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Absolute continuity in families of parametrised non-homogeneous self-similar measures

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Abstract. Let μ be a planar self-similar measure with similarity dimension exceeding 1, satisfying a mild separation condition, and such that the fixed points of the associated similitudes do not share a common line. Then, we prove that the orthogonal projections $\pi_{e\sharp}(\mu)$ are absolutely continuous for all $e \in S^1 \setminus E$, where the exceptional set E has zero Hausdorff dimension. The result is obtained from a more general framework which applies to certain parametrised families of self-similar measures on the real line. Our results extend the previous work of Shmerkin and Solomyak from 2016, where it was assumed that the similitudes associated with μ have a common contraction ratio.

1. Introduction

This paper studies the absolute continuity of parametrised non-homogeneous self-similar measures on $\mathbb{R}$. It is closely related to the works of Shmerkin [8], Shmerkin and Solomyak [9], and Saglietti, Shmerkin, and Solomyak [7].

1.1. Statement of the main result

We start by formulating the main result; we explain its connection to previous work in the next subsection, and, after that, we finish the introduction by stating and proving the application for projections of self-similar measures.

Definition 1.1 (Setting of the main result). Let $U \subset \mathbb{R}$ be an open interval and $m \geq 2$. We associate with each $u \in U$ a list of contractive similitudes on $\mathbb{R}$ of the form

$$\Psi_u := (\psi_{u,1}, \dots, \psi_{u,m}) = (\lambda_1 x + t_1(u), \dots, \lambda_m x + t_m(u)), \tag{1.2}$$

where

$$\lambda_1, \dots, \lambda_m \in (0, 1) \quad \text{and} \quad t_1(u), \dots, t_m(u) \in \mathbb{R}, \quad u \in U.$$

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So, the translations are allowed to depend on $u \in U$, but their number is constant, m. We make the following assumptions.

(A1) The map $u \mapsto t_j(u)$ is real-analytic, and the family $\{\Psi_u\}_{u\in U}$ satisfies transversality of order K for some $K \in \mathbb{N}$; see Definition 1.3.

(A2) There exist three sequences $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \{1, \ldots, m\}^{\mathbb{N}}$ such that none of the maps $u \mapsto \psi_{u,\mathbf{i}}(0)$, $u \mapsto \psi_{\mathbf{j},u}(0)$, and $u \mapsto \psi_{\mathbf{k},u}(0)$ is a convex combination of the other two. Here $u \mapsto \psi_{u,\mathbf{i}}(0)$, for example, refers to the map

$$u \mapsto \lim_{n\to\infty} \psi_{u,\mathbf{i}|_n}(0) := \lim_{n\to\infty} \psi_{u,i_1} \circ \cdots \circ \psi_{u,i_n}(0), \qquad \mathbf{i}|_n = (i_1, \ldots, i_n).$$

(A3) For some probability vector $\mathbf{p} = (p_1, \ldots, p_m) \in (0,1)^m$ with $p_1 + \cdots + p_m = 1$, the *similarity dimension* is

$$s(\bar{\lambda}, \mathbf{p}) := \frac{\sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \lambda_j},$$

where $\bar{\lambda} = (\lambda_1, \ldots, \lambda_m)$ satisfies $s(\bar{\lambda}, \mathbf{p}) > 1$.

Here is the definition of transversality mentioned in (A1).

Definition 1.3 (Transversality of order K). Let $\{\Psi_u\}_{u\in U}$ be a parametrised family of similitudes as in (1.2), let $K \in \{0, 1, 2, \ldots\}$, and assume that the map $u \mapsto t_j(u)$ is K times continuously differentiable for all $1 \le j \le m$. For $u \in U$, write

$$\Delta_{\mathbf{i},\mathbf{j}}(u) := \psi_{u,\mathbf{i}}(0) - \psi_{u,\mathbf{j}}(0), \qquad \mathbf{i}, \mathbf{j} \in \{1, \ldots, m\}^n, \; n \in \mathbb{N}.$$

The family $\{\Psi_u\}_{u\in U}$ *satisfies transversality of order* K if there exist a constant $c > 0$ and a sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that $n_j \to \infty$, and

$$\max_{k\in\{0,\ldots,K\}} |\Delta_{\mathbf{i},\mathbf{j}}^{(k)}(u)| \ge c^{n_j}, \qquad u \in U, \; \mathbf{i}, \mathbf{j} \in \{1, \ldots, m\}^{n_j}, \; \mathbf{i} \ne \mathbf{j}, \; j \in \mathbb{N}. \quad (1.4)$$

Here $\Delta_{\mathbf{i},\mathbf{j}}^{(k)}$ is the k-th derivative of $\Delta_{\mathbf{i},\mathbf{j}}$.

This notion of transversality is a variant of the one used by Hochman in [3, Definition 5.6]. The notion above is weak enough to be applied in Proposition 1.7, yet strong enough to imply the zero-dimensionality of the exceptional set E appearing in [3, Theorem 1.7]. We verify this fact in Proposition 4.4 and Appendix A.

Now we can state our main result.

Theorem 1.5. *Let μ_u, $u \in U$, be the self-similar measure associated with a pair $(\Psi_u, \mathbf{p})$ satisfying the assumptions in Definition 1.1. Then, there exists a set $E \subset U$ of Hausdorff dimension 0 such that $\mu_u \ll \mathcal{L}^1$ for all $u \in U \setminus E$.*

Here $\mathscr{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. The definition of a self-similar measure can be found in Section 2.2.

1.2. Comparison to previous work

Theorem 1.5 above is modelled after [9, Theorem A] of Shmerkin and Solomyak. We now discuss the main differences between the two theorems, and also draw connections to other related results. First, [9, Theorem A] allows for both the contraction parameters $\lambda_j = \lambda_j(u)$ and the translation vectors $t_j = t_j(u)$ (as in (1.2)) to depend on u. However, it is assumed in [9, Theorem A] that $\lambda_i(u) = \lambda_j(u)$ for all $1 \leq i, j \leq m$. In other words, the lists of similitudes are equicontractive, but the contraction ratio may vary with u. The equicontractivity assumption is convenient, because μ_u then looks like a "generalised Bernoulli convolution", and a technique pioneered by Shmerkin in [8] (in the context of classical Bernoulli convolutions) is available to study the absolute continuity of μ_u. A crucial feature of (classical and generalised) Bernoulli convolutions in the proofs of [8,9] is the property that they can be expressed as infinite convolutions of (scaled copies) of a single atomic measure.

In the non-homogeneous setting of Theorem 1.5, the measures μ_u no longer have an infinite convolution structure, and hence the method of [8,9] is not directly applicable. A way around this problem was found in [2]: Galicer, Saglietti, Shmerkin, and Yavicoli (see [2, Lemma 6.6]) discovered a way to express non-homogeneous self-similar measures as averages over measures with an infinite convolution structure. This naturally comes at a price: the infinite convolutions are no longer self-similar measures. The components of the infinite convolution are no longer rescaled copies of a single measure, but are, rather, drawn at random from a finite pool of (atomic) measures.

It turns out that the lack of strict self-similarity is not an insurmountable problem. In [7], Saglietti, Shmerkin and Solomyak used the decomposition from [2] to study the absolute continuity of parametrised self-similar measures, where the translation vectors $t_1, \ldots, t_m$ are fixed, but the contractions $\lambda_1, \ldots, \lambda_m$ vary freely in an open set. The initial motivation for our study was to understand if the technique in [7] could be adapted to give new information on the projections of planar self-similar measures – beyond the homogeneous case covered by [9, Theorem A]. The answer is affirmative. In fact, we improve the method by showing that the exceptional set has Hausdorff dimension zero, as opposed to Lebesgue measure zero as in [7, Theorem 1.1]. The reader should, thus, view Theorem 1.5 not only as a non-homogeneous variant of [9, Theorem A], but also as an adaptation of [7, Theorem 1.1] to the case where the translation parameters vary.

1.3. Application to projections of planar self-similar measures

We now describe the main application of Theorem 1.5 to projections of planar self-similar measures. Let $U = (0, 2\pi)$,

$$\Psi = (\psi_1, \dots, \psi_m) = (\lambda_1 x + t_1, \dots, \lambda_m x + t_m), \quad \lambda_j \in (0,1),\ t_j \in \mathbb{R}^2,$$

be a list of contractive homotheties on $\mathbb{R}^2$, and let μ be the self-similar measure associated with Ψ and some probability vector $\mathbf{p} \in (0,1)^m$ such that

$$s(\bar{\lambda}, \mathbf{p}) > 1. \tag{1.6}$$

Let $\pi_u \colon \mathbb{R}^2 \to \mathbb{R}$, $u \in (0, 2\pi)$, be the orthogonal projection $\pi_u(x) = x \cdot (\cos u, \sin u)$, and note that the measures $\pi_{u\sharp}\mu$ are again self-similar: they are the self-similar measures associated with the probability vector $\mathbf{p}$, and the lists of similitudes

$$\Psi_u = (\lambda_1 x + \pi_u(t_1), \dots, \lambda_m x + \pi_u(t_m)), \qquad u \in (0, 2\pi).$$

Note that the contraction ratios $\lambda_1, \dots, \lambda_m$ are independent of u, so $(\Psi_u, \mathbf{p})$ satisfies the condition (A3) by (1.6). To verify that the family of similitudes $\{\Psi_u\}_{u \in U}$ also meets the assumptions (A1) and (A2), we need to impose the following two hypotheses on Ψ:

(P1) $\limsup_{n\to\infty} \log \Delta_n / n > -\infty$, where

$$\Delta_n = \Delta_n(\Psi) = \min\{|\Delta_{\mathbf{i},\mathbf{j}}| : \mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^n,\ \mathbf{i} \neq \mathbf{j}\},$$

and $\Delta_{\mathbf{i},\mathbf{j}} = \psi_{\mathbf{i}}(0) - \psi_{\mathbf{j}}(0)$.

(P2) The fixed points of the similitudes in Ψ do not lie on a common line.

We make some remarks on the sharpness of these assumptions after the proof of the following proposition. In case the self-similar measure is generated by maps having no rotations, the proposition is new in the non-homogeneous case, and also relaxes the separation assumption compared to [9, Theorem B(i)] in the homogeneous case. If the maps have dense rotations, then the reader is referred to the works of Shmerkin and Solomyak [9, Theorem B(ii)] and Rapaport [6].

Proposition 1.7. *If the pair* $(\Psi, \mathbf{p})$ *satisfies* (1.6) *and conditions* (P1)–(P2)*, then the family* $\{\Psi_u\}_{u\in U}$ *satisfies* (A1)– (A3)*. In particular, the self-similar measure* μ *associated with the pair* $(\Psi, \mathbf{p})$ *satisfies* $\pi_{u\sharp}\mu \ll \mathcal{L}^1$ *for all* $u \in U \setminus E$*, where* $\dim_{\mathrm{H}} E = 0$.

Proof. It is easy to check (and very well known) that the projections π_u satisfy the following transversality condition for some absolute constant $\delta > 0$:

$$\max\{|\pi_u(x)|, |\partial_u \pi_u(x)|\} > \delta |x|, \qquad u \in (0, 2\pi),\ x \in \mathbb{R}^2. \tag{1.8}$$

Now, we claim that $\{\Psi_u\}_{u\in U}$ satisfies transversality of order 1, according to Definition 1.3. By (P1), there exists $c > 0$ and a sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers such that

$$|\Delta_{\mathbf{i},\mathbf{j}}| \geq c^{n_j}, \qquad \mathbf{i},\mathbf{j} \in \{1,\ldots,m\}^{n_j},\ \mathbf{i} \neq \mathbf{j}.$$

Note that

$$\psi_{u,\mathbf{k}}(0) = \pi_u(\psi_{\mathbf{k}}(0)), \qquad \mathbf{k} \in \{\mathbf{i},\mathbf{j}\},\ u \in U,$$

so $\Delta_{\mathbf{i},\mathbf{j}}(u) = \pi_u(\Delta_{\mathbf{i},\mathbf{j}})$ for $u \in U$. It follows from (1.8) and (P1) that

$$\max\{|\Delta_{\mathbf{i},\mathbf{j}}(u)|, |\Delta'_{\mathbf{i},\mathbf{j}}(u)|\} = \max\{|\pi_u(\Delta_{\mathbf{i},\mathbf{j}})|, |\partial_u \pi_u(\Delta_{\mathbf{i},\mathbf{j}})|\} \geq \delta|\Delta_{\mathbf{i},\mathbf{j}}| \geq \delta c^{n_j}$$

for all $j \in \mathbb{N}$ and distinct $\mathbf{i},\mathbf{j} \in \{1,\ldots,m\}^{n_j}$. By adjusting c slightly, this implies (1.4) with $K = 1$, and hence assumption (A1) is satisfied.

As noted above, (A3) follows immediately from (1.6). After verifying (A2), the final claim follows directly from Theorem 1.5. Thus it remains to check assumption (A2). Since the fixed points of the similitudes in Ψ do not share a common line, there exist three sequences $\mathbf{i},\mathbf{j},\mathbf{k} \in \{1,\ldots,m\}^{\mathbb{N}}$ such that $\psi_{\mathbf{i}}(0)$, $\psi_{\mathbf{j}}(0)$, and $\psi_{\mathbf{k}}(0)$ do not lie on a common line either. Then, using the relations $\psi_{u,\mathbf{i}}(0) = \pi_u(\psi_{\mathbf{i}}(0))$ and so on, it is easy to check that none of the three functions

$$u \mapsto \psi_{u,\mathbf{i}}(0), \quad u \mapsto \psi_{u,\mathbf{j}}(0), \quad \text{and} \quad u \mapsto \psi_{u,\mathbf{k}}(0)$$

can be expressed as a convex combination of the other two. This gives (A2), and the proof is complete. ■

We close the section with a few remarks on the assumptions (P1)–(P2) and (A1)–(A2).

Remark 1.9. We do not know if assumption (P1) is necessary: maybe it is possible to bundle (1.6) and (P1) to the single assumption that $\dim_{\mathrm{H}} \mu > 1$.Then, of course, (P2) would become redundant and our result would strictly generalise what Marstrand's projection theorem results in this setting. In the present circumstances, however, assumption (P2) is necessary. To see this, we apply a result of Simon and Vágó [10] concerning the projections of the standard Sierpiński carpet S, namely the self-similar set on $\mathbb{R}^2$ generated by the homotheties

$$\left\{\psi_i(x) = \frac{x}{3} + \frac{t_i}{3}\right\}_{i=1}^{8},$$

where the translation vectors t_i range in the set $\{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\}$. It is shown in [10, Theorem 14] that if ν is the self-similar measure on S determined by

$$\nu = \sum_{i=1}^{8} \frac{1}{8} \cdot \psi_{i\sharp}\nu,$$

then there exists a dense G_δ-set of directions $u \in (0, 2\pi)$ such that $\pi_{u\sharp}\nu \not\ll \mathcal{L}^1$. We note that $\pi_{u\sharp}\nu$ is again a self-similar measure on $\mathbb{R}$, associated with the family of similitudes

$$\left\{\psi_{u,i}(x) = \frac{x}{3} + \frac{\pi_u(t_i)}{3}\right\}_{i=1}^{8}.$$

Further, it follows from the argument of [3, Theorem 1.6] that for every $u \in (0, 2\pi)\setminus\mathbb{Q}$, there exist $c > 0$ (in fact, one can take $c = 1/30$) and a sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that

$$|\psi_{u,\mathbf{i}}(0) - \psi_{u,\mathbf{j}}(0)| \geq c^{n_j}, \qquad \mathbf{i}, \mathbf{j} \in \{1, \ldots, 8\}^{n_j}, \ \mathbf{i} \neq \mathbf{j}. \tag{1.10}$$

In particular, we may find $u \in (0, 2\pi)$ such that (1.10) holds, and $\pi_{u\sharp}\nu \not\ll \mathcal{L}^1$. Finally, if $\mu := \pi_{u\sharp}\nu$, for this choice of $u \in (0, 2\pi)$, is viewed as a measure on $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$, then both (1.6) and (P1) are satisfied, yet all the projections of μ are evidently also singular. Of course, (P2) fails in this case, so Proposition 1.7 is not contradicted.

