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Schrödinger equations defined by a class of self-similar measures

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Abstract. We study linear and non-linear Schrödinger equations defined by fractal measures. Under the assumption that the Laplacian has compact resolvent, we prove that there exists a unique weak solution for a linear Schrödinger equation, and then use it to obtain the existence and uniqueness of a weak solution of a non-linear Schrödinger equation. We prove that for a class of self-similar measures on $\mathbb{R}$ with overlaps, the linear Schrödinger equations can be discretized so that the finite element method can be applied to obtain approximate solutions. We also prove that the numerical solutions converge to the actual solution and obtain the rate of convergence.

1. Introduction

The Schrödinger operator in the fractal setting has been studied by a number of authors. Strichartz [29] studied the essential spectrum on the product of two copies of an infinite blowup of the Sierpiński gasket. Chen et al. ([8]) studied the spectral asymptotics of the eigenvalues and Bohr's formula on several unbounded fractals. For Schrödinger operators defined by measures, typically self-similar measures, the authors of [25] studied the bound states and Bohr's formula. It has become more and more apparent to physicists that spacetime exhibits fractal behaviour. A multifractal spacetime model has recently been proposed by a physicist, Calcagni [2–4]. This is obtained by replacing the standard Lebesgue measure on a spacetime manifold with a Borel measure which is in general not absolutely continuous with respect to Lebesgue measure. Also, solution of the Schrödinger equation could play an important role in studying related mathematical problems. In fact, Hu and Zähle obtained heat kernel upper bound in metric measure spaces by using the solution of the Schrödinger equation [18]. Motivated by these, we study the Schrödinger equation defined by a fractal measure.

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Let $U \subset \mathbb{R}^d$, $d \geq 1$ be a bounded open set, and let μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Let $L^2(U,\mu) := L^2(U,\mu,\mathbb{C})$ denote the space of measurable functions $u\colon U \to \mathbb{C}$ such that $\|u\|_\mu < \infty$ with

$$\|u\|_\mu := \Big(\int_U |u|^2 \, d\mu\Big)^{1/2}.$$

Let $H^1(U) := H^1(U,\mathbb{C})$ be the Sobolev space of complex-valued functions equipped with the norm

$$\|u\|_{H^1(U)} := \Big(\int_U |u|^2 \, dx + \int_U |\nabla u|^2 \, dx\Big)^{1/2}.$$

Let $H_0^1(U) := H_0^1(U,\mathbb{C})$ denote the completion of $C_c^\infty(U)$ in the H^1-norm, where $C_c^\infty(U)$ is the space of all complex-valued $C^\infty(U)$ functions with compact support in U. Throughout this paper, we regard $L^2(U,\mu)$ and $H_0^1(U)$ as real Hilbert spaces with the scalar product

$$(u,v)_\mu := \operatorname{Re} \int_U u\overline{v} \, d\mu \quad \text{and} \quad (u,v)_{H_0^1(U)} := \operatorname{Re} \int_U \nabla u \cdot \nabla \overline{v} \, dx,$$

respectively (see, e.g., [5,6]), where $\operatorname{Re}(z)$ denotes the real part of a complex number z and $\overline{v}$ denotes the conjugate function of v. It is known (see, e.g., [17]) that μ defines a Dirichlet Laplace operator Δ_μ^D (or simply Δ_μ), if the following *Poincaré inequality for a measure (PI)* holds: There exists some constant $C > 0$ such that

$$\int_U |u|^2 \, d\mu \leq C \int_U |\nabla u|^2 \, dx \qquad \text{for all } u \in C_c^\infty(U) \tag{1.1}$$

(see, e.g., [17,22,23]). Recall that the *lower L^∞-dimension* of μ is defined as

$$\underline{\dim}_\infty(\mu) := \varliminf_{\delta \to 0^+} \frac{\ln\big(\sup_x \mu(B_\delta(x))\big)}{\ln \delta},$$

where the supremum is taken over all $x \in \operatorname{supp}(\mu)$. We remark that if $\underline{\dim}_\infty(\mu) > d-2$, then inequality (1.1) holds (see [17]). Moreover, one can prove by using [33, Proposition 2.1] that $d - 2 \leq \dim_{\mathrm{H}}(\mu) := \inf\{\dim_{\mathrm{H}}(E)\colon \mu(\mathbb{R}^d \setminus E) = 0\} \leq d$. In particular, if $d = 1$, (PI) holds for any positive finite Borel measure μ, and thus Δ_μ is well defined. For self-similar measures satisfying the open set condition, some conditions equivalent to (PI) can be found in [17]. Validity of more general Poincaré type inequalities for measures, under various conditions, have been studied extensively in the literature (see, e.g., [1, 15] and the references therein). In particular, it is shown in [1] that if Ω is a bounded $W^{1,p}$-extension domain on $\mathbb{R}^d$, $S \subset \overline{\Omega}$ is closed, $p \in (1,d)$, $d - p < \delta < d$, and μ is δ-Ahlfors regular on S, then

$$\|\operatorname{Tr} f\|_{L^q(S,d\mu)} \leq C\|f\|_{W^{1,p}(\Omega)}$$

for every $f \in W^{1,p}(\Omega)$ and $p \leq q \leq p\delta/(d-p)$. This implies, in particular, that if $d \geq 3$, $p = q = 2$, and $d - 2 < \delta < d$, then the Poincaré inequality stated in (1.1) holds.

The main purpose of this paper is to study the following linear Schrödinger equation defined by the Dirichlet Laplacian Δ_μ:

$$\begin{cases} i\partial_t u + \Delta_\mu u = f(t) & \text{on} \quad U \times [0,T], \\ u = 0 & \text{on} \quad \partial U \times [0,T], \\ u = g & \text{on} \quad U \times \{t = 0\}, \end{cases} \tag{1.2}$$

where $u := u(t)$ is a Hilbert space valued function of t. We study the solution of equation (1.2) both theoretically and numerically.

We will describe the construction of the Laplacian $-\Delta_\mu$ in (1.2), as well as the associated non-negative bilinear form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ (see (2.1)) in Section 2.1. To give an explicit formula for the weak solution of the Schrödinger equation (1.2), we assume that $-\Delta_\mu$ has compact resolvent. Then there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ such that $-\Delta_\mu \varphi_n = \lambda_n \varphi_n$ for all $n \geq 1$, where the eigenvalues satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$, we say that a map $F\colon \mathcal{H} \to \mathcal{H}$ is *Lipschitz continuous* on $\mathcal{H}$ if there exists some constant $C > 0$ such that $\|F(u) - F(v)\|_{\mathcal{H}} \leq C\|u - v\|_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$. See Definition 2.2 and (3.2) for the definitions of $L^2([0,T], X)$ and $E_\alpha(U,\mu)$, respectively, where X is a Banach space and $\alpha \geq 0$. Using Theorem 3.1, we obtain our first main theorem.

Theorem 1.1. *Let $U \subset \mathbb{R}^d$, $d \geq 1$, be a bounded open set, and let μ be a positive finite Borel measure with* $\operatorname{supp}(\mu) \subseteq \overline{U}$ *and $\mu(U) > 0$. Assume that μ satisfies (PI), $-\Delta_\mu$ has compact resolvent, and $F(\cdot)$ is Lipschitz continuous on* $\operatorname{dom}\mathcal{E}$. *If $g = \sum_{n=1}^\infty b_n \varphi_n \in \operatorname{dom}\mathcal{E}$, then the following non-linear Schrödinger equation*

$$\begin{cases} i\partial_t u + \Delta_\mu u = F(u) & \textit{on} \quad U \times [0,T], \\ u = 0 & \textit{on} \quad \partial U \times [0,T], \\ u = g & \textit{on} \quad U \times \{t = 0\} \end{cases} \tag{1.3}$$

has a unique weak solution $u(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ satisfying

$$u(t) = \sum_{n=1}^\infty b_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^\infty \Big(\int_0^t e^{-i\lambda_n(t-\tau)}\big(F(u(\tau)), \varphi_n\big)_\mu \, d\tau\Big)\varphi_n.$$

Moreover, under the additional assumption that $F(\cdot)$ is Lipschitz continuous on $E_\alpha(U,\mu)$ and $g \in E_\alpha(U,\mu)$, where $\alpha \geq 2$, we have $u(t) \in L^\infty([0,T], E_\alpha(U,\mu))$ and $\partial_t u(t) \in L^\infty([0,T], E_{\alpha-2}(U,\mu))$.

As an example, let $F(u) = \sin u - mu$, $m \geq 0$ (see, e.g., [16]). Then $F(\cdot)$ is Lipschitz continuous on $\operatorname{dom}\mathcal{E}$.

We call a closed and connected μ-measurable subset I of $\overline{U}$ a *cell (in $\overline{U}$)* if $\mu(I) > 0$. Clearly, each connected component of $\overline{U}$ is a cell. We say that two cells I, J in U are *measure disjoint* with respect to μ if $\mu(I \cap J) = 0$. Let $I \subseteq \overline{U}$ be a cell. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of I if $J \subseteq I$ for all $J \in \mathbf{P}$, and $\mu(I) = \sum_{J \in \mathbf{P}} \mu(J)$. A sequence of μ-partitions $(\mathbf{P}_k)_{k \geq 1}$ is *refining* if for any $J_1 \in \mathbf{P}_k$ and any $J_2 \in \mathbf{P}_{k+1}$, either $J_2 \subseteq J_1$ or they are measure disjoint, i.e., each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$. Throughout this paper, $|E|$ denotes the diameter of a subset $E \subseteq \mathbb{R}^d$.

In order to discretize (1.2) and obtain numerical approximations of the weak solution, we will often impose the following additional conditions on μ: (1) μ is a continuous (i.e., atomless) measure on $\mathbb{R}$; (2) there exists a sequence of refining μ-partitions $(\mathbf{P}_k)_{k \geq 1}$ of $\overline{U}$ such that for any $k \geq 2$ and any $I \in \mathbf{P}_k$, there exist similitudes $(\tau_{I,J})_{J \in \mathbf{P}_1}$ of the form $\tau_{I,J}(x) = r_{I,J}x + b_{I,J}$ and constants $(c_{I,J})_{J \in \mathbf{P}_1}$ such that $\tau_{I,J}(J) \subseteq I$, and

$$\mu|_I = \sum_{J \in \mathbf{P}_1} c_{I,J} \cdot \mu|_J \circ \tau_{I,J}^{-1}. \tag{1.4}$$

Formula (1.4) implies that the μ measure of each cell in the partition can be computed, making it possible to discretize the Schrödinger equation (1.2).

Let $U = (a, b)$ and $f(x, t) \equiv 0$ in (1.2). Multiplying the first equation in (1.2) by $\overline{v} \in \operatorname{dom}\mathcal{E}$, integrating both sides over $[a, b]$ with respect to $d\mu$, and then taking the real parts, we obtain

$$\operatorname{Re} \int_a^b i\partial_t u(x,t)\overline{v}(x)\, d\mu = \operatorname{Re} \int_a^b \partial_x u(x,t)\overline{v}'(x)\, dx, \tag{1.5}$$

where $\partial_x u(x, t)$ and $\partial_t u(x, t)$ are the partial derivatives of u with respect to x and t, respectively. Let $u_1(x, t)$ and $u_2(x, t)$ be the real and imaginary parts of $u(x, t)$, respectively. Then (1.5) can be rewritten as

$$\int_a^b \partial_t u_2(x,t) v(x)\, d\mu = -\int_a^b \partial_x u_1(x,t) v'(x)\, dx \tag{1.6}$$

and

$$\int_a^b \partial_t u_1(x,t) v(x)\, d\mu = \int_a^b \partial_x u_2(x,t) v'(x)\, dx \tag{1.7}$$

for all real-valued functions $v \in \operatorname{dom}\mathcal{E}$.

