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Metric results for numbers with multiple q-expansions

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Abstract. Let M be a positive integer and $q \in (1, M+1]$. A q-expansion of a real number x is a sequence $(c_i) = c_1c_2\cdots$ with $c_i \in \{0, 1, \ldots, M\}$ such that $x = \sum_{i=1}^{\infty} c_i q^{-i}$. In this paper we study the set $\mathcal{U}_q^j$ consisting of those real numbers having exactly j q-expansions. Our main result is that for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M+1)$, we have

$$\dim_H \mathcal{U}_q^j \leq \max\{0, 2\dim_H \mathcal{U}_q - 1\} \text{ for all } j \in \{2, 3, \ldots\}.$$

Here q_{KL} is the Komornik–Loreti constant. As a corollary of this result, we show that for any $j \in \{2, 3, \ldots\}$, the function mapping q to $\dim_H \mathcal{U}_q^j$ is not continuous.

1. Introduction

Fix a positive integer M. For $q \in (1, M+1]$ we call a sequence $(c_i) = c_1c_2\cdots \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ a q-expansion of x in base q if

$$x = \pi_q((c_i)) := \sum_{i=1}^{\infty} \frac{c_i}{q^i}.$$

The study of q-expansions was pioneered in the papers of Rényi [28] and Parry [26]. Since these beginnings, the study of q-expansions has drawn significant attention. This is in part due to its connections with many other areas of mathematics. These areas include dynamical systems, fractal geometry, and number theory.

It is well known that for $q \in (1, M+1]$, a number x has an expansion in base q if and only if $x \in I_q := [0, M/(q-1)]$. When $q = M+1$, then $I_q = [0, 1]$ and every $x \in [0, 1]$ has a unique expansion except for a countable set of exceptions that have precisely two. When $q \in (1, M+1)$, then the situation is much more interesting. For instance, for any $q \in (1, M+1)$, it is the case that Lebesgue almost every $x \in I_q$ has a continuum of q-expansions [8, 29]. For completion, we mention that if $q > M+1$,

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then the set of points with an expansion in base q is a fractal set, and if a point has an expansion in base q, then this sequence must be unique.

For each $j \in \mathbb{N}$ let

$$\mathcal{U}_q^j := \{x \in I_q : \#\pi_q^{-1}(x) = j\}.$$

Similarly, we let

$$\mathcal{U}_q^{\aleph_0} := \{x \in I_q : \pi_q^{-1}(x) \text{ is an infinite countable set}\}.$$

We call $\mathcal{U}_q^1$ the *univoque set*. For simplicity, throughout this paper we write $\mathcal{U}_q$ instead of $\mathcal{U}_q^1$. We also let $U_q := \pi_q^{-1}(\mathcal{U}_q)$ be the corresponding set of sequences. The sets $\mathcal{U}_q$ and U_q were first properly studied by Erdős et al. in the early 1990s [12–14]. Since then these sets have received significant attention, and we now have a good understanding of their combinatorial, topological and fractal properties. See for instance the papers [1, 9, 10, 16–18, 20, 23, 24] and the results therein. For $j \in \{2, 3, \ldots\} \cup \{\aleph_0\}$, many important theorems have been obtained on the properties of the set $\mathcal{U}_q^j$, see [4–6, 13, 19, 25, 30, 31, 33]. However, our knowledge of the set $\mathcal{U}_q^j$ is significantly less than that of the set $\mathcal{U}_q$. This paper is in part motivated by a desire to address this shortcoming.