Remark 1.11. In the homogeneous analogue for our main theorem, namely [9, Theorem A], the assumptions (A1) and (A2) are elegantly bundled into a single hypothesis, which reads as follows: for any distinct $\mathbf{i}, \mathbf{j} \in \{1, \ldots, m\}^{\mathbb{N}}$, the map $u \mapsto \Delta_{\mathbf{i},\mathbf{j}}(u)$ is not identically zero. We prefer to avoid making this assumption, as it would limit the scope of the previous application; it would force us to assume that Ψ (in Proposition 1.7) satisfies the strong separation condition. Now (P1) is satisfied under – for example – the open set condition.

2. A model of random measures

As we explained in Section 1.2, a major hurdle in proving our main theorem is the fact that non-homogeneous self-similar measures do not have an "infinite convolution" structure. However, by the results in [2], a non-homogeneous self-similar measure can, nonetheless, be expressed as an average of certain "statistically self-similar" random measures with an infinite convolution structure. We will need all the details of this decomposition, and they will now be thoroughly explained for the reader's convenience.

2.1. An abstract random model

Let $\mathcal{T}$ be a finite index set; we will often refer to elements $\tau \in \mathcal{T}$ as *types*. To every $\tau \in \mathcal{T}$, we assign a list of equicontractive similitudes on $\mathbb{R}^d$

$$\Psi(\tau) := (\psi_1^\tau, \ldots, \psi_{m(\tau)}^\tau) = (\lambda(\tau)x + t_1(\tau), \ldots, \lambda(\tau) + t_{m(\tau)}(\tau)), \tag{2.1}$$

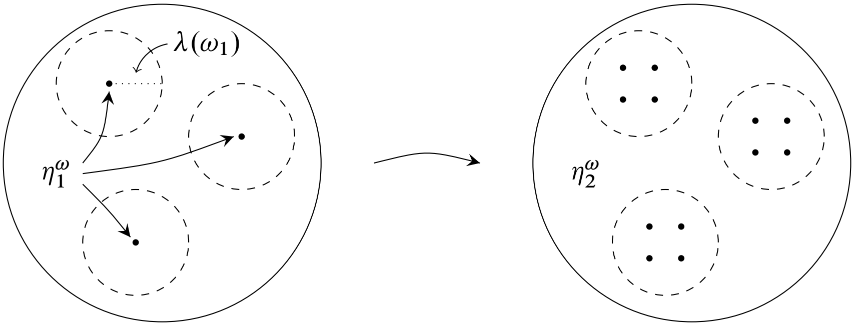

Figure 1. The transition from η_1^ω to η_2^ω in (2.3).

where $t_j(\tau) \in \mathbb{R}^d$, $\lambda(\tau) \in (0,1)$, and $m(\tau) \geq 1$. We emphasise that the contraction ratios $\lambda(\tau) \in (0,1)$ are allowed to depend on τ, but they are constant within each individual family $\Psi(\tau)$. Also, repetitions are allowed: a single similitude may appear with multiple different indices in $\Psi(\tau)$. We also allow $m(\tau) = 1$ for one or more $\tau \in \mathcal{T}$.

To each $\tau \in \mathcal{T}$, we assign the following discrete measure:

$$\eta(\tau) := \frac{1}{m(\tau)} \sum_{j=1}^{m(\tau)} \delta_{\psi_j^\tau(0)} = \frac{1}{m(\tau)} \sum_{j=1}^{m(\tau)} \delta_{t_j(\tau)}. \tag{2.2}$$

Finally, to every type $\tau \in \mathcal{T}$ we assign some probability $q(\tau) \in (0,1)$ such that

$$\sum_{\tau \in \mathcal{T}} q(\tau) = 1.$$

Next, we let $\Omega := \mathcal{T}^{\mathbb{N}}$, and we let $\mathbb{P}$ be the usual product probability (Bernoulli) measure on Ω determined by the probabilities $q(\tau)$. With each $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, we then associate the following infinite convolution:

$$\eta^\omega := \mathop{\ast}_{n \geq 1} \eta_n^\omega := \mathop{\ast}_{n \geq 1} \left[\prod_{j=1}^{n-1} \lambda(\omega_j)\right]_\sharp \eta(\omega_n). \tag{2.3}$$

Here $r_\sharp \nu$, $r > 0$ stands for the push-forward of $\nu \in \mathcal{M}(\mathbb{R}^d)$ under the dilation $x \mapsto rx$ and $\mathcal{M}(X) = \{\mu\colon \mu$ is a non-trivial Radon measure on X with compact support$\}$. To get an idea of what is happening here, consider the following. If all the families $\Psi(\tau)$ were the same, $\Psi(\tau) \equiv \Psi$ and, in particular, $\lambda(\tau) \equiv \lambda$, then (2.3) would simply give the usual self-similar measure generated by Ψ. To get an intuition of the general case, we refer to Figure 1. Now, the triple $(\Omega, \{\eta^\omega\}_{\omega \in \Omega}, \mathbb{P})$ is a probability space of "statistically self-similar" measures.

2.2. Disintegration of self-similar measures

The random measures introduced above are mostly a tool in this paper; eventually, we are interested in deterministic self-similar measures. We now explain the connection, but the reader may wish to take a look at Proposition 2.8 to see where we are heading. Let

$$m \geq 2, \quad \lambda_1, \dots, \lambda_m \in (0,1), \quad \text{and} \quad t_1, \dots, t_m \in \mathbb{R}^d.$$

Let Ψ be the corresponding list of homotheties

$$\Psi := (\psi_1, \dots, \psi_m) = (\lambda_1 x + t_1, \dots, \lambda_m x + t_m). \tag{2.4}$$

To each $j \in \{1, \dots, m\}$ we further assign a probability $p_j \in (0,1)$ such that $\sum p_j = 1$. Then, there exists a unique probability measure μ on $\mathbb{R}^d$ satisfying the relation

$$\mu = \sum_{j=1}^{m} p_j \cdot \psi_{j\sharp}\mu.$$

Writing $\mathbf{p} = (p_1, \dots, p_m)$, we call μ the *self-similar measure associated with the pair* $(\Psi, \mathbf{p})$. Now, we relate the measure μ to the random measures discussed in the previous section. Fix an integer $N \geq 1$ and write

$$\mathcal{T} := \mathcal{T}^N := \{(N_1, \dots, N_m) \in \mathbb{N}_0^m \colon N_1 + \dots + N_m = N\}. \tag{2.5}$$

The elements of $\mathcal{T}$ should be understood as the *types* from the previous section, and $N \geq 1$ should be understood as a free parameter whose role will be clarified much later. We next define the probabilities $q(\tau)$, $\tau \in \mathcal{T}$, and eventually the lists $\Psi(\tau)$, $\tau \in \mathcal{T}$. Recall that $m \in \mathbb{N}$ was the cardinality of the family Ψ. We say that an N-sequence $(n_1, \dots, n_N) \in \{1, \dots, m\}^N$ has type

$$\tau(n_1, \dots, n_N) = (N_1, \dots, N_m) \in \mathcal{T}$$

if k appears in the sequence N_k times for all $1 \leq k \leq m$. The formula above defines a map $\tau \colon \{1, \dots, m\}^N \to \mathcal{T}$.

Example 2.6. *If* $m = 3$ *and* $N = 4$*, then* $\tau(1,2,1,2) = (2,2,0)$.

Recalling the probabilities $p_1, \dots, p_m$ from above, we define the probabilities for each type in $\mathcal{T}$ as follows:

$$q(N_1, \dots, N_m) := \sum_{\substack{(n_1, \dots, n_N) \in \{1, \dots, m\}^N \\ \tau(n_1, \dots, n_N) = (N_1, \dots, N_m)}} p_{n_1} \cdots p_{n_N} = m(N_1, \dots, N_m) p_1^{N_1} \cdots p_m^{N_m}. \tag{2.7}$$

Here $m(\tau)$ is the number of N-sequences with type τ. We used the fact that the value of the product $p_{n_1} \cdots p_{n_N}$ only depends on the type of the sequence $(n_1, \ldots, n_N)$. Clearly,

$$\sum_{\tau \in \mathcal{T}} q(\tau) = 1.$$

Finally, it is time to define the lists $\Psi(\tau)$ for $\tau \in \mathcal{T}$. Recall that $N \geq 1$ is a fixed parameter. For a type $\tau \in \mathcal{T}$, we define the list

$$\Psi(\tau) := \Psi^N(\tau) := (\psi_{n_1} \circ \cdots \circ \psi_{n_N} : \tau(n_1, \ldots, n_N) = \tau).$$

Note that a single similitude may appear several times in this list, so in general

$$\Psi(\tau) \neq \{\psi_{n_1} \circ \cdots \psi_{n_N} : \tau(n_1, \ldots, n_N) = \tau\},$$

unless one interprets the right-hand side as a multiset. It is nevertheless convenient to write "$\varphi \in \Psi(\tau)$"; this simply means that φ appears at least once in the sequence $\Psi(\tau)$.

A key point of the definition of $\Psi(\tau)$ is that all the similitudes in $\Psi(\tau)$ now have the same contraction ratio. More precisely, if $\tau = (N_1, \ldots, N_m) \in \mathcal{T}$ and $\varphi \in \Psi(\tau)$, then

$$\lambda(\varphi) = \lambda_1^{N_1} \cdots \lambda_m^{N_m} =: \lambda(\tau).$$

Thus, the lists $\Psi(\tau)$, $\tau \in \mathcal{T}$, indeed have the form (2.1). With this in mind, the general framework from the previous section is applicable, and it yields the discrete measures $\eta(\tau)$ as in (2.2), the infinite convolutions η^ω, $\omega \in \Omega = \mathcal{T}^{\mathbb{N}}$, as in (2.3), and the measure $\mathbb{P}$ derived from the probabilities $q(\tau)$, $\tau \in \mathcal{T}$. Further, the self-similar measure μ is related to the measures η^ω via the following disintegration formula.

Proposition 2.8. *With the notation above, we have*

$$\mu = \int_\Omega \eta^\omega \, \mathrm{d}\mathbb{P}(\omega).$$

Proof. Although the proof can be found in [7, Lemma 6.2] or [2, (55)], we give the details for the convenience of the reader. Let T be the left shift on Ω defined by $T(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)$. By (2.3), (2.2), and the fact that $\mathbb{P}$ is the product probability measure on Ω determined by the probabilities $q(\tau)$, $\tau \in \mathcal{T}$, defined in (2.7), we have

$$\begin{aligned}\int_\Omega \eta^\omega \, \mathrm{d}\mathbb{P}(\omega) &= \int_\Omega \eta(\omega_1) * \lambda(\omega_1)_\sharp \eta^{T(\omega)} \, \mathrm{d}\mathbb{P}(\omega) \\ &= \int_\Omega \frac{1}{m(\omega_1)} \sum_{j=1}^{m(\omega_1)} \delta_{t_j(\omega_1)} * \lambda(\omega_1)_\sharp \eta^{T(\omega)} \, \mathrm{d}\mathbb{P}(\omega)\end{aligned}$$

$$= \int_\Omega \frac{1}{m(\omega_1)} \sum_{j=1}^{m(\omega_1)} \psi_{j\sharp}^{\omega_1} \eta^{T(\omega)} \, d\mathbb{P}(\omega)$$

$$= \sum_{\tau \in \mathcal{T}} q(\tau) \int_\Omega \frac{1}{m(\tau)} \sum_{j=1}^{m(\tau)} \psi_{j\sharp}^{\tau} \eta^{\omega} \, d\mathbb{P}(\omega)$$

$$= \sum_{\tau \in \mathcal{T}} \frac{q(\tau)}{m(\tau)} \sum_{j=1}^{m(\tau)} \psi_{j\sharp}^{\tau} \int_\Omega \eta^{\omega} \, d\mathbb{P}(\omega)$$

$$= \sum_{(n_1,\dots,n_N) \in \{1,\dots,m\}^N} p_{n_1} \cdots p_{n_N} \cdot (\psi_{n_1} \circ \cdots \circ \psi_{n_N})_\sharp \int_\Omega \eta^{\omega} \, d\mathbb{P}(\omega).$$

The proof is finished by the uniqueness of the self-similar measure; recall (2.4). ■

3. Fourier dimension estimates

In this section, we will work with the following hypotheses.

Definition 3.1. Let $\mathcal{T}$ be a (finite) collection of types as in Section 2.1, and let $U \subset \mathbb{R}$ be an open interval. To each $u \in U$ and $\tau \in \mathcal{T}$, assign a family of similitudes of the form

$$\Psi_u(\tau) = (\psi_1^u, \dots, \psi_{m(\tau)}^u) = (\lambda(\tau)x + t_1(\tau, u), \dots, \lambda(\tau)x + t_{m(\tau)}(\tau, u)), \quad (3.2)$$

where

$$m(\tau) \geq 1, \quad \lambda(\tau) \in (0,1), \quad \text{and} \quad t_1(\tau, u), \dots, t_{m(\tau)}(\tau, u) \in \mathbb{R}.$$

Note that since $\mathcal{T}$ is finite, we automatically have the inequalities $\min\{\lambda(\tau) \colon \tau \in \mathcal{T}\} > 0$ and $\max\{\lambda(\tau) \colon \tau \in \mathcal{T}\} < 1$. We will assume that for fixed $\tau \in \mathcal{T}$ and $1 \leq j \leq m(\tau)$ the map

$$u \mapsto t_j(\tau, u), \qquad u \in U$$

is real-analytic, and we assume that $\sup |t_j(\tau, u)| < \infty$, where the sup runs over $\tau \in \mathcal{T}$, $1 \leq j \leq m(\tau)$, and $u \in U$.

For the remainder of this section, we fix a collection of types $\mathcal{T}$, an open interval $U \subset \mathbb{R}$, and families of similitudes $\Psi_u(\tau)$, $(u, \tau) \in U \times \mathcal{T}$, as in (3.2), satisfying the assumptions of Definition 3.1. We also fix probabilities $q(\tau) \in (0,1)$, $\tau \in \mathcal{T}$, such that $\sum_{\tau \in \mathcal{T}} q(\tau) = 1$. Given these parameters, we follow the construction in Section 2.1 to generate the probability space $(\Omega, \mathbb{P})$ which is independent of u and also the measures

$$\eta_u(\tau), \ \tau \in \mathcal{T}, \quad \text{and} \quad \eta_u^{\omega}, \ \omega \in \Omega.$$

We recall the following explicit formula for the measures $\eta_u(\tau)$:

$$\eta_u(\tau) = \frac{1}{m(\tau)} \sum_{j=1}^{m(\tau)} \delta_{t_j(\tau,u)}.$$

We are interested in the Fourier transforms of the measures η_u^ω. Recall that if $\nu \in \mathcal{M}(\mathbb{R}^d)$, then

$$\dim_{\mathrm{F}} \nu := \sup\{s \in [0,d] : \exists\, C > 0 \text{ such that } |\hat{\nu}(\xi)| \leq C|\xi|^{-s/2} \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}\}.$$

Here is the main result of this section.

