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h-Laplacians on singular sets

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Abstract. Until now, the correspondence between the Alexander–Kolmogorov complex, and the de Rham one, by means of a small scale parameter, has not gone that far as passing to the limit of the resolvent of the associated Laplacian, when the small parameter tends towards zero. Along these lines, a result proving a complete Hodge decomposition was missing. We bridge this gap by means of our own rescaled h-cohomology, h being a very small parameter. Passing to the limit of the resolvent enables us to consider the extension to singular spaces, in particular, our h-differential operators also enable us to also make the connection with those of analysis on fractals, as introduced by Jun Kigami, and taken up by Robert S. Strichartz.

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1. Introduction

How could one define differentiation and integration on general topological sets? This is the problem that James Waddell Alexander and Andreï Nikolaïevitch Kolmogorov tried to solve with chains and cochains [4]. The underlying idea was given by Kolmogorov himself [4, 21]:

"*The author's goal is to construct a particular difference calculus which, on the one hand, leads to differential operators acting on antisymmetric tensors (multi-*

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vectors) by a limit process, and on the other hand is closely related to the concepts of combinatorial topology.

In particular, it is possible to define new invariants of complexes and closed sets using this difference calculus, and to prove some generalizations of the known duality theorems."

One may see the underlying perspectives, especially, defining and handling differential on non-smooth objects, by means of simplices and the associated cohomology groups.

This is only the first step. What happens when those objects are very small, either since their measure tends towards zero, or when they belong to an everywhere singular set, of fractal type?

Let us recall, first, the context and the existing works on connected subjects. It is often taken for granted that de Rham differential forms are limits of suitably rescaled Alexander–Spanier cochains. One has to be much more precise as soon as one ventures on this terrain. In the work by Alain Connes and Henri Moscovici [7] (mainly devoted to a proof of the Novikov conjecture for hyperbolic groups), the authors review the Alexander–Spanier realization of the cohomology of a smooth manifold M, as it can be found in the original work by Edwin H. Spanier [40, Chapter 6]. The main topic is the definition of an homomorphism of complexes between the (quotient) Alexander–Spanier complex associated to the cohomology, $\bar{C}^\star(M)$, and the de Rham one, $\Lambda^\star(M)$ (recall that given a cochain complex $C^\star(M) = \{C^p(M), \delta\}$ and the sub-complex $C_0^\star(M) = \{C_0^p(M), \delta\} \subset C^\star(M)$, where $C_0^p(M)$ denotes the set of functions from M^{p+1} to $\mathbb{R}$ which vanish on a neighborhood of the p-th diagonal of M, $\bar{C}^\star(M)$ is simply the complex quotient of $\bar{C}^\star(M)$ by $C_0^\star(M)$ – a very natural way of proceeding, the fact that a function vanishes in a given region necessarily implying the same feature for the differential). To the aforementioned purpose, the manifold is endowed with a Riemannian metric, while considering an open covering B which satisfies specific properties. The rescaling is obtained by means of this covering. However, the isomorphism is not made explicit. The second work one might think of is the one by Laurent Bartholdi, Thomas Schick, Stephen Smale, and Nathan Smale, on abstract and classical Hodge–de Rham theory in [2], followed up by the results from the last two authors in [39], where, given a compact Riemannian manifold M, the authors build cochain maps between the de Rham complex of M, $\Lambda^\star(M)$, and the Alexander–Spanier one $\bar{C}^\star(M)$ at a scale $\alpha > 0$, for sufficiently small values of the parameter α. The authors go as far as comparing the Hodge Laplacian on differential forms, and a suitably rescaled one on the space of cochains at scale α. It is shown, in the case of functions, and when α tends towards zero, that the rescaled Laplacian on cochains converges towards the Hodge operator.

Note that the techniques used in the aforementioned Connes and Moscovici paper are different than the ones of our work. First, A. Connes and H. Moscovici integrate differential forms on simplices. Then, they involve the exponential map, which result in numerous and heavy computations. As for the work by Smale et al., it is, also, based on the use of integral operators. Given a metric d, and the scale parameter α, they consider the α-neighborhood of the diagonal in the product manifold X^p, denoted by U_α, and the associated spaces $L^2_{\mathrm{alt}}(U^p_\alpha)$ and $C^\infty(U^p_\alpha)$ of alternating functions of respectively L^2 and C^∞_{alt} class on U^p_α. They prove, for any integer p, the isomorphism between the α-scale subspace $\mathrm{Harm}^p_\alpha(M) \subset L^2_{\mathrm{alt}}(U^p_\alpha)$ of harmonic p-forms on the manifold, and the cohomology in degree p of the respective complexes

$$0 \longrightarrow L^2(M) \xrightarrow{\delta^1} L^2_{\mathrm{alt}}(U^2_\alpha) \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{p-1}} L^2_{\mathrm{alt}}(U^p_\alpha) \xrightarrow{\delta^p} L^{p+1}_{\mathrm{alt}}(U^{p+1}_\alpha) \xrightarrow{\delta^{p+1}} \cdots$$

and

$$0 \longrightarrow C^\infty(M) \xrightarrow{\delta^1} C^\infty_{\mathrm{alt}}(U^2_\alpha) \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{p-1}} C^\infty_{\mathrm{alt}}(U^p_\alpha) \xrightarrow{\delta^p} C^\infty_{\mathrm{alt}}(U^{p+1}_\alpha) \xrightarrow{\delta^{p+1}} \cdots$$

where δ denotes the classical Alexander–Spanier coboundary operator. They also show the isomorphism with the de Rham cohomology. As is not the case in the Connes and Moscovici paper, the isomorphism is given explicitly. However, the result only concerns the cohomology, i.e., the quotiented kernels of the coboundary operator. Moreover, in so far as they solely deal with harmonic forms, they do not have the associated Hodge theory, which can only be obtained by means of a suitable renormalization. Things are easier to handle, when only dealing with harmonic forms. In particular, there is thus no result regarding the limit of the resolvent when the scale parameter tends towards zero.

As for a general theory of differential operators on fractals, the problem was tackled by Fabio Cipriani and Jean-Luc Sauvageot in [5]. The authors place themselves along the lines of noncommutative geometry *à la Connes*, where, given a compact topological space K, a continuous function f on K is represented by means of a bounded operator $\pi(f)$, which acts on a Hilbert space H. If F denotes a self-adjoint operator of square 1, acting on H, the (commutator) operator $df = i[F, \pi(f)]$, where $i^2 = -1$, stands for a "substitute" for the differential of f. In the case of postcritically finite (p.c.f) fractals[1], the authors build Fredholm modules, in relation with the self-similar Dirichlet form E associated to the self-similar fractal[2]. A key result of the Cipriani and Sauvageot paper is [5, Proposition 3.1], where they exhibit the existence of an "essentially unique derivation", denoted by ∂, defined on the Dirichlet

[1]For the reader who might not be familiar with those notions, we refer to the book of Jun Kigami [19, Definition 1.3.13].

[2]See the seminal works of Arne Beurling and Jacques Deny in [3], along with the aforementioned book [19, Chapter 2].

algebra $C(K) \cap F$, taking its values in a real Hilbert module H, and which is a differential square root of the Dirichlet form E. In other words, this means that the algebra of continuous functions on K acts in a continuous way, and that the classical Leibniz rule for the derivative of a product is true. Then, by using the Fredholm modules, the authors are able to associate, to each harmonic structure, a topological invariant of the considered compact topological space K, the "K-homology class of the Fredholm module".

In [18], Marius Ionescu, Luke G. Rogers and Alexander Teplyaev go further, and give an explicit description of the elements of the aforementioned Hilbert module H. A very interesting feature of this work is that the authors are able to give "a direct sum decomposition of this module to piecewise harmonic components that correspond to the cellular structure of the fractal". They go as far as giving an analog of the Hodge decomposition for H.

A completely different approach has been developed by Michel L. Lapidus and Machiel van Frankenhuijsen in [34–36] (see also [25]), where the authors suggest that there should exist a fractal cohomology having direct links with the theory of complex dimensions, introduced by Michel L. Lapidus and his collaborators in [13, 16, 22–34, 36]. Further results have been obtained by Michel L. Lapidus and Tim Cobler in [6], where they study the properties of the derivative operator $D = \frac{d}{dz}$ on a particular family of weighted Bergman space constituted of entire functions on $\mathbb{C}$. See, also, [26] and, especially, [27].

We hereafter place ourselves in the same kind of perspective. To begin with, we generalize the algebraic notion of chains (instead of cochains), to what we call fermions. Then, we redefine the concept of h-differentiation, where h denotes a very small real parameter. We go so far as connecting the associated h-cohomology to the classical de Rham one, much more simpler than what can be found in the existing work. If we rely on analogous tools that happen to be the same as the ones that can be found in the Smale et al. work (for instance, the diagonal of the product manifold, and the same explicit isomorphism), our approach takes a completely different turn: in fact, we are the only ones to pass to the limit of the resolvent, when the scale parameter tends towards zero.

This very powerful result enables us to consider the special cases of singular spaces. In fact, the h-differentiation, connected, as one could foresee, to the notion of boundary, leads to a local operator, equivalent to the classical Riemannian Laplacian, which can act on singular objects. When the parameter h tends towards zero, one recovers the usual Laplacian.

A natural question that may be asked is whether this Laplacian is the same as the one of fractal analysis introduced by Jun Kigami [19] and Robert S. Strichartz [42]. This question is all the more interesting, as Laplacians on fractals are defined by means of local differences – the starting point being graph Laplacians. More precisely, one uses Dirichlet forms, built by induction on a sequence of prefractals, i.e., a sequence of finite graphs which converge towards the fractal set involved. For a continuous function on this set, and subject to existence, its Laplacian is obtained as the renormalized limit of the sequence of graph Laplacians. At first sight, one cannot be sure that this operator is the same as the usual Riemannian one – one understands that it is an operator of the same nature, but further? Another concern comes from the fact that changing the measure also changes the Laplacian! The problem is even less obvious as such an operator is not of order two: existing works on the Sierpiński gasket [41], or on the Weierstrass curve [10], show that the order is greater than two.

Our differential is completely different from the one of Cipriani and Sauvageot, hence, also from the differential of Ionescu, Rogers and Teplyaev. It relies on the use of paths across the consecutive prefractal graphs. The detailed study of these differentials is the object of our following work [12].

The main results obtained in this paper can be found in the following places:

(i) In Definition 5.7, where, for the small parameter $h > 0$, we define the h-Laplacian, Δ_h, acting on the space of p-forms, for $p \in \mathbb{N}$.

(ii) In Theorem 5.8, where we pass to the limit of the resolvent of the h-Laplacian, $(z - |\Delta_h|)^{-1}$, when the scale parameter h tends towards zero. This results requires the introduction of a modified scalar product on the space of p-forms (see Proposition 5.5), a compulsory step in order to otain the full rescaled Hodge decomposition, and not only the part associated to harmonic forms as in [39].

(iii) In Section 6, where we make the connection between the h-Laplacian and random walks, namely, Markov chain Monte Carlo methods (MCMC), especially, the Metropolis algorithm. We also extend the definition of the h-Laplacian to continuous functions; see Definition 6.1.

(iv) In Section 7, where we explore the connection between the h-Laplacian, and the now classical Laplacian of fractal analysis [19]. Thanks to the result of the aforementioned Theorem 5.8, we can consider functions which are either not smooth, or are defined on spaces with singularities, namely, the fractal case; see, especially, Property 7.4. Note that in light of the results of Section 6, namely, the connection with MCMC methods, which are perfectly suited when a huge number of data is involved, appears as very interesting in the case of fractal based structures, especially, when they are approximated by means of prefractal graphs, where iterations quickly yield very large numbers of points.

Henceforth, in the light of h-cohomology, the link is obvious: the h-Laplacian can be either obtained by means of de Rham differentiation, but also by means of local differences. So, modulo a multiplicative constant, the value of which will also be discussed and questioned, this is the same operator as the Laplacian on fractals.

In doing so, one falls back on the results exposed by R. S. Strichartz et al. in [1], where the authors build k-forms and de Rham differential operators d and δ on prefractals, k-forms being considered as k-ones on graphs, a natural approach in the light that "a k-form is an object that can be integrated over k-dimensional subjects". Passing to the limit – which calls for ad hoc renormalization – shows that their Laplacian on 0-forms – functions – is the same as the one of J. Kigami.

The circle is thus complete, one is on a closed path. Strichartz et al. made the connection with Hodge–de Rham theory, the missing one with the Alexander–Kolmogorov complex reinforces the legitimacy of differential operators on fractals. And last but not least, one also falls on random walks, which occur through the normalization process required to obtain the limit of the h-Laplacian.

2. Geometric context

Notation 1. In the sequel, we will denote by A a ring of characteristic different from 2, and by X a general space.

Definition 2.1 (p-fermion). By analogy with particle physics, given a positive integer p, we will call p-fermion on X, with values in A, any antisymmetric map f from X^{p+1} to A, i.e., such that, for any transposition τ, and any $(x_0, \ldots, x_p)$ in X^{p+1},

$$f(x_0, \ldots, x_p) = -f(x_{\tau(0)}, \ldots, x_{\tau(p)}).$$

A 0-fermion on X is simply a map f from X to A.

Remark 2.1. p-fermions are simply the generalization of p-chains.

Definition 2.2 (A-module of p-fermions on X). Given a positive integer p, we will denote by $F^p(X, A)$ the A-module of p-fermions on X with values in A, which makes it an abelian group with respect to the addition, with an external law from $A \times F^p(X, A)$ to $F^p(X, A)$ where:

$$\forall (a, b) \in A^2, \quad \forall (f, g) \in (F^p(X, A))^2, \quad \begin{cases} a(f + g) = af + ag, \\ (a + b)f = af + bf. \end{cases}$$

Notation 2 (Constant c_p). In the sequel, given a positive integer p, we denote by $c_p \in A$ a constant, the value of which will be defined when necessary.

