetilde{\varepsilon})$ is the entropy function.

Proof. Choosing $N \in \mathbb{N}$ such that $|\lambda|^{-(N-1)} \le T \le |\lambda|^{-N}$ we may assume that $T = |\lambda|^{-N}$. We can write $\xi = \overline{t\lambda^{-N}}$, where $t \in \mathbb{C}$ and $|t| < 1$. We have that

$$\begin{aligned} |\widehat{\mu}_{\lambda,w}^p(\xi)| &\le \prod_{j=1}^{\infty} |\Phi(\lambda^j \overline{\xi})| = \prod_{j=1}^{\infty} |\Phi(\lambda^j t \lambda^{-N})| \\ &\le \prod_{j=1}^{N} |\Phi(\lambda^{j-N} t)| = \prod_{j=0}^{N-1} |\Phi(\lambda^{-j} t)|. \end{aligned}$$

As in Lemma 2.1, we denote by $\|x\|$, the distance of x to the closest integer. Given $\varepsilon > 0$, we let $\widetilde{\varepsilon}$ be as in the statement. Let

$$S(N, \widetilde{\varepsilon}) := \left\{t \in \mathbb{C}, |t| < 1 : \|\mathrm{Re}(\lambda^{-j} t)\| < \rho \text{ for at least } (1-\widetilde{\varepsilon})N \text{ integers } j \in [N]\right\},$$

where we denote $[N] = \{0, 1, \dots, N-1\}$, and $\rho = \rho(\lambda) = \frac{|\lambda|^2}{2(|\lambda|^2+3)}$.

Note that if $t \notin S(N, \widetilde{\varepsilon})$, by Lemma 2.1, we have

$$|\widehat{\mu}_{\lambda,w}^p(\xi)| \le (1 - \eta(2\rho, p))^{\widetilde{\varepsilon}N} = |\lambda|^{N\varepsilon} < T^{-\varepsilon},$$

and it follows that

$$\left\{\xi \in \mathbb{C}, |\xi| \le T : |\widehat{\mu}_{\lambda,w}^p(\xi)| \ge T^{-\varepsilon}\right\} \subseteq S(N, \widetilde{\varepsilon}).$$

Hence, in order to prove that $\{\xi \in \mathbb{C}, |\xi| \le T : |\widehat{\mu}_{\lambda,w}^p(\xi)| \ge T^{-\varepsilon}\}$ can be covered by a small number of squares of side-length 1, we will estimate the amount and size of squares needed to cover $S(N, \widetilde{\varepsilon})$.

For each $t \in \mathbb{C}, |t| < 1$, we define integers $r_j(t)$ and $\varepsilon_j(t) \in [-1/2, 1/2)$ such that

$$\mathrm{Re}(\lambda^{-j} t) = r_j(t) + \varepsilon_j(t). \tag{2.2}$$

Then $t \in S(N, \widetilde{\varepsilon})$ precisely when $|\varepsilon_j(t)| < \rho$ at least $(1-\widetilde{\varepsilon})N$ times among the indices $j \in [N]$. We will simply write r_j and ε_j when no confusion arises.

Let $N_1 = \lceil (1-\widetilde{\varepsilon})N \rceil$. For each $t \in S(N, \widetilde{\varepsilon})$, there is a subset $I \subseteq [N]$ with at least N_1 elements such that $|\varepsilon_j| < \rho$ for all $j \in I$. We shall estimate the size of $S(N, \widetilde{\varepsilon})$ by considering each index set I separately, and for this, we define

$$S(I, \widetilde{\varepsilon}) := \left\{ t \in \mathbb{C}, |t| < 1 : \ \|\mathrm{Re}(\lambda^{-j} t)\| < \rho \text{ for all } j \in I \right\}.$$

If $j = 0$, $\mathrm{Re}(t) = r_0 + \varepsilon_0$, so for $|t| < 1$, there are at most 3 choices for r_0 and at most 6 choices for r_1.

Next, we denote $\lambda = a + ib$, $a, b \in \mathbb{R}$, $b \neq 0$, and given $j \in [N]$,

$$\lambda^{-j} t = c_j + i d_j, \tag{2.3}$$

where c_j and d_j depend on j and t.

By (2.2), we have

$$\mathrm{Re}(\lambda^{-(j+1)} t) = r_{j+1} + \varepsilon_{j+1} \tag{2.4}$$

and, on the other hand

$$\begin{aligned} c_{j+1} := \mathrm{Re}(\lambda^{-(j+1)} t) &= \mathrm{Re}(\lambda^{-1})\mathrm{Re}(\lambda^{-j} t) - \mathrm{Im}(\lambda^{-1})\mathrm{Im}(\lambda^{-j} t) \\ &= \frac{a}{|\lambda|^2}(r_j + \varepsilon_j) + \frac{b}{|\lambda|^2} d_j, \end{aligned} \tag{2.5}$$

where in the last equality we use that $\lambda^{-1} = \frac{a-bi}{|\lambda|^2}$. Using again (2.2) and a simple calculation, we obtain

$$r_{j-1} + \varepsilon_{j-1} = a c_j - b d_j,$$

and therefore

$$d_j = \frac{a c_j - r_{j-1} - \varepsilon_{j-1}}{b} = \frac{1}{b}\big(a(r_j + \varepsilon_j) - r_{j-1} - \varepsilon_{j-1}\big), \tag{2.6}$$

where in the last equality we use that $c_j = r_j + \varepsilon_j$.