Proposition 3.3. *Assume that there exist $\tau \in \mathcal{T}$ and three indices $1 \leq i_1 < i_2 < i_3 \leq m(\tau)$ such that $u \mapsto t_{i_3}(\tau,u) - t_{i_1}(\tau,u)$ is not identically zero, and*

$$u \mapsto \frac{t_{i_2}(\tau,u) - t_{i_1}(\tau,u)}{t_{i_3}(\tau,u) - t_{i_1}(\tau,u)}, \qquad u \in U \tag{3.4}$$

is non-constant. Then, there exists a set $G \subset \Omega$ with $\mathbb{P}(G) = 1$ such that if $\omega \in G$, then

$$\dim_{\mathrm{H}}\{u \in U : \dim_{\mathrm{F}} \eta_u^\omega = 0\} = 0. \tag{3.5}$$

Notice that as the maps $u \mapsto t_j(\tau,u)$ are real-analytic by assumption, the denominator in (3.4) can vanish at most in a finite number of points for $\tau \in \mathcal{T}$ fixed. To get started, we compute an explicit expression for the Fourier transform $\widehat{\eta_u^\omega}$. Recall that for all $r > 0$,

$$\widehat{r_\sharp \nu}(\xi) = \hat{\nu}(r\xi), \qquad \nu \in \mathcal{M}(\mathbb{R}^d),\ \xi \in \mathbb{R}^d. \tag{3.6}$$

For brevity, we write

$$\lambda(\omega|_n) := \prod_{j=1}^{n} \lambda(\omega_j), \qquad n \geq 0, \tag{3.7}$$

where $\omega|_n := (\omega_1, \dots, \omega_n)$ is the initial segment of ω of length n. In particular, recalling (2.3),

$$\eta_u^\omega = \mathop{\ast}_{n\geq 1} [\lambda(\omega|_{n-1})_\sharp \eta_u(\omega_n)].$$

Then, by (3.6),

$$\begin{aligned} \widehat{\eta_u^\omega}(\xi) &= \prod_{n\geq 1} \widehat{\eta_u(\omega_n)}(\lambda(\omega|_{n-1})\xi) \\ &= \prod_{n\geq 1} \frac{1}{m(\omega_n)} \sum_{j=1}^{m(\omega_n)} \exp\bigl(-2\pi i \lambda(\omega|_{n-1}) t_j(\omega_n,u)\xi\bigr). \end{aligned} \tag{3.8}$$

Now that we have the formula (3.8) at hand, we record a continuity property for $\widehat{\eta_u^\omega}$, which will be needed to verify the measurabilty of sets defined via $\dim_{\mathrm{F}} \eta_u^\omega$.

Lemma 3.9. *For $\xi \in \mathbb{R}$ fixed, the map $(\omega, u) \mapsto \widehat{\eta_u^\omega}(\xi)$ is continuous.*

Proof. The statement concerns the product topology on $\Omega \times U$, therefore, convergence $(\omega^j, u^j) \to (\omega, u)$ means that for any $N \in \mathbb{N}$ and $\delta > 0$, the following holds for $j \in \mathbb{N}$ sufficiently large:

$$\omega^j|_N = \omega|_N \quad \text{and} \quad |u^j - u| < \delta. \tag{3.10}$$

Fix $\varepsilon > 0$. We now claim that the difference $|\widehat{\eta_{u_j}^{\omega_j}}(\xi) - \widehat{\eta_u^\omega}(\xi)|$ can be made smaller than ε by first choosing N in (3.10) large enough, depending on ε, ξ, and then δ small enough, depending on ε, N. To see this, write first

$$\widehat{\eta_{u_j}^{\omega_j}}(\xi) = \prod_{n=1}^{N} F(\omega_n^j, u_j, \xi) \times \prod_{n=N+1}^{\infty} F(\omega_n^j, u_j, \xi) =: \Pi_{\le N}(\omega^j, u^j, \xi) \cdot \Pi_{>N}(\omega^j, u^j, \xi),$$

where $F(\ldots)$ is an abbreviation for the function appearing in (3.8). Similarly, $\widehat{\eta_u^\omega}(\xi) =: \Pi_{\le N}(\omega, u, \xi) \cdot \Pi_{>N}(\omega, u, \xi)$. Then, introducing a cross-term, one may estimate

$$\begin{aligned} \left|\widehat{\eta_{u_j}^{\omega_j}}(\xi) - \widehat{\eta_u^\omega}(\xi)\right| &\le \left|\Pi_{\le N}(\omega^j, u^j, \xi)\right|\left|\Pi_{>N}(\omega^j, u^j, \xi) - \Pi_{>N}(\omega, u, \xi)\right| \\ &\quad + \left|\Pi_{>N}(\omega, u, \xi)\right|\left|\Pi_{\le N}(\omega^j, u^j, \xi) - \Pi_{\le N}(\omega, u, \xi)\right| \\ &\le \left|\Pi_{>N}(\omega^j, u^j, \xi) - \Pi_{>N}(\omega, u, \xi)\right| && (3.11) \\ &\quad + \left|\Pi_{\le N}(\omega, u^j, \xi) - \Pi_{\le N}(\omega, u, \xi)\right|, && (3.12) \end{aligned}$$

noting that all partial products are bounded by 1, and using $\omega^j|_N = \omega|_N$ upon arrival at (3.12). Since $\xi \in \mathbb{R}$ is fixed, the term $\Pi_{>N}(\omega, u, \xi)$ will converge to 1 as $N \to \infty$ at a rate independent of ω and u, and the same is true for $\Pi_{>N}(\omega^j, u^j, \xi)$. To see this, first estimate the individual factors in the product $\Pi_{>N}(\omega, u, \xi)$ for $n > N$.

$$\begin{aligned} |F(\omega_n, u, \xi) - 1| &\le \frac{1}{m(\omega_n)} \sum_{j=1}^{m(\omega_n)} \left|\exp\bigl(-2\pi i \lambda(\omega|_{n-1}) t_j(\omega_n, u)\xi\bigr) - 1\right| \\ &\le 2\pi \lambda_{\sup}^{n-1} t_{\sup}|\xi|, \end{aligned}$$

using that $x \mapsto e^{ix}$ is 1-Lipschitz, and recalling from Definition 3.1 that

$$\lambda_{\sup} := \sup\{\lambda(\tau) : \tau \in \mathcal{T}\} < 1 \quad \text{and} \quad t_{\sup} := \sup\{|t(\tau, u)| : \tau \in \mathcal{T},\ u \in U\} < \infty.$$

This implies that the factors of $\Pi_{>N}(\omega, u, \xi)$ converge to 1 rapidly enough to also ensure $\Pi_{>N}(\omega, u, \xi) \to 1$ as $N \to \infty$, uniformly in (ω, u). So, (3.11) can be made less than ε by choosing N large, in a manner depending only on ε, ξ. After this, to handle (3.12), one recalls that

$$(3.12) = \left|\prod_{n=1}^{N} F(\omega_n, u^j, \xi) - \prod_{n=1}^{N} F(\omega_n, u, \xi)\right|.$$

Since each individual function $u \mapsto F(\omega_n, u, \xi)$ is continuous, $1 \leq n \leq N$, the difference above can be made less than ε by requiring that δ in (3.10) is small enough, depending only on ε, N. This completes the proof. ■

Corollary 3.13. *The set* $\{(\omega, u) \in \Omega \times U : \dim_{\mathrm{F}} \eta_u^{\omega} = 0\}$ *is Borel.*

Proof. The set in question can be expressed as

$$\bigcap_{\varepsilon>0} \bigcap_{i=1}^{\infty} \bigcup_{|\xi|\geq i} \{(\omega, u) : |\widehat{\eta_u^{\omega}}(\xi)| > |\xi|^{-\varepsilon}\},$$

where the unions and intersections run over rational numbers, and the individual sets $\{(\omega, u) : |\widehat{\eta_u^{\omega}}(\xi)| > |\xi|^{-\varepsilon}\}$ are open by the previous lemma. ■

We now return to the proof of Proposition 3.3. We single out the type $\tau_0 \in \mathcal{T}$ such that (3.4) holds, and assume without loss of generality that $u \mapsto t_3(\tau_0, u) - t_1(\tau_0, u)$ is not identically zero and

$$u \mapsto \frac{t_2(\tau_0, u) - t_1(\tau_0, u)}{t_3(\tau_0, u) - t_1(\tau_0, u)}$$

is non-constant on U. We note that the event

$$G_0 := \left\{\omega \in \Omega : \liminf_{n\to\infty} \frac{1}{n} \left|\left\{1 \leq i \leq n : \omega_i = \tau_0\right\}\right| > \wp\right\} \tag{3.14}$$

has probability $\mathbb{P}(G_0) = 1$ by the law of large numbers for any choice of

$$0 < \wp < q(\tau_0).$$

We write $\wp := q(\tau_0)/2$. In the sequel, we will only consider points $\omega \in G_0$. We will not quite prove (3.5) for $\omega \in G_0$, but the eventual full probability set appearing in Proposition 3.3 will be contained in G_0.

We start by noting that

$$\frac{1}{m(\omega_n)} \left|\sum_{j=4}^{m(\omega_n)} \exp\bigl(-2\pi i \lambda(\omega|_{n-1}) t_j(\omega_n, u)\xi\bigr)\right| \leq 1 - \frac{3}{m(\omega_n)}.$$

With this in mind, and writing

$$f_1(u) := t_2(\tau_0, u) - t_1(\tau_0, u) \quad \text{and} \quad f_2(u) := t_3(\tau_0, u) - t_1(\tau_0, u), \quad u \in U,$$

we may rather crudely estimate as follows for all $n \geq 1$ such that $\omega_n = \tau_0$:

$$\frac{1}{m(\omega_n)} \left| \sum_{j=1}^{m(\omega_n)} \exp\bigl(-2\pi i \lambda(\omega|_{n-1}) t_j(\omega_n, u)\xi\bigr) \right| \\ \leq \frac{1}{m(\omega_n)} \left| 1 + \exp\bigl(-2\pi i \lambda(\omega|_{n-1}) f_1(u)\xi\bigr) + \exp\bigl(-2\pi i \lambda(\omega|_{n-1}) f_2(u)\xi\bigr) \right| \\ + \left(1 - \frac{3}{m(\omega_n)}\right).$$

So, if we write $\zeta_{\omega,u}(n, \xi)$ for the term on the middle line, that is,

$$\zeta_{\omega,u}(n, \xi) = \left| 1 + \exp\bigl(-2\pi i \lambda\bigl(\omega|_{n-1}\bigr) f_1(u)\xi\bigr) + \exp\bigl(-2\pi i \lambda(\omega|_{n-1}) f_2(u)\xi\bigr) \right|,$$

then, recalling (3.8), we have now shown that

$$|\widehat{\eta_u^{\omega}}(\xi)| \leq \prod_{\substack{n \geq 1 \\ \omega_n = \tau_0}} \left[\frac{\zeta_{\omega,u}(n, \xi)}{m(\omega_n)} + \left(1 - \frac{3}{m(\omega_n)}\right) \right]. \tag{3.15}$$

The indices ω_n with $\omega_n \neq \tau_0$ will be irrelevant for the estimate, but there are plenty of indices $\omega_n = \tau_0$ by the assumption $\omega \in G_0$. Note that trivially $\zeta_{\omega,u}(n, \xi) \leq 3$, and the right-hand side of (3.15) gives useful information about precisely those indices $n \geq 1$ with $\omega_n = \tau_0$ for which $\zeta_{\omega,u}(n, \xi) < 3$.

To achieve a useful estimate for $\zeta_{\omega,u}(n, r)$, we note that

$$|1 + \exp(-2\pi i x) + \exp(-2\pi i y)| = 3$$

if and only if $\|x\| = 0 = \|y\|$, where $\|x\| \in [0, 1/2]$ stands for the distance of $x \in \mathbb{R}$ to the nearest integer. Furthermore, by compactness (or a more quantitative argument if desired), for any $\rho > 0$ there exists $\alpha > 0$ such that

$$\max\{\|x\|, \|y\|\} \geq \rho \;\Rightarrow\; |1 + \exp(-2\pi i x) + \exp(-2\pi i y)| \leq 3 - \alpha.$$

Recalling the definition of $\zeta_{\omega,u}(n, r)$, it follows that

$$\max\bigl\{\|\lambda(\omega|_{n-1}) f_1(u)\xi\|, \|\lambda(\omega|_{n-1}) f_2(u)\xi\|\bigr\} \geq \rho \;\Rightarrow\; \zeta_{\omega,u}(n, \xi) \leq 3 - \alpha. \tag{3.16}$$

So, now the remaining task is to show that the quantity on the left-hand side of (3.16) is greater than ρ quite often, if $\rho > 0$ is taken sufficiently small. To formulate a more rigorous statement, a few additional pieces of notation are beneficial. First, we will write

$$\theta(\tau) := \lambda(\tau)^{-1}, \quad \theta(W) := \prod_{i=j}^{j+k} \theta(\omega_i), \quad \text{and} \quad \lambda(W) := \prod_{i=j}^{j+k} \lambda(\omega_i),$$

whenever $\tau \in \mathcal{T}$, and $W = (\omega_j, \ldots, \omega_{j+k})$ is a finite word over $\mathcal{T}$. The collection of all finite words over $\mathcal{T}$ will be denoted by $\mathcal{T}^*$. The notation above agrees with (3.7). We also define

$$\lambda(\emptyset) := 1 =: \theta(\emptyset),$$

where $\emptyset$ is the empty word. It is unpleasant that the numbers $\lambda(\omega|_{n-1}) f_j(u)\xi$ from (3.16) decrease as n increases, so we wish to reindex them in increasing order. Second, we are only interested in those $n \geq 1$ for which $\omega_n = \tau_0$, and we want to reshape our notation to reflect this. So, for $\omega \in G_0$, write

$$\omega = W_1 W_2 \cdots, \tag{3.17}$$

where each W_m has the form $W_m = W'_m \tau_0$ with $W'_m \in (\mathcal{T} \setminus \{\tau_0\})^*$ (we allow $W'_m = \emptyset$ here). We will generally use the letter m to index the words W_m.

Now, we fix $\omega \in G_0$ and a large integer $M \geq 1$, and we define

$$\Theta_m := \Theta_m^{(M,\omega)} := \theta(\tau_0)\theta(W_{M-m+1} \cdots W_M), \qquad 1 \leq m \leq M. \tag{3.18}$$

Then $\Theta_1 = \theta(\tau_0)\theta(W_M)$ and $\Theta_m \leq \Theta_{m+1}$ for $1 \leq m \leq M - 1$.

Remark 3.19. Let $M \geq 1$ be a large integer, and let $\omega \in G_0$. Let $1 \leq n(1) < n(2) < \cdots < n(M+1)$ be the first $M + 1$ indices with $\omega(n(m)) = \tau_0$. Let

$$\xi \in [\theta(\omega|_{n(M)}), \theta(\omega|_{n(M+1)})) \quad \text{and} \quad \nu := \frac{\xi}{\theta(\omega|_{n(M)})} \in [1, \theta(W_{M+1})).$$

Then, if $1 \leq m \leq M$, and the numbers Θ_m are defined as in (3.18), we have

$$\Theta_m f_j(u)\nu = \theta(\tau_0)\frac{\lambda(W_1 \cdots W_M) f_j(u)\xi}{\lambda(W_{M-m+1} \cdots W_M)} = \lambda(\omega|_{n(M-m)-1}) f_j(u)\xi. \tag{3.20}$$

So, $\Theta_m f_j(u)\nu$ is far from an integer

for all $\nu \in [1, \theta(W_{M+1}))$ *and for most* $1 \leq m \leq M$,

if and only if $\lambda(\omega|_{n(m)-1}) f_j(u)\xi$ is far from an integer

for all $\xi \in [\theta(\omega|_{n(M)}), \theta(\omega|_{n(M+1)}))$ *and for most* $1 \leq m \leq M$.