Very little is known about the metric properties of $\mathcal{U}_q^j$. A simple bifurcation argument of Sidorov [30] and the first author [5] implies that

$$\dim_H \mathcal{U}_q^j \le \dim_H \mathcal{U}_q \quad \text{for all} \quad q > 1 \quad \text{and} \quad j \in \{2, 3, \ldots\} \cup \{\aleph_0\}. \tag{1}$$

See also [32, Lemma 5.5, Proposition 5.6]. Here and throughout $\dim_H F$ denotes the Hausdorff dimension of a set F. When q is such that $\dim_H \mathcal{U}_q = 0$, then (1) immediately implies that

$$\dim_H \mathcal{U}_q^j = \dim_H \mathcal{U}_q \quad \text{for all} \quad j \in \{2, 3, \ldots\} \cup \{\aleph_0\}. \tag{2}$$

It was shown in [10, 16, 23, 24] that $\dim_H \mathcal{U}_q = 0$ for all $q \in (1, q_{\mathrm{KL}}]$, where q_{KL} is the Komornik–Loreti constant (see Section 2 for more details). Therefore, (2) holds for all $q \in (1, q_{\mathrm{KL}}]$. On the other hand, when $q = M + 1$, $\mathcal{U}_q^2$ is countable and $\mathcal{U}_q = [0, 1] \setminus \mathcal{U}_q^2$, so $\dim_H \mathcal{U}_q^2 = 0 < 1 = \dim_H \mathcal{U}_q$. When $q > M + 1$, because every expansion is unique, we have $1 > \dim_H \mathcal{U}_q > \dim_H \mathcal{U}_q^j = 0$. Therefore, (2) fails for every $q \ge M + 1$. Because of these observations, it is natural to restrict our attention to studying the metric properties of $\mathcal{U}_q^j$ for q in the interval $(q_{\mathrm{KL}}, M + 1)$. Recently Sidorov [30] showed that if $M = 1$ and $q \approx 1.83929$ is the *Tribonacci number*, i.e., the positive root of the equation $x^3 = x^2 + x + 1$, then $\dim_H \mathcal{U}_q^j = \dim_H \mathcal{U}_q$ for all $j \in \{2, 3, \ldots\}$. Motivated by Sidorov's work, the second author and her coauthors proved in [32] that there exist infinitely many $q \in (q_{\mathrm{KL}}, M + 1)$ such that $\dim_H \mathcal{U}_q^j = \dim_H \mathcal{U}_q$ for all $j \in \{2, 3, \ldots\}$. In this paper we prove that for a typical $q \in (q_{\mathrm{KL}}, M+1)$ property (2) does not hold. In particular, we show the following statement.

Theorem 1. *For Lebesgue almost every $q \in (q_{\mathrm{KL}}, M+1)$ we have*

$$\dim_H \mathcal{U}_q^j \le \max\{0, 2\dim_H \mathcal{U}_q - 1\} \text{ for all } j \in \{2,3,\ldots\}.$$

For any $q \in (1, M+1)$ it is the case that $\dim_H \mathcal{U}_q < 1$. Therefore, Theorem 1 implies that $\dim_H \mathcal{U}_q^j < \dim_H \mathcal{U}_q$ for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M+1)$. Notice that as well as establishing that $\dim_H \mathcal{U}_q^j < \dim_H \mathcal{U}_q$ holds almost everywhere in $(q_{\mathrm{KL}}, M+1)$, Theorem 1 also provides a lower bound for the difference between these quantities. We remark that our methods do not imply a version of Theorem 1 for $\mathcal{U}_q^{\aleph_0}$. This is because if $x \in \mathcal{U}_q^{\aleph_0}$, then there does not necessarily exist $(c_i) \in \pi_q^{-1}(x)$ and $n > 1$, such that $\pi_q((c_{n+i})) \in \mathcal{U}_q^2$.

The following theorem provides information on when $\mathcal{U}_q^j$ is empty.

Theorem 2. *Let*

$$O := \{q \in (q_{\mathrm{KL}}, M+1) \colon \dim_H \mathcal{U}_q < 1/2\}.$$

Then for Lebesgue almost every $q \in O$ we have $\mathcal{U}_q^j = \emptyset$ for any $j \in \{2,3,\ldots\}$.

The following corollary is a consequence of Theorem 1 and the results from [32] mentioned above.

Corollary 3. *For any $j \in \{2,3,\ldots\}$, the function $f \colon (q_{\mathrm{KL}}, M+1) \to [0,1]$ given by $f(q) = \dim_H \mathcal{U}_q^j$ is not continuous.*

Corollary 3 is contrary to the case when $j = 1$, for which it is known that the function mapping q to $\dim_H \mathcal{U}_q$ is continuous, see [2, 20].