Theorem 1.2. *Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ such that* $\operatorname{supp}(\mu) \subseteq [a, b]$. *Assume that there exists a sequence of refining μ-partitions* $(\mathbf{P}_k)_{k \geq 1}$

of $[a, b]$ *satisfying* (1.4)*, and* $g \in \operatorname{dom}\mathcal{E}$*. Then equations* (1.6) *and* (1.7) *can be discretized, and the finite element method can be applied to yield a system of first-order ordinary differential equations* (4.7)*. If, in addition,* $\int_I x^j \, d\mu$*,* $I \in \mathbf{P}_1$*,* $j = 0, 1, 2$*, can be evaluated explicitly, then the unique solution of equation* (4.7) *can be solved numerically.*

We are mainly interested in fractal measures. Let F be a non-empty compact subset of $\mathbb{R}^d$. Throughout this paper, an *iterated function system (IFS)* refers to a finite family of contractive similitudes $\{S_j\}_{j=1}^q$ defined on F, i.e.,

$$S_j(x) = \rho_j x + b_j, \qquad j = 1, \dots, q,$$

where $0 < \rho_j < 1$, and $b_j \in \mathbb{R}^d$. It is well known that for each IFS $\{S_j\}_{j=1}^q$, there exists a unique non-empty compact subset $K \subseteq F$, called the *self-similar set*, such that

$$K = \bigcup_{j=1}^q S_j(K);$$

moreover, associated to each set of probability weights $\{w_j\}_{j=1}^q$ (that is, $w_j > 0$ and $\sum_{j=1}^q w_j = 1$), there is a unique probability measure, called the *self-similar measure*, satisfying the following identity

$$\mu = \sum_{j=1}^q w_j \mu \circ S_j^{-1}$$

(see [12, 19]). An IFS $\{S_j\}_{j=1}^q$ is said to satisfy the *open set condition (OSC)* if there exists a non-empty bounded open set O such that $\bigcup_k S_k(O) \subseteq O$ and $S_k(O) \cap S_j(O) = \emptyset$ for all $k \neq j$. IFSs that do not satisfy (OSC), as well as all associated self-similar measures, are said to have overlaps.

It is worth pointing out that for general self-similar measures with overlaps, it does not seem possible to discretize the Schrödinger equations (1.2) in the way described in the paper, and thus it is not clear how numerical approximations of the weak solution can be obtained. Theorem 1.2 provides a framework under which discretization can be performed.

Based on Theorem 1.2, we solve the linear Schrödinger equation (1.2) numerically for three different one-dimensional self-similar measures with overlaps, namely, the infinite Bernoulli convolution associated with the golden ratio, the three-fold convolution of the Cantor measure, and a class of self-similar measures that we call *essentially of finite type (EFT)* (see [26]). These measures share the common property that the support can be partitioned into a sequence of arbitrarily small intervals whose measures can be computed explicitly.

The following theorem shows that the approximate solutions converge to the actual weak solution, and we also obtain a rate of convergence. See Section 2.1 and Definition 2.2 for the definitions of $\|\cdot\|_{\mathrm{dom}\,\mathcal{E}}$ and $\|\cdot\|_{2,\mathrm{dom}\,\mathcal{E}}$, respectively.

Theorem 1.3. *Assume the hypotheses of Theorem 1.2, let $U=(a,b)$, $f\equiv 0$ and $g=\sum_{n=1}^{\infty} b_n\varphi_n \in E_3((a,b),\mu)$ in equation (1.2), and fix $t\in[0,T]$. If there exist constants $r\in(0,1)$ and $c>0$ satisfying $\max\{|I|: I\in \mathbf{P}_k\}\le cr^k$ for all $k\ge 1$, then the approximate solutions u^m obtained by the finite element method converge in $L^2((a,b),\mu)$ to the actual weak solution u. Moreover,*

$$\|u^m-u\|_\mu \le 2\big(\sqrt{cT}\,\|\partial_t u\|_{2,\mathrm{dom}\,\mathcal{E}}+\|u\|_{\mathrm{dom}\,\mathcal{E}}\big)r^{m/2}.$$

The rest of this paper is organized as follows. Section 2 summarizes some notation, definitions and results that will be needed throughout the paper. We give the existence and uniqueness of solution of the Schrödinger equation (1.2) in Section 3. Section 4 is devoted to the proof of Theorem 1.2. In Section 5, we apply Theorem 1.2 to three different self-similar measures with overlaps. The proof of Theorem 1.3 is given in Section 6.

2. Preliminaries

In this section, we summarize some notation, definitions and facts that will be used throughout the rest of the paper. For a Banach space X, we denote its topological dual by X'.

Definition 2.1. Let X be a Banach space, $u\colon (a,b)\subseteq\mathbb{R}\to X$, and $t_0\in(a,b)$. Then u is said to be *differentiable* at t_0 in the norm $\|\cdot\|_X$ if there exists $v_0\in X$ such that

$$\lim_{h\to 0}\left\|\frac{u(t_0+h)-u(t_0)}{h}-v_0\right\|_X=0.$$

v_0 is called the *derivative* of u at t_0, and we write

$$v_0=\partial_t u(t_0)=\lim_{h\to 0}\frac{u(t_0+h)-u(t_0)}{h}.$$

Higher-order derivatives are defined similarly.

Note that if u is differentiable at t_0 in the norm $\|\cdot\|_X$, then it is continuous at t_0 in the norm $\|\cdot\|_X$.

Definition 2.2. Let X be a separable Banach space with norm $\|\cdot\|_X$. Denote by $L^p([0,T],X)$ the space of all measurable functions $u\colon[0,T]\to X$ satisfying

(1) $\|u\|_{L^p([0,T],X)} := \big(\int_0^T \|u(t)\|_X^p\,dt\big)^{1/p}<\infty$, if $1\le p<\infty$, and

(2) $\|u\|_{L^\infty([0,T],X)} := \operatorname{ess\,sup}_{0\le t\le T} \|u(t)\|_X < \infty$, if $p = \infty$.

If the interval $[0, T]$ is understood, we will abbreviate these norms as $\|u\|_{p,X}$ and $\|u\|_{\infty,X}$, respectively.

Remark 2.1. For each $1 \le p \le \infty$, $L^p([0, T], X)$ is a Banach space; furthermore, $L^{p_2}([0, T], X) \subseteq L^{p_1}([0, T], X)$ if $1 \le p_1 \le p_2 \le \infty$. Let X be a separable Banach space with inner product $(\cdot,\cdot)_X$. If $(X, (\cdot,\cdot)_X)$ is a separable Hilbert space, then $L^2([0, T], X)$ is a Hilbert space with the inner product

$$(u, v)_{L^2([0,T],X)} := \int_0^T (u(t), v(t))_X \, dt.$$

Definition 2.3. Let X be a Banach space. We define $C([0, T], X)$ to be the vector space of all continuous functions $u\colon [0, T] \to X$ such that

$$\|u\|_{C([0,T],X)} := \max_{0\le t\le T} \|u\|_X < \infty.$$

2.1. Dirichlet Laplacian defined by a measure

Let $U \subset \mathbb{R}^d$, $d \ge 1$, be a bounded open set and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that (PI) holds. In [17], it is shown that (PI) implies that μ defines a Dirichlet operator in

$$L^2(U, \mu, \mathbb{R}) := \Big\{u\colon U \to \mathbb{R}\colon \int_U |u|^2 \, d\mu < \infty\Big\}.$$

Similarly, we can define a Dirichlet operator in $L^2(U, \mu) := L^2(U, \mu, \mathbb{C})$, as follows. (PI) implies each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U, \mu)$ sense) member $\hat{u}$ that belongs to $L^2(U, \mu)$ and satisfies both conditions below.

(1) There exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \hat{u}$ in $H_0^1(U)$ and $u_n \to \hat{u}$ in $L^2(U, \mu)$;

(2) $\hat{u}$ satisfies inequality (1.1).

We call $\hat{u}$ the $L^2(U, \mu)$-*representative* of u. Define a mapping $\iota\colon H_0^1(U) \to L^2(U, \mu)$ by $\iota(u) = \hat{u}$. ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as $\mathcal{N} := \big\{u \in H_0^1(U)\colon \|\iota(u)\|_\mu = 0\big\}$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota\colon \mathcal{N}^\perp \to L^2(U, \mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U, \mu)$-representative $\hat{u}$ simply by u.

Consider the non-negative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2(U, \mu)$ defined by

$$\mathcal{E}(u, v) := \operatorname{Re} \int_U \nabla u \cdot \nabla \overline{v} \, dx \tag{2.1}$$

with domain $\operatorname{dom}\mathcal{E} = \mathcal{N}^{\perp}$, or more precisely, $\iota(\mathcal{N}^{\perp})$. (PI) implies that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a closed quadratic form in $L^2(U,\mu)$. Hence, there exists a non-negative self-adjoint operator A in $L^2(U,\mu)$ such that

$$\mathcal{E}(u,v) = \big(A^{1/2}u, A^{1/2}v\big)_{\mu} \quad \text{and} \quad \operatorname{dom}\mathcal{E} = \operatorname{dom}(A^{1/2})$$

(see, e.g., [13, Theorem 1.3.1]). We observe that the domain of A is a subset of $\operatorname{dom}\mathcal{E} \subset H_0^1(U)$. We write $\Delta_\mu^D = -A$ and call it the *(Dirichlet) Laplacian* with respect to μ. If no confusion is possible, we denote Δ_μ^D simply by Δ_μ.

Let $u \in \operatorname{dom}\mathcal{E}$. Then $u \in \operatorname{dom}\Delta_\mu$ if and only if there exists a unique $f \in L^2(U,\mu)$ such that $\mathcal{E}(u,v) = (f,v)_\mu$ for all $v \in \operatorname{dom}\mathcal{E}$. In this case, $-\Delta_\mu u = f$. Throughout this paper, we let

$$\operatorname{dom}\mathcal{E} := \mathcal{N}^{\perp} \quad \text{and} \quad \|\cdot\|_{\operatorname{dom}\mathcal{E}} := \sqrt{\mathcal{E}(\cdot,\cdot)}.$$

3. Extrapolation and weak solutions

In this section we consider the existence and uniqueness of weak solution of equation (1.2). Let $U \subset \mathbb{R}^d$, $d \geq 1$, be a bounded open set and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that (PI) holds. Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be defined as in Section 2.1, and $-\Delta_\mu$ be the Dirichlet operator with respect to μ. By identifying $L^2(U,\mu)$ with $(L^2(U,\mu))'$, we have the following Gelfand triple (see, e.g., [14, 32]):

$$\operatorname{dom}\mathcal{E} \hookrightarrow L^2(U,\mu) \cong (L^2(U,\mu))' \hookrightarrow (\operatorname{dom}\mathcal{E})',$$

where all the embeddings are continuous, injective, and dense. Here, the embedding $L^2(U,\mu) \hookrightarrow (\operatorname{dom}\mathcal{E})'$ is given by

$$w \in L^2(U,\mu) \mapsto (w,\cdot)_\mu \in (L^2(U,\mu))' \subset (\operatorname{dom}\mathcal{E})'.$$

It follows that for any $u \in \operatorname{dom}\mathcal{E}$, there exists a unique $w \in (\operatorname{dom}\mathcal{E})'$ such that

$$\mathcal{E}(u,v) = \langle w, v\rangle \qquad \text{for all } v \in \operatorname{dom}\mathcal{E},$$

where throughout this paper, $\langle\cdot,\cdot\rangle$ denotes the pairing between $(\operatorname{dom}\mathcal{E})'$ and $\operatorname{dom}\mathcal{E}$. On the other hand, we note that the form $\mathcal{E}$ is coercive by (PI). Hence, by the Lax–Milgram theorem, for every $w \in (\operatorname{dom}\mathcal{E})'$, there exists a unique $u \in \operatorname{dom}\mathcal{E}$ such that

$$\mathcal{E}(u,v) = \langle w, v\rangle \qquad \text{for all } v \in \operatorname{dom}\mathcal{E}.$$

Thus, we can define a bijective operator L from $\operatorname{dom}\mathcal{E}$ to $(\operatorname{dom}\mathcal{E})'$ by

$$Lu = w, \tag{3.1}$$

and equip $(\operatorname{dom}\mathcal{E})'$ with the scalar product

$$(u,v)_{(\operatorname{dom}\mathcal{E})'} := \mathcal{E}(L^{-1}u, L^{-1}v).$$

Note that $\operatorname{dom} L = \operatorname{dom}\mathcal{E}$ and $\langle w, v\rangle = (w,v)_\mu$ for all $w \in (\operatorname{dom}\mathcal{E})'$ and $v \in \operatorname{dom}\mathcal{E}$. It follows that L is an extension of $-\Delta_\mu$. Throughout this paper, we equip $(\operatorname{dom}\mathcal{E})'$ with the norm

$$\|w\|_{(\operatorname{dom}\mathcal{E})'} := \|L^{-1}w\|_{\operatorname{dom}\mathcal{E}} \qquad \text{for } w \in (\operatorname{dom}\mathcal{E})'.$$

We remark that this norm is the standard norm in $(\operatorname{dom}\mathcal{E})'$, which is equivalent to the general norm in $(\operatorname{dom}\mathcal{E})'$ (see, e.g., [6]).

Definition 3.1. Use the notation above. Let $0 < T < \infty$. Assume that we are given $f \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ and $g \in \operatorname{dom}\mathcal{E}$. A function $u \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ with $\partial_t u \in L^\infty([0,T], (\operatorname{dom}\mathcal{E})')$ is a *weak solution* of the Schrödinger equation (1.2) if the following conditions are satisfied.

(1) $\langle i\partial_t u, v\rangle - \mathcal{E}(u,v) = (f(t), v)_\mu$ for each $v \in \operatorname{dom}\mathcal{E}$ and Lebesgue a.e. $t \in [0,T]$;

(2) $u(0) = g$.