Now, combining (2.4), (2.5) and (2.6),

$$\begin{aligned} \varepsilon_{j+1} &= c_{j+1} - r_{j+1} \\ &= \frac{a}{|\lambda|^2}(r_j + \varepsilon_j) + \frac{b}{|\lambda|^2}\left(\frac{1}{b}\big(a(r_j + \varepsilon_j) - r_{j-1} - \varepsilon_{j-1}\big)\right) - r_{j+1} \\ &= \frac{2a}{|\lambda|^2}(r_j + \varepsilon_j) - \frac{r_{j-1} + \varepsilon_{j-1}}{|\lambda|^2} - r_{j+1}, \end{aligned}$$

or, equivalently,

$$\frac{2a r_j - r_{j-1}}{|\lambda|^2} - r_{j+1} = \varepsilon_{j+1} + \frac{\varepsilon_{j-1}}{|\lambda|^2} - \frac{2a\varepsilon_j}{|\lambda|^2}.$$

Taking absolute value in the previous equality and using that $|\varepsilon_j| \leq 1/2$ for all j and $|a| \leq 1$, we get

$$\left|r_{j+1} - \frac{2ar_j - r_{j-1}}{|\lambda|^2}\right| \leq \frac{1}{2}\Big(1 + \frac{1}{|\lambda|^2} + \frac{2|a|}{|\lambda|^2}\Big) \leq \frac{1}{2}\Big(1 + \frac{3}{|\lambda|^2}\Big).$$

Therefore, given r_{j-1}, r_j, we can have at most $\lceil \frac{1}{2}(1 + \frac{3}{|\lambda|^2}) \rceil$ choices of r_{j+1}.

If $j-1, j, j+1 \in I$, then $|\varepsilon_{j-1}|, |\varepsilon_j|, |\varepsilon_{j+1}| < \rho$, so that

$$|\varepsilon_{j+1}| + \frac{|\varepsilon_{j-1}|}{|\lambda|^2} + \frac{2|a||\varepsilon_j|}{|\lambda|^2} < 1/2$$

and at most one value of r_{j+1} is possible. Also note that

$$\left|\{j \in [N] : j-1, j, j+1 \in I\}\right| \geq N - 3|N \setminus N_1| - 2 \geq (1 - 3\widetilde{\varepsilon})N - 2.$$

Thus, the total number of sequences $r_1, \ldots, r_N$ is at most

$$M_N := 3 \cdot 6\Big(\Big\lceil 1 + \frac{3}{|\lambda|^2} \Big\rceil\Big)^{3\widetilde{\varepsilon}N+2}.$$

Invoking (2.2) and (2.3), we have

$$c_N = \operatorname{Re}(\lambda^{-N} t) = r_N + \varepsilon_N, \tag{2.7}$$

with $|c_N - r_N| \leq 1/2$.

On the other hand,

$$r_{N-1} + \varepsilon_{N-1} = \operatorname{Re}(\lambda^{-(N-1)} t) = \operatorname{Re}(\lambda\lambda^{-Nt}) = ac_N - bd_N,$$

and then, using (2.8), (2.7) and that $|\varepsilon_j| \leq 1/2$ for all j, we obtain

$$\left|d_N - \frac{ar_N}{b} - \frac{r_{N-1}}{b}\right| \leq \frac{|a|+1}{2|b|}. \tag{2.8}$$

From (2.7) and (2.8), we conclude that, for each pair (r_{N-1}, r_N), the complex number $\lambda^{-N} t = c_N + i d_N$ belongs to a rectangle of dimensions $\frac{|a|+1}{2|b|} \times 1$. Then, t is contained in a rectangle of dimensions $\frac{|a|+1}{2|b|}|\lambda|^N \times |\lambda|^N$, and we obtain that $S(I, \widetilde{\varepsilon})$ can be covered by M_N rectangles of the mentioned size.

By Stirling's formula, we can estimate $\binom{N}{N_1}$ and we see that the number of index sets I is at most $e^{h(\widetilde{\varepsilon})N}$ for large enough N. Therefore, $S(N, \widetilde{\varepsilon})$ can be covered by

$$M_N e^{h(\widetilde{\varepsilon})N}$$

rectangles of dimensions $\frac{|a|+1}{2|b|}|\lambda|^N \times |\lambda|^N$. Finally, rescaling, we have that

$$\{\xi \in \mathbb{C}, |\xi| \leq T : |\widehat{\mu}^p_{\lambda,w}(\xi)| \geq T^{-\varepsilon}\}$$

can be covered by the above number of rectangles of dimensions $\frac{|a|+1}{2|b|} \times 1$. Moreover, since each rectangle can be covered by a finite number of squares of side-length 1, we conclude the proof. ∎