The rest of the paper is arranged as follows. In Section 2 we recall some relevant definitions and results from expansions in non-integer bases. In Section 3 we prove a number of technical results that will assist in our proof of Theorems 1 and 2. In Section 4 we prove Theorems 1 and 2.

2. Preliminaries

Fix a positive integer M. We will denote an element of $\{0,\ldots,M\}^{\mathbb{N}}$ by (c_i) or $c_1c_2\ldots$. We call a finite string of digits $w = c_1\cdots c_n$ with $c_i \in \{0,1,\ldots,M\}$ a *word*. For convenience, we let $\{0,\ldots,M\}^0$ denote the set consisting of the empty word. Given two finite words $w = c_1\cdots c_n$ and $v = d_1\cdots d_m$, we denote by $wv = c_1\cdots c_n d_1\cdots d_m$ their concatenation. Accordingly, for $k \in \mathbb{N}$ and a finite word w, we denote by w^k or $(w)^k$ the concatenation of w with itself k times, and by w^∞ or

$(w)^\infty$ the concatenation of w with itself infinitely many times. For a sequence (c_i) we denote by $\overline{(c_i)} = (M - c_1)(M - c_2)\cdots$ its reflection.

We will use the lexicographic ordering on sequences. If (c_i) and (d_i) are two sequences, then we write $(c_i) \prec (d_i)$ if there exists $k \in \mathbb{N}$ such that $c_i = d_i$ for $i = 1, \ldots, k-1$ and $c_k < d_k$. Similarly, we write $(c_i) \preceq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. We also write $(d_i) \succ (c_i)$ if $(c_i) \prec (d_i)$, and $(d_i) \succeq (c_i)$ if $(c_i) \preceq (d_i)$.

For any fixed base $q \in (1, M+1]$, every $x \in I_q$ has a lexicographically largest expansion $b(x,q) = (b_i)$ obtained by the greedy algorithm, and a lexicographically largest infinite expansion $a(x,q) = (a_i)$; see [3, 11]. Such expansions are called the *greedy* and *quasi-greedy expansions* of x in base q, respectively. A sequence (c_i) is called *finite* if it ends with an infinite string of zeros, and it is called *infinite* otherwise. The case $x = 1$ is particularly important. In this special case we introduce the simpler notation $\alpha(q) := a(1,q) = (\alpha_i)$.

We recall from [21,22] that there exists a smallest base $q_{\mathrm{KL}} \in (1, M+1)$ (depending on M) for which $x = 1$ has a unique expansion. This number is called the Komornik–Loreti constant and is defined using the classical Thue–Morse sequence $(\tau_i)_{i=0}^\infty$.

$$\alpha(q_{\mathrm{KL}}) := \begin{cases} (k + \tau_i)_{i=1}^\infty & \text{if } M = 2k+1, \\ (k + \tau_i - \tau_{i-1})_{i=1}^\infty & \text{if } M = 2k, \end{cases} \tag{3}$$

where $(\tau_i)_{i=1}^\infty = 1101\ 0011\ \cdots$ denotes the truncated Thue–Morse sequence. Then the sequence $\alpha(q_{\mathrm{KL}})$ begins with

$$\begin{cases} (k+1)(k+1)k(k+1)kk(k+1)(k+1)\cdots & \text{if } M = 2k+1, \\ (k+1)k(k-1)(k+1)(k-1)k(k+1)k\cdots & \text{if } M = 2k. \end{cases} \tag{4}$$

The following lexicographic characterisation of the quasi-greedy and greedy expansions was given in [3].