Recalling (3.16), we need exactly the latter kind of information to treat the product (3.15), while the next lemma will give information of the former kind.

Lemma 3.21. *There is a set $G \subset G_0$ with $\mathbb{P}(G) = 1$ such that the following holds for all*

$$\omega = W_1 W_2 \cdots \in G \qquad \text{(as in (3.17))},$$

$M \geq 1$, $c > 0$, and $\delta \in (0,1)$. If $\rho > 0$ is sufficiently small, depending on $\delta, \wp, \theta_{\max}$ and $\log \theta_{\max} / \log \theta_{\min}$, where

$$\theta_{\min} := \inf\{\theta(\tau) : \tau \in \mathcal{T}\} > 1 \quad \text{and} \quad \theta_{\max} := \sup\{\theta(\tau) : \tau \in \mathcal{T}\} < \infty,$$

then the set

$$E_{\rho,\delta,M,\omega,c} := \Big\{ \frac{z_1}{z_2} : |z_i| \in [c, 2c] \text{ and } \exists\, \nu \in \big[1, \theta(W_{M+1})\big) \text{ such that} \\ \frac{1}{M} \Big| \big\{ 1 \leq m \leq M : \max\{\|\Theta_m z_1 \nu\|, \|\Theta_m z_2 \nu\|\} < \rho \big\} \Big| \geq 1 - \delta \Big\}$$

can be covered by $\lesssim_{\omega,c} \exp(H \cdot \log(1/\delta) \cdot \delta M)$ intervals of length $\lesssim_c \lambda_{\max}^M$, where $\lambda_{\max} = \theta_{\min}^{-1}$, and $H \geq 1$ depends on $\theta_{\min}, \theta_{\max}$ and $(\Omega, \mathbb{P})$. Here $\Theta_m = \Theta_m^{(M,\omega)}$ as in (3.18).

The notation $a \lesssim_p b$ above means that there exists a constant $C \geq 1$ depending only on the parameter p such that $0 \leq a \leq Cb$. The proof of the lemma is an "Erdős–Kahanes"-type argument, and is very similar to [7, Proposition 5.4] – so similar, in fact, that we can use many estimates from [7, Proposition 5.4] verbatim. The best way to describe the difference between Lemma 3.21 and [7, Proposition 5.4] is perhaps to say that Lemma 3.21 is a combination of [7, Proposition 5.4] and [9, Lemma 3.2]. The argument originates back to the works of Erdős [1] and Kahane [4]. If the reader is not familiar with the general scheme of the proof, then we recommend [5, Proposition 6.1] for a neat version of the argument in a simpler setting.

Before proving the lemma, we use it to prove Proposition 3.3.

Proof of Proposition 3.3. We claim that the set G appearing in the statement of Lemma 3.21 also works here. In other words, if $\omega \in G$, then

$$\dim_{\mathrm{H}}\{u \in U : \dim_{\mathrm{F}} \eta_u^\omega = 0\} = 0. \tag{3.22}$$

Assume that (3.22) fails, define a Borel set $B := \{u \in U : \dim_{\mathrm{F}} \eta_u^\omega = 0\}$, and let $\sigma \in \mathcal{M}(B)$ be an ε-Frostman measure for some $\varepsilon > 0$ (i.e., $\sigma([a,b]) \leq (b-a)^\varepsilon$ for all $a < b$). One can show that B is Borel as in Corollary 3.13. We will reach a contradiction by showing that $\sigma(B) = 0$. To do so, it suffices to show that $\sigma(B \cap I) = 0$ for all intervals $I \subset \mathbb{R}$ such that

$$u \mapsto \zeta(u) := \frac{f_1(u)}{f_2(u)} = \frac{t_2(\tau,u) - t_1(\tau,u)}{t_3(\tau,u) - t_1(\tau,u)}$$

is C_I-bilipschitz on I. Indeed, by analyticity, there is only a discrete set of values $u \in U$ where either $t_3(\tau,u) - t_1(\tau,u) = 0$ or $\zeta'(u) = 0$. We now fix such an interval I. Then, we also fix $\delta \in (0,1)$ and $M \geq 1$. We assume without loss of generality that there exists $c = c_I > 0$ such that

$$c \leq \inf_{u \in I} \min\{|f_1(u)|, |f_2(u)|\} \leq \sup_{u \in I} \max\{|f_1(u)|, |f_2(u)|\} \leq 2c. \tag{3.23}$$

The maps f_1, f_2 are real-analytic and non-constant by assumption, so I can, up to a countable set, be further partitioned into intervals where (3.23) holds. Thus, it suffices to show that $\sigma(B \cap I) = 0$ for all such intervals I.

Next, we find $\rho > 0$ so small that the conclusion of Lemma 3.21 holds for $E = E_{\rho,\delta M,\omega,c}$. As the lemma says, the set $E \cap \zeta(I)$ can be covered by $\lesssim_{\omega,c} \exp(H \cdot \log(1/\delta) \cdot \delta M)$ intervals of length $\lesssim_c \lambda_{\max}^M$, where $c = c_I$ is the constant appearing in (3.23). Since ζ is C_I-bilipschitz on I, the same conclusion (up to a change of constants) is true for the following set:

$$\tilde{E}_{M,\delta} := \Big\{ u \in I : \exists\, \nu \in \big[1, \theta(W_{M+1})\big) \text{ such that} \\ \frac{1}{M}\Big|\big\{1 \leq m \leq M : \max\big\{\|\Theta_m f_1(u)\nu\|, \|\Theta_m f_2(u)\nu\|\big\} < \rho\big\}\Big| \geq 1 - \delta \Big\}.$$

From the ε-Frostman property of σ, we infer that

$$\sigma(\tilde{E}_{M,\delta}) \lesssim_{\omega,c_I,C_I} \exp(H \cdot \log(1/\delta) \cdot \delta M) \cdot \lambda_{\max}^{\varepsilon M}. \tag{3.24}$$

Taking $\delta > 0$ sufficiently small, depending on ε, H, and $\lambda_{\max}$, we see from (3.24) that

$$\sum_{M \geq 1} \sigma(\tilde{E}_{M,\delta}) < \infty,$$

and consequently $\tilde{E} := \limsup_{M\to\infty} \tilde{E}_{M,\delta}$ has vanishing σ measure by the Borel–Cantelli lemma. To complete the proof, it remains to show that

$$B \cap I \subset \tilde{E}.$$

Pick $u \in I \setminus \tilde{E}$. We wish to show that $u \notin B$, or in other words,

$$\dim_{\mathrm{F}} \eta_u^{\omega} > 0.$$

Pick any $M \geq 1$ so large that $u \notin \tilde{E}_{M,\delta}$, and, as in Remark 3.19 above, let $1 \leq n(1) < n(2) < \cdots < n(M+1)$ be an enumeration of the first $M+1$ indices for which $\omega(n(m)) = \tau_0$. Recall from (3.20) the relationship

$$\Theta_m f_j(u)\left(\frac{\xi}{\theta(\omega|_{n(M)})}\right) = \lambda(\omega|_{n(M-m)-1}) f_j(u)\xi,$$

valid for $j \in \{1,2\}$, $1 \le m \le M$, and $\xi \in [\theta(\omega|_{n(M)}), \theta(\omega|_{n(M+1)}))$. Since

$$\nu := \frac{\xi}{\theta(\omega|_{n(M)})} \in [1, \theta(W_{M+1}))$$

for any such choice of ξ, the assumption $u \notin \tilde{E}_{M,\delta}$ states that

$$\begin{aligned}&\left|\left\{1 \le m \le M \colon \max\left\{\|\lambda(\omega|_{n(m)-1}) f_1(u)\xi\|, \|\lambda(\omega|_{n(m)-1}) f_2(u)\xi\|\right\} \ge \rho\right\}\right| \\ &\quad = \left|\left\{1 \le m \le M \colon \max\{\|\Theta_m f_1(u)\nu\|, \|\Theta_m f_2(u)\nu\|\} \ge \rho\right\}\right| \ge \delta M\end{aligned}$$

for all $\xi \in [\theta(\omega|_{n(M)}), \theta(\omega|_{n(M+1)}))$. Recalling (3.16) and then (3.15), we infer that

$$\left|\widehat{\eta_u^{\omega}}(\xi)\right| \le \left(1 - \frac{\alpha}{m(\tau_0)}\right)^{\delta M}, \qquad \xi \in \left[\theta(\omega|_{n(M)}), \theta(\omega|_{n(M+1)})\right), \tag{3.25}$$

where $\alpha = \alpha(\rho) > 0$. But since $\wp n(M) \lesssim M \le n(M)$ for $M \ge 1$ sufficiently large (recall the parameter $\wp$ from (3.14) and that $\omega \in G \subset G_0$), and also

$$\theta_{\min}^{n(M)} \le \theta(\omega|_{n(M)}) \le \theta(\omega|_{n(M+1)}) \le \theta_{\max}^{n(M+1)},$$

the estimate in (3.25) yields $\dim_{\mathrm{F}} \eta_u^{\omega} > 0$. The proof is complete. ■

It remains to establish Lemma 3.21.

Proof of Lemma 3.21. Fix $\omega = W_1 W_2 \cdots \in G_0$, $M \ge 1$, $c > 0$, $\delta, \rho \in (0,1)$. Assume that

$$z_1/z_2 \in E := E_{\rho,\delta,M,\omega,c}$$

with $|z_1|, |z_2| \in [c, 2c]$, so by definition there exists $\nu \in [1, \theta(W_{M+1}))$ such that

$$\left|\{1 \le m \le M \colon \max\{\|\Theta_m z_1 \nu\|, \|\Theta_m z_2 \nu\|\} < \rho\}\right| \ge (1-\delta)M. \tag{3.26}$$

We only consider the case $z_1, z_2 \in [c, 2c]$. Now, for $1 \le m \le M$, we write

$$\Theta_m z_1 \nu =: K_m + \varepsilon_m \quad \text{and} \quad \Theta_m z_2 \nu =: L_m + \delta_m, \tag{3.27}$$

where $K_m, L_m \in \mathbb{N}$, and $\varepsilon_m, \delta_m \in [-1/2, 1/2)$. To emphasise the obvious, all the numbers K_m, L_m, ε_m and δ_m depend on the parameters M, z_j, ν, ω, u even if we suppress this from the notation – whenever the reader sees K_m, say, we ask them to think of $K_m^{M,z_1,z_2,\nu,\omega,u}$. We note that

$$\min\{K_M, L_M\} \gtrsim \Theta_M \min\{z_1, z_2\}\nu \ge c\lambda_{\max}^{-|W_1 \cdots W_M|} \ge c\lambda_{\max}^{-M} \tag{3.28}$$

by the definition of Θ_M. Now, we discuss the rest of the proof in a heuristic manner. By (3.28), we have

$$\frac{z_1}{z_2} = \frac{\Theta_M z_1 \nu}{\Theta_M z_2 \nu} = \frac{K_M + \varepsilon_M}{L_M + \delta_M} \in B\left(\frac{K_M}{L_M}, C\lambda_{\max}^M\right), \quad C = C_c \geq 1. \tag{3.29}$$

To cover the ratios z_1/z_2, we will use the balls above, and hence we need to estimate the number of possible ratios K_M/L_M for all admissible choices of z_1, z_2, ν, ω. This number will be, in fact, estimated by finding an upper bound on the number of possible sequences

$$(K_m)_{m=1}^M \quad \text{and} \quad (L_m)_{m=1}^M. \tag{3.30}$$

We will use the fact that these sequences arise from the (real) sequences $(\Theta_m z_j \nu)_{j=1}^M$ satisfying (3.26). This will imply the following useful property on both sequences in (3.30). If $\rho > 0$ is picked sufficiently small in (3.26), then for most indices $1 \leq m \leq M-2$ (depending on $\delta > 0$ in (3.26)), the number K_{m+2} (resp. L_{m+2}) is determined by K_m and K_{m+1} (resp. L_m and L_{m+1}). And even for those values of m for which this fails, there are $\lesssim 1$ options for K_{m+2} and L_{m+2}. These properties will be established in Lemma 3.44 below. So, at the end of the day, estimating the number of sequences (3.30) boils down to the following combinatorial question. How many sequences $(n_m)_{m=1}^M$ of natural numbers are there such that

- for most indices m the number n_{m+2} is determined by (n_m, n_{m+1}), and
- for the remaining indices there are $\lesssim 1$ choices for n_{m+2}.

Note that this problem no longer contains any reference to u, ν, ω. The answer turns out to be so small that the proof can be concluded.

We turn to the details, and the first main task is to quantify the dependence of K_{m+2} on K_m, K_{m+1}. This estimate is verbatim the same as the one obtained in the proof of [7, Proposition 5.4], but we repeat the details for the reader's convenience. We start by observing that

$$\frac{\Theta_{m+1}}{\Theta_m} = \frac{\theta(W_{M-m} \cdots W_M)}{\theta(W_{M-m+1} \cdots W_M)} = \theta(W_{M-m}), \qquad 1 \leq m \leq M-1 \tag{3.31}$$

by (3.18). On the other hand, as a direct computation based on (3.27) shows, the ratio Θ_{m+1}/Θ_m is quite close to K_{m+1}/K_m as the difference of the two quantities is

$$\frac{\Theta_{m+1}}{\Theta_m} - \frac{K_{m+1}}{K_m} = \frac{\varepsilon_{m+1}}{K_m} - \left(\frac{\Theta_{m+1}}{\Theta_m}\right)\frac{\varepsilon_m}{K_m}, \tag{3.32}$$

as a direct computation based on (3.27) shows. In the sequel we will write

$$\theta := \theta(\tau_0) > 1.$$

We also define $\beta(\tau) > 0$, $\tau \in \mathcal{T}$ such that $\theta(\tau) = \theta^{\beta(\tau)}$ (in particular, $\beta(\tau_0) = 1$), and we write

$$\beta(W) := \sum_{i=j}^{j+k} \beta(\omega_j), \qquad W = (\omega_j, \dots, \omega_{j+k}) \in \mathcal{T}^*.$$

Then (3.31) can be rewritten as

$$\frac{\Theta_{m+1}}{\Theta_m} = \theta^{\beta(W_{M-m})}, \qquad 1 \le m \le M-1. \tag{3.33}$$

Next, combining (3.32) and (3.33), we obtain

$$\left|\theta^{\beta(W_{M-m})} - \frac{K_{m+1}}{K_m}\right| \le \frac{\theta^{\beta(W_{M-m})}|\varepsilon_m| + |\varepsilon_{m+1}|}{K_m}, \quad 1 \le m \le M-1. \tag{3.34}$$

Noting that $\beta(W_{M-m})^{-1} \le \beta(\tau_0)^{-1} = 1$ since W_{M-m} ends in τ_0, we may infer from (3.34) further that

$$\begin{aligned}\left|\theta - \left(\frac{K_{m+1}}{K_m}\right)^{\beta(W_{M-m})^{-1}}\right| &= \left|(\theta^{\beta(W_{M-m})})^{\beta(W_{M-m})^{-1}} - \left(\frac{K_{m+1}}{K_m}\right)^{\beta(W_{M-m})^{-1}}\right| \\ &\le \left|\theta^{\beta(W_{M-m})} - \frac{K_{m+1}}{K_m}\right| \le \frac{\theta^{\beta(W_{M-m})}|\varepsilon_m| + |\varepsilon_{m+1}|}{K_m},\end{aligned} \tag{3.35}$$

using also the inequality $|x^s - y^s| \le |x - y|$, valid for $x, y \ge 1$ and $0 \le s \le 1$. Similarly, we have

$$\left|\theta - \left(\frac{K_{m+2}}{K_{m+1}}\right)^{\beta(W_{M-(m+1)})^{-1}}\right| \le \frac{\theta^{\beta(W_{M-(m+1)})}|\varepsilon_{m+1}| + |\varepsilon_{m+2}|}{K_{m+1}}. \tag{3.36}$$