Remark 2.3. Clearly, if $\text{Im}(\lambda) = 0$, the above proof fails, but the exclusion of $\lambda \in \mathbb{R}$ may appear artificial. When $m = 2$, the case $\lambda \in \mathbb{R}$ reduces back to the family of real Bernoulli convolutions, treated in [9]. However, it is interesting to consider the case when $m \geq 3$ and the vectors w_i are not collinear. More precisely, consider an IFS of the form $\{\lambda z + w_i\}_{i=1}^m$, $\lambda \in (0, 1)$ and $m \geq 3$. In these cases, after an affine change of coordinates, that does not affect the results we want to obtain, we can assume without loss of generality that $w_1 = 0, w_2 = 1$ and $w_3 = i$. For example, here we can include the self-similar measure supported on the Sierpiński gasket associated to the IFS $\{\frac{1}{2}z + w_i\}_{i=1}^3$, where w_i are the vertices of an equilateral triangle centered at the origin.

To deal with this case, first we need to prove a similar estimation as in Lemma 2.1 but considering the distance of a complex number to the lattice $\mathbb{Z}^2$ instead of the distance of its real part to $\mathbb{Z}$.

Lemma 2.4. *The following holds for all $z \in \mathbb{C}$, and $c \in (0, 1)$: If $d(z, \mathbb{Z}^2) > \frac{c}{2}$ then $|\Phi(z)| < 1 - \eta(c, p)$, where $\eta(c, p)$ is a positive constant and $d(z, \mathbb{Z}^2)$ denotes the distance of z to the closest point on the lattice $\mathbb{Z}^2$.*

Proof. Recalling the definition of Φ and using that $w_1 = 0, w_2 = 1$ and $w_3 = i$, we have

$$\begin{aligned}
|\Phi(z)| &= |p_1 + p_2 \exp(2\pi i \text{Re}(z)) + p_3 \exp(2\pi i \text{Re}(iz)) + (1 - p_1 - p_2 - p_3)| \\
&\leq 1 - c_1 \max(\|\text{Re}(z)\|, \|\text{Re}(iz)\|)^2 \\
&= 1 - c_1 \max(\|\text{Re}(z)\|, \| - \text{Im}(z)\|)^2 \\
&= 1 - c_1 \max(\|\text{Re}(z)\|, \|\text{Im}(z)\|)^2,
\end{aligned}$$

for some constant $c_1 > 0$ depending on p and, as before, $\|x\|$ denotes the distance of x to the closest integer. Since

$$d(z, \mathbb{Z}^2) = \|\text{Re}(z)\|^2 + \|\text{Im}(z)\|^2 \leq 2 \max(\|\text{Re}(z)\|, \|\text{Im}(z)\|)^2,$$

and $d(z, \mathbb{Z}^2) > c/2$, we obtain

$$|\Phi(z)| \leq 1 - \frac{c_1}{2} d^2(z, \mathbb{Z}^2) \leq 1 - \eta(c, p).$$ ∎

Next, we present the analogue of Proposition 2.2 for the case $\lambda \in \mathbb{R}$, with $|\lambda| < 1$. The statement is similar, but of course the dependence of δ on ε is different. Since the proof follows directly from the one given in [9] for the real case, we omit it.

Proposition 2.5. *Given $\lambda \in (0,1)$ and a probability vector $p = (p_1, \dots, p_m)$, $m \geq 3$, there is a constant $C = C_\lambda > 0$ such that for each $\varepsilon > 0$ small enough (depending continuously on λ) the following holds for all T large enough: The set of frequencies $|\xi| \leq T$ such that $|\widehat{\mu}^p_{\lambda,w}(\xi)| \geq T^{-\varepsilon}$ can be covered by $C_\lambda T^\delta$ squares of side-length 1, where*

$$\delta = \frac{4\widetilde{\varepsilon}\log\left(\lceil 2 + \frac{1}{\lambda}\rceil\right) + h(\widetilde{\varepsilon})}{\log\left(\frac{1}{\lambda}\right)}, \qquad \widetilde{\varepsilon} = \frac{\varepsilon \log(\lambda)}{\log\left(1 - \eta\left(\frac{\lambda}{2(\lambda+1)}, p\right)\right)},$$

and $h(\widetilde{\varepsilon}) = -\widetilde{\varepsilon}\log(\widetilde{\varepsilon}) - (1-\widetilde{\varepsilon})\log(1-\widetilde{\varepsilon})$ is the entropy function.

3. A Kaufman-type theorem in two dimensions

As an application of the results obtained in the previous section, we present a version of Kaufman's theorem in the complex plane.

Theorem 3.1. *Let μ be an homogeneous self-similar measure on $\mathbb{C}$ which is not a single atom and let $F : \mathbb{C} \to \mathbb{C}$ be an analytic function with $F'' \neq 0$ in a neighborhood of* supp μ. *Then there exist $\sigma = \sigma(\mu) > 0$ and $C = C(F,\mu) > 0$ such that*

$$|\widehat{F\mu}(\xi)| \leq C|\xi|^{-\sigma}.$$

For the proof we need the following well-known result (see, for example, [3]).