Definition 2.3 (p-differential δ^p). Given a positive integer p, we define the p-differential δ^p from $F^p(X,A)$ to $F^{p+1}(X,A)$, for any f in $F^p(X,A)$, as follows:

$$\forall (x_0,\dots,x_{p+1}) \in X^{p+2},$$
$$\delta^p(f)(x_0,\dots,x_{p+1}) = c_p\left(\sum_{q=0}^{p}(-1)^q f\big(\dots,x_{q-1},x_{q+1},\dots\big)\right).$$

As for the 0-differential δ^0, from $F^0(X,A)$ to $F^1(X,A)$, it is defined, for any f in $F^0(X,A)$, as follows:

$$\forall (x_0,x_1) \in X^2, \quad \delta^0(f)(x_0,x_1) = c_0(f(x_1) - f(x_0)).$$

Remark 2.2. The kernel of the 0-differential δ^0 is the subset

$$F^0_{\text{constant}}(X,A) \subset F^0(X,A)$$

of constant 0-fermions on X. For the sake of simplicity, we will from now on identify this kernel with A:

$$\ker \delta^0 \equiv A.$$

Property 2.1. *For all $p \in \mathbb{N}$,*

$$\delta^{p+1} \circ \delta^p = 0.$$

Proof. (i) For $p = 0$, given f in $F^0(X,A)$, and $(x_0,x_1) \in X^2$, we have that

$$\delta^0(f)(x_0,x_1) = c_0(f(x_0) - f(x_1)),$$

which yields, for any $(x_0,x_1,x_2) \in X^3$,

$$\begin{aligned}
&\delta^1(\delta^0(f))(x_0,x_1,x_2)\\
&\quad = c_1\big\{\delta^0(f)(x_0,x_1) - \delta^0(f)(x_0,x_2) + \delta^0(f)(x_1,x_2)\big\}\\
&\quad = c_0c_1\big\{f(x_0) - f(x_1) - f(x_0) + f(x_2) + f(x_1) - f(x_2)\big\}\\
&\quad = 0.
\end{aligned}$$

(ii) For $p > 0$, given f in $F^p(X,A)$, and $(x_0,\dots,x_{p+2}) \in X^{p+3}$:

$$\begin{aligned}
&\delta^{p+1}(\delta^p(f))(x_0,\dots,x_{p+2})\\
&\quad = c_{p+1}\left\{\sum_{q=0}^{p+1}(-1)^q \delta^p(f)\big(\dots,x_{q-1},x_{q+1},\dots\big)\right\}\\
&\quad = c_pc_{p+1}\sum_{q=0}^{p+1}(-1)^q\left\{\sum_{q'=0}^{p}(-1)^{q'} f\big(\dots,x_{q-1},x_{q+1},\dots,x_{q'-1},x_{q'+1},\dots\big)\right\}.
\end{aligned}$$

To understand how things are going in this double sum: this amounts, in the $(p+3)$-uple $(x_0, \dots, x_{p+2})$, in suppressing two terms x_q, and $x_{q'}$. So, the following configurations occur:

- Either $q < q'$, in which case, one first takes out $x_{q'}$, which occupies the place number q'. One then takes out x_q, which still occupies its original place number q. The resulting term is thus:

$$(-1)^q(-1)^{q'} f\left(\dots, x_{q-1}, x_{q+1}, \dots, x_{q'-1}, x_{q'+1}, \dots\right)$$

which can be illustrated as:

  – Step 0

...	x_{q-1}	~~x_q~~	x_{q-1}	...	$x_{q'-1}$	~~$x_{q'}$~~	$x_{q'-1}$	...

↑ place number q ↑ place number q'

  – Step 1

place number $q'-1$ ↓

...	x_{q-1}	~~x_q~~	x_{q-1}	...	$x_{q'-1}$	$x_{q'+1}$	...

↑ place number q ↑ place number q'

- Either $q > q'$, in which case, one first takes out $x_{q'}$, which occupies the place number q'. One then takes out x_q, which this time occupies the place number $q-1$, due to the shift induced by suppressing $x_{q'}$.
The resulting term is thus exactly the opposite of the previous one:

$$(-1)^{q-1}(-1)^{q'} f(\dots, x_{q-1}, x_{q+1}, \dots, x_{q'-1}, x_{q'+1}, \dots)$$

which can be illustrated as:

  – Step 0

...	$x_{q'-1}$	x'_q	$x_{q'+1}$	...	x_{q-1}	~~x_q~~	x_{q-1}	...

↑ place number q' ↑ place number q

  – Step 1

place number $q-2$ ↓

...	$x_{q'-1}$	$x_{q'+1}$	...	x_{q-1}	x_q	x_{q+1}	...

↑ place number $q'-1$ ↑ place number $q-1$

All quantities of the above double sum are thus simplified two by two. ■

Remark 2.3. The above definition can be understood in the following sense: p-fermions act on a collection $(x_0, \dots, x_p)$ of points in X^{p+1}, which are the vertices of n-simplices. Those simplices are themselves n-faces of $(n+1)$-simplices, the vertices $(x_0, \dots, x_{p+1})$ of which are then in X^{p+2}. Thus, the p-differential stands out as a map acting on the co-boundary of the elements of X^{p+1}.

It could seem strange that the p-differential takes values in $F^{p+1}(X, A)$: in classical analysis, one loses information in the differentiation process. In our case, it is just a generalization, in order to enable one to handle all (oriented) paths between given extremities, bearing in mind that differentiation is deeply linked to the rate of increase. The definition makes all the more sense that one can introduce a metric, and consider very close points, as we will do further.

Definition 2.4 (p-cycle, closed p-fermion). A p-fermion f will be called a *p-cycle*, or a *closed p-fermion*, if

$$\delta^p f = 0.$$

Definition 2.5 (Exact p-fermion). A p-fermion f will be called *exact* if there exists a $(p-1)$-fermion g such that

$$f = \delta^p g.$$

Definition 2.6 (p-homology group). Given a positive integer p, the quotient group $\ker \delta^p / \operatorname{Im} \delta^{p+1}$ will be called a p-homology group of X over A. It thus corresponds to the equivalence classes of closed p-fermions, modulo exact p-fermions.

Definition 2.7 (p-cohomology group). Given a positive integer p, the quotient group $\ker \delta^p / \operatorname{Im} \delta^{p-1}$ will be called *p-cohomology group* of X over A.

Since $\operatorname{Im} \delta^{-1} = \{0\}$, the zero cohomology quotient group $\ker \delta^0 / \operatorname{Im} \delta^{-1}$ is simply $\ker \delta^0$.

Definition 2.8 (Complex of fermions). The complex $(F^\bullet(X, A), \delta^\bullet)$ is

$$F^0 \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{p-1}} F^p \xrightarrow{\delta^p} F^{p+1} \xrightarrow{\delta^{p+1}} \cdots$$

where, for any natural integer p,

$$\delta^{p+1} \circ \delta^p = 0.$$

Notation 3. We set

$$F^\bullet(X, A) = \bigoplus_{p=0}^{\infty} F^p(X, A).$$

The associated cohomology, i.e., the set constituted of $\ker \delta^0$ and of the p-cohomology groups $\ker \delta^{p+1} / \operatorname{Im} \delta^p$, $p \in \mathbb{N}$, will be denoted by

$$H^\bullet\big(F^\bullet(X, A), \delta^\bullet\big).$$

Property 2.2 (Acyclic complex of fermions). *The complex $F^\bullet(X, A)$ is acyclic: its cohomology is constant, i.e.,*

$$\forall p \in \mathbb{N}, \quad \ker \delta^{p+1} / \operatorname{Im} \delta^p = \{0\},$$

which amounts to

$$\forall p \in \mathbb{N}, \quad \ker \delta^{p+1} = \operatorname{Im} \delta^p.$$

We set

$$H^0(F^\bullet(X, A), \delta^\bullet) = \ker \delta^0,$$

i.e.,

$$H^0(F^\bullet(X, A), \delta^\bullet) = A.$$

This implies, for the associated general cohomology, that

$$H^\bullet(F^\bullet(X, A), \delta^\bullet) = H^0(F^\bullet(X, A), \delta^\bullet) = A.$$

Remark 2.4. Since, for any $p \in \mathbb{N}^\star$,

$$\delta^p \circ \delta^{p-1} = 0 \quad \text{and} \quad \operatorname{Im} \delta^{p-1} \subset \ker \delta^p,$$

this simply amounts to

$$\ker \delta^p = \operatorname{Im} \delta^{p-1}.$$

When $p \geqslant 1$, the p-cohomology group of X over A reduces to the trivial quotient group

$$\ker \delta^p / \operatorname{Im} \delta^{p-1} = \{0\}.$$

Thus, for $p \geqslant 1$, the p-cohomology groups $\ker \delta^p / \operatorname{Im} \delta^{p-1}$ do not play any part in the complex $F^\bullet(X, A)$. Hence:

$$H^\bullet(F^\bullet(X, A), \delta^\bullet) = H^0(F^\bullet(X, A), \delta^\bullet) = A.$$

Proof. (i) For $p = 0$, $\operatorname{Im} \delta^0$ is the set of 1-fermions f^1 such that there exists a 0-fermion f^0 such that:

$$\forall (x, y) \in X^2, \quad f^1(x, y) = c_1 \big\{ f^0(x) - f^0(y) \big\}.$$

Recalling now that the 1-differential δ, from $F^1(X, A)$ to $F^2(X, A)$, is defined, for any f^1 in $F^1(X, A)$, by

$$\forall (x, y, z) \in X^3, \quad \delta(f^1)(x, y, z) = c_2 \big\{ f^1(y, z) - f^1(x, z) + f^1(x, y) \big\},$$

its kernel is thus the set of 1-fermions f^1 such that

$$\forall (x, y, z) \in X^3, \quad f^1(x, y) = f^1(x, z) - f^1(y, z),$$

which can also be written as

$$\forall (x,y) \in X^2, \quad \forall z \in X, \quad f^1(x,y) = f^1(x,z) - f^1(y,z).$$

One can see that, given a pair (x,y) in X^2, $f^1(x,y)$ does not depend on the third variable. Given z in X, let us set

$$\tilde{f}(x) = f^1(x,z) \quad \text{and} \quad \tilde{f}(y) = f^1(y,z)$$

Then, $\tilde{f}$ is a 1-fermion, and

$$f^1(x,y) = \tilde{f}(x) - \tilde{f}(y).$$

Thus, we have that

$$\ker \delta \subset \operatorname{Im} \delta^0,$$

which yields

$$\ker \delta = \operatorname{Im} \delta^0.$$

(ii) For a given integer $p \geqslant 1$, let us prove that

$$\ker \delta^{p+1} = \operatorname{Im} \delta^p.$$

Hence, $\operatorname{Im} \delta^p$ is the set of $(p+1)$-fermions f^{p+1} such that there exists a p-fermion f^p such that

$$\forall (x_0, \dots, x_{p+1}) \in X^{p+2},$$
$$f^{p+1}(x_0, \dots, x_{p+1}) = c_{p+1} \left(\sum_{q=0}^{p} (-1)^q f^p\big(\dots, x_{q-1}, x_{q+1}, \dots\big) \right).$$

Recalling that the $(p+1)$-differential δ^{p+1}, from $F^{p+1}(X,A)$ to $F^{p+2}(X,A)$, is defined, for any f in $F^{p+3}(X,A)$, by

$$\forall (x_0, \dots, x_{p+2}) \in X^{p+3},$$
$$\delta^{p+1}(f)(x_0, \dots, x_{p+2}) = c_p \left(\sum_{q=0}^{p+1} (-1)^q f\big(\dots, x_{q-1}, x_{q+1}, \dots\big) \right),$$

its kernel is thus the set of $(p+2)$-fermions f^{p+2} such that

$$\forall (x_0, \dots, x_{p+2}) \in X^{p+3}, \quad \sum_{q=0}^{p+1} (-1)^q f^{p+2}\big(\dots, x_{q-1}, x_{q+1}, \dots\big) = 0,$$

which can also be written as

$$\forall (x_0, \ldots, x_{p+1}) \in X^{p+2},$$
$$(-1)^{p+2} f^{p+2}(x_0, \ldots, x_{p+1}) = -\sum_{q=0}^{p} (-1)^q f^{p+2}(\ldots, x_{q-1}, x_{q+1}, \ldots).$$

One can see that given a $(p+2)$-uple $(x_0, \ldots, x_{p+1})$ in X^{p+2}, the image $f^{p+2}(x_0, \ldots, x_{p+1})$ does not depend on the variable x_{p+2}. It can thus be written in the following form

$$\begin{aligned} f^{p+2}(x_0, \ldots, x_{p+1}) &= (-1)^{p+3} \sum_{q=0}^{p} (-1)^q f^{p+2}(\ldots, x_{q-1}, x_{q+1}, \ldots) \\ &= \tilde{c}_{p+1} \sum_{q=0}^{p} (-1)^q \tilde{f}^{p+1}(\ldots, x_{q-1}, x_{q+1}, \ldots), \end{aligned}$$

where $\tilde{f}^{p+1}$ denotes a $(p+1)$-fermion.

Thus, we thave that

$$\ker \delta^{p+1} \subset \operatorname{Im} \delta^p,$$

which yields

$$\ker \delta^{p+1} = \operatorname{Im} \delta^p. \qquad \blacksquare$$

3. De Rham cohomology

For the benefit of the reader who may not be familiar with mathematical notions devoted to the de Rham cohomology, we shall first recall several definitions.

3.1. A few reminders

Notation 4. In the sequel, X denotes a smooth manifold, of dimension $n \in \mathbb{N}^\star$. We will hereafter use the classical $\wedge$ notation for exterior derivatives.

Definition 3.1. Given a natural integer p, we will denote by $\Omega^p(X)$ the space of p-forms on X.