Remark 3.1. Here we comment on Definition 3.1.

(a) The boundary condition $u|_{\partial U} = 0$ in (1.2) is included in the assumption $u(t) \in \operatorname{dom}\mathcal{E}$. If $u \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ and $\partial_t u \in L^\infty([0,T], (\operatorname{dom}\mathcal{E})')$, then $u \in C([0,T], L^2(U,\mu))$, and thus the initial condition $u(0) = g$ makes sense.

(b) Condition (1) is equivalent to

$$i\partial_t u - Lu = f(t) \quad \text{in } (\operatorname{dom}\mathcal{E})' \quad \text{for Lebesgue a.e. } t \in [0,T],$$

where L is defined as in (3.1).

We now assume that $-\Delta_\mu$ has compact resolvent and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(U,\mu)$ such that $-\Delta_\mu\varphi_n = \lambda_n\varphi_n$ for all $n \ge 1$, where $0 < \lambda_1 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \cdots$ and $\lim_{n\to\infty}\lambda_n = \infty$. Some sufficient conditions for $-\Delta_\mu$ to have compact resolvent can be found in [10, 17, 22]. In particular, if $n = 1$, then $-\Delta_\mu$ has compact resolvent for any such μ. We remark that

$$\operatorname{dom}\mathcal{E} = \Big\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty |a_n|^2\lambda_n < \infty\Big\} \text{ and } \operatorname{dom}\Delta_\mu = \Big\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty |a_n|^2\lambda_n^2 < \infty\Big\}.$$

Since $\operatorname{dom}\mathcal{E} \hookrightarrow L^2(U,\mu) \hookrightarrow (\operatorname{dom}\mathcal{E})'$, $\{\varphi_n\}_{n=1}^\infty$ is also a complete orthogonal set of $\operatorname{dom}\mathcal{E}$. Note that $w = \sum_{n=1}^\infty a_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$ if and only if there exists a unique $L^{-1}w = \sum_{n=1}^{\infty} b_n\varphi_n \in \operatorname{dom}\mathcal{E}$ such that $\mathcal{E}(L^{-1}w, v) = \langle w, v\rangle$ for all $v \in \operatorname{dom}\mathcal{E}$. Substituting $v = \varphi_n$ for $n \geq 1$, we get $a_n = \langle w, \varphi_n\rangle = \mathcal{E}(L^{-1}w, \varphi_n) = b_n\lambda_n$, and so $w = \sum_{n=1}^{\infty} a_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$ if and only if $\|w\|^2_{(\operatorname{dom}\mathcal{E})'} = \|L^{-1}w\|^2_{\operatorname{dom}\mathcal{E}} = \sum_{n=1}^{\infty} a_n^2/\lambda_n < \infty$. Therefore, for every $u = \sum_{n=1}^{\infty} a_n\varphi_n \in \operatorname{dom}\mathcal{E}$, we have $Lu = \sum_{n=1}^{\infty} a_n\lambda_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$, and

$$(\operatorname{dom}\mathcal{E})' = \left\{\sum_{n=1}^{\infty} a_n\varphi_n \colon \sum_{n=1}^{\infty} a_n^2/\lambda_n < \infty\right\}.$$

In order to state the regularity result of equation (1.2), we introduce the spaces $E_\alpha(U,\mu)$, $\alpha \geq 0$, to which the initial data g and f belong. For $\alpha \geq 0$, define

$$E_\alpha(U,\mu) := \left\{\sum_{n=1}^{\infty} b_n\varphi_n \colon \sum_{n=1}^{\infty} |b_n|^2\lambda_n^\alpha < \infty\right\} \tag{3.2}$$

with the norm $\|\cdot\|_{E_\alpha(U,\mu)}$ given by

$$\|u\|_{E_\alpha(U,\mu)} := \left(\sum_{n=1}^{\infty} |b_n|^2\lambda_n^\alpha\right)^{1/2} \qquad \text{for } u = \sum_{n=1}^{\infty} b_n\varphi_n.$$

We remark that $(E_\alpha(U,\mu), \|\cdot\|_{E_\alpha(U,\mu)})$ is a Hilbert space (see, e.g., [16, Proposition 2.4]) and that $E_{\alpha_2}(U,\mu) \subseteq E_{\alpha_1}(U,\mu)$ if $\alpha_1 \leq \alpha_2$. In particular, $E_0(U,\mu) = L^2(U,\mu)$, $E_1(U,\mu) = \operatorname{dom}\mathcal{E}$, and $E_2(U,\mu) = \operatorname{dom}\Delta_\mu$.

Let

$$g = \sum_{n=1}^{\infty} b_n\varphi_n \in L^2(U,\mu) \quad \text{and} \quad f(t) = \sum_{n=1}^{\infty} \beta_n(t)\varphi_n \in L^2([0,T], L^2(U,\mu)),$$

where $\beta_n(t) = (f(t), \varphi_n)_\mu$ for $n \geq 1$. Define

$$u(t) := \sum_{n=1}^{\infty} b_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^{\infty}\Big(\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n$$

and

$$K(t) := -i\sum_{n=1}^{\infty} b_n\lambda_n e^{-i\lambda_n t}\varphi_n - if(t) - \sum_{n=1}^{\infty}\lambda_n\Big(\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n. \tag{3.3}$$

Theorem 3.1. *Let $U \subset \mathbb{R}^d$, $d \geq 1$, be a bounded open set, and let μ be a positive finite Borel measure with* $\operatorname{supp}(\mu) \subseteq \overline{U}$ *and* $\mu(U) > 0$. *Assume that μ satisfies (PI) and $-\Delta_\mu$ has compact resolvent. Let g, $f(t)$, $u(t)$ and $K(t)$ be defined as above. If $g \in \operatorname{dom}\mathcal{E}$ and $f(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$, then the following hold.*

(a) *$\partial_t u = K(t)$ in $(\operatorname{dom}\mathcal{E})'$ for Lebesgue a.e. $t \in [0,T]$.*

(b) *$u(t)$ is the unique weak solution of the Schrödinger equation* (1.2).

(c) *If, in addition, $g \in E_\alpha(U,\mu)$ and $f \in L^\infty([0,T], E_\alpha(U,\mu))$, where $\alpha \geq 2$, then $u(t) \in L^\infty([0,T], E_\alpha(U,\mu))$ and $\partial_t u(t) \in L^\infty([0,T], E_{\alpha-2}(U,\mu))$.*

(d) *If $f \equiv 0$, then*

$$\|u(t)\|_\mu = \|g\|_\mu \quad \text{and} \quad \mathcal{E}(u,u) = \mathcal{E}(g,g) \qquad \text{for all } t \in [0,T].$$

Proof. Since $g \in \operatorname{dom}\mathcal{E}$, $f(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$, we have $u(t) \in C([0,T], \operatorname{dom}\mathcal{E})$ and $K(t) \in L^\infty([0,T], (\operatorname{dom}\mathcal{E})')$. In fact, using Hölder's inequality, we obtain

$$\begin{aligned}
\|u(t)\|^2_{C([0,T],\operatorname{dom}\mathcal{E})} &= \max_{t\in[0,T]} \|u(t)\|^2_{\operatorname{dom}\mathcal{E}} \\
&\leq 2\Big(\sum_{n=1}^\infty |b_n|^2\lambda_n + T\sum_{n=1}^\infty \lambda_n \int_0^T \left|\beta_n(\tau)\right|^2 d\tau\Big) \\
&= 2\big(\|g\|^2_{\operatorname{dom}\mathcal{E}} + T\|f(t)\|^2_{2,\operatorname{dom}\mathcal{E}}\big) < \infty, \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
\|K(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} &= \operatorname*{ess\,sup}_{t\in[0,T]} \|K(t)\|^2_{(\operatorname{dom}\mathcal{E})'} \\
&\leq 3\Big(\sum_{n=1}^\infty |b_n|^2\lambda_n + \|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} \\
&\qquad + \operatorname*{ess\,sup}_{t\in[0,T]} \sum_{n=1}^\infty \lambda_n \Big|\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big|^2\Big) \\
&\leq 3\big(\|g\|^2_{\operatorname{dom}\mathcal{E}} + \|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} + T\|f(t)\|^2_{2,\operatorname{dom}\mathcal{E}}\big) < \infty. \qquad (3.5)
\end{aligned}$$

(a) Let δ satisfy $0 < 2\delta < T$ and write $u(t) =: \sum_{n=1}^\infty c_n(t)\varphi_n$. For all $t \in [\delta, T-\delta]$ and each $h \in (-\delta,\delta)$, we have

$$\begin{aligned}
u(t+h) - u(t) &= \sum_{n=1}^\infty b_n\big(e^{-i\lambda_n(t+h)} - e^{-i\lambda_n t}\big)\varphi_n \\
&\quad - i\sum_{n=1}^\infty \Big(\int_t^{t+h} e^{-i\lambda_n(t+h-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n \\
&\quad - i\sum_{n=1}^\infty \Big(\int_0^t \big(e^{-i\lambda_n(t+h-\tau)} - e^{-i\lambda_n(t-\tau)}\big)\beta_n(\tau)\,d\tau\Big)\varphi_n.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&|c_n(t+h)-c_n(t)|^2\\
&\quad=\Big|b_n\big(e^{-i\lambda_n(t+h)}-e^{-i\lambda_n t}\big)-i\int_t^{t+h}e^{-i\lambda_n(t+h-\tau)}\beta_n(\tau)\,d\tau\\
&\qquad-i\int_0^t\big(e^{-i\lambda_n(t+h-\tau)}-e^{-i\lambda_n(t-\tau)}\big)\beta_n(\tau)\,d\tau\Big|^2\\
&\quad\le 3\Big(|b_n|^2\cdot\big|e^{-i\lambda_n(t+h)}-e^{-i\lambda_n t}\big|^2+\Big|\int_t^{t+h}e^{-i\lambda_n(t+h-\tau)}\beta_n(\tau)\,d\tau\Big|^2\\
&\qquad+\Big|\int_0^t\big(e^{-i\lambda_n(t+h-\tau)}-e^{-i\lambda_n(t-\tau)}\big)\beta_n(\tau)\,d\tau\Big|^2\Big)\\
&\quad\le 3\Big(h^2|b_n|^2\lambda_n^2+h\int_t^{t+h}|\beta_n(\tau)|^2\,d\tau\\
&\qquad+T\int_0^T\big|e^{-i\lambda_n(t+h-\tau)}-e^{-i\lambda_n(t-\tau)}\big|^2\cdot|\beta_n(\tau)|^2\,d\tau\Big)\\
&\quad\le 3h^2\Big(|b_n|^2\lambda_n^2+\operatorname*{ess\,sup}_{t\in[0,T]}|\beta_n(t)|^2+T\lambda_n^2\int_0^T|\beta_n(\tau)|^2\,d\tau\Big)\\
&\quad=:3h^2\lambda_n M_n,
\end{aligned}
\tag{3.6}
$$

where the fact $|e^{-i\theta}-1|\le\theta$ for $\theta>0$ is used in the second and third inequalities. We remark that $\sum_{n=1}^\infty M_n=\|g\|^2_{\operatorname{dom}\mathcal{E}}+\|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'}+T\|f(t)\|^2_{2,\operatorname{dom}\mathcal{E}}<\infty$. Write $K(t)=:\sum_{n=1}^\infty d_n(t)\varphi_n$. Using (3.3) and Hölder's inequality, we have

$$
\begin{aligned}
|d_n(t)|^2&=\Big|-ib_n\lambda_n e^{-i\lambda_n t}-i\beta_n(t)-\lambda_n\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big|^2\\
&\le 3\Big(|b_n|^2\lambda_n^2+\operatorname*{ess\,sup}_{t\in[0,T]}|\beta_n(t)|^2+T\lambda_n^2\int_0^T|\beta_n(\tau)|^2\,d\tau\Big)=3\lambda_n M_n.
\end{aligned}
\tag{3.7}
$$

Note that the classical derivative $c_n'(t)=d_n(t)$ for Lebesgue a.e. $t\in[0,T]$. It follows that

$$
s_n(t,h):=\frac{c_n(t+h)-c_n(t)}{h}-d_n(t)\to 0\qquad\text{as}\qquad h\to 0 \tag{3.8}
$$

for Lebesgue a.e. $t\in[\delta,T-\delta]$ and $h\in(-\delta,\delta)$. Combining (3.6) and (3.7), we have for Lebesgue a.e. $t\in[\delta,T-\delta]$ and each $h\in(-\delta,\delta)$,

$$
\frac{|s_n(t,h)|^2}{\lambda_n}\le\frac{2}{\lambda_n}\Big(\frac{|c_n(t+h)-c_n(t)|^2}{h^2}+|d_n(t)|^2\Big)\le 12M_n. \tag{3.9}
$$

Using (3.9) and Weierstrass' M-test, we see the series $\sum_{n=1}^\infty|s_n(t,h)|^2/\lambda_n$ converges uniformly for all $h\in(-\delta,\delta)$ and Lebesgue a.e. $t\in[\delta,T-\delta]$. Thus, for Lebesgue a.e.