Lemma 4. *The following statements are true:*

(i) *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection between the interval $(1, M+1]$ and the set of infinite sequences (α_i) satisfying the lexicographic inequalities*
$$(\alpha_{n+i}) \preceq (\alpha_i) \quad \textit{for all} \quad n \geq 0.$$

(ii) *For a fixed $q \in (1, M+1]$, the map $x \mapsto a(x,q)$ is a strictly increasing bijection between the interval $(0, M/(q-1)]$ and the infinite sequences (a_i) satisfying the lexicographic inequalities*
$$(a_{n+i}) \preceq \alpha(q) \quad \textit{whenever} \quad a_n < M.$$

(iii) *For a fixed* $q \in (1, M+1]$, *the map* $x \mapsto b(x,q)$ *is a strictly increasing bijection between the interval* $[0, M/(q-1)]$ *and the sequences* (b_i) *satisfying the lexicographic inequalities*

$$(b_{n+i}) \prec \alpha(q) \quad \textit{whenever} \quad b_n < M.$$

Let

$$V := \left\{q \in (1, M+1] \colon \overline{\alpha(q)} \preceq \alpha_{i+1}\alpha_{i+2}\cdots \preceq \alpha(q) \text{ for all } i \geq 0\right\}.$$

The set V has zero Lebesgue measure, see [10]. For each $M \in \mathbb{N}$ we define the generalised golden ratio q_{GR} to be the unique $q \in (1, M+1)$ for which $(0, M/(q-1)) \cap \mathcal{U}_q \neq \emptyset$ for $q > q_{\mathrm{GR}}$ and $(0, M/(q-1)) \cap \mathcal{U}_q = \emptyset$ for $q < q_{\mathrm{GR}}$. In [4] it was shown that for each $M \in \mathbb{N}$ a generalised golden ratio exists, and is given by the following formula:

$$q_{\mathrm{GR}} = \begin{cases} k+1 & \text{if } M = 2k; \\ \frac{k+1+\sqrt{k^2+6k+5}}{2} & \text{if } M = 2k+1. \end{cases}$$

Moreover, q_{GR} is the smallest element of V [11] and $M+1$ is the largest element of V. By [11, Theorem 1.3], we have

$$[q_{\mathrm{GR}}, M+1] \setminus V = (q_{\mathrm{GR}}, M+1) \setminus V = \bigcup (q_l, q_r),$$

where the union on the right-hand side is pairwise disjoint and countable. The open intervals (q_l, q_r) are referred to as the basic intervals of $[q_{\mathrm{GR}}, M+1] \setminus V$. The following property of basic intervals was established in [10].

Lemma 5. *Let* (q_l, q_r) *be a basic interval. For any* $q_1, q_2 \in (q_l, q_r]$ *we have* $U_{q_1} = U_{q_2}$.

We now recall some technical results on $\mathcal{U}_q^j$. The following lemma follows from [6, Lemma 1.6]. We remark that this lemma is phrased in the case when $M = 1$, but the same argument applies for arbitrary M.

Lemma 6. *Let* $1 < q < M+1$ *and* $x \in \mathcal{U}_q^j$ *for some* $j \geq 3$. *Then there exists* (c_i) *a* q*-expansion of* x *and an integer* $k > 0$ *such that* $\pi_q((c_{k+i})) \in \mathcal{U}_q^2$.

Lemma 6 implies the following statement.

Lemma 7. *For any* $1 < q < M+1$ *and* $j \geq 3$, *we have*

$$\dim_H \mathcal{U}_q^j \leq \dim_H \mathcal{U}_q^2.$$

The subsequent lemma follows from results proved in [4]. See the discussion after Lemma 2.8 in this paper.

Lemma 8. *Assume* $x = \pi_q((c_i)) \in \mathcal{U}_q \subset I_q$.

(i) $c_1 = 0$ *if and only if* $x \in [0, 1/q)$.

(ii) *For* $i = 1, \ldots, M-1$, $c_1 = i$ *if and only if* $x \in \big(\frac{(i-1)(q-1)+M}{q^2-q}, \frac{i+1}{q}\big)$.

(iii) $c_1 = M$ *if and only if* $x \in \big(\frac{(M-1)(q-1)+M}{q^2-q}, \frac{M}{q-1}\big]$.

With Lemma 8 in mind, we define the switch region as follows:

$$S_q := \bigcup_{i=1}^{M}\Big[\frac{i}{q}, \frac{(i-1)(q-1)+M}{q^2-q}\Big].$$

Note that S_q is the complement to the intervals listed in items (i), (ii), and (iii) in Lemma 8. The following properties of the switch region were established in [4].