Notation 5 (Partial derivative). Given a strictly positive integer p, a smooth p-form f on X, and k in $\{0, \ldots, p\}$, the partial derivative $\partial_k f$ is defined, for any

$$(x_0, \ldots, x_p) = \big((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}\big) \in X^{p+1},$$

by

$$\partial_k f(x_0, \dots, x_p) = \sum_{i_k=1}^{n} \frac{\partial f}{\partial x_{k,i_k}} \big((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \dots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}\big)\, dx^{k,i_k}.$$

Definition 3.2 (De Rham differential). Given a p-form $\omega \in \Omega^p(X)$ such that, for any $x = (x_1, \dots, x_n) \in X$,

$$\omega(x) = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} f_{i_1,\dots,i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

and where, for any $(i_1, \dots, i_p) \in \{1, \dots, n\}^p$, the $f_{i_1,\dots,i_p}$ denote smooth functions on X, the de Rham differential $d\omega$ is defined by

$$d\omega(x) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} \frac{\partial f_{i_1,\dots,i_p}}{\partial x_k}(x)\, dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Definition 3.3 (Diagonal). Given a natural integer p, the *diagonal* of X^{p+1} is defined as the set

$$\Delta_X = \big\{\underline{x} = (x, \dots, x)\big\} \subset X^{p+1}.$$

Definition 3.4 (De Rham complex on X). The *de Rham complex* on X is the cochain complex of differential forms

$$0 \xrightarrow{d} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots$$

that we will denote by $\Omega^{\bullet,d}$.

Property 3.1. *We have*

$$d^2 = 0.$$

3.2. Natural correspondence between fermions and differential forms

Definition 3.5 (p-linear forms on the tangent space TX). Given a strictly positive integer p, a smooth p-fermion f on X, and $x \in X$, we define a p-linear form $r_p(f)(x)$ on T_xX as follows:

$$\begin{aligned} &\forall (u_1, \dots, u_p) \in (T_xX)^p, \\ &\qquad r_p(f)(x)(u_1, \dots, u_p) = \partial_1 \cdots \partial_p f(x, \dots, x)\big(u_1, \dots, u_p\big). \end{aligned}$$

In the case where $p = 0$, we simply set

$$r_0(f)(x) = f(x, \dots, x).$$

Proposition 3.2. *Given a strictly positive integer p, a smooth p-fermion f on X, and x in X, we have that*

$$r_p(f) \in \Omega^p(X) \qquad \text{and} \qquad (p+1)\partial_0 \cdots \partial_p f|_{\Delta_X} = d r_p(f),$$

i.e., for $(u_0, u_1, \dots, u_p) \in (T_x X)^{p+1}$,

$$(p+1)\partial_0 \cdots \partial_p f(x, \dots, x)(u_0, u_1, \dots, u_p) = d(r_p(f))(x)(u_0, u_1, \dots, u_p).$$

Proof. As introduced in Notation 5, for any

$$(x_0, \dots, x_p) = \big((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \dots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}\big) \in X^{p+1},$$

we have that

$$\begin{aligned}\partial_0 \cdots \partial_p f(x_0, \dots, x_p) \\ = \sum_{i_0=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{0,i_0} \cdots \partial x_{p,i_p}} \big((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \dots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}\big) \\ dx^{0,i_0} \wedge \cdots \wedge dx^{p,i_p},\end{aligned}$$

which yields, on the diagonal,

$$\begin{aligned}\partial_0 \cdots \partial_p f(x, \dots, x) \\ = \sum_{i_0=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \cdots \partial x_{i_p}} ((x_i)_{1 \leqslant i \leqslant n}, \dots, (x_i)_{1 \leqslant i \leqslant n}) dx^{i_0} \wedge \cdots \wedge dx^{i_p}.\end{aligned}$$

As previously, for any

$$(x_0, \dots, x_p) = \big((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \dots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}\big) \in X^{p+1},$$

we have that

$$\begin{aligned}\partial_1 \cdots \partial_p f(x_0, \dots, x_p) \\ = \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p} f}{\partial x_{1,i_1} \cdots \partial x_{p,i_p}} \big((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \dots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}\big) \\ dx^{1,i_1} \wedge \cdots \wedge dx^{p,i_p}.\end{aligned}$$

Thus,

$$\begin{aligned}d\big[\partial_1 \cdots \partial_p f(x_0, \dots, x_p)\big] \\ = \sum_{k=0}^{p} \sum_{i_k=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{k,i_k} \partial x_{1,i_1} \cdots \partial x_{p,i_p}} ((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \dots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n}) \\ dx^{k,i_k} \wedge dx^{1,i_1} \wedge \cdots \wedge dx^{p,i_p},\end{aligned}$$

which yields, on the diagonal,

$$
\begin{aligned}
d[\partial_1 \ldots \partial_p f(x, \ldots, x)] \\
= \sum_{k=0}^{p} \sum_{i_k=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_k} \partial x_{i_1} \cdots \partial x_{i_p}} ((x_{i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, (x_{i_p})_{1 \leqslant i_p \leqslant n}) \\
dx^{i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
\end{aligned}
$$

One may note that, given an integer k in $\{0, \ldots, p\}$, the exterior product

$$dx^{i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

vanishes for $i_k = i_1, \ldots, i_p$. Thus, the nonzero terms depend on the values of $i_1, \ldots, i_p$ and not on k, which enables us to write

$$
\begin{aligned}
d[\partial_1 \cdots \partial_p f(x, \ldots, x)] \\
= \sum_{k=0}^{p} \sum_{i_0=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \partial x_{i_1} \cdots \partial x_{i_p}} ((x_{i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, (x_{i_p})_{1 \leqslant i_p \leqslant n}) \\
dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\
= (p+1) \sum_{i_0=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \partial x_{i_1} \cdots \partial x_{i_p}} ((x_{i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, (x_{i_p})_{1 \leqslant i_p \leqslant n}) \\
dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\
= (p+1) \partial_0 \cdots \partial_p f(x, \ldots, x).
\end{aligned}
$$

■

Corollary 3.3 (Correspondence between fermions and differential forms). *By choosing $c_p = p + 1$, we thus obtain that*

$$dr_p = r_{p+1} \delta^p.$$

Proof. We have

$$
\begin{array}{ccc}
F^p & \xrightarrow{\delta^p} & F^{p+1} \\
\Big\downarrow{\scriptstyle r_p} & & \Big\downarrow{\scriptstyle r_{p+1}} \\
\Omega^p & \xrightarrow{d} & \Omega^{p+1}.
\end{array}
$$

Let us consider a p-fermion f, which operates on a $(x_0, \ldots, x_p)$ of X^{p+1}. Indifferently, one may handle the variables as $(x_0, \ldots, x_p)$, or as $(x_1, \ldots, x_{p+1})$, thus, writing $(\partial_0, \ldots, \partial_p)$ or $(\partial_1, \ldots, \partial_{p+1})$ is equivalent.

Then, we trivially have that

$$\partial_1 \ldots \partial_p \partial_{p+1} f = \partial_0 \partial_1 \cdots \partial_p f.$$

Due to our previous result, we also have that

$$(dr_p)(f)(x,\dots,x) = (p+1)\partial_0 \cdots \partial_p f(x,\dots,x).$$

At the same time, for any $(x_0,\dots,x_{p+1}) \in X^{p+2}$, we have that

$$\begin{aligned}
&(r_{p+1}\delta^p)(f)(x_0,\dots,x_{p+1}) \\
&\quad = \partial_1 \cdots \partial_p \partial_{p+1}\delta^p(f)(x_0,\dots,x_{p+1}) \\
&\quad = \partial_1 \cdots \partial_p \partial_{p+1} c_p \left\{ \sum_{q=0}^{p} (-1)^q f(\dots,x_{q-1},x_{q+1},\dots) \right\} \\
&\quad = c_p \sum_{i_1=1}^{n} \cdots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \cdots \partial x_{p+1,i_{p+1}}} \left[\sum_{q=0}^{p} (-1)^q f(\dots,x_{q-1},x_{q+1},\dots) \right] \\
&\qquad\qquad dx^{1,i_1} \wedge \cdots \wedge dx^{p+1,i_{p+1}} \\
&\quad = c_p \sum_{q=0}^{p} (-1)^q \sum_{i_1=1}^{n} \cdots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \cdots \partial x_{p+1,i_{p+1}}} \left[f(\dots,x_{q-1},x_{q+1},\dots) \right] \\
&\qquad\qquad dx^{1,i_1} \wedge \cdots \wedge dx^{p+1,i_{p+1}}.
\end{aligned}$$

One may note that, given an integer q in $\{1,\dots,p\}$, the derivative

$$\sum_{i_1=1}^{n} \cdots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \cdots \partial x_{p+1,i_{p+1}}} \left[f(\dots,x_{q-1},x_{q+1},\dots) \right]$$

takes the value zero, since there is no x_q!

Thus,

$$\begin{aligned}
&(r_{p+1}\delta^p)(f)(x_0,\dots,x_{p+1}) \\
&\quad = c_p(-1)^0 \sum_{i_1=1}^{n} \cdots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \cdots \partial x_{p+1,i_{p+1}}} \left[f(x_1,\dots,x_{p+1}) \right] \\
&\qquad\qquad dx^{1,i_1} \wedge \cdots \wedge dx^{p+1,i_{p+1}},
\end{aligned}$$

which yields, on the diagonal,

$$\begin{aligned}
&(r_{p+1}\delta^p)(f)(x,\dots,x) \\
&\quad = c_p \sum_{i_1=1}^{n} \cdots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_1} \cdots \partial x_{i_{p+1}}}(x,\dots,x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}},
\end{aligned}$$

which, by means of a change of indices, can also be written as

$$\begin{aligned}(r_{p+1}\delta^p)(f)(x,\dots,x) \\ = c_p \sum_{i_0=1}^{n} \cdots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \cdots \partial x_{i_p}}(x,\dots,x) dx^{i_0} \wedge \cdots \wedge dx^{i_p},\end{aligned}$$

i.e.,

$$(r_{p+1}\delta^p)(f)(x,\dots,x) = c_p \partial_0 \cdots \partial_p f. \qquad \blacksquare$$

Remark 3.1. This result is all the more important, since it enables us to make a connection between the Alexander–Kolmogorov cohomology, based upon differences, and the de Rham one, which is the usual one.

Notation 6 (Complex of smooth fermions on X). We will denote by

$$(F^\bullet(X), \delta^\bullet) = \bigoplus_{p=0}^{\infty} F^p(X)$$

the acyclic complex of smooth fermions on X, and by $\delta^\bullet$ the associated differential (which means that, in practice, one deals with a δ^p, for a given value of the integer p).

4. (h, p)-fermions

Notation 7. In the sequel, we will denote by (X, dist) a metric space.

Definition 4.1 ((h, p)-fermions on X). Given a strictly positive real number h, and a natural integer p, we will denote by $F_h^p(X, A)$ the *set of p-fermions* on X, with values in A, defined on

$$X_h^{p+1} = \left\{(x_0,\dots,x_p) \in X^{p+1}, \forall (i,j) \in \{0,\dots,p\}^2 \;:\; \mathrm{dist}(x_i, x_j) < h\right\}$$

and by

$$(F_h^\bullet(X), \delta^\bullet) = \bigoplus_{p=0}^{\infty} F_h^p(X),$$

the associated complex.

Definition 4.2 (h-cohomology). We will call $H_h^\bullet(X, \mathrm{dist}, A)$ the *h-cohomology* of (X, dist) at scale h, with values in A.

Definition 4.3 (Radius of injectivity). Given a Riemannian manifold (M, g), the *injectivity radius* on M is defined as

$$\rho(M, g) = \mathrm{inj}(M, g) = \inf_{x\in M} \mathop{\mathrm{inj}}_{x\in M}(M, g),$$

where, for any $x \in M$,

$$\operatorname*{inj}_{x\in M}(M,g) = \sup\big\{r \geqslant 0:\ \exp_x \text{ is a diffeomorphism on the ball } B(x,r) \subset T_xM\big\}.$$

5. h-Hodge theory

5.1. Geometric context

Notation 8. In the sequel, we denote by:

(i) X a Riemannian manifold, of dimension d_X, equipped with the natural Riemannian distance dist; we denote by μ the associated measure on Borel sets. We obviously have that

$$\forall x \in X, \quad \forall \varepsilon > 0, \quad \mu(B(x,\varepsilon)) > 0.$$

(ii) $h > 0$ a real parameter.

(iii) μ^{p+1} the product measure on X^{p+1}.

Definition 5.1 (Measure on X_h^{p+1}). We define a measure μ_h^{p+1} on X_h^{p+1} as follows:

$$\mu_h^{p+1} = C_p(\cdot,h)\mu^{p+1},$$

where the normalization factor $= C_p(\cdot,h)$ stands as a parameter.

Definition 5.2. We set

$$L^2F_h^p = L^2(X_h^{p+1},\mu^{p+1}).$$

Theorem 5.1. *Given a compact analytic Riemannian manifold (X,g), there exists a finite number of real numbers*

$$0 = h_0 < h_1 < \cdots < h_{\max} = \operatorname{diam} X$$

such that the fibration

$$H_h^\bullet(L^2,X,g) \mapsto h$$

is constant on each interval $]h_i,h_{i+1}[$.

Moreover,

(i) *For* $h > \operatorname{diam} X$,

$$H_h^\bullet(L^2,X,g) = \mathbb{C}.$$

(ii) *For* $h < h_1$,

$$H_h^\bullet(L^2,X,g) \simeq H^\bullet.$$

Property 5.2. *The p-differential δ^p is a bounded operator from $L^2 F_h^p$ to $L^2 F_h^{p+1}$, the norm of which obviously depends on h.*

Proof. This immediately comes from the fact that the space X is compact, while δ^p is a difference operator acting on continuous functions on X. ∎

Notation 9 (Normalized differential). From now on, given a strictly positive real number h, we will denote by δ_h the normalized differential

$$\delta_h = h^{-1}\delta.$$

Remark 5.1. As explained above, the differential δ is bounded independently of h. The interesting point is that in the normalized one δ_h, the h^{-1} coefficient allows us to let h to tend towards zero, which enables one to recover the usual de Rham differential and infinitesimal calculus.