Proposition 3.2. *Let μ be a self-similar measure on $\mathbb{C}$ which is not a single atom. Then there exist positive constants C and s, depending on μ, such that $\mu(B(x,r)) \leq Cr^s$, for all $x, r > 0$.*

Proof of Theorem 3.1. Fix ξ such that $|\xi| \gg 1$ and choose $N \in \mathbb{N}$ such that $1 < |\lambda|^N|\xi|^{2/3} \leq |\lambda|^{-1}$.

Let us decompose μ in the following way:

$$\mu = \mu_N * \nu_N$$

where $\mu_N = \ast_{n=1}^N (\sum_{j=1}^m p_j \delta_{\lambda^n w_j})$ and ν_N is a rotated and scaled down copy of μ by a factor λ^N.

For the next calculation, we will write $e(z) = e^{2\pi i \mathrm{Re}(z)}$ for simplicity.

$$\begin{aligned}\widehat{F\mu}(\xi) &= \int_{\mathbb{C}} e^{2\pi i \mathrm{Re}(F(v)\bar{\xi})}\, d\mu(v) \\ &= \int_{\mathbb{C}}\int_{\mathbb{C}} e(F(z+w)\bar{\xi})\, d\mu_N(z) d\nu_N(w)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{C}} \int_{\mathbb{C}} e((F(z) + F'(z)w + O(|w|^2))\bar{\xi})\, d\mu_N(z)\, d\nu_N(w) \\
&= \int_{\mathbb{C}} \int_{\mathbb{C}} e(F(z)\bar{\xi} + F'(z)w\bar{\xi})e(O(|\xi||w|^2))\, d\mu_N(z) d\nu_N(w) \\
&= \int_{\mathbb{C}} \int_{\mathbb{C}} e(F(z)\bar{\xi} + F'(z)w\bar{\xi})(1 + O(|\xi||w|^2))\, d\mu_N(z)\, d\nu_N(w) \\
&= \int_{\mathbb{C}} e(F(z)\bar{\xi})\Big(\int_{\mathbb{C}} e(F'(z)w\bar{\xi})\, d\nu_N(w)\Big)\, d\mu_N(z) \\
&\qquad + \int_{\mathbb{C}} \int_{\mathbb{C}} e(F'(z)w\bar{\xi} + F(z)\bar{\xi})O(|\xi||w|^2)\, d\mu_N(z)\, d\nu_N(w) \\
&= \int_{\mathbb{C}} e(F(z)\bar{\xi})\Big(\int_{\mathbb{C}} e(F'(z)w\bar{\xi})\, d\nu_N(w)\Big)\, d\mu_N(z) + O(|\xi||\lambda|^{2N}),
\end{aligned}
$$

where in the third equality we replace F by its linear approximation (where, as usual, $O(X)$ denotes a quantity bounded by CX in modulus) and in the the fifth equality we use that $|e(\delta) - 1| = O(\delta)$.

Then, by the assumptions made at the beginning of the proof on ξ and N, we have $|\lambda|^N \approx |\xi|^{-2/3}$ and $|\xi||\lambda|^{2N} \approx |\xi|^{-1/3}$. Here, $x \approx y$ means that $C^{-1}x \leq y \leq Cx$. Then,

$$
\begin{aligned}
|\widehat{F\mu}(\xi)| &\leq \Big|\int_{\mathbb{C}} e(F(z)\bar{\xi})\Big(\int_{\mathbb{C}} e(F'(z)w\bar{\xi})\, d\nu_N(w)\Big)\, d\mu_N(z)\Big| + O(|\xi|^{-1/3}) \\
&\leq \int_{\mathbb{C}} |\widehat{\nu}_N(F'(z)\bar{\xi})|\, d\mu_N(z) + O(|\xi|^{-1/3}) \\
&= \int_{\mathbb{C}} |\widehat{\mu}(\lambda^N F'(z)\bar{\xi})|\, d\mu_N(z) + O(|\xi|^{-1/3}).
\end{aligned}
$$

Consider $T = M|\lambda|^N|\xi|$, where $M := \sup_{z\in \operatorname{supp}\mu} |F'(z)|$ and fix $\varepsilon > 0$ to be determined later. Then, by Proposition 2.2, there is $C = C_\lambda > 0$ such that the set of frequencies $|\xi| \leq T$ for which $|\widehat{\mu}(\xi)| \geq T^{-\varepsilon}$ can be covered by CT^δ squares with side-length 1. Let $I_1, \ldots, I_{CT^\delta}$ be these squares. Observe that if $\xi \notin \bigcup_{j=1}^{CT^\delta} I_j$, then $|\widehat{\mu}(\xi)| \leq T^{-\varepsilon}$.