Lemma 9. *Let* $q \in (q_{\mathrm{GR}}, M+1]$. *Then the following statements are true:*

(i) *Let* $x \in I_q \setminus \mathcal{U}_q$. *Then there exist* $(a_i), (b_i) \in \{0, \ldots, M\}^{\mathbb{N}}$ *such that* $\pi_q((a_i)) = \pi_q((b_i)) = x$ *and* $a_1 \neq b_1$ *if and only if* $x \in S_q$.

(ii) *Let* $k \in \{1, \ldots, M\}$ *and* $x \in I_q \setminus \mathcal{U}_q$. *Then there exist* $(a_i), (b_i) \in \{0, \ldots, M\}^{\mathbb{N}}$ *such that* $a_1 = k$, $b_1 = k-1$ *and* $\pi_q((a_i)) = \pi_q((b_i)) = x$ *if and only if* $x \in \big[\frac{k}{q}, \frac{(k-1)(q-1)+M}{q^2-q}\big]$.

(iii) *If* $i \neq j$, *then* $\big[\frac{i}{q}, \frac{(i-1)(q-1)+M}{q^2-q}\big] \cap \big[\frac{j}{q}, \frac{(j-1)(q-1)+M}{q^2-q}\big] = \emptyset$.

Lemma 9 implies the useful fact that if $x \in I_q$ and there exist sequences $(a_i), (b_i) \in \{0, \ldots, M\}^{\mathbb{N}}$ such that $\pi_q((a_i)) = \pi_q((b_i)) = x$ and $a_1 \neq b_1$, then a_1 and b_1 are successive digits in $\{0, \ldots, M\}$, and if (c_i) is another sequence such that $\pi_q((c_i)) = x$, then either $c_1 = a_1$ or $c_1 = b_1$.

3. Properties of $\mathcal{U}_q^2$

In this section we prove some properties of the set $\mathcal{U}_q^2$ and associated power series.

Lemma 10. *Let* $q \in (q_{\mathrm{GR}}, M+1]$. *Then* $x \in \mathcal{U}_q^2$ *if and only if there exist* $(a_i), (b_i) \in U_q$, *a finite word* $w \in \bigcup_{n=0}^{\infty}\{0, 1, \ldots, M\}^n$ *and* $0 \leq m < M$ *such that*

$$x = \pi_q(wm(a_i)) = \pi_q(w(m+1)(b_i)) \tag{5}$$

and

$$\pi_q(w_{j+1}\ldots w_n m(a_i)) \notin S_q \text{ for all } 0 \leq j < n. \tag{6}$$

Proof. Let $x \in I_q$ and suppose that (5) and (6) hold for some $(a_i), (b_i) \in U_q$. Then (5) implies that x has at least two different q-expansions, which are $wm(a_i)$ and

$w(m+1)(b_i)$. Furthermore, Lemma 9, (6), and the fact that $(a_i), (b_i) \in U_q$ imply that these are the only q-expansions. Therefore, $x \in \mathcal{U}_q^2$.

Now we prove the necessity. Take $x \in \mathcal{U}_q^2$ having exactly two q-expansions (c_i) and (d_i). Let $n \geq 1$ be the least integer such that $c_1 \cdots c_{n-1} = d_1 \cdots d_{n-1}$ and $c_n \neq d_n$. Without loss of generality we assume $c_n < d_n$. By Lemma 9 we know that there exists $0 \leq m < M$ such that

$$\pi_q((c_{n-1+i})) = \pi_q((d_{n-1+i})) \in \left[\frac{m+1}{q}, \frac{m(q-1)+M}{q^2-q}\right] \text{ and } c_n = m, d_n = m+1.$$