Definition 5.3 (Reminder: Hodge star operator on a finite-dimensional oriented Euclidean space). Let E be a finite-dimensional oriented Euclidean space, endowed with a nondegenerate symmetric bilinear form $\wedge$. We set

$$\dim E = n \in \mathbb{N}^\star.$$

Given a natural integer $p \leqslant n$, $\bigwedge^p E$ and $\bigwedge^{n-p} E$ respectively denote the subspaces of p and $n-p$ vectors. One trivially has

$$\dim \bigwedge^p E = \dim \bigwedge^{n-p} E = \binom{n}{p}.$$

(The choice of a basis amounts to choosing p vectors among the n elements of any basis of E.)

The Hodge star operator $\star$ is simply the natural isomorphism between $\bigwedge^p E$ and $\bigwedge^{n-p} E$. For any orthonormal basis $\{e_1, \dots, e_n\}$, we have that

$$\star(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_n.$$

Property 5.3. *Given a natural integer $p \leqslant n$, and a p-vector $\eta \in \bigwedge^p E$:*

$$\star \star \eta = (-1)^{k(n-p)}\eta.$$

Remark 5.2. We thus have that

$$\begin{array}{ccc}
\Lambda^p & \xrightarrow{\quad\star\quad} & \Lambda^{n-p} \\
 & & \Big\downarrow d \\
\Lambda^{p-1} & \xleftarrow{\quad\star\quad} & \Lambda^{n-p+1}.
\end{array}$$

Now, in the case of our smooth manifold X, the involved space E is simply the tangent space $T_x X$ at some point $x \in X$, as is given in Definition 5.4 just below.

Definition 5.4 (Hodge star operator on the de Rham complex). The above definition 5.3 of the Hodge star operator naturally extends to the de Rham complex $\Omega^{\bullet,d}$ on the smooth manifold X, as the natural isomorphism between Ω^p and Ω^{n-p} through

$$\star(\partial_1 \cdots \partial_p) = \partial_{p+1} \cdots \partial_n .$$

Definition 5.5 ($d^\star$ operator on the de Rham complex). Given a strictly positive integer $p \leqslant n$, we define the codifferential $d^\star$ by

$$d^\star : \Omega^p \longrightarrow \Omega^{p-1}$$

via

$$d^\star = (-1)^{n(p-1)+1} \star d \star .$$

Thus, we have the following diagram:

$$\begin{array}{ccc} \Omega^p & \xrightarrow{\quad\star\quad} & \Omega^{n-p} \\ {\scriptstyle d^\star}\downarrow & & \downarrow{\scriptstyle d} \\ \Omega^{p-1} & \xleftarrow{\quad\star\quad} & \Omega^{n-p+1} . \end{array}$$

Definition 5.6 (Hodge Laplacian). The *Hodge Laplacian* on $\Omega^\bullet(X)$ is given by

$$\square = (d + d^\star)^2 = d d^\star + d^\star d .$$

Notation 10 (Space of harmonic forms). For any positive integer p, we will denote by $H_{|\Omega^p}$ the space of harmonic forms on Ω^p, i.e., the forms f such that

$$\square f = 0 .$$

Theorem 5.4 (Hodge decomposition). *Given a compact analytic Riemannian manifold X, then, for any strictly positive integer p, we have that*

$$\Omega^{p-1} \xrightarrow{\ d\ } \Omega^p \xrightarrow{\ d\ } \Omega^{p+1} ,$$

and

$$\Omega^{p+1} \xrightarrow{\ d^\star\ } \Omega^p \xrightarrow{\ d^\star\ } \Omega^{p-1} .$$

To facilitate understanding, the following diagram might be helpful:

$$\begin{array}{ccc} \Omega^{p-1} & \xrightarrow{\quad d\quad} & \Omega^p \\ {\scriptstyle d^\star}\uparrow & \swarrow{\scriptstyle dd^\star} & \downarrow{\scriptstyle d} \\ \Omega^p & \xleftarrow{\quad d^\star\quad} & \Omega^{p+1} . \end{array}$$

Also, we have the following orthogonal, direct sum decompositions,

$$\begin{cases} \ker d_{|\Omega^p} = \operatorname{Im} d_{|\Omega^{p-1}} \oplus H_{|\Omega^p}, \\ \ker d^\star_{|\Omega^p} = \operatorname{Im} d^\star_{|\Omega^{p+1}} \oplus H_{|\Omega^p}, \end{cases}$$

and

$$\begin{cases} \Omega^p(X) = \operatorname{Im} d_{|\Omega^{p-1}} \oplus H_{|\Omega^p} \oplus (\ker d_{|\Omega^p})^\perp, \\ \Omega^p(X) = \operatorname{Im} d^\star_{|\Omega^{p+1}} \oplus H_{|\Omega^p} \oplus (\ker d^\star_{|\Omega^p})^\perp, \end{cases}$$

which naturally yields

$$\square = \begin{pmatrix} dd^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^\star d \end{pmatrix}.$$

Moreover, $d_{|\Omega^p}$ *induces an isomorphism* j_p *from* $(\ker d_{|\Omega^p})^\perp$ *onto* $\operatorname{Im} d_{|\Omega^p}$*:*

$$(d_{|\Omega^p})\Big|_{(\ker d_{|\Omega^p})^\perp} = j_p.$$

At the same time, $d^\star_{|\Omega^p}$ *induces an isomorphism* $j^\star_p$ *from* $\operatorname{Im} d_{|\Omega^p} \subset \Omega^{p+1}$ *onto* $\operatorname{Im} d^\star_{|\Omega^{p+1}}$*:*

$$(d^\star_{|\Omega^p})\Big|_{\operatorname{Im} d_{|\Omega^p}} = j^\star_p$$

In the same way, $d^\star_{|\Omega^p}$ *induces an isomorphism* $j^\star_{p-1}$ *from* $\operatorname{Im} d_{|\Omega^{p-1}}$ *onto* $\operatorname{Im} d^\star_{|\Omega^p}$*:*

$$(d^\star_{|\Omega^p})\Big|_{\operatorname{Im} d_{|\Omega^p}} = j^\star_{p-1}$$

while $d_{|\Omega^p}$ *induces an isomorphism* j_{p-1} *from* $(\ker d_{|\Omega^{p-1}})^\perp = \operatorname{Im} d^\star_{|\Omega^p} \subset \Omega^{p-1}$ *onto* $\operatorname{Im} d_{|\Omega^{p-1}}$*:*

$$(d_{|\Omega^p})\Big|_{\operatorname{Im} d^\star_{|\Omega^p}} = j_{p-1}.$$

This yields the Hodge decomposition

$$\square = \begin{pmatrix} j_{p-1} j^\star_{p-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & j^\star_p j_p \end{pmatrix}.$$

5.2. Main result: Limit of the resolvent of the h-Laplacian

Proposition 5.5 (Modified scalar product on $\Omega^p(X)$, $p \in \mathbb{N}$). *In the sequel, given a natural integer* p*, we modify the usual scalar product* $(\cdot,\cdot)$ *on* $\Omega^p(X)$ *(see Notation 8 for* X*) by means of a multiplicative strictly positive constant* α_p*, setting, for any pair* (u, v) *of smooth* p*-fermions on* X*:*

$$\widetilde{(u, v)}_p = \alpha_p (u, v)_p.$$

Since

$$d^\star : \Omega^{p+1} \longrightarrow \Omega^p,$$

we can naturally introduce the operator

$$\widetilde{d}^\star = \frac{\alpha_{p+1}}{\alpha_p} d^\star.$$

(The multiplicative constant α_{p+1} *comes from the modified scalar product on the set* $\Omega^{p+1}(X)$*, while the division by* α_p *stands as a normalization constant.)*

We then set

$$\Delta_0 = (d + \widetilde{d}^\star)^2 = \bigoplus_{p\in\mathbb{N}} \begin{pmatrix} \frac{\alpha_p}{\alpha_{p-1}} j_p j_p^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_{p+1}}{\alpha_p} j_p^\star j_p \end{pmatrix}.$$

This compulsory step is required in order to obtain a full rescaled Hodge decomposition, when one passes to the limit of the resolvent when the scale parameter h tends to zero; see Theorem 5.8 below.

Definition 5.7 (h-Laplacian). Let us recall that, in the above, given a strictly positive real number h, we have introduced the normalized differential

$$\delta_h = h^{-1}\delta.$$

The natural correspondence of Corollary 3.3, by means of p and $(p+1)$-linear forms,

$$\forall p \in \mathbb{N}, \quad d r_p = r_{p+1}\delta,$$

naturally induces the existence of the operator $\delta^\star$,

$$\forall p \in \mathbb{N}, \quad d^\star r_p = r_{p+1}\delta^\star,$$

and its normalized version

$$\delta_h^\star = h^{-1}\delta^\star.$$

We now define the h-Laplacian by

$$\Delta_h = (\delta_h + \delta_h^\star)^2.$$

Notation 11 (Spectrum of a Laplacian operator). Given a Laplacian operator among the ones previously encountered, $\square$, Δ_0, Δ_h, we denote by $\mathrm{Spec}(\cdot)$ its *spectrum*:

$$\mathrm{Spec}(\square), \quad \mathrm{Spec}(\Delta_0) \quad \text{or} \quad \mathrm{Spec}(\Delta_h).$$

Notation 12 (Canonical projections). In the sequel, we will denote by:

(i) $\Pi_{F_h^\bullet,\Omega^\bullet}$ the canonical projection from $L^2F_h^\bullet$ onto $L^2(X,\Omega^\bullet)$.

(ii) $\Pi_{\Omega^\bullet,F_h^\bullet}$ the canonical injection from $L^2(X,\Omega^\bullet)$ on $L^2F_h^\bullet$ such that, on smooth functions,

$$\Pi_{F_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,F_h^\bullet} = \mathrm{Id}_{L^2(X,\Omega^\bullet)},$$

and such that $\Pi_{\Omega^\bullet,F_h^\bullet} \circ \Pi_{F_h^\bullet,\Omega^\bullet}$ is an orthogonal projection, for the Hilbert structure of $L^2F_h^\bullet$ – which simply comes from the fact that

$$\Pi_{\Omega^\bullet,F_h^\bullet} \circ \underbrace{\Pi_{F_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,F_h^\bullet}}_{\mathrm{Id}_{L^2(X,\Omega^\bullet)}} \circ \Pi_{F_h^\bullet,\Omega^\bullet} = \Pi_{\Omega^\bullet,F_h^\bullet} \circ \Pi_{F_h^\bullet,\Omega^\bullet}.$$

Proposition 5.6. *Let us denote by $r_{p,h}$ the restriction to F_h of the p-linear form r_p introduced in Definition 3.5. Since $F_h^p \subset F^p$, we can then use the diagram given in the proof of Corollary 3.3, which yields:*

$$F_h^p \xrightarrow{\ r_{p,h}\ } \Omega^p$$

This provides a further understanding of the aforementioned canonical operators:

(i) *The first one simply arises as*

$$\Pi_{F_h^\bullet,\Omega^\bullet} = r_{p,h}.$$

(ii) *As for the second one, it is uniquely determined by the following condition:*

$$\Pi_{F_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,F_h^\bullet} = \mathrm{Id}_{L^2(X,\Omega^\bullet)},$$

along with the fact that $\Pi_{\Omega^\bullet,F_h^\bullet} \circ \Pi_{F_h^\bullet,\Omega^\bullet}$ is self-adjoint.

It also happens that the restriction $r_{p,h}$ to $F_h^p \subset F^p$ of course does not depend on h.

Property 5.7. *Given a strictly positive real number h, $|\Delta_h|$ is bounded, self-adjoint, and non-negative on $L^2F_h^\bullet$.*

Proof. This directly comes from the definition of Δ_h:

$$\Delta_h = (\delta_h + \delta_h^\star)^2,$$

where the differential δ_h is bounded, as shown in Property 5.2. ■

Theorem 5.8 (Limit of the h-Laplacian). *Given a compact subset $K \subset \mathbb{C} \setminus \mathrm{Spec}(\Delta_0)$, there exists a strictly positive constant h_K such that, for any h in $]0,h_K[$, the resolvent $(z-|\Delta_h|)^{-1}$ exists, and*

$$\lim_{h\to 0} (z-|\Delta_h|)^{-1} = \lim_{h\to 0} \Pi_{\Omega^\bullet,F_h^\bullet} (z-|\Delta_0|)^{-1} \Pi_{F_h^\bullet,\Omega^\bullet}.$$

Remark 5.3. This very strong result enables us to consider the extension to *singular spaces*, i.e., when the functions involved are (locally) *not smooth*, due to the presence of *singularities*, as will be done in forthcoming Section 7.

Proof of Theorem 5.8. In the case of smooth functions, a direct computation by means of a Taylor expansion on the diagonal of the involved matrices, shows that the result is true.

Now, and what interests us, in the more general case where there exist *singularities* (i.e., in the case of a non-smooth function), given a strictly positive real number h, we obviously have that $(z - |\Delta_h|)^{-1}$ is defined for $z \in \mathbb{C} \setminus \mathrm{Spec}(\Delta_h)$.