Using trivial estimates, it follows from (3.34) that

$$\frac{K_{m+1}}{K_m} \le \theta^{\beta(W_{M-m})} + \frac{\theta^{\beta(W_{M-m})} + 1}{2K_m} \le 2\theta^{\beta(W_{M-m})} \le (2\theta_{\max})^{\beta(W_{M-m})} \tag{3.37}$$

and we have a similar estimate for K_{m+2}/K_{m+1}, with the only difference of replacing m by $m+1$. Note that

$$\beta(W_{M-(m+1)}) \le 1 + \beta(W_{M-(m+1)}) \le \theta^{\beta(W_{M-(m+1)})/\log\theta} \le \theta_{\max}^{k\beta(W_{M-(m+1)})},$$

where $k \in \mathbb{N}$ is such that $e^{1/k} \le \theta_{\min}$, and consequently, by (3.37),

$$\begin{aligned}&\beta(W_{M-(m+1)}) \max\left\{\left(\frac{K_{m+2}}{K_{m+1}}\right)^{\frac{\beta(W_{M-(m+1)})^{-1}}{\beta(W_{M-(m+1)})}}, \left(\frac{K_{m+1}}{K_m}\right)^{\frac{\beta(W_{M-(m+1)})^{-1}}{\beta(W_{M-m})}}\right\} \\ &\le \theta_{\max}^{k\beta(W_{M-(m+1)})}(2\theta_{\max})^{\beta(W_{M-(m+1)})^{-1}} =: (C\theta_{\max}^{k+1})^{\beta(W_{M-(m+1)})}.\end{aligned} \tag{3.38}$$

Thus, using the inequality $|x^s - y^s| \le s \max\{x^{s-1}, y^{s-1}\}|x-y|$ valid for $x, y > 0$ and $s \ge 1$, we get (note that $s = \beta(W_{M-(m+1)}) \ge 1$ since $W_{M-(m+1)}$ ends in τ_0)

$$\left|\frac{K_{m+2}}{K_{m+1}} - \left(\frac{K_{m+1}}{K_m}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-m})}}\right|$$
$$= \left|\left(\frac{K_{m+2}}{K_{m+1}}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-(m+1)})}} - \left(\frac{K_{m+1}}{K_m}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-m})}}\right|$$
$$\le \beta(W_{M-(m+1)}) \max\left\{\left(\frac{K_{m+2}}{K_{m+1}}\right)^{\frac{\beta(W_{M-(m+1)})-1}{\beta(W_{M-(m+1)})}}, \left(\frac{K_{m+1}}{K_m}\right)^{\frac{\beta(W_{M-(m+1)})-1}{\beta(W_{M-m})}}\right\}$$
$$\cdot \left|\left(\frac{K_{m+2}}{K_{m+1}}\right)^{\beta(W_{M-(m+1)})^{-1}} - \left(\frac{K_{m+1}}{K_m}\right)^{\beta(W_{M-m})^{-1}}\right|$$
$$\le (C\theta_{\max}^{k+1})^{\beta(W_{M-(m+1)})}\left[\frac{\theta^{\beta(W_{M-(m+1)})}|\varepsilon_{m+1}|+|\varepsilon_{m+2}|}{K_{m+1}} + \frac{\theta^{\beta(W_{M-m})}|\varepsilon_m|+|\varepsilon_{m+1}|}{K_m}\right]$$

by applying (3.37), (3.38), (3.35), and (3.36). Finally, this yields

$$\left|K_{m+2} - K_{m+1}\left(\frac{K_{m+1}}{K_m}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-m})}}\right|$$
$$\le (C\theta_{\max}^{k+2})^{\beta_{\max}(|W_{M-m}|+|W_{M-(m+1)}|)} \cdot \max\{|\varepsilon_m|, |\varepsilon_{m+1}|, |\varepsilon_{m+2}|\},$$
$$1 \le m \le M-2. \quad (3.39)$$

Here $|W|$ denotes the length of the word $W \in \mathcal{T}^*$, $C \ge 1$ is an absolute constant, and

$$\beta_{\max} := \sup\{\beta(\tau) : \tau \in \mathcal{T}\} \le \frac{\log \theta_{\max}}{\log \theta_{\min}}. \quad (3.40)$$

As far as the argument above is concerned, there is no difference between the numbers K_m and L_m (recall (3.27)). Hence also

$$\left|L_{m+2} - L_{m+1}\left(\frac{L_{m+1}}{L_m}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-m})}}\right|$$
$$\le (C\theta_{\max}^{k+2})^{\beta_{\max}(|W_{M-m}|+|W_{M-(m+1)}|)} \cdot \max\{|\delta_m|, |\delta_{m+1}|, |\delta_{m+2}|\},$$
$$\le m \le M-2. \quad (3.41)$$

For $1 \le m \le M-2$, we write

$$B_m := (C\theta_{\max}^{k+2})^{\beta_{\max}(|W_{M-m}|+|W_{M-(m+1)}|)} \quad \text{and} \quad \rho_m := (2B_m)^{-1}. \quad (3.42)$$

Then, it is immediate from (3.39) and (3.41) that whenever $1 \le m \le M-2$ and

$$\max\{|\delta_m|, |\delta_{m+1}|, |\delta_{m+2}|, |\varepsilon_m|, |\varepsilon_{m+1}|, |\varepsilon_{m+2}|\} < \rho_m, \quad (3.43)$$

we have

$$\max\left\{\left|K_{m+2}-K_{m+1}\left(\frac{K_{m+1}}{K_m}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-m})}}\right|,\left|L_{m+2}-L_{m+1}\left(\frac{L_{m+1}}{L_m}\right)^{\frac{\beta(W_{M-(m+1)})}{\beta(W_{M-m})}}\right|\right\}<\frac{1}{2}$$

Since K_{m+2} and L_{m+2} are integers, this implies that the pair (K_{m+2}, L_{m+2}) is uniquely determined by the pairs (K_m, L_m) and (K_{m+1}, L_{m+1}). This proves (b) of the following lemma, which is a modification of [7, Lemma 5.5] to the case of two sequences.

Lemma 3.44. *Let* $1 \le m \le M-2$.

(a) *Given* (K_m, L_m), (K_{m+1}, L_{m+1}), *there are at most* $(2B_m+1)^2$ *possible choices for the pair* (K_{m+2}, L_{m+2}). *Further, there are* $\lesssim_{c,\tau_0} B_0^4$ *choices for the quadruple* (K_1, L_1, K_2, L_2), *where*

$$B_0 := (\theta_{\max})^{|W_{M-1}W_M W_{M+1}|}.$$

(b) *If* (3.43) *holds, then the pair* (K_{m+2}, L_{m+2}) *is uniquely determined by the pairs* (K_m, L_m) *and* (K_{m+1}, L_{m+1}).

The first statement in (a) follows from the estimates (3.39) and (3.41). We justify the second statement in (a). The number of possible choices for K_j, $j \in \{1,2\}$ is the cardinality of natural numbers, K_j satisfying the equation $\Theta_j z_j \nu = K_j + \varepsilon_j$ with parameters Θ_j, z_j and $\nu \in [1, \theta(W_{M+1}))$. By definition,

$$\Theta_1 = \theta(\tau_0)\theta(W_M) \lesssim_{\tau_0} B_0 \quad \text{and} \quad \Theta_2 = \theta(\tau_0)\theta(W_{M-1}W_M) \lesssim_{\tau_0} B_0.$$

Since $|z_j| \in [c, 2c]$ by assumption, we see that $|K_1| \lesssim_{c,\tau_0} B_0$ and $|K_2| \lesssim_{c,\tau_0} B_0$. In other words, the number of possible values of K_1 and K_2 is bounded from above by $\lesssim_{c,\tau_0} B_0$. A similar estimate holds for the number of possible values for L_1 and L_2.

Heuristic digression. Before giving the final details, we make a little heuristic digression. Assume for a moment (completely unrealistically) that (3.43) holds for all $1 \le m \le M-2$. Then, by Lemma 3.44 (b), the pair (K_{m+2}, L_{m+2}) would always be uniquely determined by (K_m, L_m) and (K_{m+1}, L_{m+1}). This would imply that the total number of sequences $(K_m, L_m)_{m=1}^M$ is equal to the number of initial quadruples (K_1, L_1, K_2, L_2), that is, $\lesssim_{c,\tau_0} B_0^4$. So, how large is B_0^4 actually? Recall that $\omega \in G_0$ (as in (3.14)), so

$$\liminf_{n\to\infty} \frac{1}{n}\left|\{1 \le i \le n : \omega_i = \tau_0\}\right| > \wp. \tag{3.45}$$

In particular, the gap $|W_{M+1}| = n(M+1) - n(M)$ between two consecutive indices $n(j)$ with $\omega(n(j)) = \tau_0$ becomes arbitrarily short relative to $n(M)$, as $M \to \infty$. It follows that for any $\delta > 0$, we have

$$|W_{M-1}W_M W_{M+1}| \leq \delta |W_1 \cdots W_M|$$

for $M \gg_{\delta,\omega} 1$, and hence $B_0^4 = (\theta_{\max}^4)^{|W_{M-1}W_M W_{M+1}|} \leq \exp(C\delta|W_1 \cdots W_M|)$. Since $|W_1 \cdots W_M|$ is comparable to M for $M \gg_\omega 1$ by (3.45), this would complete the proof under the assumption that (3.43) holds for all $1 \leq m \leq M-2$.

The remaining details. We shall now continue the rigorous proof of Lemma 3.21. Recall from (3.26) that $z_1, z_2 \in [-2c, -c] \cup [c, 2c]$ and $\nu \in [1, \theta(W_{M+1}))$ are such that

$$|\{1 \leq m \leq M : \max\{\|\Theta_m z_1 \nu\|, \|\Theta_m z_2 \nu\|\} < \rho\}| \geq (1-\delta)M,$$

and note that this can be re-written as

$$|\{1 \leq m \leq M : \max\{|\varepsilon_m|, |\delta_m|\} \geq \rho\}| < \delta M.$$

Consequently,

$$|\{1 \leq m \leq M-2 : \max\{|\varepsilon_m|, |\varepsilon_{m+1}|, |\varepsilon_{m+2}|, |\delta_m|, |\delta_{m+1}|, |\delta_{m+2}|\} \geq \rho\}| \leq 3\delta M. \tag{3.46}$$

The property in (3.46) may look similar to the useful condition (3.43), except that there is now a fixed number ρ instead of ρ_m. Fortunately, it turns out that if $\rho > 0$ is taken small enough, depending on δ, $|\mathcal{T}|$, $\theta_{\max}$, then actually $\rho \leq \rho_m$ for most choices of m, and (3.46) does provide useful information.

Let $N := |W_1 \cdots W_M|$, and pick $M \geq 1$ (depending on ω) so large that

$$\frac{M}{N} = \frac{1}{N}\big|\{1 \leq n \leq N : \omega_n = \tau_0\}\big| \geq \wp. \tag{3.47}$$

This is possible by (3.45). Since $N = \sum_{1 \leq m \leq M} |W_m|$, we infer from Chebyshev's inequality and (3.47) that

$$\left|\left\{1 \leq m \leq M : |W_m| \geq \frac{2}{\wp\delta}\right\}\right| \leq \frac{\wp\delta N}{2} \leq \frac{\delta M}{2}. \tag{3.48}$$

Then set

$$\rho := \frac{1}{2}(C\theta_{\max})^{-4\beta_{\max}/(\wp\delta)},$$

where $\beta_{\max} \leq \log\theta_{\max}/\log\theta_{\min}$ is familiar from (3.40). Now is also a good time to recall the number ρ_m, $1 \leq m \leq M-2$ from (3.40). We next claim that

$$|\{1 \leq m \leq M-2 : \rho \geq \rho_m\}| \leq \delta M. \tag{3.49}$$

To see this, re-write the inequality $\rho \geq \rho_m$ as

$$(C\theta_{\max})^{-4\beta_{\max}/(\wp\delta)} \geq (C\theta_{\max})^{-\beta_{\max}(|W_{m-m}|+|W_{M-m-1}|)}.$$

This is equivalent to

$$|W_{M-m}| + |W_{M-m-1}| \geq 4/(\wp\delta),$$

which implies $\max\{|W_{M-m}|, |W_{M-m-1}|\} \geq 2/(\wp\delta)$. By (3.48), this is only possible for at most δM indices $m \in \{1, \dots, M-2\}$, as claimed.

Now, note that if $\max\{|\varepsilon_m|, |\varepsilon_{m+1}|, |\varepsilon_{m+2}|, |\delta_m|, |\delta_{m+1}|, |\delta_{m+2}|\} \geq \rho_m$, then either

$$\max\{|\varepsilon_m|, |\varepsilon_{m+1}|, |\varepsilon_{m+2}|, |\delta_m|, |\delta_{m+1}|, |\delta_{m+2}|\} \geq \rho \quad \text{or} \quad \rho \geq \rho_m.$$

Thus, combining (3.46) and (3.49), we find that the index set

$$\begin{aligned} \mathcal{I} &:= \mathcal{I}_{M,z_1,z_2,\nu,\omega} \\ &:= \{1 \leq m \leq M-2 \colon \max\{|\varepsilon_m|, |\varepsilon_{m+1}|, |\varepsilon_{m+2}|, |\delta_m|, |\delta_{m+1}|, |\delta_{m+2}|\} \geq \rho_m\} \end{aligned}$$

has cardinality

$$|\mathcal{I}| \leq 4\delta M. \tag{3.50}$$

Now, it is time to set aside the parameters ω, ν for a moment. Let us just consider the following combinatorial question. Fix an index set $\mathcal{J} \subset \{1, \dots, M-2\}$ and consider all possible sequences of pairs of natural numbers $(k_m, l_m)_{m=1}^M$ with the properties that

(i) there are $A_0 \in \mathbb{N}$ choices for the initial quadruple (k_1, l_1, k_2, l_2),

(ii) for (k_m, l_m) and (k_{m+1}, l_{m+1}) fixed, the pair (k_{m+2}, l_{m+2}) can be chosen in at most $A_m \in \mathbb{N}$ different ways, and

(iii) for $m \in \{1, \dots, M-2\} \setminus \mathcal{J}$, the pair (k_{m+2}, l_{m+2}) is uniquely determined by the pairs (k_m, l_m) and (k_{m+1}, l_{m+1}).

How many sequences $(k_m, l_m)_{m=1}^M$ are there satisfying (i)–(iii)? The answer is: at most

$$A_0 \cdot \prod_{m \in \mathcal{J}} A_m$$

sequences.

Returning to the main line of the proof, we recall from Lemma 3.44, combined with (3.50), that the sequence $(K_m, L_m)_{m=1}^M$ satisfies the conditions (i)–(iii) with constants $A_0 \lesssim_{c,\tau_0} B_0^4$ and $A_m = (2B_m + 1)^2$, and with index set $\mathcal{J} = \mathcal{I}_{M,z_1,z_2,\nu,\omega} = \mathcal{I}$. Thus, there are at most

$$\lesssim_{c,\tau_0} \mathcal{B}_M := \mathcal{B}_{M,\omega,\nu} := B_0^4 \cdot \prod_{m \in \mathcal{I}} (2B_m + 1)^2 \tag{3.51}$$

sequences $(K_m, L_m)_{m=1}^M$ corresponding to this $\mathcal{I}$.