$t \in [\delta, T-\delta]$,

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-K(t)\right\|^2_{(\mathrm{dom}\,\mathcal{E})'} = \lim_{h\to 0}\sum_{n=1}^{\infty}|s_n(t,h)|^2/\lambda_n$$
$$= \sum_{n=1}^{\infty}\lim_{h\to 0}|s_n(t,h)|^2/\lambda_n = 0,$$

where (3.8) is used in the last equality. It follows that $\partial_t u(t) = K(t)$ in $(\mathrm{dom}\,\mathcal{E})'$ for Lebesgue a.e. $t \in [\delta, T-\delta]$. The desired result follows by letting $\delta \to 0^+$.

(b) We first note that $u(0) = g$ and

$$Lu = \sum_{n=1}^{\infty} b_n\lambda_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^{\infty}\lambda_n\Big(\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n, \tag{3.10}$$

where L is defined as in (3.1). Combining (3.10) and part (a), we have that $i\partial_t u(t) - Lu(t) = f(t)$ on $(\mathrm{dom}\,\mathcal{E})'$ for Lebesgue a.e. $t \in [0,T]$. It follows from Remark 3.1, that $u(t)$ is a weak solution of (1.2). To prove uniqueness, it suffices to show that the only solution of (1.2) with $f(t) \equiv g \equiv 0$ is $u(t) \equiv 0$. Let u be a weak solution of (1.2) with $f(t) \equiv g \equiv 0$. To verify this, note that

$$\langle i\partial_t u, -i\overline{u}\rangle + \mathcal{E}(u, -i\overline{u}) = 0 \qquad \text{for Lebesgue a.e. } t \in [0,T].$$

Since $\mathcal{E}(u,-i\overline{u}) = \mathrm{Re}\int_U \nabla u\cdot\nabla\overline{(-i\overline{u})}\,dx = 0$, $u(0) = g \equiv 0$, and

$$\langle i\partial_t u, -i\overline{u}\rangle = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_\mu,$$

we obtain $\|u(t)\|^2_\mu = 0$ for Lebesgue a.e. $t \in [0,T]$. It follows that $u = 0$, which proves (b).

(c) As in (3.4) and (3.5), we can deduce from the additional assumptions that

$$\begin{aligned}\|u(t)\|^2_{C([0,T],E_\alpha(U,\mu))} &= \max_{t\in[0,T]}\|u(t)\|^2_{E_\alpha(U,\mu)}\\ &\le 2\Big(\sum_{n=1}^{\infty}|b_n|^2\lambda_n^\alpha + \sum_{n=1}^{\infty}\Big(\int_0^T|\beta_n(\tau)|\,d\tau\Big)^2\lambda_n^\alpha\Big)\\ &\le 2\Big(\sum_{n=1}^{\infty}|b_n|^2\lambda_n^\alpha + T\sum_{n=1}^{\infty}\lambda_n^\alpha\int_0^T\big|\beta_n(\tau)\big|^2\,d\tau\Big)\\ &= 2\big(\|g\|^2_{E_\alpha(U,\mu)} + T\|f(t)\|^2_{2,E_\alpha(U,\mu)}\big) < \infty\end{aligned}$$

and

$$\begin{aligned}\|\partial_t u(t)\|^2_{\infty,E_{\alpha-2}(U,\mu)} &= \operatorname*{ess\,sup}_{t\in[0,T]} \|\partial_t u(t)\|^2_{E_{\alpha-2}(U,\mu)} \\ &\le 3\Bigg(\sum_{n=1}^{\infty} |b_n|^2\lambda_n^{\alpha} + \|f(t)\|^2_{\infty,E_{\alpha-2}(U,\mu)} \\ &\qquad + \operatorname*{ess\,sup}_{t\in[0,T]} \sum_{n=1}^{\infty} \lambda_n^{\alpha}\bigg|\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\bigg|^2\Bigg) \\ &\le 3\big(\|g\|^2_{E_\alpha(U,\mu)} + \|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} + T\|f(t)\|^2_{2,E_\alpha(U,\mu)}\big) < \infty.\end{aligned}$$

Hence, the assertions hold.

(d) Since $f \equiv 0$, we have by parts (a) and (b) that

$$u(t) = \sum_{n=1}^{\infty} b_n e^{-i\lambda_n t}\varphi_n \qquad \text{and} \qquad \partial_t u(t) = -i\sum_{n=1}^{\infty} b_n\lambda_n e^{-i\lambda_n t}\varphi_n.$$

It follows that

$$\|u(t)\|^2_{\mu} = \sum_{n=1}^{\infty} |b_n|^2 = \|g\|^2_{\mu} \text{ and } \mathcal{E}(u,u) = \sum_{n=1}^{\infty} |b_n|^2\lambda_n = \mathcal{E}(g,g) \text{ for all } t \in [0,T]. \quad \blacksquare$$

Now we prove Theorem 1.1. The main ingredients are Banach's fixed point theorem and Theorem 3.1.

Proof of Theorem 1.1. Given a function $u \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$, set $h(t) := F(u(t))$. By the assumption on $F(\cdot)$, we see that $h \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$. Consequently, Theorem 3.1 ensures that the linear Schrödinger equation

$$\begin{cases} i\partial_t w + \Delta_\mu w = h & \text{on} \quad U \times [0,T], \\ w = 0 & \text{on} \quad \partial U \times [0,T], \\ w = g & \text{on} \quad U \times \{t = 0\} \end{cases} \tag{3.11}$$

has a unique weak solution $w(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ given by

$$w(t) := \sum_{n=1}^{\infty} \alpha_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^{\infty}\bigg(\int_0^t e^{-i\lambda_n(t-\tau)}\big(h(\tau),\varphi_n\big)_\mu\,d\tau\bigg)\varphi_n. \tag{3.12}$$

Define $A\colon L^\infty([0,T], \operatorname{dom}\mathcal{E}) \to L^\infty([0,T], \operatorname{dom}\mathcal{E})$ by $A[u] = w$.

We now claim that if $T > 0$ is small enough, then A is a contraction mapping from $L^\infty([0,T], \operatorname{dom}\mathcal{E})$ to $L^\infty([0,T], \operatorname{dom}\mathcal{E})$. Let $u(t), v(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$. Since

$F(\cdot)$ is Lipschitz continuous on $\operatorname{dom}\mathcal{E}$, $F(u(t)) - F(v(t)) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$. It follows that

$$\|F(u(t)) - F(v(t))\|^2_{\operatorname{dom}\mathcal{E}} = \sum_{n=1}^\infty \big|\big(F(u(t)) - F(v(t)), \varphi_n\big)_\mu\big\|^2 \lambda_n \tag{3.13}$$

for Lebesgue a.e. $t \in [0,T]$. We first obtain from (3.12) and (3.13) that for Lebesgue a.e. $0 \le t \le T$,

$$\begin{aligned}\|A[u(t)] - A[v(t)]\|^2_{\operatorname{dom}\mathcal{E}} &= \sum_{n=1}^\infty \lambda_n \left| \int_0^t e^{-i\lambda_n(t-\tau)} \big(F(u(\tau)) - F(v(\tau)), \varphi_n\big)_\mu \, d\tau \right|^2 \\ &\le t \int_0^t \sum_{n=1}^\infty \lambda_n \big|\big(F(u(\tau)) - F(v(\tau)), \varphi_n\big)_\mu\big|^2 \, d\tau \\ &= t \int_0^t \|F(u(\tau)) - F(v(\tau))\|^2_{\operatorname{dom}\mathcal{E}} \, d\tau \qquad \text{(by (3.13))} \\ &\le CT \int_0^t \|u(\tau) - v(\tau)\|^2_{\operatorname{dom}\mathcal{E}} \, d\tau \le CT^2 \|u - v\|^2_{\infty,\operatorname{dom}\mathcal{E}},\end{aligned}$$

where the assumption that $F(\cdot)$ is Lipschitz is used in the second inequality. It follows that

$$\big\|A[u] - A[v]\big\|_{\infty,\operatorname{dom}\mathcal{E}} \le \sqrt{C}T \|u - v\|_{\infty,\operatorname{dom}\mathcal{E}}.$$

Thus, $A[\cdot]$ is a strict contraction, provided $T > 0$ is so small that $\sqrt{C}T = \gamma < 1$.

Given any $T > 0$, we select $T_1 > 0$ so small that $\sqrt{C}T_1 < 1$. We can then apply Banach's fixed point theorem to obtain a weak solution u of the non-linear Schrödinger equation (1.3) that exists on the time interval $[0, T_1]$. Since $u(t) \in \operatorname{dom}\mathcal{E}$ for a.e. $0 \le t \le T_1$, we can find some $T_2 \in (T_1/2, T_1)$ such that $u(T_2) \in \operatorname{dom}\mathcal{E}$. We can then repeat the argument above to extend our solution u to the time interval $[T_2, T_3]$ such that $u(T_3) \in \operatorname{dom}\mathcal{E}$ and $T_3 \in [2T_2, T_1 + T_2]$. Repeating this process for a finite number of steps, we obtain a weak solution that exists on the entire interval $[0, T]$.

To prove the uniqueness, suppose that u and v are two weak solutions of the non-linear Schrödinger equation (1.3). Then we have $A[u] = u$ and $A[v] = v$. It follows from the uniqueness of the fixed point of A that $u(t) = v(t)$ in $L^\infty([0, T_1], \operatorname{dom}\mathcal{E})$. Combining this argument with the extension argument above shows that $u(t) = v(t)$ in $L^\infty([0,T], \operatorname{dom}\mathcal{E})$.

Assume that $F(\cdot)$ is Lipschitz continuous on $E_\alpha(U,\mu)$ and $g \in E_\alpha(U,\mu)$ for some $\alpha \ge 2$. Then we have that $h(t) = F(u(t)) \in L^\infty([0,T], E_{\alpha-2}(U,\mu))$ for all $u(t) \in L^\infty([0,T], E_\alpha(U,\mu))$. For any $u(t) \in L^\infty([0,T], E_\alpha(U,\mu))$, Theorem 3.1 (c) implies that equation (3.11) has a unique solution $w(t) \in L^\infty([0,T], E_\alpha(U,\mu))$ with $\partial_t w(t) \in L^\infty([0,T], E_{\alpha-2}(U,\mu))$ satisfying (3.12). Similarly, we can show that there exists a unique $u(t) \in L^\infty([0,T], E_\alpha(U,\mu))$ with $\partial_t u(t) \in L^\infty([0,T], E_{\alpha-2}(U,\mu))$ such that $A[u] = u$, which completes the proof. ∎

4. The finite element method for linear Schrödinger equations

In this section, we let $U = (a, b)$ and $f \equiv 0$ in equation (1.2), and use the finite element method to solve the homogeneous Schrödinger equation (1.2). Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq [a, b]$. Assume that there exists a sequence of refining μ-partitions $(\mathbf{P}_m)_{m\geq 1} = \big(\{I_{m,\ell}\}_{\ell=0}^{N(m)}\big)_{m\geq 1}$ satisfying (1.4) in Section 1. Without loss of generality, we can write $I_{m,\ell} = [x_{m,\ell}, y_{m,\ell}]$ with

$$a = x_{m,0} < y_{m,0} \leq x_{m,1} < y_{m,1} \leq \cdots \leq x_{m,N(m)} < y_{m,N(m)} = b$$

for all $m \geq 1$ and $0 \leq \ell \leq N(m)$. Moreover, we have

$$[a, b] \setminus \Big(\bigcup_{\ell=0}^{N(m)} I_{m,\ell}\Big) = \bigcup_{\ell=0}^{N(m)-1} (y_{m,\ell}, x_{m,\ell+1}) \subseteq [a, b] \setminus \operatorname{supp}(\mu) \qquad \text{for all } m \geq 1.$$

In particular, if $\operatorname{supp}(\mu) = [a, b]$, then $y_{m,\ell} = x_{m,\ell+1}$ for all $m \geq 1$ and $\ell = 0, \ldots, N(m) - 1$.