We thus have to determine the spectrum of the h-Laplacian Δ_h, which can be done by using the definition, i.e.,

$$\Delta_h = (\delta_h + \delta_h^\star)^2.$$

We can naturally write that

$$\Pi_{F_h^\bullet,\Omega^\bullet}(\delta_h + \delta_h^\star)^2 = \bigoplus_{p\in\mathbb{N}} \begin{pmatrix} \dfrac{\alpha_p}{\alpha_{p-1}} j_p j_p^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{\alpha_{p+1}}{\alpha_p} j_p^\star j_p \end{pmatrix},$$

the resolvent of which exists. One then goes back to the resolvent of the h-Laplacian by applying the projection $\Pi_{F_h^\bullet,\Omega^\bullet}$.

A delicate point is to ensure the existence of the limit

$$\lim_{h\to 0} \Pi_{\Omega^\bullet,F_h^\bullet}(z - |\Delta_0|)^{-1}\Pi_{F_h^\bullet,\Omega^\bullet}.$$

This directly comes from Proposition 5.6, since the canonical projections involved do not depend on h. Thus, the operator

$$\Pi_{\Omega^\bullet,F_h^\bullet}(z - |\Delta_0|)^{-1}\Pi_{F_h^\bullet,\Omega^\bullet} \tag{5.1}$$

does not depend on h. In fact, since, on smooth functions,

$$\Pi_{F_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,F_h^\bullet} = \mathrm{Id}_{L^2(X,\Omega^\bullet)},$$

we can observe that the operator defined in (5.1) above is (uniformly in h) continuous on $L^2(F_h^\bullet)$. Hence, we obtain the sought for result on $L^2(F_h^\bullet)$. ∎

Property 5.9 (h-normalization constant). *Given a natural integer p, the normalization constant C_p introduced in Definition 5.1 enables us to connect the measure μ_h^{p+1} on X_h^{p+1} to the one on X^{p+1}. It is simply given by*

$$C_p = h^{-p d_X}.$$

Proof. This just comes from the definition of X_h^{p+1}. In fact, since

$$X_h^{p+1} = \left\{(x_0, \ldots, x_p) \in X^{p+1}, \forall (i,j) \in \{0, \ldots, p\}^2 \ : \ \mathrm{dist}(x_i, x_j) < h\right\},$$

we can switch from X_h^{p+1} to X^{p+1} by means of a change of variables, of the following form

$$(x_0, \ldots, x_p) \in X_h^{p+1} \mapsto (h\tilde{x}_0, \ldots, h\tilde{x}_p), \quad \text{with } (\tilde{x}_0, \ldots, \tilde{x}_p) \in X^{p+1}.$$

The associated Jacobian, which connects μ^{p+1} to μ_h^{p+1}, is thus obviously equal to h^{-pd_X}. ∎

6. h-Laplacian and random walks

Notation 13. In the sequel, we denote by (X, d, μ) a metric measure space such that

$$\forall x \in X, \quad \forall h > 0, \quad \mu(B(x,h)) > 0.$$

Notation 14 (Continuous functions on X). We denote by $C^\infty(X, \mathbb{C})$ the *set of smooth (or infinitely differentiable) functions* on X, which take values in $\mathbb{C}$.

The *subset of continuous functions* on X, which take values in $\mathbb{R}^+ \times \mathbb{R}^+$, will be denoted by $C^0(X, \mathbb{R}^+ \times \mathbb{R}^+)$.

Definition 6.1 (h-Laplacian). Given a strictly positive real number h, we define the h-Laplacian as the operator

$$|\Delta_h| = \delta_h^\star \delta_h$$

which acts on continuous functions f on X as follows:

$$\forall x \in X, \quad |\Delta_h|(f)(x) = \frac{2c_0^2}{h^2} \int_{B(x,h)} \{f(y) - f(x)\} C_0(x, y, h)\, d\mu_X(y),$$

where C_0 denotes a function defined on $X^2 \times]0, +\infty[$, and where c_0^2 denotes a strictly positive constant.

Remark 6.1. (i) The $\frac{1}{h^2}$ term comes from the definition of $\delta_h = \frac{1}{h}\delta$.

(ii) In Proposition 2.2, we showed that the complex $(F^\bullet(X, \mathbb{C}), \delta^\bullet)$ is acyclic, and

$$H^0(F^\bullet(X, \mathbb{C}), \delta^\bullet) = \mathbb{C}.$$

Thus, in the Laplacian decomposition given in Proposition 5.5, the sole term that plays a part now is the one that corresponds to $p = 0$, from which one gets the term involving C_0.

Remark 6.2. Following a comment in our introduction, we may note that the h-Laplacian depends on the choice of the measure μ_X. Somewhere this is not surprising – at the very beginning, which means, the cohomology, there were sums. Now, as we will see it in the sequel, the role played by the normalization factor C_0 will, in a certain sense, counterbalance this choice. So, finally, we perfectly fall back on our feet, with an operator that is just defined up to multiplicative constants.

Definition 6.2 (h-Markov operator). Given a strictly positive real number h, we introduce the operator M_h, given by

$$\frac{\mathrm{Id} - M_h}{h^2} = \frac{1}{2c_0^2} |\Delta_h|.$$

Property 6.1. *Given a strictly positive real number h, we trivially have, for the constant function on X which takes the value* 1*, that*

$$M_h(1) = 1.$$

Given $x \in X$, let us denote by $M_h(x, \cdot)\, d\mu_X(\cdot)$ the measure such that, for any continuous function f on X:

$$M_h(f)(x) = \int_X f(y) M_h(x, y)\, d\mu_X(y).$$

For any $(y, z) \in X^2$, we have that

$$\begin{aligned} &M_h(x, y)\, d\mu_X(y) \\ &\quad = \big\{\mathbb{1}_X - \mathbb{1}_{B(x,h)} C_0(x, y, h)\, d\mu_X(y)\big\}\delta_x + \mathbb{1}_{B(x,h)} C_0(x, y, h)\, d\mu_X(y). \end{aligned}$$

Since the M_h operator is Markov if and only if, for any $x \in X$,

$$0 \leqslant \int_X M_h(x, y)\, d\mu(y) = 1,$$

a necessary condition is thus that, for any $x \in X$,

$$\int_X \mathbb{1}_{B(x,h)} C_0(x, z, h)\, d\mu(z) = \int_{B(x,h)} C_0(x, z, h)\, d\mu(z) \leqslant 1.$$

Property 6.2 (Metropolis–Hastings algorithm [15,37]). *We recall that the Metropolis–Hastings algorithm is a Markov chain Monte Carlo method (MCMC), which enables one to generate a collection of sample states from a probability distribution $P(x)$, by means of a Markov process, which enables one to asymptotically reach a unique stationary distribution $\mathbb{P}_M(x) = \mathbb{P}(x)$.*

The transition probabilities, from a given state x, to another y one, which are involved in the Markov process, have to satisfy the following necessary conditions:

(i) *Existence of a stationary distribution $P_M(x)$, which requires the so-called detailed balance condition, in terms of conditional probabilities:*

$$\mathbb{P}[y|x]\mathbb{P}[x] = \mathbb{P}[x|y]\mathbb{P}[y]$$

which means that the process involved is a reversible one.

(ii) *Uniqueness of stationary distribution, which directly comes from the ergodicity (aperiodicity and positive recurrence in time) of the Markov process. One easily sees that the aperiodicity guarantees that the system does not return to the same state at fixed intervals, while the positive recurrence ensures that the expected number of steps for returning to the same state is finite.*

Remark 6.3. We can note that since

$$\frac{\mathbb{P}[y|x]}{\mathbb{P}[y]} = \frac{\mathbb{P}[x|y]}{\mathbb{P}[x]},$$

the transition is thus separated to, first, the proposal of a transition state, second, its acceptance/or rejection. The proposal distribution $\mathbb{P}\text{rop}[y|x]$ is thus the conditional probability of proposing a state y, given the original one x. It is naturally connected to the probability of acceptance of the new state y with regard to x by

$$\mathbb{P}[y|x] = \mathbb{P}\text{rop}[y|x]\mathbb{A}[y|x].$$

At the same time, since one deals with a reversible process, we can write that

$$\mathbb{P}[x|y] = \mathbb{P}\text{rop}[x|y]\mathbb{A}[x|y].$$

Those two relations yield that

$$\frac{\mathbb{A}[y|x]}{\mathbb{A}[x|y]} = \frac{\mathbb{P}[y|x]}{\mathbb{P}[x|y]}\frac{\mathbb{P}\text{rop}[x|y]}{\mathbb{P}\text{rop}[y|x]}.$$

One of the probabilities of acceptance has to take the value 1 (either one stays in x, either one moves to y). States x and y playing symmetric parts, we can concentrate on the probability of acceptance of y:

$$\mathbb{A}[y|x] = \min\left\{1, \frac{\mathbb{P}[y|x]}{\mathbb{P}[x|y]}\frac{\mathbb{P}\text{rop}[x|y]}{\mathbb{P}\text{rop}[y|x]}\right\}.$$

One clearly sees a very useful advantage of such a method: bypassing the determination of normalization constants.

The algorithm itself is implemented according to the following steps:

(i) At time $t = 0$, one chooses an initial state x_0.

(ii) At time $t > 0$, one generates a random candidate y, and compute the acceptance probability:

$$\min\{1, \frac{\mathbb{P}[y|x]}{\mathbb{P}[x|y]} \frac{\mathbb{P}\mathrm{rop}[x|y]}{\mathbb{P}\mathrm{rop}[y|x]}\}$$

and accept, or reject.

Remark 6.4. Why Markov chain Monte Carlo methods (MCMC)? As recalled in the generalization paper by W. K. Hastings [15], such methods appear as more efficient than conventional ones once one deals with problems in "a large number of dimensions". Such a choice thus seems interesting for an upcoming potential application to fractal based structures, especially, when they are approximated by means of prefractal graphs, where iterations quickly yield very large number of points.

Property 6.3 (h-Metropolis operator). *Given a strictly positive number h, a natural choice for the normalization factor C_0 involved in Definition 6.1 of the h-Laplacian is such that, for any $(x, y) \in X^2$,*

$$C_0(x, y, h) = \min\Big\{\frac{1}{\mu(B(x,h))}, \frac{1}{\mu(B(y,h))}\Big\}.$$

One thus recovers the Metropolis operator associated to the Markov kernel $(\mathrm{Id} - M_h)$.

The associated random walk is the following: if the walk is at x, one chooses y in $B(x,h)$ for the probability

$$\mathbb{1}_{B(x,h)} \frac{1}{\mu(B(x,h))} \, d\mu_X(y).$$

Then, depending whether $\mu(B(y,h)) \geqslant \mu(B(x,h))$ or not, one moves to y, or stay in x.

7. Singular spaces: Connections with previous works

In this section, we explore the connections between the extension of the h-Laplacian to singular spaces (by means of Theorem 5.8), and previous results of analysis on fractals, as introduced by Jun Kigami, and taken up by Robert S. Strichartz.

We hereafter place ourselves in the Euclidean plane of dimension 2, equipped with a direct orthonormal frame.

7.1. Framework of the study: Prefractal graph approximation

Notation 15. In the sequel, we will denote by S a singular set, of fractal type. Examples of such sets are the classical Sierpiński gasket, the Koch curve, the Weierstrass curve.

By following the method developed by J. Kigami [20], we approximate S by a sequence $(S_m)_{m\in\mathbb{N}}$ of finite graphs, the so-called *prefractals*. In classical cases, those graphs can be built through an iterative process, by means of an iterated function system (i.f.s) $T = \{T_0, \dots, T_{N-1}\}$ of N maps, $N \in \mathbb{N}$, such that

$$S = \bigcup_{i=0}^{N-1} T_i(S).$$

When the maps of the i.f.s. are contractive, this latter property is the so-called *collage theorem* [17]. When the maps are not contractive, one can, under specific conditions, have an equivalent result (see [9]).

The process is more or less complicated, depending on wether the maps of the i.f.s. are affine (Sierpiński gasket and Koch curve), or not (Weierstrass curve).

Example 7.1. (i) In the case of Sierpiński gasket, the iterated function system is constituted of three affine contractive maps (similarities), all with the same contraction ratio $\frac{1}{2}$, and fixed points P_0, P_1, P_2 located at the vertices of the initial equilateral triangle (see [42], and Figure 1 in the sequel):

$$\forall j \in \{0,1,2\}, \quad \forall x \in \mathbb{R}^2, \quad T_j(x) = \frac{1}{2}(x - P_j) + P_j.$$

(ii) In the case of a non-affine fractal curve, such as the Weierstrass one, the iterated function system is constituted of $N_b \geqslant 3$ nonlinear maps, which, if they cannot be said to be contractive in the classical sense, bear an equivalent property (see [9]). The fixed points are located at the vertices of the initial graph.

Definition 7.1 (Prefractal graph approximation). Let us consider a sequence of finite discrete graphs $(S_m)_{m\in\mathbb{N}}$. For any natural integer m, we denote by V_m the set of vertices of S_m. The initial set of points V_0 stands as the boundary of any ∂S_m, $m \in \mathbb{N}$.

We suppose that:

(i) The sequence $(V_m)_{m\in\mathbb{N}}$ is increasing, i.e.,

$$\forall m \in \mathbb{N}, \quad V_m \subset V_{m+1}.$$

(ii) For any natural integer m, the graph S_m is equipped with an edge relation $\underset{m}{\sim}$: two vertices x and y of S_m, i.e., two points belonging to V_m, will be said adjacent (or neighboring points) (see Figure 2) if and only if the line segment $[x, y]$ is an edge of S_m. Note that this edge relation depends on m, which means that points connected in V_m might not stay connected in V_{m+1}.

(iii) The Euclidean distance between adjacent points tends towards zero when m goes to infinity, and the union $\bigcup_{m\in\mathbb{N}} V_m$ is dense in S.