The proof so far has only used the assumption $\omega \in G_0$, but the rest of the argument only works for ω in a slightly smaller set $G \subset \Omega$ (which still has full probability). This is because of the quantity on the right-hand side of (3.51), which depends on ω; recall the definitions of B_0 and B_m from Lemma 3.44 and (3.42). The quantity would be too large if the lengths of the words $W_1, \dots, W_{M+1}$ were very unevenly distributed. At the end of the proof of [7, Proposition 5.4] (see also [7, Lemma 5.2]), the following estimate is obtained, which holds for all $\omega \in G_0$ in a set of full probability (this set is finally the set G), and for all $M \geq 1$ sufficiently large (depending on ω):

$$\max_{\substack{\mathcal{I} \subset \{0,\dots,M-2\} \\ |\mathcal{I}| \leq 4\delta M}} \sum_{m \in \mathcal{I}} (|W_{M-n}| + |W_{M-n-1}|) \leq C \cdot \log(1/4\delta) \cdot \delta M, \tag{3.52}$$

where $C \geq 1$ is a constant depending on $(\Omega, \mathbb{P})$. In particular, for these sequences $\omega \in G_0$, and recalling from (3.50) that $|\mathcal{I}| \leq 4\delta M$, one obtains the following estimate from the definition of the numbers B_0, B_m, and (3.52):

$$\mathcal{B}_M \leq \exp(H \cdot \log(1/\delta) \cdot \delta M). \tag{3.53}$$

Here $H \geq 1$ is a constant depending only on $\theta_{\min}$, $\theta_{\max}$, and $(\mathbb{P}, \Omega)$, as desired. In fact, the contribution from the lonely factor B_0^4 could be handled in a more elementary way, as explained in the heuristic digression earlier, and only requires $\omega \in G_0$.

Now we have argued that the number of sequences $(K_m, L_m)_{m=1}^{M}$ arising from the fixed index set $\mathcal{I}_{M,z_1,z_2,\omega,\nu}$ is bounded by a constant times the right-hand side of (3.53). To wrap up, we use Stirling's formula to observe that the number of subsets of $\{0, \dots, M-2\}$ of cardinality at most $4\delta M$ is bounded from above by $\exp(C\delta M)$. So, the previous estimate for the number of sequences only changes by a constant factor if we take all relevant index sets into account!

Recalling (3.29) and the discussion following (3.29), the proof of Lemma 3.21 is now complete. ∎

4. Proof of the main result

This section contains the proof of Theorem 1.5. The argument is very similar to that in [7, Section 6]. However, from a technical perspective, many steps in the proof in [7] seem to require slight adjustment in our setting. Such adjustments would be difficult to explain properly without repeating virtually all of the details from [7] – even where no adjustments are necessary.

Here are the assumptions of the main theorem once more.

Definition 4.1. Let $U \subset \mathbb{R}$ be an open interval and $m \geq 2$. We assign to each $u \in U$ a list of contractive similitudes on $\mathbb{R}$ of the form

$$\Psi_u := (\psi_{u,1}, \dots, \psi_{u,m}) = (\lambda_1 x + t_1(u), \dots, \lambda_m x + t_m(u)), \tag{4.2}$$

where

$$\lambda_1, \dots, \lambda_m \in (0,1) \quad \text{and} \quad t_1(u), \dots, t_m(u) \in \mathbb{R}, \quad u \in U.$$

We make the following assumptions:

(A1) The map $u \mapsto t_j(u)$ is real-analytic, and the family $\{\Psi_u\}_{u \in U}$ satisfies transversality of order K for some $K \in \mathbb{N}$, recall Definition 1.3.

(A2) There exist three sequences $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \{1, \dots, m\}^{\mathbb{N}}$ such that none of the maps $u \mapsto \psi_{u,\mathbf{i}}(0)$, $u \mapsto \psi_{\mathbf{j},u}(0)$, and $u \mapsto \psi_{\mathbf{k},u}(0)$ is a convex combination of the other two.

(A3) For some probability vector $\mathbf{p} = (p_1, \dots, p_m) \in (0,1)^m$ with $p_1 + \cdots + p_m = 1$, the *similarity dimension* is

$$s(\bar{\lambda}, \mathbf{p}) := \frac{\sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \lambda_j},$$

where $\bar{\lambda} = (\lambda_1, \dots, \lambda_m)$, satisfies $s(\bar{\lambda}, \mathbf{p}) > 1$.

Here is the main result again.

Theorem 4.3. *Let μ_u, $u \in U$, be the self-similar measure associated with a pair $(\Psi_u, \mathbf{p})$ satisfying the assumptions in Definition 4.1. Then, there exists a set $E \subset U$ of Hausdorff dimension 0 such that $\mu_u \ll \mathcal{L}^1$ for all $u \in U \setminus E$.*

We start by recording the following consequence of assumption (A1).

Proposition 4.4. *Assume (A1), and define the numbers*

$$\Delta_n(u) := \min\{|\psi_{u,\mathbf{i}}(0) - \psi_{u,\mathbf{j}}(0)| : \mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^n,\ \mathbf{i} \neq \mathbf{j}\}.$$

Then, there exists a set $E \subset U$ with $\dim_{\mathrm{H}} E = 0$ such that

$$\limsup_{n \to \infty} \frac{\log \Delta_n(u)}{n} > -\infty, \qquad u \in U \setminus E.$$

The statement above is superficially the same as [3, Theorem 5.9], but recall that we are using a definition of transversality somewhat different from Hochman's. We postpone the proof to the appendix, see Proposition A.3.

Now we start the proof of Theorem 4.3 by fixing a number $N \geq 1$. We recall the types $\mathcal{T} = \mathcal{T}^N$ defined in (2.5). Then, for every $u \in U$, we follow the procedure of Sections 2.1–2.2 to write

$$\mu_u = \int_\Omega \eta_u^\omega \, d\mathbb{P}(\omega), \tag{4.5}$$

where

$$\eta_u^\omega = \mathop{\ast}_{n \geq 1} \left[\prod_{j=1}^{n-1} \lambda(\omega_j) \right]_\sharp \eta_u(\omega_n)$$

and

$$\eta_u(\tau) = \frac{1}{m(\tau)} \sum_{j=1}^{m(\tau)} \delta_{\psi^{\tau}_{u,j}(0)}.$$

We recall that the maps in

$$\Psi_u(\tau) = (\psi^{\tau}_{u,1}, \ldots, \psi^{\tau}_{u,m(\tau)}) = (\lambda(\tau)x + t_1(\tau, u), \ldots, \lambda(\tau)x + t_{m(\tau)}(\tau, u)),$$
$$u \in U,\ \tau \in \mathcal{T},$$

were obtained (via the procedure described in Section 2.2) as N-fold compositions of the maps in Ψ_u in (4.2), and they all have a common contraction ratio $\lambda(\tau)$, depending only on $\tau \in \mathcal{T}$.

Next, as in [7], we fix another integer parameter $s \geq 1$. Then, for $\omega \in \Omega$ and $u \in U$ fixed, we split the infinite convolution defining η^{ω}_u as $\eta^{\omega}_u = \eta^{\omega}_{\mathrm{small},u} * \eta^{\omega}_{\mathrm{big},u}$, where

$$\eta^{\omega}_{\mathrm{small},u} := \left(\mathop{\ast}_{s \text{ divides } n} \left[\prod_{j=1}^{n-1} \lambda(\omega_j) \right]_{\sharp} \eta_u(\omega_n) \right) \tag{4.6}$$

and

$$\eta^{\omega}_{\mathrm{big},u} := \left(\mathop{\ast}_{s \text{ does not divide } n} \left[\prod_{j=1}^{n-1} \lambda(\omega_j) \right]_{\sharp} \eta_u(\omega_n) \right).$$

The plan will be to show that, for generic choices of ω, u, the measure $\eta^{\omega}_{\mathrm{small},u}$ has positive Fourier dimension, whereas $\eta^{\omega}_{\mathrm{big},u}$ has Hausdorff dimension one (if N and s were chosen large enough). These observations are eventually combined in Section 4.3 to complete the proof of Theorem 4.3. If the reader is not familiar with the argument in [7], then it might be a good idea to start with reading (the short) Section 4.3 to see where we are heading.

4.1. Fourier decay for $\eta^{\omega}_{\mathrm{small},u}$

We infer the following corollary from Proposition 3.3.

Corollary 4.7. *Assume the same notation as in the previous section. Assume that there exist $\tau_0 \in \mathcal{T}$ and three indices $1 \leq i_1 < i_2 < i_3 \leq m(\tau_0)$ such that the map $u \mapsto t_{i_3}(\tau_0, u) - t_{i_1}(\tau_0, u)$ is not identically zero, and*

$$u \mapsto \frac{t_{i_2}(\tau_0, u) - t_{i_1}(\tau_0, u)}{t_{i_3}(\tau_0, u) - t_{i_1}(\tau_0, u)}, \quad u \in U, \tag{4.8}$$

is non-constant. Then, there exists a set $G \subset \Omega$ with $\mathbb{P}(G) = 1$ such that if $\omega \in G$, then

$$\dim_{\mathrm{H}}\{u \in U : \dim_{\mathrm{F}} \eta^{\omega}_{\mathrm{small},u} = 0\} = 0.$$

Here $t_j(\tau, u)$, $1 \le j \le m(\tau)$, are the translation vectors of the similitudes in $\Psi_u(\tau)$. For Proposition 3.3 to be applicable, we first need to realise $\eta^{\omega}_{\mathrm{small},u}$ as a typical measure arising from a random model as in Section 2.1. Here we mostly follow the proof of [7, Lemma 6.4].

Proof of Corollary 4.7. We first define a new set of types $\mathcal{T}' := \mathcal{T}^s$. For any choice of $\tau' := (\omega_1, \dots, \omega_s) \in \mathcal{T}'$, we define the contraction ratio

$$\lambda(\tau') := \lambda(\omega_1) \cdots \lambda(\omega_s). \tag{4.9}$$

We also define the probabilities

$$q'(\tau') := q(\omega_1) \cdots q(\omega_s), \qquad \tau' = (\omega_1, \dots, \omega_s) \in \mathcal{T}',$$

where $q(\tau) > 0$ are the probabilities associated with the initial types $\tau \in \mathcal{T}$. Clearly,

$$\sum_{\tau' \in \mathcal{T}'} q'(\tau') = 1.$$

We let $\mathbb{P}'$ be the product probability measure on the space $\Omega' := (\mathcal{T}')^{\mathbb{N}}$ induced by the probabilities $q'(\tau')$. Then, we define the similitudes

$$\Psi_u(\tau') := \{\lambda(\tau')x + t_1(\omega_s, u), \dots, \lambda(\tau')x + t_{m(\omega_s)}(\omega_s, u)\} \tag{4.10}$$

for $\tau' = (\omega_1, \dots, \omega_s) \in \mathcal{T}'$. Now that these types and similitudes have been defined, the formulae in Section 2.1 give rise to the measures

$$\eta_u(\omega_1, \dots, \omega_s) = \frac{1}{m(\omega_s)} \sum_{j=1}^{m(\omega_s)} t_j(\omega_s, u) = \eta_u(\omega_s), \quad (\omega_1, \dots, \omega_s) \in \mathcal{T}',$$

and finally,

$$\eta^{\omega'}_u = \mathop{\ast}_{n \ge 1} \Big[\prod_{j=1}^{n-1} \lambda(\omega'_j) \Big]_\sharp \eta_u(\omega'_n), \tag{4.11}$$

where $\omega'_j, \omega'_n \in \mathcal{T}'$ for $j, n \ge 1$.

Next, we "embed" the random measures $\eta^{\omega}_{\mathrm{small},u}$ inside the family of random measures defined in (4.11). To this end, if $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, we define the sequence $F(\omega) \in \Omega'$ by the obvious formula

$$F(\omega) = ((\omega_1, \dots, \omega_s), (\omega_{s+1}, \dots, \omega_{2s}), \dots). \tag{4.12}$$

Then, it follows from the definitions (4.6) and (4.9)–(4.11) that

$$\eta^{F(\omega)}_u = \eta^{\omega}_{\mathrm{small},u}, \qquad \omega \in \Omega,$$

where the left-hand side refers to the measure defined in (4.11). Further, we note that $F_\sharp \mathbb{P} = \mathbb{P}'$, where $\mathbb{P}$ is the probability on $\Omega = \mathcal{T}^{\mathbb{N}}$ induced by the probabilities $q(\tau)$. Hence, the conclusion of Corollary 4.7 will follow once we manage to produce a set $G' \subset \Omega'$ of full $\mathbb{P}'$-probability such that

$$\dim_{\mathrm{H}}\{u \in U : \dim_{\mathrm{F}} \eta_u^{\omega'} = 0\} = 0, \qquad \omega' \in G'.$$

Here we finally use Proposition 3.3. All we need to find is a type $\tau' \in \mathcal{T}'$ and three indices $1 \le i_1 < i_2 < i_3 \le m(\tau')$ such that the map $u \mapsto t'_{i_3}(\tau', u) - t'_{i_1}(\tau', u)$ is not identically zero and

$$u \mapsto \frac{t'_{i_2}(\tau', u) - t'_{i_1}(\tau', u)}{t'_{i_3}(\tau', u) - t'_{i_1}(\tau', u)}, \quad u \in U, \tag{4.13}$$

is non-constant. (We also note that the assumption $\sup\{|t_j(\tau', u)| : u \in U, \tau \in \mathcal{T}', 1 \le j \le m(\tau')\} < \infty$ from Definition 3.1 can be arranged by splitting U to countably many intervals, since the maps $u \mapsto t_j(\tau', u) \in (0, 1)$ are continuous each, and $\mathcal{T}'$ is finite.)

Returning to (4.13), we recall from (4.10) that the translation vectors associated with the type $(\omega_1, \dots, \omega_s) \in \mathcal{T}'$ coincide with the translation vectors of the type $\omega_s \in \mathcal{T}$. Thus, we can – for example – take $\tau' := (\tau_0, \tau_0, \dots, \tau_0) \in \mathcal{T}^s$, where $\tau_0 \in \mathcal{T}$ is the type appearing in (4.8). The proof of Corollary 4.7 is complete. ■

In order to use Corollary 4.7 in the proof of Theorem 4.3, we need to secure its main hypothesis. This is the content of the next lemma.

Lemma 4.14. *Under the assumptions (A1)–(A2), there are arbitrarily large values of $N \ge 1$ such that the following holds. There exist a type $\tau_N \in \mathcal{T}^N$ and three values $1 \le i_1 < i_2 < i_3 \le m(\tau_N)$ such that the map $u \mapsto t_{i_3}(\tau_N, u) - t_{i_1}(\tau_N, u)$ is not identically zero, and*

$$u \mapsto \frac{t_{i_2}(\tau_N, u) - t_{i_1}(\tau_N, u)}{t_{i_3}(\tau_N, u) - t_{i_1}(\tau_N, u)}, \qquad u \in U,$$

is non-constant.