We first note that equations (1.6) and (1.7) are derived from the integral form of the homogeneous Schrödinger equation (1.2) with $U = (a, b)$. Now, we apply the finite element method to approximate the solution $u(x, t)$ satisfying (1.6) and (1.7) by

$$u^m(x, t) := \sum_{j=0}^{N(m)} (w_{1,j}(t) + i w_{2,j}(t))\phi_j(x), \tag{4.1}$$

where, for $j = 0, 1, \ldots, N(m)$, $w_{1,j}(t) := w^m_{1,j}(t)$ and $w_{2,j}(t) := w^m_{2,j}(t)$ are real-valued functions to be determined, and $\phi_j(x) := \phi_{m,j}(x)$ are the standard piecewise linear *finite element basis functions* (also called *tent functions*) defined by

$$\begin{aligned} \phi_j(x) &:= \phi_{m,j}(x) \\ &= \begin{cases} \frac{x - x_{m,j-1}}{y_{m,j-1} - x_{m,j-1}} & \text{if } x \in I_{m,j-1},\ j = 1, 2, \ldots, N(m), \\ 1 & \text{if } x \in [y_{m,j-1}, x_{m,j}],\ j = 1, 2, \ldots, N(m)-1, \\ \frac{x - y_{m,j}}{x_{m,j} - y_{m,j}} & \text{if } x \in I_{m,j},\ j = 0, 1, \ldots, N(m) - 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2}$$

Let $u^m_1(x, t)$ and $u^m_2(x, t)$ be the real and complex parts of $u^m(x, t)$, respectively. We require $u^m(x, t)$ to satisfy equations (1.6) and (1.7) as follows:

$$\int_a^b \partial_t u^m_2(x, t)\phi_j(x)\, d\mu = -\int_a^b \partial_x u^m_1(x, t)\phi'_j(x)\, dx \tag{4.3}$$

and

$$\int_a^b \partial_t u_1^m(x,t)\phi_j(x)\,d\mu = \int_a^b \partial_x u_2^m(x,t)\phi_j'(x)\,dx. \tag{4.4}$$

Moreover, we require $u^m(x,t)$ to satisfy the Dirichlet boundary condition $u^m(a,t) = u^m(b,t) = 0$. We note that $\phi_\ell(a) = \phi_\ell(x_{m,0}) = 0$ and $\phi_j(b) = \phi_j(y_{m,N(m)}) = 0$ for all $\ell = 1,\dots,N(m)$ and $j = 0,1,\dots,N(m)-1$. Thus, $w_{k,0}(t) = w_{k,N(m)}(t) = 0$ for all $t \in [0,T]$ and $k = 1,2$. Using this and (4.1), we can express (4.3) and (4.4) in matrix form as

$$\widehat{\mathbf{M}}\widehat{\mathbf{w}}' := \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}' = - \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} =: -\widehat{\mathbf{K}}\widehat{\mathbf{w}}, \tag{4.5}$$

where $\mathbf{M} = \mathbf{M}^{(m)} = (M_{\ell j}^{(m)})$ and $\mathbf{K} = \mathbf{K}^{(m)} = (K_{\ell j}^{(m)})$ are, respectively, the *mass* and *stiffness matrices*, defined by

$$M_{\ell j}^{(m)} := \int_a^b \phi_\ell(x)\phi_j(x)\,d\mu \quad \text{and} \quad K_{\ell j}^{(m)} := \int_a^b \phi_\ell'(x)\phi_j'(x)\,dx$$

for all $1 \le \ell, j \le N(m)-1$, and

$$\mathbf{w}_1(t) = \mathbf{w}_{1,m}(t) := \begin{bmatrix} w_{1,1}(t) \\ \vdots \\ w_{1,N(m)-1}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2(t) = \mathbf{w}_{2,m}(t) := \begin{bmatrix} w_{2,1}(t) \\ \vdots \\ w_{2,N(m)-1}(t) \end{bmatrix}.$$

This gives us a system of first-order linear ODEs with constant coefficients. To solve it, we need to impose initial conditions. Based on the initial condition $u(x,0) = g(x) \in \operatorname{dom}\mathcal{E}$, we require $u^m(x,t)$ to satisfy the initial condition $u^m(x_{m,j},0) = g(x_{m,j})$ for all $1 \le j \le N(m)-1$. This leads to the initial condition

$$\widehat{\mathbf{w}}(0) = \widehat{\mathbf{w}}_m(0) := \left[g_1(x_{m,1}),\dots,g_1(x_{m,N(m)-1}), g_2(x_{m,1}),\dots,g_2(x_{m,N(m)-1})\right]^T, \tag{4.6}$$

where $g_1(x)$ and $g_2(x)$ are the real and complex parts of $g(x)$, respectively. Consequently, we obtain the linear system

$$\widehat{\mathbf{M}}\widehat{\mathbf{w}}' = -\widehat{\mathbf{K}}\widehat{\mathbf{w}}, \quad t > 0, \quad \text{and} \quad \widehat{\mathbf{w}}(0) = \widehat{\mathbf{w}}_{m,0}. \tag{4.7}$$

It is well known that $\mathbf{K}$ is invertible (see, e.g., [31]). Here, $\mathbf{M}$ depends on the measure μ and its μ-partitions $(\mathbf{P}_m)_{m\ge1} = (\{I_{m,\ell}\}_{\ell=0}^{N(m)})_{m\ge1}$. We prove below that $\mathbf{M}$ is invertible. It follows that $\widehat{\mathbf{M}}$ is also invertible. Thus, the system in (4.7) has a unique solution. More precisely, the following proposition holds.

Proposition 4.1. *Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ such that* $\operatorname{supp}(\mu) \subseteq [a,b]$. *Assume moreover, that there exists a sequence of refining μ-partitions $(\mathbf{P}_m)_{m\geq 1} = (\{I_{m,\ell}\}_{\ell=0}^{N(m)})_{m\geq 1}$ of $[a,b]$. Then the mass matrix $\mathbf{M}$ defined as above is positive definite. Consequently, (4.7) has a unique solution $\widehat{\mathbf{w}}(t)$. Furthermore, $w_{1j}(t) \in C(0,T)$ and $w_{2j}(t) \in C(0,T)$ for $j = 1,\ldots,N(m)-1$.*

Proof. Suppose, on the contrary, that $\mathbf{M}$ were not positive definite. Then there would exist some $\mathbf{c} = (c_1,\ldots,c_{N(m)-1}) \in \mathbb{R}^{N(m)-1} \setminus \{\mathbf{0}\}$ such that $\mathbf{c}^T\mathbf{M}\mathbf{c} \leq 0$. Let $\mathbf{v} = \sum_{\ell=1}^{N(m)-1} c_\ell \phi_{m,\ell}(x)$, where each $\phi_{m,\ell}(x)$ is defined by (4.2). Then $\mathbf{c}^T\mathbf{M}\mathbf{c} = \langle \mathbf{M}\mathbf{v}, \mathbf{v}\rangle$. Since $\langle \mathbf{M}\mathbf{v}, \mathbf{v}\rangle = \|\mathbf{v}\|_\mu^2$, we have $\|\mathbf{v}\|_\mu^2 \leq 0$. On the other hand, since $\mu(I_{m,\ell}) > 0$ for all $0 \leq \ell \leq N(m)$, the definition of $\phi_{m,\ell}(x)$ implies that $\|\phi_{m,\ell}\|_\mu^2 > 0$ for all $0 \leq \ell \leq N(m)$, and thus $\|\mathbf{v}\|_\mu^2 > 0$, a contradiction. Hence, $\mathbf{M}$ is positive definite. The continuity of $w_{1j}(t)$ and $w_{2j}(t)$ follows from standard theory. ■

Proof of Theorem 1.2. The assertions hold by combining the derivations above and Proposition 4.1. ■

As in the classical case, the matrix $\mathbf{K}$ can be computed directly. In order to compute $\mathbf{M}$, we use the assumption that $(\mathbf{P}_m)_{m\geq 1} = \big(\{I_{m,\ell}\}_{\ell=0}^{N(m)}\big)_{m\geq 1}$ satisfies (1.4). In the following, the constants $c_{I,J}$ and similitudes $\tau_{I,J}$ come from (1.4). From the definition of the $\phi_{m,j}$ and (1.4), for $1 \leq \ell \leq N(m)-1$, we have

$$\begin{aligned} M_{\ell,\ell}^{(m)} &= (y_{m,\ell-1} - x_{m,\ell-1})^{-2} \cdot \sum_{J\in\mathbf{P}_1} c_{I_{m,\ell-1},J} \int_J (\tau_{I_{m,\ell-1},J}(x) - x_{m,\ell-1})^2 \, d\mu \\ &\quad + (x_{m,\ell} - y_{m,\ell})^{-2} \cdot \sum_{J\in\mathbf{P}_1} c_{I_{m,\ell},J} \int_J (\tau_{I_{m,\ell},J}(x) - y_{m,\ell})^2 \, d\mu. \end{aligned}$$

For $2 \leq \ell \leq N(m)-1$, we obtain

$$\begin{aligned} M_{\ell,\ell-1}^{(m)} &= -(y_{m,\ell-1} - x_{m,\ell-1})^{-2} \\ &\quad \cdot \sum_{J\in\mathbf{P}_1} c_{I_{m,\ell-1},J} \int_J (\tau_{I_{m,\ell-1},J}(x) - x_{m,\ell-1})(\tau_{I_{m,\ell-1},J}(x) - y_{m,\ell-1}) \, d\mu, \end{aligned}$$

and $M_{\ell-1,\ell}^{(m)} = M_{\ell,\ell-1}^{(m)}$. For the special case $\operatorname{supp}(\mu) = [a,b]$, the authors have given the explicit formula for $\mathbf{M}$ in [31].

Define

$$\mathcal{J}_{k,j} := \int_{I_{1,j}} x^k \, d\mu, \quad k = 0,1,2, \text{ and } j = 0,\ldots,N(1).$$

Since each $\tau_{I,J}$ is of the form $\tau_{I,J}(x) = r_{I,J}x + b_{I,J}$, we see that the matrix $\mathbf{M}$ is completely determined by the integrals $\mathcal{J}_{k,j}$, where $k = 0,1,2$ and $j = 0,\ldots,N(1)$.

Hereafter, we assume that the constant $\mathcal{J}_{k,j}$ can be evaluated explicitly for $k = 0, 1, 2$ and $j = 0, \dots, N(1)$.

We now use the central difference method to solve equation (4.7). Let $t_n = n\Delta t$ and $\widehat{\mathbf{w}}_n := \widehat{\mathbf{w}}(t_n)$ for all $n \geq 0$ and some $\Delta t > 0$. We approximate the derivative as

$$\widehat{\mathbf{w}}'(t_n) \approx \frac{\widehat{\mathbf{w}}_{n+1} - \widehat{\mathbf{w}}_n}{\Delta t} \quad \text{and} \quad \widehat{\mathbf{w}}(t_n) \approx \frac{\widehat{\mathbf{w}}_{n+1} + \widehat{\mathbf{w}}_n}{2}. \tag{4.8}$$

Since $\mathbf{M}$ and $\mathbf{K}$ are positive definite, so is $2\widehat{\mathbf{M}} + (\Delta t)\widehat{\mathbf{K}}$ for all $\Delta t > 0$. Substituting (4.8) into (4.7), we can rewrite (4.7) as

$$\begin{cases} \widehat{\mathbf{w}}_{n+1} = (2\widehat{\mathbf{M}} + (\Delta t)\widehat{\mathbf{K}})^{-1}(2\widehat{\mathbf{M}} - (\Delta t)\widehat{\mathbf{K}})\widehat{\mathbf{w}}_n, \quad n = 0, 1, 2, \dots, \\ \widehat{\mathbf{w}}_0 = \widehat{\mathbf{w}}(t_0) = \widehat{\mathbf{w}}(0), \\ t_n = n\Delta t. \end{cases} \tag{4.9}$$

To solve this system, fix Δt and substitute the initial condition $\widehat{\mathbf{w}}_0$ from (4.6) into the first equation in (4.9) to get $\widehat{\mathbf{w}}_1$. Then $\widehat{\mathbf{w}}_{n+1}$ can be computed recursively.

5. Fractal measures defined by iterated function systems

In this section, we solve the homogeneous Schrödinger equation numerically for three different measures. These measures are defined by IFSs with overlaps and satisfy (1.4) (see [31, Proposition 5.1 and Section 5.3]). In the first and second cases, the measures satisfy a family of second-order self-similar identities. These identities were first introduced by Strichartz et al. [30] to approximate the density of the infinite Bernoulli convolution associated with the golden ratio.