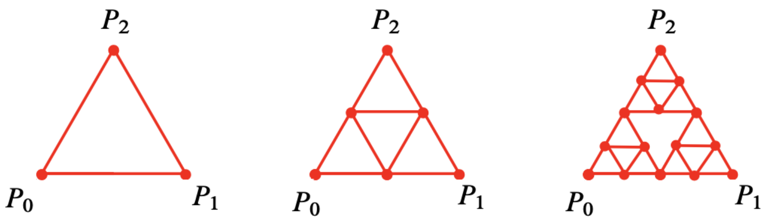

Figure 1. In the case of the Sierpiński gasket, the graphs S_0, S_1, S_2, with $\partial S_0 = V_0 = \{P_0, P_1, P_2\}$.

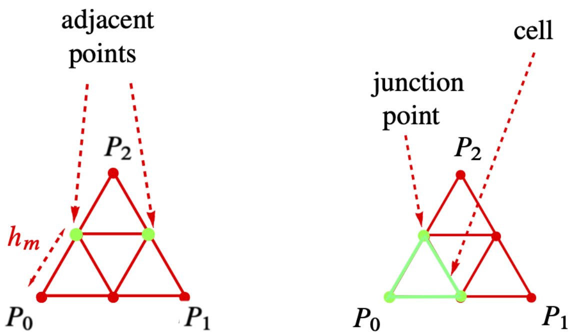

Figure 2. In the case of the Sierpiński gasket, adjacent points, junction points, cells.

The sequence $(S_m)_{m\in\mathbb{N}}$ will then be called a *prefractal graph approximation* to S (see Figure 1 for an example, in the case of the Sierpiński gasket).

Notation 16 (Adjacent consecutive vertices of the m-th level prefractal approximation, $m \in \mathbb{N}$). For the sake of clarity, given a natural integer m, two adjacent, consecutive vertices of the m-th level prefractal approximation S_m will be denoted in the following form

$$x_{m,k} \quad \text{and} \quad x_{m,k+1}, \quad 0 \leqslant k \leqslant N-1,$$

where N is the number of maps of the iterated function system.

The qualifier "consecutive" is to be understood in the sense that such points are obtained by means of consecutive maps of the iterated function system. We refer to [8, 42] for further details and examples.

Definition 7.2 (m-radius (or m-height)). Given a natural integer m, we will call m-radius (or m-height) of S_m the maximal Euclidean distance between two connected vertices of S_m, which we will denote by

$$h_m = \max_{(x,y)\in V_m^2,\, x \underset{m}{\sim} y} d_{\text{Eucl}}(x, y).$$

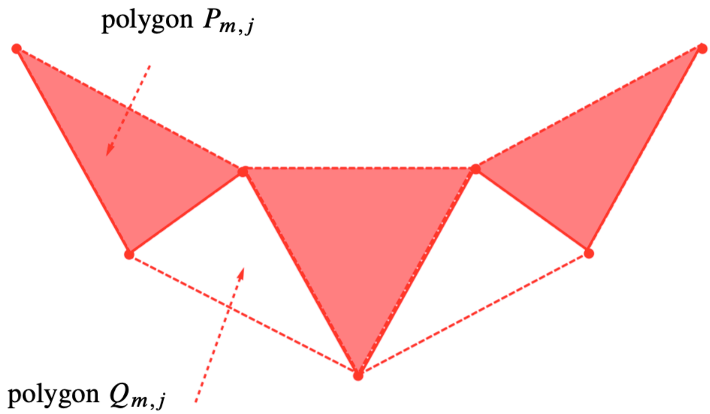

Figure 3. $P_{m,j}$ and $Q_{m,j}$ polygons/cells, in the case of the Weierstrass curve.

Property 7.1 (Polygonal domain [11]). *For any natural integer m, the* $\#V_m$ *consecutive vertices of the graph* S_m *are, also, the vertices of* N^m *simple polygons* $P_{m,j}$*, for* $0 \leqslant j \leqslant N^m - 1$*, with* N *sides (see Figure 3). For any integer* j *such that* $0 \leqslant j \leqslant N^m - 1$*, one obtains each polygon* $P_{m,j}$ *by connecting the point number* j *to the point number* $j+1$ *if* $j = i \pmod N$*, for* $0 \leqslant i \leqslant N-2$*, and the point number* j *to the point number* $j - N + 1$ *if* $j = -1 \pmod N$*.*

To go further, and as required in the specific case of a fractal curve (in order to have a complete polygonal neighborhood of the curve), the $\#V_m - 1$ *consecutive vertices of the graph* S_m*, distinct of* P_0 *and* P_{N-1}*, are the vertices of* $N^m - 1$ *simple polygons* $Q_{m,j}$*,* $1 \leqslant j \leqslant N^m - 2$*, with maximum* N *sides. For any integer* j *such that* $1 \leqslant j \leqslant N^m - 2$*, one obtains each polygon* $Q_{m,j}$ *by linking the point number* j *to the point number* $j+1$ *if* $j = i \pmod N$*, for* $1 \leqslant i \leqslant N-1$*, and the point number* j *to the point number* $j - N + 1$ *if* $j = 0 \pmod N$*.*

Of course, those latter polygons are not to be taken into account when the considered singular set is not a fractal curve. If such is the case, we have that

$$\left\{Q_m^j,\ 1 \leqslant j \leqslant N^m - 2\right\} = \emptyset.$$

Example 7.2. (i) In the case of the Sierpiński gasket, the polygonal domain is constituted of equilateral triangles, as can be seen in Figure 1.

(ii) In the case of the Weierstrass curve, the polygonal domain is constituted of N-gons, as it can be seen in Figure 3.

Definition 7.3 (m-cell). Given a natural integer m, we call m-cell any simple polygon $P_{m,j}$, $0 \leqslant j \leqslant N^m - 1$, or, when necessary, $Q_{m,j}$, $1 \leqslant j \leqslant N^m - 2$.

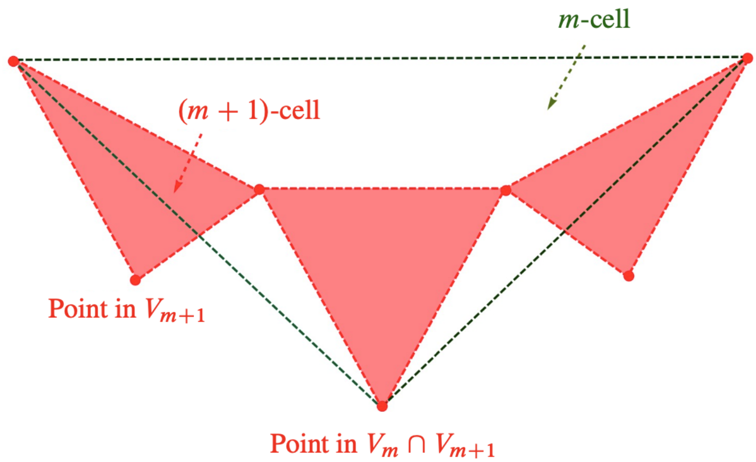

Figure 4. $(m+1)$-cells and m-cells, in the case of the Weierstrass curve, for $N = 3$.

Notation 17. For the sake of simplicity, given a natural integer m, the set of cells of S_m will be denoted by C_m.

Remark 7.1. (i) Except for the intersection points (i.e., junction points), m-cells are disjoint.

(ii) In spite of the fact that the sequence $(V_m)_{m\in\mathbb{N}}$ is increasing, S_m is not necessarily contained in S_{m+1}. For instance, one clearly see it is the case for the Sierpiński gasket, since an $(m+1)$-cell is obtained by dividing a m-one into three. In a different configuration, let us say, the Weierstrass curve (see Figure 4 – we refer to [8] for further details), this is not the case.

Definition 7.4 (Power of a vertex of the prefractal graph $\mathcal{S}_m$, $m \in \mathbb{N}^\star$ with regard to the polygonal family $\{C_m^j, 0 \leqslant j \leqslant \#C_m^j - 1\}$). Given a strictly positive integer m, a vertex x of the prefractal graph $\mathcal{S}_m$ will be said:

(i) of power one with regard to the polygonal family $\{C_m^j, 0 \leqslant j \leqslant \#C_m^j\}$ if x belongs to one and only one m-cell $C_{m,j}$, $0 \leqslant j \leqslant \#C_m^j - 1$;

(ii) of power $\frac{1}{k}$, $k \in \mathbb{N}^\star$, with regard to the polygonal family $\{C_m^j, 0 \leqslant j \leqslant \#C_m^j - 1\}$ if x is a common vertex to k cells $C_{m,j}$, $0 \leqslant j \leqslant \#C_m^j - 1$.

Remark 7.2. (i) The above power is required when defining a measure (see [10], in the case of the Weierstrass curve, or [42], in the case of the Sierpiński gasket). It acts as a kind of weight.

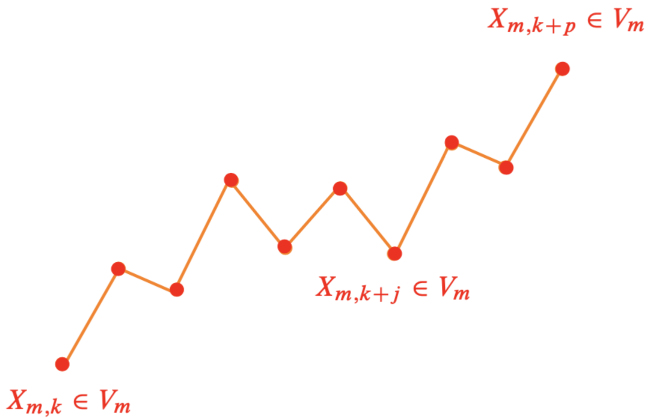

Figure 5. An m-path.

(ii) In the case of the Sierpiński gasket, except for boundary points (the fixed points of the affine maps of the associated i.f.s., P_0, P_1, P_2,), each vertex point at a given level $m \in \mathbb{N}^\star$ belongs to exactly two m-cells, and thus has power $\frac{1}{2}$. As explained in [42], one can get rid of the part played by the boundary points when computing a measure, since the sum involved goes to zero when the integer m tends towards infinity.

(iii) In the case of the Weierstrass curve, except again for boundary points, each vertex point at a given level $m \in \mathbb{N}^\star$ belongs to at most two m-cells, in which case it also has power $\frac{1}{2}$.

(iv) The associated power coefficient $\frac{1}{2}$ thus plays the part of a multiplicative constant. For the sake of simplicity, we will consider it as contained in the one involved in the definition of our Laplacians (r^{-m} in Theorem 7.3).

Definition 7.5 (m-path). Given two vertices in $\bigcup_{m\in\mathbb{N}} V_m$, i.e., two vertices $x_{m,k}$ and $x_{m,k+p}$, for $m \in \mathbb{N}$, $0 \leqslant k \leqslant \#V_m$ and $0 \leqslant p \leqslant \#V_m - k$, we call *$m$-path* between $x_{m,k}$ and $x_{m,k+p}$ the ordered set of vertices given by

$$P_m(x_{m,k}, x_{m,k+p}) = \left\{x_{m,k+j},\ 0 \leqslant k \leqslant p\right\}.$$

An example is given in Figure 7.5.

Definition 7.6 ((m,n)-path). (i) Given a natural integer m, and two adjacent vertices $x_{m,k}$ and $x_{m,k+1} \underset{m}{\sim} x_{m,k}$ of V_m, for $0 \leqslant k \leqslant \#V_m - 1$, we call (m,m)-path between $x_{m,k}$ and $x_{m,k+1}$ the ordered set of vertices

$$P_{m,n}(x_{m,k}, x_{m,k+1}) = \left\{x_{m+n,k+j},\ 0 \leqslant j \leqslant N^{n-m}\right\},$$

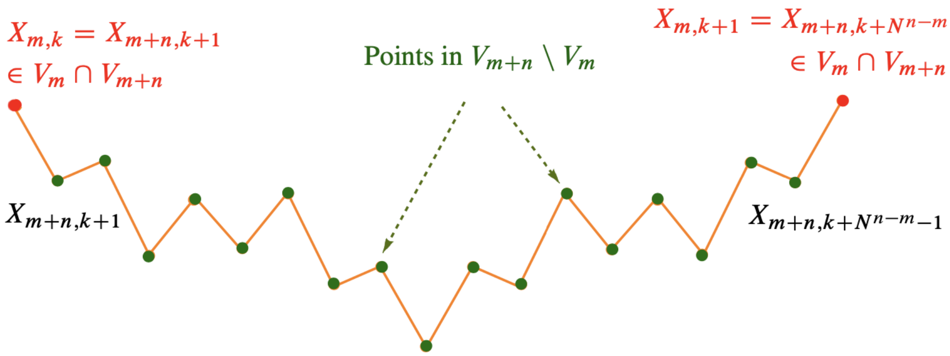

Figure 6. An (m, n)-path.

where

$$x_{m+n,k} = x_{m,k} \quad \text{and} \quad x_{m+n,k+n} = x_{m,k+1}.$$

(We recall that N denotes the number of maps of the iterated function system introduced at the beginning of Section 7.1. N^{n-m} simply means that N^{n-m} new points have been introduced between $x_{m,k}$ and $x_{m,k+1}$.)

An example is given in Figure 6.

(ii) Given a natural integer m, and two vertices $x_{m,k}$ and $x_{m,k+p}$ of V_m, for $0 \leqslant p \leqslant \# V_m$ and $0 \leqslant k \leqslant \# V_m - p$, we call (m, m)-*path* between $x_{m,k}$ and $x_{m,k+p}$ the ordered set of vertices given by

$$\mathcal{P}_{m,n}(x_{m,k}, x_{m,k+p}) = \bigcup_{j=0}^{p-1} \mathcal{P}_{m,n}(x_{m,k+j}, x_{m,k+j+1}).$$

Remark 7.3. Given two vertices x and y in $\bigcup_{m\in\mathbb{N}} V_m$, i.e., two vertices x and y in V_m, for a given value of the integer m, there exists an infinity of (m, n)-paths between x and y. It is clear that the minimal one – the simplest one, is the m-one.