Moreover, because $x \in \mathcal{U}_q^2$, we know that $(c_{n+i}), (d_{n+i}) \in U_q$. Taking $w = c_1 \cdots c_{n-1}$, $(a_i) = (c_{n+i})$, and $(b_i) = (d_{n+i})$, we see that $x = \pi_q(wm(a_i)) = \pi_q(w(m+1)(b_i))$, and so (5) holds. To see that (6) holds, we remark that if it did not hold, then Lemma 9 would imply that we have a choice of digit before the n-th position. This would imply that x has at least three different q-expansions and so would contradict that $x \in \mathcal{U}_q^2$. Therefore, (6) holds. ∎

As we will see later on in the paper, two sequences $(a_i), (b_i) \in U_q$ give rise to a point with exactly two q-expansions if and only if q is the zero of some appropriate power series with coefficients given by (a_i) and (b_i). We now set out to show that such a power series has at most one zero in $(q_{\mathrm{KL}}, M+1)$. To prove this, we will make use of ideas from [27].

Fix $M \geq 1$. We consider functions of the form

$$g(x) = 1 + \sum_{i=1}^{\infty} b_i x^i \quad \text{with } b_i \in \{0, \pm 1, \ldots, \pm M\}. \tag{7}$$

We say that the *δ-transversality condition* holds on the interval I for some $\delta > 0$ if for any $x \in I$ and g of the form (7), whenever $g(x) < \delta$ we have $g'(x) < -\delta$. A power series h is called a *$(*)$-function* if for some $n \geq 1$ and $a_n \in [-M, M]$ we have

$$h(x) = 1 - M\sum_{i=1}^{n-1} x^i + a_n x^n + M \sum_{i=n+1}^{\infty} x^i.$$

The following lemma connects $(*)$-functions and δ-transversality.

Lemma 11. *If h is a $(*)$-function such that*

$$h(x_M) > \delta \quad \textit{and} \quad h'(x_M) < -\delta$$

for some $x_M \in (0,1)$ and $\delta \in (0,1)$, then the δ-transversality condition holds on the interval $[0, x_M]$.

Proof. The case where $M = 1$ was proved in [27]. The case where $M > 1$ is proved by an analogous argument. ∎

The following lemma will allow us to establish δ-transversality within the interval $[0, q_{\mathrm{KL}}^{-1}]$ for some $\delta > 0$.

Lemma 12. *For each $M \in \mathbb{N}$ let the $(*)$-function h and x_M be defined as follows.*

- *Assume M is of the form $M = 2k + 1$. Let*

$$h(x) = \begin{cases} 1 - x - x^2 - x^3 + 0.5x^4 + \sum_{i=5}^{\infty} x_i & \text{when } k = 0, \\ 1 - 3x - 0.5x^2 + 3\sum_{i=3}^{\infty} x^i & \text{when } k = 1, \\ 1 - (k+3)x + (2k+1)\sum_{i=2}^{\infty} x^i & \text{when } k \geq 2. \end{cases}$$

  Let

$$x_M = \begin{cases} 2^{-2/3} & \text{when } k = 0, \\ \frac{-(k+1)+\sqrt{(k+1)^2+4(k+1)}}{2(k+1)} & \text{when } 1 \leq k < 3, \\ \frac{1}{(k+1)} & \text{when } k \geq 3. \end{cases}$$

- *Assume M is of the form $M = 2k$. Let*

$$h(x) = \begin{cases} 1 - 2x - 0.5x^2 + 2\sum_{i=3}^{\infty} x^i & \text{when } k = 1, \\ 1 - (k+2)x + 2k\sum_{i=2}^{\infty} x^i & \text{when } k \geq 2. \end{cases}$$

  Let

$$x_M = \begin{cases} \frac{-(k+1)+\sqrt{(k+1)^2+4k}}{2k} & \text{when } 1 \leq k < 3, \\ \frac{1}{(k+1)} & \text{when } k \geq 3. \end{cases}$$

Then $h(x_M) > 0$ and $h'(x_M) < 0$ and $[1/(M+1), 1/q_{\mathrm{KL}}] \subset [0, x_M]$.

Proof. We begin by remarking that the fact $h(x_M) > 0$ and $h'(x_M) < 0$ was established when $M = 1$ in [27]. Moreover, $[1/2, 1/q_{\mathrm{KL}}] \subset [0, 2^{-2/3}]$ follows from the fact $q_{\mathrm{KL}} \approx 1.787\ldots$.