5.1. Infinite Bernoulli convolution associated with the golden ratio

We consider the infinite Bernoulli convolution associated with the golden ratio:

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1},$$

where

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho), \qquad \rho = \frac{\sqrt{5} - 1}{2}.$$

We note that $\operatorname{supp}(\mu) = [0, 1]$. Strichartz et al. [30] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_1(x) := \rho^2 x, \qquad T_2(x) := \rho^3 x + \rho^2, \qquad T_3(x) := \rho^2 x + \rho.$$

Moreover, μ satisfies the following second-order identities [20]. For each Borel set $A \subseteq [0,1]$,

$$\begin{bmatrix} \mu(T_1 T_j A) \\ \mu(T_2 T_j A) \\ \mu(T_3 T_j A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \qquad j = 1,2,3,$$

where

$$M_1 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad M_2 = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 0 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

These identities can be used to compute the measure of suitable subintervals of [0, 1]. Define

$$\mathbf{P}_k := \left\{ T_{\boldsymbol{j}}([0,1]) \colon \boldsymbol{j} \in \{1,2,3\}^k \right\} \qquad \text{for } k \geq 1. \tag{5.1}$$

It follows from [31, Proposition 5.1] that $(\mathbf{P}_k)_{k\geq 1}$ is a sequence of refining μ-partitions of [0, 1] satisfying (1.4). Moreover, the integrals $\int_0^1 x^k \, d\mu \circ T_j$, $k = 0, 1, 2$, $j = 1, 2, 3$ have been calculated in [7, Section 4.2]. We remark that

$$\int_0^1 x \, d\mu \circ T_3 = \frac{1}{2(3+\rho)} \quad \text{and} \quad \int_0^1 x^2 \, d\mu \circ T_3 = \frac{2-\rho}{2(\rho+8)};$$

the calculations of these integrals in [7, 9] are incorrect. Let $(a, b) = (0, 1)$, and thus we can calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (4.5). The result is shown in Figure 1. Here we choose g to have small support so that it models the Dirac delta function.

5.2. Three-fold convolution of the Cantor measure

We consider the following three-fold convolution of the Cantor measure studied in [20, 24, 26]. The three-fold convolution of the Cantor measure μ is the self-similar measure defined by the following IFS with overlaps (see [24]):

$$S_j(x) = \frac{1}{3}x + \frac{2}{3}(j-1), \qquad j = 1,2,3,4,$$

together with probability weights $\{1/8, 3/8, 3/8, 1/8\}$. That is,

$$\mu = \frac{1}{8}\mu \circ S_1^{-1} + \frac{3}{8}\mu \circ S_2^{-1} + \frac{3}{8}\mu \circ S_3^{-1} + \frac{1}{8}\mu \circ S_4^{-1}.$$

Note that $\operatorname{supp}(\mu) = [0, 3]$. It is shown in [20] that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_1(x) = \frac{1}{3}x, \qquad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2.$$

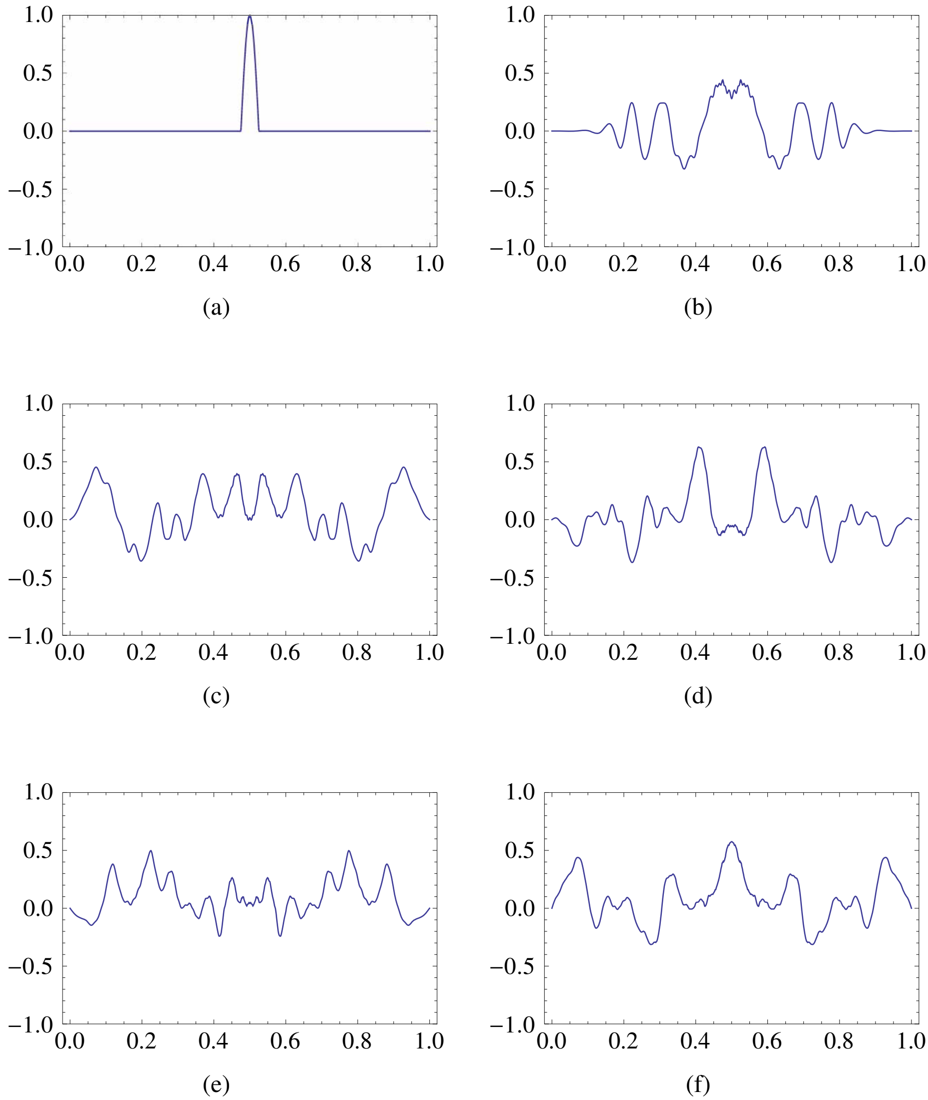

Figure 1. Figure for numerical solutions of the homogeneous Schrödinger equation (1.2) with $U = (0,1)$ and μ being the infinite Bernoulli convolution associated with the golden ratio. $(\mathbf{P}_k)_{k\geq 1}$ is defined by (5.1). The initial condition is given by the function $g(x) := \sin(20\pi(x-0.475)) + i\sin(20\pi(x-0.475))$ for $x \in (0.475, 0.525)$, and $g(x) := 0$ otherwise. Here $\Delta t = 0.0001$. From (a) to (f), the values of t are 0.0, 0.001, 0.002, 0.004, 0.008, 0.02, respectively.

In fact, for each Borel $A \subseteq [0,3]$,

$$\begin{bmatrix} \mu(T_1 T_j A) \\ \mu(T_2 T_j A) \\ \mu(T_3 T_j A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad j = 1,2,3,$$

where M_1, M_2, M_3 are given by

$$M_1 = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define

$$\mathbf{P}_k := \left\{T_{\boldsymbol{j}}([0,3]) \colon \boldsymbol{j} \in \{1,2,3\}^k\right\} \qquad \text{for } k \geq 1. \tag{5.2}$$

It follows from [31, Proposition 5.1] that $(\mathbf{P}_k)_{k\geq 1}$ is a sequence of refining μ-partitions of $[0,3]$ satisfying (1.4). The integrals $\int_0^3 x^k \, d\mu \circ T_j$, $k = 0,1,2$, $j = 1,2,3$ have been calculated in [7, Section 4.3]. Let $(a,b) = (0,3)$, and thus we can calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (4.5). The result is shown in Figure 2.

5.3. A class of self-similar measures that are essentially of finite type

In this subsection, we consider the following family of IFSs:

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + r_1(1 - r_2), \quad S_3(x) = r_2 x + 1 - r_2, \tag{5.3}$$

where the contraction ratios $r_1, r_2 \in (0,1)$ satisfy the inequality $r_1 + 2r_2 - r_1 r_2 \leq 1$, i.e., $S_2(1) \leq S_3(0)$. The Hausdorff dimension of the self-similar sets is computed in [21]. The multifractal properties and spectral dimension of the corresponding self-similar measures are recently studied in [11, 26, 27].

Let μ be a self-similar measure defined by an IFS in (5.3) and a probability vector $(p_i)_{i=1}^3$. Let $I_{1,1} := S_1([0,1]) \cup S_2([0,1])$ and $I_{1,0} := S_3([0,1])$. In order to define a sequence of refining μ-partitions of $[0,1]$, we adopt the definition of an island from [26]. Let $\mathcal{M}_k := \{1,2,3\}^k$ for $k \geq 1$ and $\mathcal{M}_0 := \emptyset$. A closed subset $I \subseteq [0,1]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold.

(1) There exists a finite sequence of indexes $\boldsymbol{i}_0, \boldsymbol{i}_1, \ldots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ with the properties $S_{\boldsymbol{i}_k}(0,1) \cap S_{\boldsymbol{i}_{k+1}}(0,1) \neq \emptyset$ for all $k = 0, \ldots, n-1$, and $I = \bigcup_{k=0}^n S_{\boldsymbol{i}_k}([0,1])$.

(2) For any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \ldots, \boldsymbol{i}_n\}$ and any $k \in \{0, \ldots, n\}$, $S_{\boldsymbol{j}}(0,1) \cap S_{\boldsymbol{i}_k}(0,1) = \emptyset$.

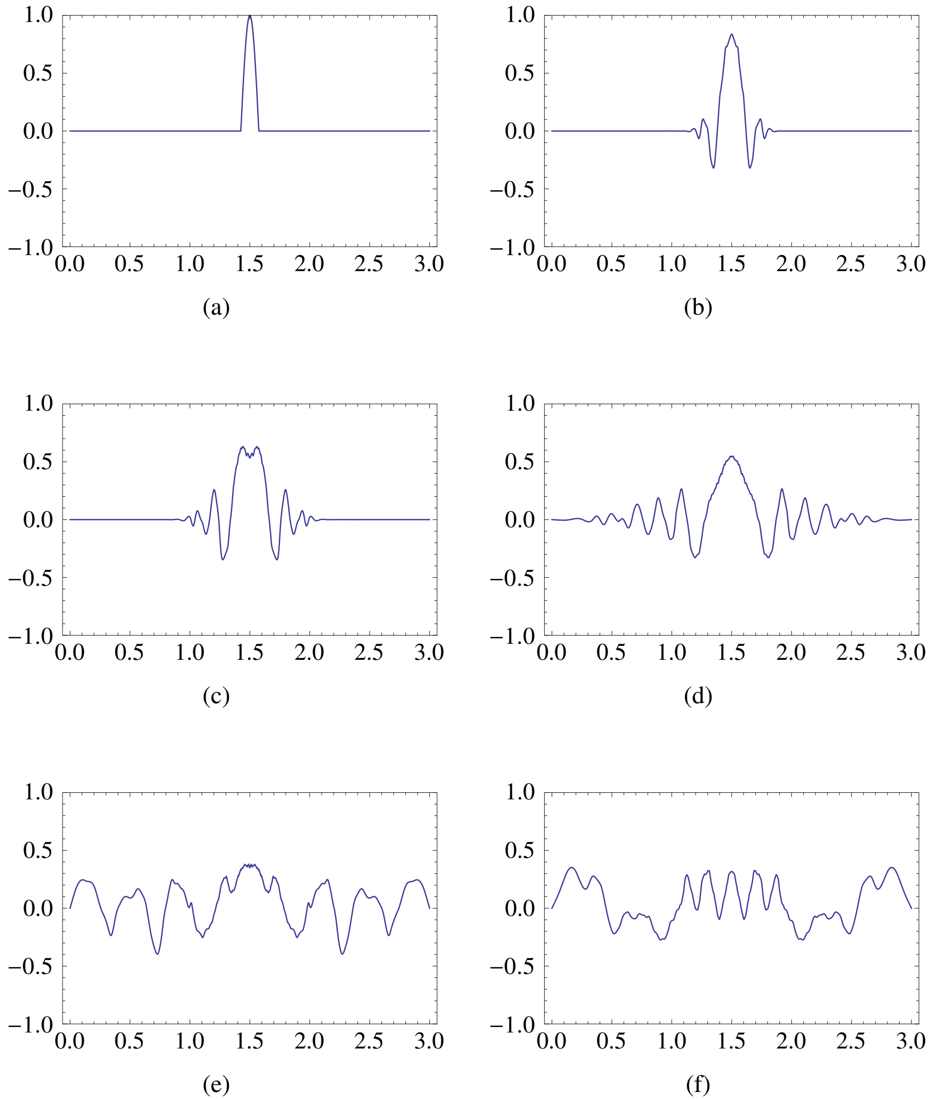

Figure 2. Figure for numerical solutions of the homogeneous Schrödinger equation (1.2) with $U = (0,3)$ and μ being the three-fold convolution of the Cantor measure. $(\mathbf{P}_k)_{k\geq 1}$ is defined by (5.2). The initial condition is given by the function $g(x) = \sin(20\pi(x/3 - 0.475)) + i\sin(20\pi(x/3 - 0.475)), x \in (1.425, 1.575)$, and $g(x) = 0$ otherwise. Here $\Delta t = 0.0001$. From (a) to (f), the values of t are the same as those in Figure 1.