Definition 7.7 (m-edge distance). Given a natural integer m, and two vertices $x_{m,k}$ and $x_{m,k+p}$ in $\bigcup_{m\in\mathbb{N}} V_m$, for $0 \leqslant k \leqslant \# V_m$ and $0 \leqslant p \leqslant \# V_{m-k}$, the m-edge distance between $x_{m,k}$ and $x_{m,k+p}$ is defined as the length of the minimal path connecting $x_{m,k}$ and $x_{m,k+p}$ in V_m, i.e.,

$$d_{m,\,\text{edge}}(x_{m,k}, x_{m,k+p}) = \sum_{k=0}^{p-1} d_{\text{Eucl}}(x_{m,k+j}, x_{m,k+j+1}).$$

In the case of adjacent vertices $x_{m,k}$ and $x_{m,k+1}$, we simply have that

$$d_{m,\,\text{edge}}(x_{m,k}, x_{m,k+1}) = d_{\text{Eucl}}(x_{m,k+j}, x_{m,k+1}).$$

Remark 7.4. (i) This edge distance between two vertices corresponds, in a sense, to the distance at a given level m of the prefractal graph approximation. Adjacent points at the same level are close, but become very distant as far as the level increases.

(ii) Defining (m, n)-paths enables one to switch, when necessary, from a level m to higher $n > m$. Such a situation happens when handling our forthcoming m-balls.

The next problem that arises now is: how can one define balls in our context?

Of course, Euclidean ones could do the job – namely, the important point is that given a radius $r > 0$, and a point x, we still have that

$$\forall \varepsilon \in]0, 1[, \quad B(x, \varepsilon r) \subset B(x, r),$$

i.e., bigger balls contain smaller ones.

An important thing is that we deal with discrete balls. This specific point has to be taken to account when defining balls – in so far as we will further consider random walks moving on a given state m of the sequence of prefractal graphs, which ends by switching from V_m to V_{m+1}, in a lack of memory process. The change of state – the m-th to the $(m + 1)$-th state – comes from the fact that $V_m \subset V_{m+1}$ and that $\#(V_{m+1} \setminus V_m) > \# V_m$ – in a sense, the probability of reaching the new state $m + 1$ is higher.

What we would like, thus, is that the definition of balls could account for this specificity. Bearing in mind that when m increases, the edge distance between adjacent vertices become smaller and smaller, the solution is that balls could have more points near their origin, i.e., with a distribution of points proportional to their position.

Definition 7.8 (m-ball). Given a natural integer m, a strictly positive number r, and a vertex x of V_m, the m-ball of center x and radius r is defined by

$$B_m(x, r) = \left\{ y \in V_m,\ d_{m,\text{edge}}(x, y) < r \right\}.$$

The associated closed ball will be denoted $\bar{B}_m(x, r)$.

Remark 7.5. The above definition 7.8 enable us to deal with the best suited ball, depending on the considered structure:

(i) In the case of Sierpiński gasket, we will handle m-balls of radius $\frac{1}{2^m}$, which coincide with m-cells.

(ii) In the case of the Weierstrass curve, we will handle m-balls of radius $j \times h_m$, for $1 \leqslant j \leqslant N - 1$ (see Definition 7.2). If the center of the ball is located at a junction point x_m (between m-cells), m-balls of radius $(N - 1) \times h_m$ enable us to encompass the m-cells with the same vertex x_m. One can also simply want to take into account the immediate (adjacent) neighbors of a vertex, in which case m-balls of radius h_m suffice.

Remark 7.6. Another interesting point that may be noted is that our definition of m-balls yields, for any vertex x of V_m, inclusion relations of the form

$$B_{m+1}(x, h_{m+1}) \subset B_{m+1}(x, h_m).$$

Property 7.2. *Since the sequence $(V_m)_{m\in\mathbb{N}}$ is increasing, we of course have, for any strictly positive number r, any natural integer m, and any vertex x of V_m, that*

$$B_m(x, r) \subset B_{m+1}(x, r).$$

This can be refined, for $r' < r$, as

$$B_m(x, r') \subset B_{m+1}(x, r).$$

Definition 7.9 (Regular probability measure on S [42]). A regular probability measure on S is a measure μ that assigns weights $\mu(C_m^j)$ to any m-cell of S_m, $m \in \mathbb{N}$, for $\{C_m^j, 0 \leqslant j \leqslant \#C_m - 1\}$, in an additive way:

(i) $\forall m \in \mathbb{N},\ \forall j \in \{0, \dots, \#C_m - 1\},\ \mu(C_m^j) > 0.$

(ii) Given two m-cells C_m^j and C_m^{j+1}, $\{C_m^j,\ 0 \leqslant j \leqslant \#C_m - 2\}$ which intersect only at junction points:

$$\mu(C_m^j \cup C_m^{j+1}) = \mu(C_m^j) + \mu(C_m^{j+1}).$$

(iii) $\lim_{m\to+\infty}(\mu(C_m^j))_{0\leqslant j\leqslant \#C_m-1} = 0.$

(iv) $\mu(S) = \lim_{m\to\infty} \sum_{j=0}^{\#C_m-1} \mu(C_m^j) = 1.$

Given a continuous function f on S, we set, from now on,

$$\int_S f\, d\mu = \lim_{m\to\infty} \sum_{j=0}^{\#C_m-1} \sum_{x \text{ vertex of } C_m^j} \frac{\mu(C_m^j)}{\#\text{vertices of } C_m^j} f(x).$$

Notation 18. From now on, we will denote by μ a measure on S.

7.2. h_m-Laplacian

Definition 7.10 (h_m-Laplacian, $m \in \mathbb{N}$). Following Definition 5.7, given a natural integer m, we define the h_m-Laplacian as the operator

$$|\Delta_{h_m}| = \delta_{h_m}^\star \delta_{h_m},$$

which acts on functions f defined on V_m as follows:

$$\forall x \in V_m, \quad |\Delta_{h_m}|(f)(x) = \frac{2c_{0,m}^2}{h_m^2} \int_{\bar{B}_m(x,h_m)} \{f(y) - f(x)\} C_0(x, y, m)\, d\mu(y),$$

where

$$C_0(x, y, m) = \min\left\{\frac{1}{\mu(\bar{B}_m(x, h_m))}, \frac{1}{\mu(\bar{B}_m(y, h_m))}\right\},$$

where μ is a regular probability measure on S (see Definition 7.9) and where $c_{0,m}^2$ denotes a strictly positive constant.

Remark 7.7. It is clear that, when $m \to \infty$,

$$\frac{1}{\mu(\bar{B}_m(x, h_m))} \gg 1 \quad \text{and} \quad \frac{1}{\mu(\bar{B}_m(y, h_m))} \gg 1.$$

Definition 7.11 (Topological Laplacian of order $m \in \mathbb{N}^\star$). For any strictly positive integer m, and any real-valued function f, defined on the set V_m of the vertices of the graph S_m, we introduce the topological Laplacian of order m, $\Delta_m^\tau(f)$, by

$$\forall x \in V_m \setminus \partial V_m, \quad \Delta_m^\tau f(x) = \sum_{y \in V_m, y \underset{m}{\sim} x} (f(y) - f(x)).$$

Theorem 7.3 (Pointwise formula, Kigami–Strichartz Laplacian [42]). *Given a strictly positive integer m, and a vertex $x \in V_m \setminus V_0$, we introduce the piecewise harmonic (with respect to the topological Laplacian Δ_m^τ) spline function $\psi_X^m \in S(H_0, V_k)$ such that*

$$\psi_x^m(y) = \begin{cases} \delta_{xy}, & \forall y \in V_m, \\ 0, & \forall y \notin V_m, \end{cases} \quad \textit{where} \quad \delta_{xy} = \begin{cases} 1, & \textit{if } x = y, \\ 0, & \textit{else}. \end{cases}$$

Provided the fractal S is self-similar, we the obtain a Laplacian, defined, for any continuous function f on S, which belongs to its domain domΔ*, by*

$$\forall x \notin V_0, \quad \Delta f(x) = \lim_{m \to \infty} \frac{r^{-m}}{\displaystyle\int_S \psi_x^m \, d\mu} \Delta_m^\tau f(X),$$

where, for any strictly positive integer m, r^{-m} is a normalization constant.

Property 7.4 (Back to the h_m-Laplacian). *The definition of the measure on S yields, for any vertex $x \in V_m \setminus V_0$:*

$$\begin{aligned} &|\Delta_{h_m}|(f)(x) \\ &= \frac{2c_{0,m}^2}{h_m^2} \int_{\bar{B}_m(x, h_m)} \{f(y) - f(x)\} \min\left\{\frac{1}{\mu(\bar{B}_m(x, h_m))}, \frac{1}{\mu(\bar{B}_m(y, h_m))}\right\} d\mu(y) \\ &= \frac{2c_{0,m}^2}{h_m^2} \sum_{y \in C_m^j, y \underset{m}{\sim} x} \frac{\mu(C_m^j)\{f(y) - f(x)\}}{\#\text{vertices of } C_m^j} \min\left\{\frac{1}{\mu(\bar{B}_m(x, h_m))}, \frac{1}{\mu(\bar{B}_m(y, h_m))}\right\}. \end{aligned}$$

Up to a positive multiplicative constant that depends on the geographic position of x, which impacts the number of its neighbors and, thus, the measures of the (closed) balls $\bar{B}_m(x, h_m)$ and $\bar{B}_m(y, h_m)$, we have that

$$\mu(\bar{B}_m(x, h_m)) = \mu(\bar{B}_m(y, h_m)) = \mu(C_m^j),$$

which yields

$$\begin{aligned} |\Delta_{h_m}|(f)(x) &= \frac{2c_{0,m}^2}{h_m^2 \#\text{vertices of } C_m^j} \sum_{y \in C_m^j,\, y \underset{m}{\sim} x} \{f(y) - f(x)\} \\ &= \frac{2c_{0,m}^2}{h_m^2 \#\text{vertices of } C_m^j} \Delta_m^\tau(f)(x). \end{aligned}$$

Since $\lim_{m \to \infty} h_m = 0$, *we have that*

$$\lim_{m \to \infty} |\Delta_{h_m}|(f)(x) = \lim_{h \to 0} |\Delta_h|(f)(x) = |\Delta_0|(f)(x).$$

Henceforth, under the condition

$$\frac{r^{-m}}{\int_S \psi_x^m \, d\mu} = \frac{2c_{0,m}^2}{h_m^2 \#\text{vertices of } C_m^j},$$

it can also be written as

$$\frac{r^{-m}}{\frac{\mu(C_m^j)}{\#\text{vertices of } C_m^j}} = \frac{2c_{0,m}^2}{h_m^2 \#\text{vertices of } C_m^j},$$

i.e.,

$$c_{0,m}^2 = \frac{r^{-m} h_m^2 (\#\text{vertices of } C_m^j)^2}{2\mu(C_m^j)}$$

in order to recover the same Laplacian, i.e., the one of classical analysis.

Remark 7.8. The above condition makes sense, in so far as

$$\mu(C_m^j) \lesssim \frac{1}{h_m^2}.$$

Then, one just has, up to a positive multiplicative constant, the equality of the normalization constants.

Remark 7.9. Henceforth, Laplacians on singular sets can be equivalently obtained, either through the now classical analysis tools on fractals introduced by J. Kigami, or by using our h-Laplacians. There is here an interesting point to note, due to the fact that the sequence $(V_m)_{m\in\mathbb{N}}$ is increasing. It thus happens that the h_{m+1}-Laplacian can be obtained if one considers the modified MCMC method (with respect to the initial one, see Property 6.2) where, given a state $x \in V_m \subset V_{m+1}$, the transition probability towards a new state $y \in V_m \subset V_{m+1}$ depends on wether $y \sim_{m+1} x$, or not (i.e., an edge relation between x and y can only exist at level m):

$$\mathbb{P}[y|x] = \mathbb{P}\Big[y \underset{m+1}{\sim} x \,\Big|\, x\Big] + \mathbb{P}\Big[y \underset{m+1}{\not\sim} x \,\Big|\, x\Big].$$

The acceptance probability is then given by

$$\min\left\{1, \frac{\#(V_{m+1} \setminus V_m)}{\#V_{m+1}} \frac{\mu(B_m(y,h_m))}{\mu(B_m(x,h_m))}, \frac{\#V_m}{\#V_{m+1}} \frac{\mu(B_{m+1}(y,h_m))}{\mu(B_{m+1}(x,h_m))}\right\}.$$

Since

$$\#(V_{m+1} \setminus V_m) > \#V_m,$$

and, more precisely, when $m \to \infty$,

$$\#(V_{m+1} \setminus V_m) \gg \#V_m,$$

when, at the same time, $h_m \to 0$, which means that the random walk will naturally ends in switching to the $(m+1)$-th level of the prefractal graph approximation.

As seen previously (see Property 7.2), we cannot write a comparison-inclusion relation between the balls $B_m(x,h_m)$ and $B_{m+1}(x,h_{m+1})$ similar to the one that exists for the corresponding Euclidean balls, i.e.,

$$B_{\text{Eucl}}(x,h_{m+1}) \subset B_{\text{Eucl}}(x,h_m).$$

Yet, the switching is natural, since

$$B_m(x,h_m) \subset B_{m+1}(x,h_m) \quad \text{and} \quad B_{m+1}(x,h_{m+1}) \subset B_{m+1}(x,h_m).$$

In fact, the random walk is initially in $B_m(x,h_m)$, but already in $B_{m+1}(x,h_m)$. It then naturally switches to $B_{m+1}(x,h_{m+1})$.

Henceforth, the h_{m+1}-Laplacian can be seen as an extension of the h_m-one to V_{m+1}. We can then draw a parallel with the decimation process of Fukushima and Shima [14, 38], where, given an eigenfunction u_m on $V_m \setminus \partial V_m$, for the eigenvalue Λ_m, one extends u_m on $V_{m+1} \setminus \partial V_{m+1}$ to a function u_{m+1}, which will itself be an eigenfunction of the $(m+1)$-th graph Laplacian Δ_{m+1}, for the eigenvalue Λ_m. In other words, this can be seen as a sort of "continuity" of the sequence of discrete Laplacians.