We now prove that $x_M^{-1} < q_{\mathrm{KL}}$ for all $M \geq 2$. This implies our final assertion $[1/(M+1), 1/q_{\mathrm{KL}}] \subset [0, x_M]$ for the remaining values of M. When $M = 3$ or 5, it follows from the definition that x_M is the solution to $(k+1)x^2 + (k+1)x - 1 = 0$ for the appropriate value of k. Using this algebraic relation it can be shown that the quasi-greedy expansion of 1 in base x_M^{-1} is $((k+1)k)^{\infty}$. Similarly, it can be shown when $M = 2$ or $M = 4$ that the quasi-greedy expansion of 1 in base x_M^{-1} is $((k+1)(k-1))^{\infty}$ for the appropriate value of k. For the remaining values of M the quasi-greedy expansion of 1 in base x_M^{-1} is k^{∞}. By inspection of (4), we see that the quasi-greedy expansion of 1 in base x_M^{-1} is lexicographically smaller than $\alpha(q_{\mathrm{KL}})$ for any $M \in \mathbb{N}$. Lemma 4 (i) then implies $x_M^{-1} < q_{\mathrm{KL}}$.

It remains to show that $h(x_M) > 0$ and $h'(x_M) < 0$. For $M = 2, 3, 4, 5$ this can be checked by a direct computation. For $M = 2k + 1$ and $k \geq 3$, we have

$$h'(x) = -3 - k + (2k + 1)\frac{2x - x^2}{(x - 1)^2}.$$

Substituting in our value for x_M, we have

$$h(x_M) = \frac{1}{k + k^2} \quad \text{and} \quad h'(x_M) = 1 + \frac{1}{k^2} + \frac{4}{k} - k.$$

We always have $h(x_M) > 0$, and it is simple to show that $h'(x_M) < 0$ for all $k \geq 3$.

For $M = 2k$ and $k \geq 3$ we have

$$h'(x) = -2 - k + 2k\frac{2x - x^2}{(x - 1)^2}.$$

Substituting in our value of x_M, we have

$$h(x_M) = \frac{1}{k + 1} \quad \text{and} \quad h'(x_M) = 2 + \frac{2}{k} - k.$$

We always have $h(x_M) > 0$, and it is easy to show that $h'(x_M) < 0$ for all $k \geq 3$. ■

Proposition 13. *Let $(a_i), (b_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$. Then there exists at most one $q \in [q_{\mathrm{KL}}, M + 1]$ such that $\pi_q((a_i)) = \pi_q((b_i)) + 1$.*

Proof. The equation $\pi_q((a_i)) = \pi_q((b_i)) + 1$ can be rewritten in the form of $0 = 1 + \sum_{i=1}^{\infty}(b_i - a_i)/q^i$. The zeros of this function can be mapped to the zeros of $0 = 1 + \sum_{i=1}^{\infty}(b_i - a_i)x^i$ by taking this reciprocal. Importantly, this map is a bijection. By Lemmas 11 and 12 we know that the δ-transversality condition holds on the interval $[1/(M + 1), q_{\mathrm{KL}}^{-1}]$ for $\delta = \min\{h(x_M)/2, |h'(x_M)|/2\}$. Therefore, there exists at most one $x^* \in [1/(M + 1), q_{\mathrm{KL}}^{-1}]$ such that $0 = 1 + \sum_{i=1}^{\infty}(b_i - a_i)(x^*)^i$. Using the bijection mentioned above, we may conclude our result. ■

4. Proofs of Theorems 1 and 2

The main focus of this section will be to prove Theorem 1. We will then explain how the argument can be adapted to prove Theorem 2. We start by remarking that by Lemma 7 we know that for any $q > 1$ we have

$$\dim_H \mathcal{U}_q^j \leq \dim_H \mathcal{U}_q^2$$

for every $j \in \{3, 4, \ldots\}$. Because of this lemma, to prove Theorem 1, it suffices to show that for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M + 1)$ we have

$$\dim_H \mathcal{U}_q^2 \leq \max\{2 \dim_H \mathcal{U}_q - 1, 0\}.$$

Moreover, to prove this statement, it is sufficient to show that for any $\varepsilon > 0$

$$\dim_H \mathcal{U}_q^2 \leq \max\{2\dim_H \mathcal{U}_q - 1 + \varepsilon, 0\} \tag{8}$$

for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M+1)$. We now fix such an $\varepsilon > 0$ and set out to show that this is the case.