Intuitively, for each level-k island I, I° is a connected component of $S_{\mathcal{M}_k}(0,1) := \bigcup_{\boldsymbol{i}\in\mathcal{M}_k} S_{\boldsymbol{i}}(0,1)$ (see Figure 3). For $k \ge 1$, define

$$\mathbf{P}_k := \big\{I : I \text{ is a level-}k \text{ island with respect to } \{\mathcal{M}_k\}\big\}. \tag{5.4}$$

It follows from [31, Section 5.3] that $(\mathbf{P}_k)_{k\ge1}$ is a sequence of refining μ-partitions of $[0,1]$ satisfying (1.4).

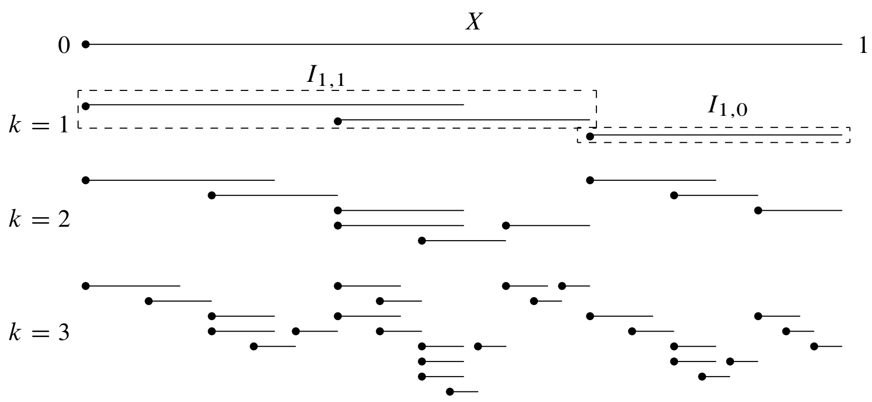

Figure 3. μ-partitions $\mathbf{P}_k$ for $k = 1, 2, 3$, where $\mathbf{P}_k$ is defined as in (5.4). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/2$ and $r_2 = 1/3$.

For every continuous function φ on $[0,1]$, we have

$$\int_0^1 \varphi\, d\mu = \sum_{i=1}^{3} p_i \int_0^1 \varphi(S_i(x))\, d\mu. \tag{5.5}$$

Using (5.5) repeatedly, we can obtain

$$\int_0^1 d\mu = 1,$$

$$\int_0^1 x\, d\mu = \frac{(p_2 r_1 + p_3)(1-r_2)}{1 - p_1 r_1 - p_2 r_2 - p_3 r_2},$$

$$\int_0^1 x^2\, d\mu = \frac{2(p_3 + p_2 r_1) r_2 (1-r_2) \int_0^1 x\, d\mu + (1-r_2)^2 (p_3 + p_2 r_1^2)}{1 - p_1 r_1^2 - p_2 r_2^2 - p_3 r_2^2}.$$

Moreover, we can calculate the integrals $\int_{I_{1,j}} x^k \, d\mu$, $j = 0, 1$ and $k = 0, 1, 2$, as follows:

$$\int_{I_{1,0}} d\mu = p_3,$$

$$\int_{I_{1,0}} x \, d\mu = r_2 p_3 \int_0^1 x \, d\mu + p_3(1 - r_2),$$

$$\int_{I_{1,0}} x^2 \, d\mu = r_2^2 p_3 \int_0^1 x^2 \, d\mu + 2 p_3 r_2 (1 - r_2) \int_0^1 x \, d\mu + p_3 (1 - r_2)^2,$$

$$\int_{I_{1,1}} d\mu = p_1 + p_2,$$

$$\int_{I_{1,1}} x \, d\mu = (p_1 r_1 + r_2 p_2) \int_0^1 x \, d\mu + p_2 r_1 (1 - r_2),$$

$$\int_{I_{1,1}} x^2 \, d\mu = (p_1 r_1^2 + p_2 r_2^2) \int_0^1 x^2 \, d\mu + 2 p_1 r_1 r_2 (1 - r_2) \int_0^1 x \, d\mu + p_2 r_1^2 (1 - r_2)^2.$$

Let $(a, b) = (0, 1)$, and thus we can calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (4.5). The results are shown in Figures 4–6. We point out that in Figure 5, $\mathrm{supp}(\mu)$ is a proper subset of $[0, 1]$.

6. Convergence of numerical approximations for linear Schrödinger equations

In this section we prove the convergence of numerical approximations of the homogeneous Schrödinger equation (1.2) with $U = (a, b)$. Some of our results are obtained by modifying similar ones in [28]. Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\mathrm{supp}(\mu) \subseteq [a, b]$. In this case, there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2((a, b), \mu)$ such that $-\Delta_\mu \varphi_n = \lambda_n \varphi_n$ for all $n \geq 1$, where the eigenvalues satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ with $\lim_{n \to \infty} \lambda_n = \infty$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k \geq 1}$ satisfying (1.4). Let V_m be the set of endpoints of all level-m sub-intervals, and arrange its elements in such a way that $V_m = \{x_{m,\ell} : \ell = 0, \ldots, N(m)\} \cup \{y_{m,\ell} : \ell = 0, \ldots, N(m)\}$ with $x_{m,\ell} < y_{m,\ell+1} \leq x_{m,\ell+1}$ for $\ell = 0, 1, \ldots, N(m) - 1$, $x_{m,0} = a$ and $y_{m,N(m)} = b$. Let S^m be the space of continuous piecewise linear functions on $[a, b]$ with nodes V_m, and let

$$S_D^m = \{u \in S^m : u(a) = u(b) = 0\}$$

be the subspace of S^m consisting of functions satisfying the Dirichlet boundary condition.

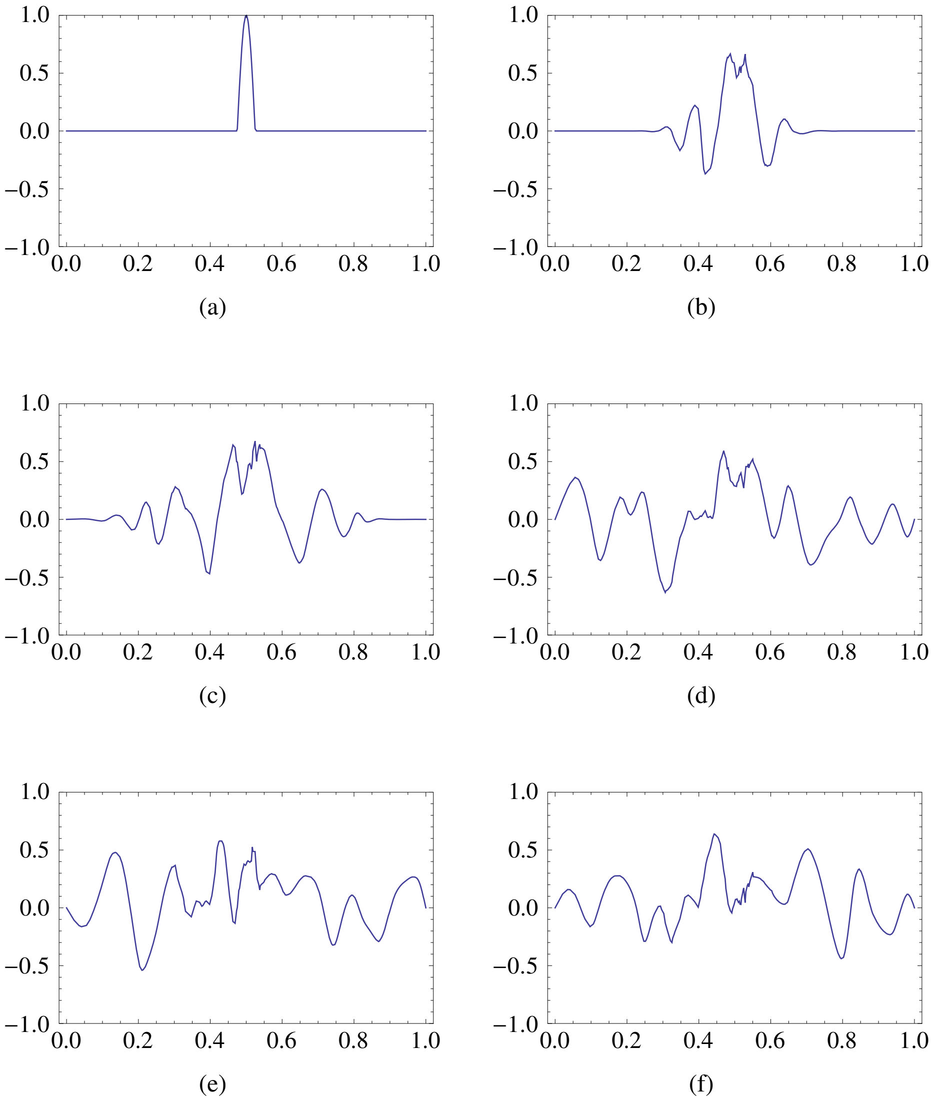

Figure 4. Numerical solutions of the homogeneous Schrödinger equation (1.2) with $U = (0, 1)$ and μ being the self-similar measure defined by the IFS in (5.3) with probability weights $p_1 = p_2 = p_3 = 1/3$ and contraction ratios $r_1 = 1/2$ and $r_2 = 1/3$. $(\mathbf{P}_k)_{k\geq 1}$ is defined by (5.4). The initial condition and the values of Δt and t are the same as those in Figure 1.

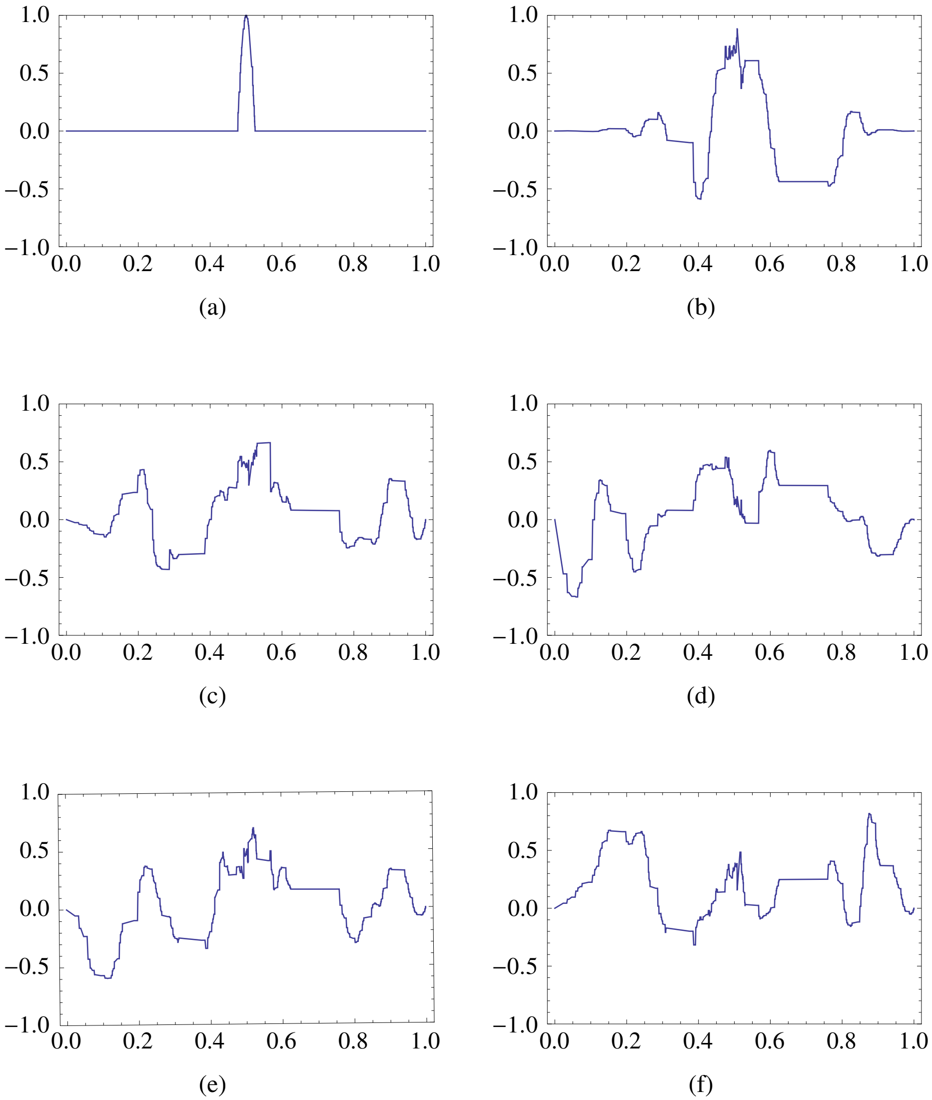

Figure 5. Numerical solutions of the homogeneous Schrödinger equation (1.2) with $U = (0, 1)$ and μ being the self-similar measure defined by the IFS in (5.3) with probability weights $p_1 = p_2 = p_3 = 1/3$ and contraction ratios $r_1 = 1/2$ and $r_2 = 1/4$. $(\mathbf{P}_k)_{k\geq 1}$ is defined by (5.4). The initial condition, and the values of Δt and t are the same as those in Figure 1. Unlike the other examples, $\mathrm{supp}(\mu)$ in this example is a proper subset of $(0, 1)$.