Proof. Let $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \{1, \dots, m\}^{\mathbb{N}}$ be the sequences specified in (A2). In other words, none of the maps $u \mapsto \psi_{u,\mathbf{i}}(0)$, $u \mapsto \psi_{u,\mathbf{j}}(0)$, and $u \mapsto \psi_{u,\mathbf{k}}(0)$ can be expressed as a convex combination of the other two. In particular,

$$\psi_{u,\mathbf{i}}(0) \not\equiv \psi_{u,\mathbf{j}}(0) \quad \text{and} \quad \psi_{u,\mathbf{i}}(0) \not\equiv \psi_{u,\mathbf{k}}(0).$$

Thus, by analyticity, $u \mapsto \psi_{u,\mathbf{k}}(0) - \psi_{u,\mathbf{i}}(0)$ has a discrete set of zeroes on U, and

$$u \mapsto \zeta(u) := \frac{\psi_{u,\mathbf{j}}(0) - \psi_{u,\mathbf{i}}(0)}{\psi_{u,\mathbf{k}}(0) - \psi_{u,\mathbf{i}}(0)}$$

is well defined and analytic in the complement of those points. Moreover, ζ is non-constant, because if $\zeta \equiv C$ for some $C \in [0,1]$, one can solve

$$\psi_{u,\mathbf{j}}(0) \equiv C \cdot \psi_{u,\mathbf{k}}(0) + (1-C) \cdot \psi_{u,\mathbf{i}}(0),$$

violating the choice of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The cases $C < 0$ and $C > 1$ are also ruled out by similar calculations. For example, if $\zeta \equiv C \in (-1, 0)$, then one can solve

$$\psi_{u,\mathbf{i}}(0) \equiv \frac{1}{1-C} \cdot \psi_{u,\mathbf{j}}(0) + \frac{-C}{1-C} \cdot \psi_{u,\mathbf{k}}(0)$$

instead, again violating the choice of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. We now pick $u_1, u_2 \in U$ such that $\zeta(u_1), \zeta(u_2)$ are finite and distinct.

Then, we note that for any $u \in U$, in particular, $u \in \{u_1, u_2\}$, it holds that

$$\sup\{|\psi_{u,\mathbf{i}}(0) - \psi_{u,\mathbf{w}}(0)| \colon \mathbf{w} \in \{1,\dots,m\}^*, \ \mathbf{w}|_n = \mathbf{i}|_n\} \to 0, \tag{4.15}$$

as $n \to \infty$, where $\{1,\dots,m\}^* = \bigcup_{n \in \mathbb{N}} \{1,\dots,m\}^n$. The same holds with $\mathbf{i}$ replaced by $\mathbf{j}$ or $\mathbf{k}$. Applying (4.15) at the points $u_1, u_2 \in U$, we infer that there exists $M \in \mathbb{N}$ such that the following holds. If $\mathbf{i}', \mathbf{j}', \mathbf{k}' \in \{1,\dots,m\}^*$ are any finite sequences with

$$\mathbf{i}'|_M = \mathbf{i}|_M =: \mathbf{i}_0, \quad \mathbf{j}'|_M = \mathbf{j}|_M =: \mathbf{j}_0, \quad \text{and} \quad \mathbf{k}'|_M = \mathbf{k}|_M =: \mathbf{k}_0,$$

then $u \mapsto \psi_{u,\mathbf{k}'}(0) - \psi_{u,\mathbf{i}'}(0)$ is not identically zero, and the map

$$u \mapsto \frac{\psi_{u,\mathbf{j}'}(0) - \psi_{u,\mathbf{i}'}(0)}{\psi_{u,\mathbf{k}'}(0) - \psi_{u,\mathbf{i}'}(0)}, \qquad u \in U, \tag{4.16}$$

is non-constant (it suffices to check that the map takes different values at u_1 and u_2).

We apply this to sequences $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ of the form

$$\mathbf{i}' := (\mathbf{i}_0\mathbf{j}_0\mathbf{k}_0)^N, \quad \mathbf{j}' := (\mathbf{j}_0\mathbf{k}_0\mathbf{i}_0)^N, \quad \text{and} \quad \mathbf{k}' := (\mathbf{k}_0\mathbf{i}_0\mathbf{j}_0)^N,$$

which have common length $3MN$, and more importantly common type in $\mathcal{T}^{3MN}$, say τ, recalling the definition (2.5). Then the numbers $\psi_{u,\mathbf{i}'}(0), \psi_{u,\mathbf{j}'}(0)$, and $\psi_{u,\mathbf{k}'}(0)$ coincide with certain translation vectors $t_{i_1}(\tau, u)$, $t_{i_2}(\tau, u)$, and $t_{i_3}(\tau, u)$, with $1 \le i_1 < i_2 < i_3 \le m(\tau)$. Thus, the non-constancy of the map in (4.16) is equivalent to the claim of the lemma. ∎

Combining the previous lemma with Corollary 4.7 finally gives the following consequence, which can be applied – eventually – in the proof of Theorem 4.3.

Corollary 4.17. *Under the assumptions (A1)–(A2), and if $N \ge 1$ is chosen as in Lemma 4.14, there exists a set $G \subset \Omega$ with $\mathbb{P}(G) = 1$ such that if $\omega \in G$, then*

$$\dim_{\mathrm{H}}\{u \in U \colon \dim_{\mathrm{F}} \eta^{\omega}_{\mathrm{small},u} = 0\} = 0.$$

4.2. Dimension of $\eta^{\omega}_{\mathrm{big},u}$

In this section, we study the dimension of the measures $\eta^{\omega}_{\mathrm{big},u}$, again following [7] closely. Here is the goal.

Proposition 4.18. *If the parameters $N, s \geq 1$ are chosen large enough, then there exists a set $E \subset U$ of Hausdorff dimension zero such that for all $u \in U \setminus E$*

$$\dim_{\mathrm{H}} \eta^{\omega}_{\mathrm{big},u} = 1 \quad \textit{for } \mathbb{P} \textit{ a.e. } \omega \in \Omega.$$

In fact, the set E coincides with the set from Proposition 4.4.

The first task is, again, to realise $\eta^{\omega}_{\mathrm{big},u}$ as a typical measure arising from a random model, as in Section 2.1. The details are the same as in the proof of [7, Lemma 6.5], but we record most of them here for completeness. As in the previous section, we define $\mathcal{T}' := (\mathcal{T})^s$, and we also define

$$\lambda(\tau') := \lambda(\omega_1) \cdots \lambda(\omega_s) \quad \text{and} \quad q(\tau') := q(\omega_1) \cdots q(\omega_s) \tag{4.19}$$

for $\tau' = (\omega_1, \ldots, \omega_s) \in \mathcal{T}'$, as before. We also let $\mathbb{P}'$ be the product probability measure on $\Omega' = (\mathcal{T}')^{\mathbb{N}}$ induced by the numbers $q(\tau')$. Defining the translation vectors for the similitudes in $\Psi_u(\tau')$ is a little trickier in this case. Here is how to do it. For $\tau' = (\omega_1, \ldots, \omega_s) \in \mathcal{T}'$ fixed, we first let

$$\mathcal{I}(\tau') := \prod_{l=1}^{s-1} \{1, \ldots, m(\omega_l)\}.$$

Then, for any $\mathbf{i} = (i_1, \ldots, i_{s-1}) \in \mathcal{I}(\tau')$, we define the translation vector

$$t_{\mathbf{i}}(\tau', u) := \sum_{l=1}^{s-1} \Big[\prod_{j=1}^{l-1} \lambda(\omega_j) \Big] t_{i_l}(\omega_l, u),$$

where $t_{i_l}(\omega_l, u)$, $i_l \in \{1, \ldots, m(\omega_l)\}$, is the i_l-th translation vector of the family $\Psi_u(\omega_l)$. Then, we set

$$\Psi_u(\tau') := \{\lambda(\tau')x + t_{\mathbf{i}}(\tau', u) : \mathbf{i} \in \mathcal{I}(\tau')\}. \tag{4.20}$$

As in the previous section, we define the map $F: \Omega \to \Omega'$ by the formula (4.12). Then, one can check, see [7, (61)], that

$$\eta_u^{F(\omega)} = \eta^{\omega}_{\mathrm{big},u}, \qquad \omega \in \Omega,$$

where the left-hand side now refers to the measures generated by the model with the types and similitudes introduced in this section. Since $F_{\sharp}\mathbb{P} = \mathbb{P}'$, we can now proceed to study the $\mathbb{P}$-almost sure dimension of the measures $\eta^{\omega}_{\text{big},u}$, $\omega \in \Omega$, by studying the $\mathbb{P}'$-almost sure dimension of the measures $\eta^{\omega'}_{u}$, $\omega' \in \Omega'$.

Before doing this, however, we record an observation that requires staring at the precise structure of $\Psi_u(\tau')$.

Remark 4.21. Let $n \geq 1$, and let $(\omega'_1, \ldots, \omega'_n) \in (\mathcal{T}')^N$. For each ω_j, $1 \leq j \leq n$, pick two similitudes

$$\psi^{\omega'_j}_{u,\mathbf{v}_j}, \psi^{\omega'_j}_{u,\mathbf{w}_j} \in \Psi_u(\omega'_j), \qquad \mathbf{v}_j, \mathbf{w}_j \in \mathcal{I}(\omega'_j),$$

and consider their n-fold compositions

$$f_{u,\mathbf{v}} = \psi^{\omega'_1}_{u,\mathbf{v}_1} \circ \cdots \circ \psi^{\omega'_n}_{u,\mathbf{v}_n} \quad \text{and} \quad f_{u,\mathbf{w}} = \psi^{\omega'_1}_{u,\mathbf{w}_1} \circ \cdots \circ \psi^{\omega'_n}_{u,\mathbf{w}_n}.$$

For reasons to become apparent a little later, we are interested in relating the quantity $|f_{u,\mathbf{v}}(0) - f_{u,\mathbf{w}}(0)|$ to the numbers $\Delta_n(u)$ defined in Proposition 4.4. This would be completely straightforward if $f_{u,\mathbf{v}}$, $f_{u,\mathbf{w}}$ were obtained as certain compositions of mappings in Ψ_u, but this is not quite the case.

To understand the problem better, consider first $\tau' = (\omega_1, \ldots, \omega_s) \in \mathcal{T}'$, pick $\mathbf{i} = (i_1, \ldots, i_{s-1}) \in \mathcal{I}(\tau')$, and note that the map

$$x \mapsto \lambda(\omega_1) \cdots \lambda(\omega_{s-1})x + t_{\mathbf{i}}(\tau', u) \tag{4.22}$$

is, in fact, the composition

$$\psi^{\omega_1}_{u,i_1} \circ \cdots \circ \psi^{\omega_{s-1}}_{u,i_{s-1}},$$

where $\psi^{\omega_j}_{u,i_j}$ is the i_j-th similitude in $\Psi_u(\omega_j)$. Unfortunately, the contraction ratio of the map in (4.22) differs by a factor of $\lambda(\omega_s)$ from the contraction ratio of the map $x \mapsto \lambda(\tau')x + t_{\mathbf{i}}(\tau', u) \in \Psi_u(\tau')$.

Despite this issue, the difference $f_{u,\mathbf{v}} - f_{u,\mathbf{w}}$ can be expressed as the difference of compositions in Ψ_u. We explain this in the case $n = 1$, that is, when

$$f_{u,\mathbf{v}}(0) - f_{u,\mathbf{w}}(0) = \psi^{\tau'}_{u,\mathbf{i}}(0) - \psi^{\tau'}_{u,\mathbf{j}}(0), \qquad \mathbf{i}, \mathbf{j} \in \mathcal{I}(\tau'),$$

for some $\tau' = (\omega_1, \ldots, \omega_s) \in \mathcal{T}'$. We write $\mathbf{i} = (i_1, \ldots, i_{s-1})$ and $\mathbf{j} = (j_1, \ldots, j_{s-1})$, where $1 \leq i_l, j_l \leq m(\omega_l)$, and we let $\psi^{\omega_s}_u$ be any similitude in $\Psi_u(\omega_s)$. Then,

$$\psi^{\tau'}_{u,\mathbf{i}} - \psi^{\tau'}_{u,\mathbf{j}} = (\psi^{\omega_1}_{u,i_1} \circ \cdots \circ \psi^{\omega_{s-1}}_{u,i_{s-1}} \circ \psi^{\omega_s}_u) - (\psi^{\omega_1}_{u,j_1} \circ \cdots \circ \psi^{\omega_{s-1}}_{u,j_{s-1}} \circ \psi^{\omega_s}_u),$$

where both the maps on the right-hand side are (Ns)-fold compositions of maps in Ψ_u. For general $n \geq 1$, the difference $f_{u,\mathbf{v}} - f_{u,\mathbf{w}}$ can always be expressed as the difference of (Nns)-fold of compositions of maps in Ψ_u, by repeating the above idea n times and hence, adding altogether n "dummy" maps instead of one; for more details, see the proof of Lemma 6.5 (and the equation (62)) in [7].

In particular, we have

$$|f_{u,\mathbf{v}}(0) - f_{u,\mathbf{w}}(0)| \geq \Delta_{Nns}(u), \qquad \mathbf{v}, \mathbf{w} \in \prod_{j=1}^{n} \mathcal{I}(\omega_j'), \ \mathbf{v} \neq \mathbf{w}, \tag{4.23}$$

by the above observations.

To study the $\mathbb{P}'$-almost sure dimension of the measures $\eta_u^{\omega'}$, $\omega' \in \Omega'$, we need to import more technology from [7]. First, it follows from [7, Theorem 1.2] that the measures $\eta_u^{\omega'}$ are exact-dimensional $\mathbb{P}'$-almost surely. For $u \in U$, there exists a constant $\alpha_u \in [0, 1]$ such that

$$\exists \lim_{r \to 0} \frac{\log \eta_u^{\omega'}(B(x, r))}{\log r} = \alpha_u$$

for $\mathbb{P}'$-almost all $\omega' \in \Omega'$, and for $\eta_u^{\omega'}$-almost every $x \in \mathbb{R}$. In particular,

$$\dim_{\mathrm{H}} \eta_u^{\omega'} = \alpha_u$$

for $\mathbb{P}'$-almost every $\omega' \in \Omega'$. Another concept we need to recall from [7, Section 1.3] is the *similarity dimension* of a random model. Given a collection of types $\mathcal{T}''$, equipped with contraction ratios $\lambda(\tau'') \in (0, 1)$ and probabilities $q(\tau'') \in (0, 1)$, the *similarity dimension* of the family of random measures $\eta^{\omega''}$ generated by this data (through the procedure described in Section 2.1) is the number

$$s(\{\eta^{\omega''}\}_{\omega'' \in \Omega''}) := \left(\int_{\Omega''} \log(\lambda(\omega_1'')) \, d\mathbb{P}''(\omega'') \right)^{-1} \int_{\Omega''} \log \frac{1}{m(\omega_1'')} \, d\mathbb{P}''(\omega'').$$

Here $\mathbb{P}''$ is the product probability measure on $\Omega'' := (\mathcal{T}'')^{\mathbb{N}}$ induced by the probabilities $q(\tau'')$, $\tau'' \in \mathcal{T}''$. In fact, we have no use for the explicit expression above (which can be found in [7, Section 1.3]), but we need the concept – twice.

First, it follows from [7, Lemma 6.2 (v)] that if $\delta > 0$, and the parameter $N \geq 1$ is chosen large enough, depending only on δ and the probability vectors $\mathbf{p}$, then

$$s(\{\eta_u^{\omega}\}_{\omega \in \Omega}) \geq (1 - \delta) s(\bar{\lambda}, \mathbf{p}), \qquad u \in U. \tag{4.24}$$

Here $s(\bar{\lambda}, \mathbf{p})$ and $\{\eta_u^{\omega}\}_{\omega \in \Omega}$ were introduced around the statement of Theorem 4.3. We note, as is clear from the proof of [7, Lemma 6.2 (v)], that the choice of N in (4.24) depends only on $\delta > 0$, and the fixed probability vector $\mathbf{p}$. In particular, recalling our main assumption $1 < s(\bar{\lambda}, \mathbf{p}) =: 1 + \varepsilon$, we may choose $N \geq 1$ so large that also

$$s(\{\eta_u^{\omega}\}_{\omega \in \Omega}) > 1 + \varepsilon/2, \qquad u \in U, \tag{4.25}$$

where $\varepsilon > 0$ does not depend on the choice of $u \in U$.