Consider the set

$$
\Gamma := \Big\{z \in \operatorname{supp}\mu : \lambda^N F'(z)\bar{\xi} \in \bigcup_{j=1}^{CT^\delta} I_j\Big\}.
$$

Then

$$
\int_{\mathbb{C}} |\widehat{\mu}(\lambda^N F'(z)\bar{\xi})|\, d\mu_N(z) = \int_{\Gamma} + \int_{\Gamma^c} \leq \mu_N(\Gamma) + T^{-\varepsilon} \leq \mu_N(\Gamma) + O(|\xi|^{-1/3}).
$$

In order to conclude the proof, we need to prove that $\mu_N(\Gamma) \leq |\xi|^{-\beta}$ for some $\beta > 0$.

First, observe that Γ can be rewritten as

$$\Big\{z \in \operatorname{supp}\mu : F'(z) \in \bigcup_{j=1}^{CT^\delta} J_j\Big\},$$

where J_j are squares of side-length $|\lambda|^{-N}|\xi|^{-1} \approx |\lambda|^{N/2}$, or which is the same, $\Gamma = \bigcup_{j=1}^{CT^\delta} J_j'$, where $J_j' = (F')^{-1}(J_j) \cap \operatorname{supp}\mu$. Using that F'' is non-zero in a neighborhood of $\operatorname{supp}\mu$, we have that

$$|z_1 - z_2| \le L\,|F'(z_1) - F'(z_2)|$$

for all $z_1, z_2 \in J_j'$ and L is a positive constant depending on F. Then, for each j,

$$\operatorname{diam} J_j' \le L \operatorname{diam} J_j \lesssim L|\lambda|^{N/2}. \tag{3.1}$$

On the other hand, since ν_N is a rotated and scaled down copy of μ by a factor λ^N, if the support of μ is contained in a ball $B_C(0)$, for some $C = C(\lambda, w_1, \dots, w_m)$, the support of ν_N is contained in a ball $B_{|\lambda|^N C}(0)$. Then, since $\mu = \mu_N * \nu_N$, one has that for any ball B,

$$\mu_N(B) \le \mu(B + B_{|\lambda|^N C}(0)). \tag{3.2}$$

Invoking (3.1) and (3.2), for each j there exists a ball B_j with the same diameter of J_j' such that $J_j' \subseteq B_j$ and $\mu_N(B_j) \le C|\lambda|^{\frac{Ns}{2}}$.

Therefore

$$\mu_N(\Gamma) \le CT^\delta|\lambda|^{\frac{Ns}{2}} \le C|\xi|^{\delta/3}|\xi|^{-s/3} \le C|\xi|^{\frac{(\delta-s)}{3}}.$$

Choosing ε small enough such that $\delta < s(\mu)$ we obtain that

$$|\widehat{F\mu}(\xi)| \le C|\xi|^{\frac{(\delta-s)}{3}} + C|\xi|^{-\varepsilon/3} + C|\xi|^{-1/3} \le C|\xi|^{-\min\left\{\frac{(s-\delta)}{3}, \frac{\varepsilon}{3}\right\}}. \qquad \blacksquare$$

Remark 3.3. The above theorem allow us to have uniform explicit power decay for the Fourier transform of $F\mu_\lambda$, the push-forward measure of complex Bernoulli convolutions, even if the measure μ_λ does not have decay at all. For example when $\lambda = 1/\theta$ and θ is a complex Pisot number such that $1 < |\theta| < \sqrt{2}$, it is known that $|\widehat{\mu_\lambda}(\xi)| \nrightarrow 0$ when $|\xi| \to \infty$, see [16].

4. Applications

4.1. Improving the L^2 dimension under convolution

We begin by recalling the definition of L^q dimensions. Let $q \in (1, +\infty)$, and set $s_n(\mu, q) := \sum_{Q \in \mathcal{D}_n} \mu(Q)^q$, with $\{\mathcal{D}_n\}_n$ the partition of $\mathbb{R}^d$ into dyadic intervals of length 2^{-n}. Define

$$\dim_q(\mu) := \liminf_{n\to+\infty} \frac{\log(s_n(\mu,q))}{(q-1)\log(2^{-n})}.$$

The L^2 dimension of a measure is also known as correlation dimension. Note that the Frostman exponent $\dim_\infty$ can also be defined as

$$\dim_\infty(\mu) := \liminf_{n\to+\infty} \frac{\log\max\{\mu(Q) : Q \in \mathcal{D}_n\}}{\log(2^{-n})}.$$

It is well known that the function $q \mapsto \dim_q(\mu)$ is continuous and non-increasing on $(1,+\infty]$ and that $\dim_q(\mu) \le \dim_{\mathrm{H}}(\mu)$ for any $q \in (1,+\infty]$, where $\dim_{\mathrm{H}}$ is the lower Hausdorff dimension of a measure, defined as

$$\dim_{\mathrm{H}}(\mu) := \inf\big\{\dim_{\mathrm{H}}(A) : \mu(A) > 0\big\}.$$

We refer the reader to [2] for the proofs of these facts and further background on dimensions of measures.

The proofs of the results in this section are similar to the ones given in [9] for the real case, but we will include them here for the sake of completeness.