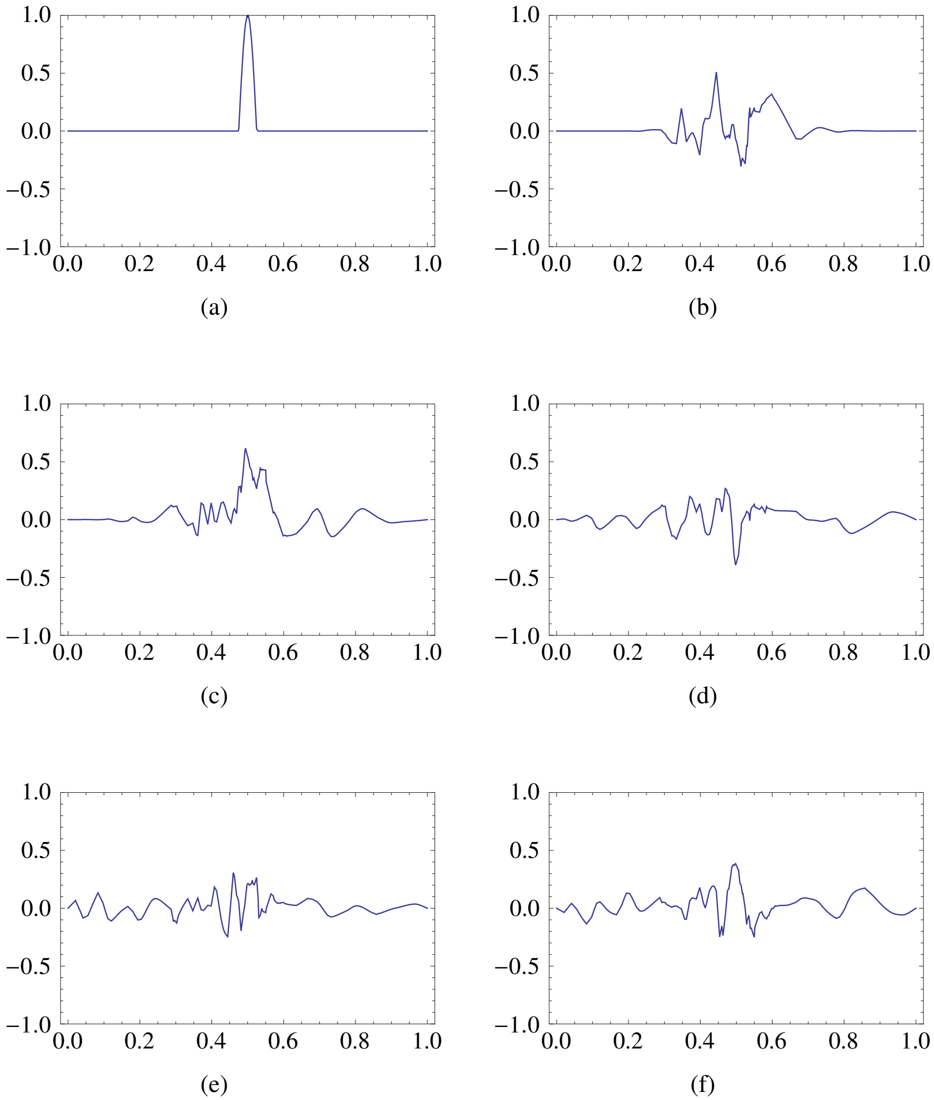

Figure 6. Numerical solutions of the homogeneous Schrödinger equation (1.2) with $U = (0, 1)$ and μ being the self-similar measure defined by the IFS in (5.3) with probability weights $p_1 = 2/3$, $p_2 = p_3 = 1/6$ and contraction ratios $r_1 = 1/2$ and $r_2 = 1/3$. $(\mathbf{P}_k)_{k\geq 1}$ is defined by (5.4). The initial condition, and the values of Δt and t are the same as those in Figure 1.

We choose the basis of S^m consisting of the tent functions $\{\phi_\ell\}_{\ell=0}^{N(m)}$ defined in (4.2) and choose the basis $\{\phi_\ell\}_{\ell=1}^{N(m)-1}$ for S_D^m. The linear map $\mathcal{F}_m\colon \operatorname{dom}\mathcal{E} \to S_D^m$ defined by

$$\mathcal{F}_m v = \sum_{\ell=1}^{N(m)-1} v(x_{m,\ell})\phi_\ell(x), \qquad v \in \operatorname{dom}\mathcal{E},$$

is called the *Rayleigh–Ritz projection* with respect to V_m. Let

$$\|V_m\| := \max\big\{|y_{m,\ell} - x_{m,\ell}| : 0 \le \ell \le N(m)\big\}$$

be the *norm* of V_m for $m \ge 1$.

Lemma 6.1. *For $m \ge 1$, let V_m and $\mathcal{F}_m$ be defined as above. Then for any $u \in \operatorname{dom}\mathcal{E}$, $\mathcal{F}_m u$ is the component of u in the subspace S_D^m, $u - \mathcal{F}_m u$ vanishes on the boundary $\{a, b\}$, and*

$$\mathcal{E}(u - \mathcal{F}_m u, v) = 0 \qquad \textit{for all } v \in S_D^m.$$

Proof. The proof can be found in [28]. ∎

Throughout the rest of this section, let $g = \sum_{n=1}^{\infty} \alpha_n \varphi_n \in E_3((a,b),\mu)$, $f = 0$, and u be the solution of the corresponding homogeneous Schrödinger equation (1.2). Then

$$(i\partial_t u, v)_\mu - \mathcal{E}(u, v) = 0 \qquad \text{for all } v \in \operatorname{dom}\mathcal{E}. \tag{6.1}$$

By Theorem 3.1 (c), $\partial_t u \in \operatorname{dom}\mathcal{E}$. As in Section 4,

$$u^m(x,t) = \sum_{j=1}^{N(m)-1} \big(w_{1j}(t) + i w_{2j}(t)\big)\phi_j(x).$$

Thus, it follows from the derivations in Section 4 that u^m satisfies

$$(i\partial_t u^m, v^m)_\mu + \mathcal{E}(u^m, v^m) = 0 \qquad \text{for all } v^m \in S_D^m, \tag{6.2}$$

and $u^m(x,0) = \sum_{\ell=1}^{N(m)-1} g(x_{m,\ell})\phi_\ell(x)$. Finally, define

$$e(x,t) := e^m(x,t) = \mathcal{F}_m u(x,t) - u^m(x,t).$$

Lemma 6.2. *Let u, u^m, e be as above. Then*

$$(\partial_t e, e)_\mu = (\mathcal{F}_m \partial_t u - \partial_t u, e)_\mu. \tag{6.3}$$

Proof. We first note that the functions e, $\partial_t e$, and $\partial_t(\mathcal{F}_m u) = \mathcal{F}_m \partial_t u$ all belong to S_D^m. Thus, substituting ie for v in (6.1) and for v^m in (6.2), we get

$$(i\partial_t u, ie)_\mu + \mathcal{E}(u, ie) = 0 \quad \text{and} \quad (i\partial_t u^m, ie)_\mu + \mathcal{E}(u^m, ie) = 0.$$

Subtracting these equations gives $(i(\partial_t u - \partial_t u^m), ie)_\mu + \mathcal{E}(u - u^m, ie) = 0$. Using the fact that $(i(\partial_t u - \partial_t u^m), ie)_\mu = (\partial_t u - \partial_t u^m, e)_\mu$, we get

$$(\partial_t u - \mathcal{F}_m \partial_t u + \mathcal{F}_m \partial_t u - \partial_t u^m, e)_\mu + \mathcal{E}(u - \mathcal{F}_m u + \mathcal{F}_m u - u^m, ie) = 0,$$

which, together with the fact $\mathcal{E}(u - \mathcal{F}_m u, ie) = 0$ (see Lemma 6.1), yields

$$(\mathcal{F}_m \partial_t u - \partial_t u^m, e)_\mu + \mathcal{E}(\mathcal{F}_m u - u^m, ie) = (\mathcal{F}_m \partial_t u - \partial_t u, e)_\mu.$$

The desired result follows from the equalities $\mathcal{E}(\mathcal{F}_m u - u^m, ie) = \mathcal{E}(e, ie) = 0$ and $\mathcal{F}_m \partial_t u - \partial_t u^m = \partial_t e$. ∎

Lemma 6.3 ([7, Lemma 5.3]). *Assume the hypotheses of Lemma 6.1, and let* $v \in \operatorname{dom}\mathcal{E}$. *Then*

$$\|\mathcal{F}_m v - v\|_\mu \le 2\|V_m\|^{1/2}\|v\|_{\operatorname{dom}\mathcal{E}} \qquad \textit{for all } m \ge 1.$$

Theorem 6.4. *Assume the hypotheses of Lemma 6.2. If there exist constants* $r \in (0,1)$ *and* $c > 0$ *satisfying* $\max\{|I| : I \in \mathbf{P}_k\} \le c r^k$ *for all* $k \ge 1$*, then*

$$\|\mathcal{F}_m u - u^m\|_\mu \le 2\sqrt{cT} r^{m/2} \|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}}.$$

Proof. The proof is similar to that of [31, Theorem 6.4]; we include it here for completeness. The left side of (6.3) can be rewritten as

$$(\partial_t e, e)_\mu = \frac{1}{2}\frac{d}{dt}\big(\|e\|_\mu^2\big) = \|e\|_\mu \cdot \frac{d}{dt}\big(\|e\|_\mu\big).$$

Thus, (6.3) leads to

$$\|e\|_\mu \cdot \frac{d}{dt}\big(\|e\|_\mu\big) = (\mathcal{F}_m \partial_t u - \partial_t u, e)_\mu \le \|\mathcal{F}_m \partial_t u - \partial_t u\|_\mu \cdot \|e\|_\mu,$$

and hence

$$\frac{d}{dt}\big(\|e\|_\mu\big) \le \|\mathcal{F}_m \partial_t u - \partial_t u\|_\mu. \tag{6.4}$$

Integrating the left side of (6.4) with respect to τ from 0 to t, we get

$$\int_0^t \frac{d}{d\tau}\big(\|e(\tau)\|_\mu\big)\, d\tau = \|e(t)\|_\mu - \|e(0)\|_\mu = \|e(t)\|_\mu, \tag{6.5}$$

where the fact $e(0) = \mathcal{F}_m u(x,0) - u^m(x,0) = \mathcal{F}_m g(x) - \sum_{\ell=1}^{N(m)-1} g(x_{m,\ell})\phi_\ell(x) = 0$ is used in the last equality. Combining (6.4), (6.5), Lemma 6.3, and Hölder's inequality, we have

$$\begin{aligned}\|e(t)\|_\mu &\le \int_0^t \|\mathcal{F}_m \partial_t u(\tau) - \partial_t u(\tau)\|_\mu \, d\tau && \text{(by (6.4) and (6.5))} \\ &\le \int_0^T 2\|V_m\|^{1/2} \|\partial_t u\|_{\operatorname{dom}\mathcal{E}} \, d\tau && \text{(by Lemma 6.3)} \\ &\le 2\sqrt{T}\|V_m\|^{1/2}\|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}} && \text{(by Hölder's inequality)} \\ &\le 2\sqrt{cT} r^{m/2} \|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}},\end{aligned}$$

proving the desired result. ■

Proof of Theorem 1.3. Combining Theorem 6.4 and Lemma 6.3, we have, for each fixed $t \in [0, T]$,

$$\begin{aligned}\|u^m - u\|_\mu &\le \|u^m - \mathcal{F}_m u\|_\mu + \|\mathcal{F}_m u - u\|_\mu \\ &\le 2\sqrt{cT} r^{m/2} \|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}} + 2r^{m/2}\|u\|_{\operatorname{dom}\mathcal{E}} \\ &\le 2\big(\sqrt{cT}\|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}} + \|u\|_{\operatorname{dom}\mathcal{E}}\big) r^{m/2},\end{aligned}$$

which completes the proof. ■

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