Definition 7.12 (Modified h_m-Laplacian, $m \in \mathbb{N}$). Following Definition 7.10, given a natural integer m, we define the modified h_m-Laplacian as the operator $|\widetilde{\Delta}_{h_m}|$, which acts on functions f defined on V_m, via

$$\forall x \in V_m : \quad |\widetilde{\Delta}_{h_m}|(f)(x) = \frac{2\widetilde{c}_{0,m}^2}{h_m^2} \int_{\bar{B}_m(x,h_m)} \{f(y) - f(x)\} \widetilde{C}_0(x,y,m) \, d\mu(y),$$

where

$$\begin{aligned} &\widetilde{C}_0(x,y,m) \\ &\quad = \min\Big\{ \frac{\#(V_{m+1} \setminus V_m)}{\#V_{m+1}} \frac{1}{\mu(B_m(x,h_m))}, \frac{\#(V_{m+1} \setminus V_m)}{\#V_{m+1}} \frac{1}{\mu(B_m(y,h_m))}, \\ &\qquad\qquad \frac{\#V_m}{\#V_{m+1}} \frac{1}{\mu(B_{m+1}(x,h_m))}, \frac{\#V_m}{\#V_{m+1}} \frac{1}{\mu(B_{m+1}(y,h_m))} \Big\}, \end{aligned}$$

and where $\widetilde{c}_{0,m}^2$ denotes a strictly positive constant.

As for the correspondence of Property 7.4, it is obtained thanks to the following property.

Property 7.5 (Recovering the modified h_m-Laplacian, $m \in \mathbb{N}$). *The definition of the measure on S yields, for any vertex $x \in V_m \setminus V_0$ and for $m \in \mathbb{N}$, that*

$$\begin{aligned} |\Delta_{h_m}|(f)(x) &= \frac{2\widetilde{c}_{0,m}^2}{h_m^2} \int_{\bar{B}_m(x,h_m)} \{f(y) - f(x)\} \widetilde{C}_0(x,y,m) \, d\mu(y) \\ &= \frac{2\widetilde{c}_{0,m}^2}{h_m^2} \sum_{y \in C_m^j, \, y \underset{m}{\sim} x} \frac{\mu(C_m^j)\{f(y) - f(x)\}}{\#\text{vertices of } C_m^j} \widetilde{C}_0(x,y,m). \end{aligned}$$

Under the condition

$$r^{-m} = \frac{2\widetilde{c}_{0,m}^2}{h_m^2} \sum_{\substack{y \in C_m^j, \, y \underset{m}{\sim} x \\ \text{or } y \in C_{m+1}^{j'}, \, y \underset{m+1}{\sim} x}} \left(\frac{\mu(C_m^j)}{\#\text{vertices of } C_m^j} \right)^2 \widetilde{C}_0(x,y,m),$$

we then obtain the correspondence between the modified h_m-Laplacian, and the Kigami–Strichartz Laplacian (see [19,42]).

7.3. Prefractal cohomology

At the beginning of our study (see Definition 2.3), given a natural integer p, we have introduced the concept of p-differential δ^p, from the set of p-fermions $F^p(X, A)$ to the set of $(p+1)$-fermions $F^{p+1}(X, A)$, by means of differences.

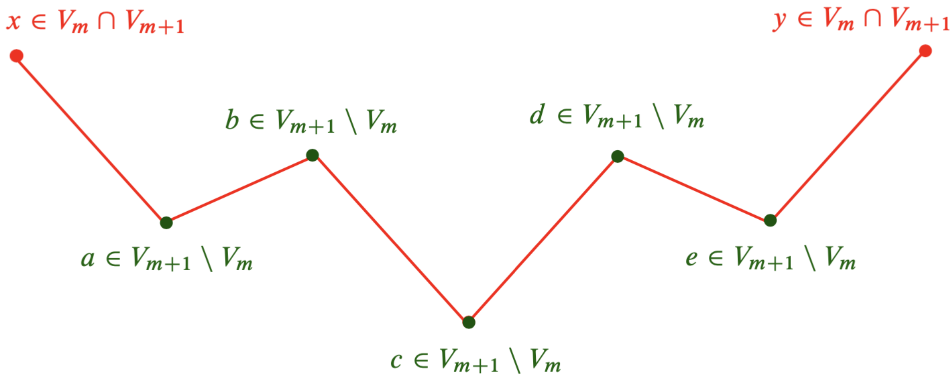

Figure 7. A prefractal path between two consecutive vertices x and y of V_m: the points a, b, c, d, and e are consecutive vertices of V_{m+1}.

In the case of prefractals, if differential operators – local ones, are also defined by means of differences, we have to be more subtle, in so far as it depends on edge relations. For instance, given $m \in \mathbb{N}^\star$, and a real-valued function f, defined on the set of vertices V_m, the topological Laplacian of order m is defined as follows:

$$\forall x \in V_m \setminus \partial V_m, \quad \Delta_m^\tau f(x) = \sum_{y \in V_m,\, y \underset{m}{\sim} x} (f(y) - f(x)).$$

Thus, local differences between adjacent points–vertices are involved.

Now, since the sequence $(V_m)_{m\in\mathbb{N}}$ is increasing, a local difference of the form

$$f(x) - f(y) \quad \text{for } y \underset{m}{\sim} x,$$

can be more explicitly written as

$$f(x_{m,k}) - f(x_{m,k+1}),$$

or, thanks to an analog of a Chasles relation along the path $P_{m,N}(x, y)$, as

$$f(x) - f(y) = \sum_{(z,t)\in(P_{m,N}(x,y))^2} (f(z) - f(t)).$$

It can then be explicited, in the case of the example displayed in Figure 7.

$$\begin{aligned} f(x) - f(y) = \{f(x) - f(a)\} + \{f(a) - f(b)\} + \{f(b) - f(c)\} \\ + \{f(c) - f(d)\} + \{f(d) - f(e)\} + \{f(e) - f(y)\}. \end{aligned}$$

Thus, p-differentials map the set of m-fermions to the set of $N \times p$ fermions:

$$F^p \xrightarrow{\delta^p} F^{Np}.$$

It is then legitimate to question the real meaning of the associated cohomology.

As in [1], we consider that p-fermions act on p-dimensional structures. Given a natural integer m, the set of vertices V_m has $\#V_m$ points, and can be considered as $\#V_m$-dimensional. In fact, the kernel $\ker \delta^{\#V_m}$ corresponds to the fermions that stay on the m-th-level approximation to the prefractal sequence $(S_m)_{m \in \mathbb{N}}$. The image $\mathrm{Im}\, \delta^{\#V_{m-1}}$ consists in the fermions coming from the $(m-1)$-th-level approximation to the prefractal sequence. The cohomology is thus constituted of the quotient groups

$$\ker \delta^{\#V_m} / \mathrm{Im}\, \delta^{\#V_{m-1}}, \quad m \in \mathbb{N}^\star.$$

In a sense, this amounts to a kind of "hierarchy" in the structure.

At this point, we would like to focus on the fact that, in [1], the authors mainly deal with low dimensional forms (0-,1-,2-). We find it interesting to handle $\#V_m$-fermions, acting on the whole set of vertices of V_m, which appears as rather natural, in so far as the points belong to the same m-th-order prefractal graph.

This is of course a mathematics paper. Yet, the following quote seems very appropriate to close this point:

"It is by no means obvious how to realize these intuitions in a precise theory, and there are perhaps more than one way to do this." [1]

Example 7.3 (The specific case of the Sierpiński gasket). In the case of the Sierpiński gasket, given a natural integer m, we have that

$$h_m = \frac{1}{2^m}.$$

For the natural probability measure μ, which assigns the value 1 to the gasket, the measure of an m-cell of the prefractal graph S_m is given, for any integer j in $\{0, \ldots, N^m - 1\}$, by

$$\mu(C_m^j) = A_m = \frac{1}{3^m}.$$

For any vertex $x \in V_m$, the number of points in the closed ball $\bar{B}_m(x, h_m)$ depends on the geographic location of x:

(i) If x belongs to V_0: x has exactly two neighbors, at distance h_m. The ball $B_m(x, h_m)$ contains exactly three points, x and its two neighbors.
The measure of the ball is then exactly the measure of an m-cell, i.e.,

$$\mu(B_m(x, h_m)) = A_m.$$

(ii) If x does not belong to V_0: x has exactly four neighbors, at distance h_m. The ball $B_m(x, h_m)$ contains exactly five points, x and its four neighbors, and, thus, three m-cells. The measure of the ball is then

$$\mu(B_m(x, h_m)) = 3A_m.$$

Meanwhile, for an m-th-order triangular cell of the gasket, with respective vertices x, y, z, we have that

$$\int_S \{\psi_x^m + \psi_y^m + \psi_z^m\}\, d\mu = A_m.$$

Thus,

$$\int_S \psi_x^m \, d\mu = \frac{1}{3} A_m.$$

Since (we refer to [42]),

$$r^{-m} = \left(\frac{5}{3}\right)^m,$$

we then obtain that

$$\frac{r^{-m}}{A_m} = \frac{2c_{0,m}^2}{9h_m^2},$$

i.e.,

$$c_{0,m}^2 = \frac{9 \times 3^m r^{-m}}{2 \times 4^m} = \frac{9}{2} \frac{5^m}{4^m}.$$

As for the detailed Hodge–de Rham calculus, one may find it, in an explicit way, in [1].

Now, as for the modified h_m-Laplacian, we have that

$$\# V_m = \frac{3^{m+1} + 3}{2} \quad \text{and} \quad \# V_{m+1} = \frac{3^{m+2} + 3}{2}.$$

At the same time, for any integer j' in $\{0, \dots, N^{m+1} - 1\}$, we also have that

$$\mu(C_m^{j'}) = A_{m+1} = \frac{1}{3^{m+1}}.$$

This yields

$$\tilde{c}_{0,m} = \frac{3^{m+2} + 3}{3^{m+1} + 3} \frac{3^{m+1}}{2 \times 4^m} r^{-m} > c_{0,m}^2.$$

Example 7.4 (The specific case of the Weierstrass curve). This case is slightly different from the one of the preceding gasket, in so far as we deal with a curve. The existing results [8, 10] enable us to handle a specific two-dimensional measure, in so far as the curve is approached by means of a polygonal neighborhood.

We hereafter denote by $N = N_b \geqslant 3$ the number of maps of the involved iterated function system (see 7.1), and by $D_{\mathcal{W}}$ the box-dimension of the curve.

Given a natural integer m, we have that

$$h_m = \frac{N_b^{(D_{\mathcal{W}}-2)m}}{(N_b-1)^{2-D_{\mathcal{W}}}}.$$

An m-cell has N_b vertices, while its measure is given by (we refer to [9, 10])

$$\mu_m \lesssim N_b^{(D_{\mathcal{W}}-3)m}.$$

For a continuous function f on the curve, belonging to the domain of the Laplacian, its Laplacian is obtained, for any $x \notin V_0$, through

$$\Delta f(x) = \lim_{m\to\infty} \Delta_m f(x) = \lim_{m\to\infty} \frac{c_m}{h_m^2} \Delta_m^\tau f(X),$$

where

$$c_m = h_m^{-2\left(\frac{D_W-1}{2-D_W}\right)}.$$

So, in a sense, the definition of the Laplacian already resembles the one of the h_m-Laplacian, which is thus obtained when

$$c_m = \frac{2c_{0,m}^2}{N_b},$$

i.e.,

$$c_{0,m}^2 = \frac{N_b}{2} h_m^{-2\left(\frac{D_W-1}{2-D_W}\right)}.$$

Now, as for the modified h_m-Laplacian, we have that

$$\#V_m = N_b^{m+1} + 1 - N_b^m \quad \text{and} \quad \#V_{m+1} = N_b^{m+2} + 1 - N_b^{m+1}.$$

At the same time, for any integer j' in $\{0, \dots, N^{m+1}-1\}$,

$$\mu(C_m^{j'}) = A_{m+1} = \frac{1}{3^{m+1}}.$$

This yields

$$c_m = 2\tilde{c}_{0,m} \sum_{\substack{y\in C_m^j,\, y \underset{m}{\sim} x \\ \text{or } y\in C_{m+1}^{j'},\, y \underset{m+1}{\sim} x}} \frac{\mu(C_m^j)}{\#\text{vertices of } C_m^j}$$

$$\min\left\{\frac{\#(V_{m+1}\setminus V_m)}{\#V_{m+1}} \frac{1}{\mu(C_m^j)}, \frac{\#V_m}{\#V_{m+1}} \frac{1}{\mu(C_{m+1}^{j'})}\right\}.$$

If we cannot presently have the exact value, we can nonetheless write

$$c_m \sim \frac{2\tilde{c}_{0,m} N_b^{(D_{\mathcal{W}}-3)m}}{N_b} \min\Big\{ \frac{N_b^{m+2}+1-N_b^{m+1}-N_b^{m+1}-1+N_b^m}{N_b^{m+2}+1-N_b^{m+1}} \frac{1}{N_b^{(D_{\mathcal{W}}-3)m}}, \frac{N_b^{m+1}+1-N_b^m}{N_b^{m+2}+1-N_b^{m+1}} \frac{1}{N_b^{(D_{\mathcal{W}}-3)(m+1)}} \Big\},$$

which yields

$$\tilde{c}_{0,m}^2 \sim \frac{N_b^{m+2}+1-N_b^{m+1}}{2(N_b^{m+1}-2N_b^m+N_b^{m-1})} c_m,$$

and

$$\tilde{c}_{0,m} > c_{0,m}.$$

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