Theorem 4.1. *Let μ be an homogeneous self-similar measure on the complex plane. Given any $\kappa > 0$, there is $\sigma = \sigma(\lambda, p, \kappa) > 0$ such that the following holds: Let ν be any Borel probability measure with $\dim_2(\nu) \le 2-\kappa$. Then*

$$\dim_2(\mu * \nu) > \dim_2(\nu) + \sigma.$$

More precisely, one can take $\sigma = 2\varepsilon$, where $\varepsilon = \varepsilon(\lambda, p, \kappa)$ is such that the value of $\delta = \delta(\varepsilon, \lambda, p)$ given in Proposition 2.2 satisfies

$$\kappa - 2\varepsilon = \delta. \tag{4.1}$$

Proof. First, note that it is possible to choose $\varepsilon < 1/2$ such that $\kappa - 2\varepsilon = \delta$, using continuity arguments.

For any Borel probability measure η on $\mathbb{C}$ we have that $\dim_2(\eta) = 2 - \alpha(\eta)$, where

$$\alpha(\eta) = \limsup_{T\to\infty} \frac{\log \int_{|\xi|<T} |\widehat{\eta}(\xi)|^2\, d\xi}{\log T}.$$

In [4, Lemma 2.5], the authors prove the above result in the real line, but the same argument can be extended to higher dimensions. So we omit the proof. Then it is enough to prove that $\alpha(\nu) \ge \kappa$ implies $\alpha(\mu * \nu) < \alpha(\nu) - \sigma$.

Denote $\kappa_0 = \alpha(\nu) \geq \kappa$. For any $\varepsilon_0 > 0$ and taking $T = 2^N$ for some $N \in \mathbb{N}$, by definition of α,

$$\int_{|\xi|\leq 2^N} |\widehat{\nu}(\xi)|^2 \, d\xi \leq O_{\varepsilon_0}(1) 2^{N(\kappa_0+\varepsilon_0)}.$$

Split the frequencies into two groups

$$E_N = \{\xi : |\xi| \leq 2^N, |\widehat{\mu}(\xi)| \leq 2^{-\varepsilon N}\} \quad \text{and} \quad F_N = \{\xi : |\xi| \leq 2^N, |\widehat{\mu}(\xi)| > 2^{-\varepsilon N}\}.$$

Then, applying Proposition 2.2, we have that F_N can be covered by $C_\lambda 2^{\delta N}$ squares of side-length 1 and, in consequence, it has Lebesgue measure bounded by $C_\lambda 2^{\delta N}$.

Using all this, we have

$$\begin{aligned}
\int_{|\xi|\leq 2^N} |\widehat{\mu * \nu}(\xi)|^2 \, d\xi &= \int_{E_N \cup F_N} |\widehat{\mu}(\xi)|^2 |\widehat{\nu}(\xi)|^2 \, d\xi \\
&\leq \int_{E_N} 2^{-2\varepsilon N} |\widehat{\nu}(\xi)|^2 \, d\xi + \int_{F_N} 1 \, d\xi \\
&\leq O_{\varepsilon_0}(1) 2^{-2\varepsilon N} 2^{(\kappa_0+\varepsilon_0)N} + C_\lambda 2^{\delta N} \\
&\leq O_{\varepsilon_0,\lambda}(1) 2^{(\kappa_0-2\varepsilon+\varepsilon_0)N},
\end{aligned}$$

using that $\kappa_0 \geq \kappa$ and the definition of ε in the last line. Since this holds for all $\varepsilon_0 > 0$, it follows from the definition of α that

$$\alpha(\mu * \nu) \leq \kappa_0 - 2\varepsilon,$$

which gives the claim since $\sigma = 2\varepsilon$. ∎

4.2. Frostman exponent for complex Bernoulli convolutions

Let $\lambda \in \mathbb{D}$ and $p \in (0, 1)$. We denote with μ_λ^p the biased complex Bernoulli convolution, i.e., μ_λ^p is the self-similar measure associated with the IFS $\{\lambda z - 1, \lambda z + 1\}$ with probability vector $(p, 1 - p)$. When $p = 1/2$, we just write μ_λ to denote the usual Bernoulli convolution.

Theorem 4.2. *Given $p_0 < 1/2$, there exists a constant $C = C(p_0)$ such that*

$$\dim_\infty(\mu_\lambda^p) \geq 2 - C(1 - |\lambda|) \log\Big(\frac{1}{1 - |\lambda|}\Big)$$

for all $p_0 \leq p \leq 1 - p_0$.

Proof. Fix $\lambda \in \mathbb{D}$ with modulus close to 1. We define $N = N(\lambda)$ to be the smallest integer such that $|\lambda|^N < 1/\sqrt{2}$. Then,

$$\frac{|\lambda|}{\sqrt{2}} \leq |\lambda|^N < \frac{1}{\sqrt{2}}.$$

In particular, assuming $|\lambda| > 1/\sqrt{2}$, we see that $|\lambda|^N \in (1/2, 1/\sqrt{2})$.