Now we have fixed $N \geq 1$, and next we fix $s \geq 1$. On the very last page of [7], the following relationship between the similarity dimensions of $\{\eta_u^\omega\}_{\omega \in \Omega}$ and $\{\eta_u^{\omega'}\}_{\omega' \in \Omega'}$ is established:

$$s(\{\eta_u^{\omega'}\}_{\omega' \in \Omega'}) = \Big(1 - \frac{1}{s}\Big) s(\{\eta_u^\omega\}_{\omega \in \Omega}), \qquad u \in U.$$

Here $\{\eta_u^{\omega'}\}_{\omega' \in \Omega'}$ is the random model discussed in this section, recall (4.19)-(4.20). So, by taking $s \geq 1$ large enough, depending on $\varepsilon > 0$ alone, we can ensure that

$$s(\{\eta_u^{\omega'}\}_{\omega' \in \Omega'}) \geq 1 + \varepsilon/3, \qquad u \in U. \tag{4.26}$$

We summarise the previous conclusions for a fixed $u \in U$:

- To show that
$$\dim_{\mathrm{H}} \eta_{\mathrm{big},u}^\omega = 1 \quad \text{for } \mathbb{P} \text{ a.e. } \omega \in \Omega,$$
it suffices to prove that
$$\dim_{\mathrm{H}} \eta_u^{\omega'} = 1 \quad \text{for } \mathbb{P}' \text{ a.e. } \omega' \in \Omega'.$$
- The map $\omega' \mapsto \dim_{\mathrm{H}} \eta_u^{\omega'}$ has $\mathbb{P}'$-almost surely constant value α_u.
- The similarity dimension of the model $\{\eta_u^{\omega'}\}_{\omega' \in \Omega'}$ exceeds one.

So, to wrap up the proof of Proposition 4.18, it remains to argue that

$$\alpha_u = \min\{s(\{\eta_u^{\omega'}\}_{\omega' \in \Omega'}), 1\} = 1, \qquad u \in U \setminus E, \tag{4.27}$$

where $\dim_{\mathrm{H}} E = 0$. This will follow from a combination of [7, Theorem 1.3] and [3, Theorem 1.8].

For $\omega' = (\omega_1', \omega_2', \ldots) \in \Omega'$ and a fixed $n \geq 1$, define the index set

$$\mathcal{I}_n'(\omega') := \prod_{j=1}^{n} \mathcal{I}(\omega_j').$$

Here $\mathcal{I}(\omega_j')$ is the index set used in (4.20) to define the similitudes $\Psi_u(\omega_j')$, $\omega_j' \in \mathcal{T}'$. Now, given $u \in U$, and a word $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathcal{I}_n'(\omega')$, consider the map $f_{u,\mathbf{v}}$, obtained as the n-fold composition

$$f_{u,\mathbf{v}} = \psi_{u,\mathbf{v}_1}^{\omega_1'} \circ \cdots \circ \psi_{u,\mathbf{v}_n}^{\omega_n'}, \tag{4.28}$$

where $\psi_{u,\mathbf{v}_j}^{\omega_j'}(x) = \lambda(\omega_j')x + t_{\mathbf{v}_j}(\omega_j', u) \in \Psi_u(\omega_j')$, as defined in (4.20). Then, we define the quantity

$$\Delta_n(u,\omega') := \begin{cases} \min\{|f_{u,\mathbf{v}}(0) - f_{u,\mathbf{w}}(0)| \colon \mathbf{u},\mathbf{w} \in \mathcal{I}_n'(\omega'),\ \mathbf{v} \neq \mathbf{w}\}, & \text{if } |\mathcal{I}_n'(\omega')| \geq 2, \\ 0, & \text{if } |\mathcal{I}_n'(\omega')| = 1. \end{cases}$$

Now, (4.26) and [7, Theorem 1.3] show that

$$\alpha_u < 1 \;\Rightarrow\; \mathbb{P}\left\{\omega' \in \Omega' \colon \frac{\log \Delta_n(u,\cdot)}{n} \leq -M\right\} \to 1 \text{ for all } M > 0. \tag{4.29}$$

So, to prove (4.27), it suffices to show that the right-hand side of (4.29) can occur only for u in a zero-dimensional set. This is an easy consequence of Proposition 4.4 and (4.23). Indeed, (4.23) shows that $\Delta_n(u,\omega') \geq \Delta_{Nns}(u)$ whenever $|\mathcal{I}'_n(\omega')| \geq 2$.

Evidently, for $\mathbb{P}'$-almost every $\omega' \in \Omega'$ we have $|\mathcal{I}'_n(\omega')| \geq 2$ for all $n \geq 1$ sufficiently large, depending on ω'. It follows that $\mathbb{P}'(G'_n) \to 1$ as $n \to \infty$, where

$$G'_n := \{\omega' \in \Omega' \colon |\mathcal{I}'_n(\omega')| \geq 2\}.$$

Recall the exceptional E from Proposition 4.4: if $u \in U \setminus E$, it follows that there exists $M > 0$, and a sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers, depending on u, such that

$$\frac{\log \Delta_{Nn_js}(u)}{n_j} \geq -M, \qquad j \in \mathbb{N}.$$

Consequently,

$$\begin{aligned}\mathbb{P}'\left\{\omega' \in \Omega' \colon \frac{\log \Delta_{n_j}(u,\omega')}{n_j} \geq -M\right\} &\geq \mathbb{P}\left\{\omega' \in G'_{n_j} \colon \frac{\log \Delta_{Nn_js}(u)}{n_j} \geq -M\right\} \\ &= \mathbb{P}'(G'_{n_j}) \to 1.\end{aligned}$$

We conclude that the right-hand side of (4.29) does not hold, and hence $\alpha_u = 1$ for all $u \in U \setminus E$. The proof of Proposition 4.18 is complete.

4.3. Concluding the proof of the main theorem

We now conclude the proof of Theorem 4.3 (also known as Theorem 1.5). We start by making a counter-assumption that

$$\dim_{\mathrm{H}} E > \varepsilon > 0,$$

where $E := \{u \in U \colon \mu_u \not\ll \mathcal{L}^1\}$. We record that E is a G_δ-set. Indeed, following [5, Proposition 8.1] we first consider

$$E_\beta := \left\{u \in U \colon \exists \text{ open } V_u \subset \mathbb{R} \text{ such that } \mu_u(V_u) > 1-\beta \text{ and } \mathcal{L}^1(V_u) < \beta\right\}.$$

Since $\mu_{u'} \rightharpoonup \mu_u$ as $u' \to u$ by the continuity of the function $u \mapsto t_j(u)$, the sets E_β are open. We have thus shown the claim since $E = \bigcap_{\beta>0} E_\beta$ and, by [5, Proposition 3.1], self-similar measures are of pure type. We may now use Frostman's lemma to pick $\sigma \in \mathcal{M}(E)$ such that $\sigma(B(x,r)) \leq r^\varepsilon$ for all $x \in \mathbb{R}$ and $r > 0$.

Now, recall the decomposition of μ_u to the measures η_u^ω from (4.5) and the subsequent decomposition of the measures η_u^ω to the pieces $\eta_{\mathrm{small},u}^\omega$ and $\eta_{\mathrm{large},u}^\omega$. From Corollary 4.17 and Fubini's theorem we infer that for σ-almost every $u \in U$,

$$\dim_{\mathrm{F}} \eta_{\mathrm{small},u}^\omega > 0 \tag{4.30}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. The use of Fubini's theorem is legitimate, because the set

$$\{(\omega, u) \in \Omega \times U : \dim_{\mathrm{F}} \eta_{\mathrm{small},u}^\omega = 0\}$$

is Borel by same the argument we used in Corollary 3.13. Also, from Proposition 4.18, we deduce that for σ-almost every $u \in U$,

$$\dim_{\mathrm{H}} \eta_{\mathrm{big},u}^\omega = 1 \tag{4.31}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$ (here Fubini's theorem was not used, so we do not to need check that $\{(\omega, u) : \dim_{\mathrm{H}} \eta_{\mathrm{big},u}^\omega = 1\}$ is Borel). It follows that for σ-almost every $u \in U$, the conclusions (4.30)–(4.31) hold simultaneously for $\mathbb{P}$-almost every $\omega \in \Omega$. But whenever (4.30)–(4.31) both hold, [8, Lemma 2.1 (2)] implies that

$$\eta_u^\omega = \eta_{\mathrm{big},u}^\omega * \eta_{\mathrm{small},u}^\omega \ll \mathcal{L}^1.$$

In particular, for σ-almost all $u \in U$, we have $\eta_u^\omega \ll \mathcal{L}^1$ for $\mathbb{P}$-almost all $\omega \in \Omega$, and then $\mu_u \ll \mathcal{L}^1$ by the decomposition (4.5). So, we have now argued that $\mu_u \ll \mathcal{L}^1$ for σ-almost every $u \in U$, which contradicts the choice of σ. The proof of Theorem 4.3 is complete. ■

A. Order K transversality and the size of exceptions

Recall that we used a notion of order K transversality somewhat different from Hochman's convention in [3, Definition 5.6]. We recall our definition.

Definition A.1 (Transversality of order K). Let $U \subset \mathbb{R}$ be an open interval, and let $\{\Psi_u\}_{u \in U}$ be a parametrised family of similitudes of the form

$$\Psi_u := (\psi_{u,1}, \ldots, \psi_{u,m}) = (\lambda_1(u)x + t_1(u), \ldots, \lambda_m(u)x + t_m(u)).$$

Note that we allow also the contraction parameters $\lambda_j(u)$ to depend on $u \in U$. Let $K \in \mathbb{N}$, and assume that the maps $u \mapsto \lambda_j(u)$ and $u \mapsto t_j(u)$ are K times continuously differentiable for all $1 \le j \le m$. For $u \in U$, write

$$\Delta_{\mathbf{i},\mathbf{j}}(u) := \psi_{u,\mathbf{i}}(0) - \psi_{u,\mathbf{j}}(0), \qquad \mathbf{i}, \mathbf{j} \in \{1, \ldots, m\}^n,\ n \in \mathbb{N}.$$

The family $\{\Psi_u\}_{u\in U}$ *satisfies transversality of order* K if there exists a constant $c > 0$ and a sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that $n_j \to \infty$, and

$$\max_{k\in\{0,\dots,K\}} |\Delta^{(k)}_{\mathbf{i},\mathbf{j}}(u)| \geq c^{n_j}, \qquad u \in U,\ \mathbf{i},\mathbf{j} \in \{1,\dots,m\}^{n_j},\ \mathbf{i} \neq \mathbf{j},\ j \in \mathbb{N}.$$

Here $\Delta^{(k)}_{\mathbf{i},\mathbf{j}}$ is the k-th derivative of $\Delta_{\mathbf{i},\mathbf{j}}$.

Recall from Proposition 4.4 that we need to show that the following set has Hausdorff dimension zero:

$$E := \left\{u \in U\colon \lim_{n\to\infty} \frac{\log \Delta_n(u)}{n} = -\infty\right\}, \tag{A.2}$$

where $\Delta_n(u) := \min\{|\Delta_{\mathbf{i},\mathbf{j}}(u)|\colon \mathbf{i},\mathbf{j} \in \{1,\dots,m\}^n,\ \mathbf{i} \neq \mathbf{j}\}$. This follows from transversality of order K, as in Definition A.1.

Proposition A.3. *Assume that* $\{\Psi_u\}_{u\in\mathbb{N}}$ *is a parametrised family of similitudes satisfying transversality of some finite order* $K \in \mathbb{N}$*, as in Definition* A.1*. Then, the set* E *in* (A.2) *has Hausdorff dimension zero.*

Proof. We follow the proof of [3, Theorem 5.9], which seems to work fine with our definition of transversality. Without change in notation, we replace U by a compact subinterval; it clearly suffices to show that the part of E in any such subinterval has Hausdorff dimension zero. In particular, then we have

$$C := C_U := \max_{0\leq k\leq K} \sup_{n\geq 1} \max_{\mathbf{i},\mathbf{j}\in\{1,\dots,m\}^n} \|\Delta^{(k)}_{\mathbf{i},\mathbf{j}}\|_{L^\infty(U)} < \infty,$$

noting that the contraction parameters $\lambda_j(u)$ are uniformly bounded away from 1 on U. We observe that $E \subset \bigcap_{\varepsilon>0} E_\varepsilon$, where

$$E_\varepsilon := \bigcup_{N\in\mathbb{N}} \bigcap_{j\geq N} \bigcup_{\substack{\mathbf{i},\mathbf{j}\in\{1,\dots,m\}^{n_j} \\ \mathbf{i}\neq\mathbf{j}}} \{u \in U\colon |\Delta_{\mathbf{i},\mathbf{j}}(u)| < \varepsilon^{n_j}\} =\colon \bigcup_{N\in\mathbb{N}} E^N_\varepsilon,$$

and $(n_j)_{j\in\mathbb{N}}$ is the sequence from the definition of transversality. So, it suffices to argue that $\underline{\dim}_{\mathrm{B}} E^N_\varepsilon \to 0$ uniformly in $N \in \mathbb{N}$ as $\varepsilon \to 0$, where $\underline{\dim}_{\mathrm{B}}$ denotes the lower box dimension, an upper bound for Hausdorff dimension. Fix $N \in \mathbb{N}$, pick $0 < \varepsilon < c$, and then choose $j \geq N$ so large that $\varepsilon^{n_j} < c^{n_j}/2^K$. By [3, Lemma 5.8], the sets

$$E^{\mathbf{i},\mathbf{j}}_\varepsilon := \{u \in U\colon |\Delta_{\mathbf{i},\mathbf{j}}(u)| < \varepsilon^{n_j}\}, \qquad \mathbf{i},\mathbf{j} \in \{1,\dots,m\}^{n_j},\ \mathbf{i} \neq \mathbf{j},$$

can be covered, each, by $\lesssim_C c^{-2n_j}$ intervals of length at most

$$2(\varepsilon^{n_j}/c^{n_j})^{1/2^K} =\colon r_{n_j}.$$

Given that there are only m^{2n_j} options for the pair $\mathbf{i}, \mathbf{j} \in \{1, \ldots, m\}^{n_j}$, this implies that

$$N(E_\varepsilon^N, r_{n_j}) \leq N\Bigg(\bigcup_{\substack{\mathbf{i},\mathbf{j} \in \{1,\ldots,m\}^{n_j} \\ \mathbf{i} \neq \mathbf{j}}} E_\varepsilon^{\mathbf{i},\mathbf{j}}, r_{n_j}\Bigg) \lesssim_C \Big(\frac{m}{c}\Big)^{2n_j},$$

where $N(A, r)$ is the least number of intervals of length $r > 0$ needed to cover a bounded set $A \subset \mathbb{R}$. It follows that

$$\underline{\dim}_{\mathrm{B}} E_\varepsilon^N \leq \liminf_{j \to \infty} \frac{\log N(E_\varepsilon^N, r_{n_j})}{-\log r_{n_j}} \leq \liminf_{j \to \infty} \frac{O(C) + 2n_j \log(m/c)}{(n_j/2^K)\log(c/\varepsilon) - \log 2}.$$

The right-hand side evidently tends to 0 as $\varepsilon \to 0$, with rate of convergence independent of $N \in \mathbb{N}$, as claimed. The proof is complete. ■

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