Fix $\kappa \in (0,1)$, and suppose that $\dim_2(\mu_\lambda^p) \le 2-\kappa$. We have the following decomposition. Let us write $S_\lambda(x) = \lambda x$ for the map that scales by λ, and recall that

$$\mu_\lambda^p = \mu_{\lambda^N}^p * S_\lambda \mu_{\lambda^N}^p * \cdots * S_{\lambda^{N-1}} \mu_{\lambda^N}^p. \tag{4.2}$$

This is a well-known fact that can be seen from expressing μ_λ^p as an infinite convolution.

Since the associated IFS satisfies the open set condition,

$$\dim_2(\mu_{\lambda^N}^p) = \frac{\log(p^2 + (1-p)^2)}{\log(|\lambda|^N)}.$$

In particular, we have that $\dim_2(\mu_{\lambda^N}^p) \ge 0$. Now, using that $\dim_2(\mu_\lambda^p) \le 2-\kappa$ and (4.2), we get $\dim_2(\mu_{\lambda^N}^p) \le 2-\kappa$. By Theorem 4.1, there is $\sigma = \sigma(\lambda, p, \kappa) > 0$ such that

$$\dim_2(\mu_{\lambda^N}^p * S_\lambda \mu_{\lambda^N}^p) \ge \sigma.$$

Proceeding inductively according to (4.2), after $N-1$ steps, we obtain that if $\dim_2(\mu_\lambda^p) \le 2-\kappa$, then

$$\dim_2(\mu_\lambda^p) \ge (N-1)\sigma.$$

It follows that if κ is such that $\sigma = \sigma(\lambda, p, \kappa) = 1/(N-1)$, then

$$\dim_2(\mu_\lambda^p) \ge 2-\kappa.$$

Thus, we just need to estimate such κ. By (4.1), we have that $\kappa = \delta + \sigma$, where $\delta = \delta(\sigma/2)$ is given by (2.1). Note that $\widetilde{\varepsilon} = C(|\lambda|^N, p)\sigma$, where $C > 0$ depends continuously on $|\lambda|^N$ and p. In what follows, C_j will denote a positive constant depending only of p_0. Since $|\lambda|^N \in (1/2, 1/\sqrt{2})$ and $p \in [p_0, 1-p_0]$, a calculation using (2.1) shows that there is a constant C_1 such that $\delta \le C_1 \sigma \log(1/\sigma)$ provided σ is small enough (which we may assume).

We deduce that

$$\dim_2(\mu_\lambda^p) \ge 2-\kappa \ge 2-\sigma - C_1\sigma \log\Big(\frac{1}{\sigma}\Big) \ge 2 - C_2\sigma \log\Big(\frac{1}{\sigma}\Big), \tag{4.3}$$

if σ is small enough. On the other hand, since $|\lambda|^{1/\sigma} = |\lambda|^{N-1} < |\lambda|^{-1}/\sqrt{2} < 2/3$ (say), we have $\sigma \le \log(1/|\lambda|)/\log(3/2)$. Finally, using that $\log(1/|\lambda|) \le 2(1-|\lambda|)$ for $1-|\lambda|$ small, we deduce that

$$\sigma \le C_4(1-|\lambda|).$$

Together with (4.3), this yields

$$\dim_2(\mu_\lambda^p) \ge 2 - C_5(1-|\lambda|)\log\Big(\frac{1}{1-|\lambda|}\Big).$$

Since we have the decomposition

$$\mu_\lambda^p = \mu_{\lambda^2}^p * S_\lambda \mu_{\lambda^2}^p,$$

and scalings do not change L^2 dimension, using Young's lemma (see [9, Lemma 5.2]), we can conclude that

$$\dim_\infty(\mu_\lambda^p) \geq 2 - C_5(1-|\lambda|^2)\log\Big(\frac{1}{1-|\lambda|^2}\Big) \geq 2 - C_6(1-|\lambda|)\log\Big(\frac{1}{1-|\lambda|}\Big). \quad \blacksquare$$

Corollary 4.3. *There is an absolute constant $C > 0$ such that*

$$\dim_\infty(\mu_\lambda) \geq 2 - C(1-|\lambda|)^2 \log\Big(\frac{1}{1-|\lambda|}\Big).$$

Proof. Again, fix $\lambda \in \mathbb{D}$ with $|\lambda|$ close to 1 and let $N = N(\lambda)$ the smallest integer such that $|\lambda|^N < 1/\sqrt{2}$ and then, $|\lambda|^N > |\lambda|/\sqrt{2}$.

Since the associated IFS satisfy the open set condition,

$$\dim_2(\mu_{\lambda^N}) = \frac{\log(\frac{1}{2})}{\log|\lambda|^N} \geq \frac{\log(\frac{1}{2})}{\log(\frac{|\lambda|}{\sqrt{2}})} = 1 - \frac{\log(\frac{1}{|\lambda|})}{\log(\frac{2}{|\lambda|})}.$$

Proceeding as in the proof of the Theorem 4.2, we obtain that if $\dim_2(\mu_\lambda) \leq 2-\kappa$, then there exists $\sigma = \sigma(\kappa,\lambda) > 0$ such that

$$\begin{aligned}\dim_2(\mu_\lambda) &\geq \dim_2(\mu_{\lambda^N}) + (N-1)\sigma \\ &\geq 1 - \frac{\log(\frac{1}{|\lambda|})}{\log(\frac{2}{|\lambda|})} + (N-1)\sigma.\end{aligned}$$

Now, if κ is such that $\sigma = \frac{\log(1/|\lambda|)}{\log(2/|\lambda|)}\frac{1}{(N-1)}$, then

$$\dim_2(\mu_\lambda) \geq 2-\kappa.$$

Then we want to estimate such κ. Proceeding as in the proof of the above theorem, we get

$$\dim_2(\mu_\lambda) \geq 2-\kappa \geq 2 - C_1\sigma\log\Big(\frac{1}{\sigma}\Big).$$

On the other hand, using that $|\lambda|/\sqrt{2} < |\lambda|^N < 1/\sqrt{2}$ and that $\log(1/|\lambda|) \leq 2(1-|\lambda|)$ for $1-|\lambda|$ small, we obtain $\frac{1}{(N-1)} \leq C_2(1-|\lambda|)$ and then

$$\sigma \leq C_3(1-|\lambda|)^2.$$

Thus,

$$\dim_2(\mu_\lambda) \geq 2 - C_4(1-|\lambda|)^2\log\Big(\frac{1}{1-|\lambda|}\Big).$$

Using again Young's lemma as in the proof of Theorem 4.2 we finish the proof. ∎

5. A particular extension to higher dimensions

In this section, we present a generalization to the results obtained in Section 2 in higher dimensions, but for a particular class of self-similar measures.

Given $p = (p_1, \dots, p_m)$ a probability vector, $w = (w_1, \dots, w_m)$ a sequence of digit vectors in $\mathbb{R}^d$ and $\lambda \in (0,1)$, let $\mu^p_{\lambda,w}$ be the self-similar measure associated to the IFS $\{f_i\}_{i=1}^m$, with $f_i = \lambda O x + w_i$ and where O is an orthogonal matrix diagonalizable over $\mathbb{R}$.

Since O is a diagonalizable matrix over $\mathbb{R}$, we can assume without loss of generality that O is the identity matrix. In fact, if we iterate the IFS (replacing f_i by $(f_i f_j)_{i,j=1}^m$), the matrix O is replaced by O^2 which has all its eigenvalues equal to 1 and then, there exist some orthonormal basis where it can be written as the identity matrix. Also, after an affine change of coordinates we can always assume that $w_1 = (0, \dots, 0)$ and $w_2 = (1, \dots, 1)$. By definition of the Fourier transform and the condition of being a self-similar measure, we can write

$$\widehat{\mu}^p_{\lambda,w}(\xi) = \prod_{n=0}^{\infty} \Phi(\lambda^n \xi),$$

where $\Phi(y) = \sum_{j=1}^m p_j \exp(2\pi i \langle y, w_j \rangle)$ for all $y \in \mathbb{R}^d$.

Next, we present a lemma that is analogous to Lemma 2.1, and also the higher dimensional version of Proposition 2.2.

Lemma 5.1. *The following holds for all $y \in \mathbb{R}^d$, $y = (y_1, \dots, y_d)$ and $c \in (0,1)$: If $||y_1 + \ldots + y_d|| > \frac{c}{2}$, then $\Phi(y) < 1 - \eta(c, p)$, where $\eta(c, p)$ is a positive constant and $||\cdot||$ denotes the distance of a real number to the closest integer.*

Proposition 5.2. *Given $\lambda \in (0,1)$ and a probability vector $p = (p_1, \dots, p_m)$, $m \geq 3$, there is a constant $C = C_\lambda > 0$ such that for each $\varepsilon > 0$ small enough (depending continuously on λ) the following holds for all T large enough: The set of frequencies $\{\|\xi\|_\infty \leq T \colon |\widehat{\mu}^p_{\lambda,w}(\xi)| \geq T^{-\varepsilon}\}$ can be covered by $C_\lambda T^\delta$ squares of side-length 1, where*

$$\delta = \frac{\log\left(\left\lceil 1 + \frac{1}{\lambda} \right\rceil\right)\widetilde{\varepsilon} + h(\widetilde{\varepsilon})}{\log\left(\frac{1}{\lambda}\right)}, \qquad \widetilde{\varepsilon} = \frac{\log(\lambda)}{\log\left(1 - \eta\left(\frac{\lambda}{\lambda+1}, p\right)\right)}\varepsilon,$$

and $h(\widetilde{\varepsilon}) = -\widetilde{\varepsilon}\log(\widetilde{\varepsilon}) - (1 - \widetilde{\varepsilon})\log(1 - \widetilde{\varepsilon})$ is the entropy function.

The proof of Lemma 5.1 is analogous to that of Lemma 2.1, and the proof of Proposition 5.2 is similar to that of the real case in [9].

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