To prove (8) holds for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M+1)$, we will make use of a countable collection of closed intervals $\{J_i\}$ that are chosen in a way that depends upon ε and M. These intervals satisfy the following properties:

(J-i) $\{J_i\}$ cover $(q_{\mathrm{KL}}, M+1)$ up to a set of Lebesgue measure zero.

(J-ii) Each J_i is small in a way that depends upon ε and M.

(J-iii) Each J_i is contained in a basic interval (q_l, q_r) of $[q_{\mathrm{GR}}, M+1] \setminus V$.

(J-iv) For any J_i, if $q_1, q_2 \in J_i$, then $|\dim_H \mathcal{U}_{q_1} - \dim_H \mathcal{U}_{q_2}| < \varepsilon/2$.

Item (J-ii) is important because we will soon show that if the $\{J_i\}$ are chosen to be small in an appropriate way that depends upon ε and M, then certain useful properties hold. We remark that item (J-iii) in the above, together with Lemma 5, implies that for any $q_1, q_2 \in J_i \subset (q_l, q_r)$, we have

$$U_{q_1} = U_{q_2}.$$

Item (J-iv) follows from the continuity of the function which maps q to $\dim_H \mathcal{U}_q$. This fact was established in [2, 20].

From the collection $\{J_i\}$ we now make an arbitrary choice that we will denote by J. For the rest of the proof of Theorem 1 the interval J is fixed. By property (J-i) of the collection $\{J_i\}$, and the arbitrariness of J, to prove (8) holds for Lebesgue almost every $q \in (q_{\mathrm{KL}}, M+1)$, it is sufficient to show that it holds for Lebesgue almost every $q \in J$. To do this, we need to introduce two new sets and a function between them.

Let $q_0 := \min\{q : q \in J\}$. To each $0 \leq m < M$ we assign

$$A_m := \bigcup_{\substack{a,b \in U_{q_0} \\ \exists q^* \in J : \pi_{q^*}(a) = \pi_{q^*}(b)+1}} \left\{\left(\pi_{q_0}(ma), \pi_{q_0}((m+1)b)\right)\right\}$$

and

$$A := \bigcup_{m=0}^{M-1} A_m. \tag{9}$$

We emphasise that for each m, the terms in the union in A_m are elements of $\mathbb{R}^2$ and not intervals. Even though A consists of elements of $\mathbb{R}^2$ obtained by applying π_{q_0} to a collection of sequences from U_{q_0}, because $U_q = U_{q_0}$ for each $q \in J$, the set A, in fact, contains information about the whole interval J and not just the specific base q_0. This fact is crucial in our proof. By Proposition 13, we know that if $a, b \in U_{q_0}$ are such that there exists $q^* \in J$ for which $\pi_{q^*}(a) = \pi_{q^*}(b) + 1$, then q^* is unique. When such a q^* exists, then we will denote it by $q^*(a, b)$.

To each $0 \leq m < M$ we assign

$$B_m := \bigcup_{\substack{a,b \in U_{q_0} \\ \exists q^* \in J : \pi_{q^*}(a) = \pi_{q^*}(b)+1}} \{(q^*(a, b), \pi_{q^*}(ma))\}$$

and

$$B := \bigcup_{m=0}^{M-1} B_m.$$

As above, for each m the terms in the union in B_m are elements of $\mathbb{R}^2$ and are not intervals. We emphasise that both A and B depend upon our choice of interval J. However, because J is fixed, we suppress this dependence within our notation.

The following lemma connects vertical slices through B with the set $\mathcal{U}_q^2$.

Lemma 14. *For any $q \in J$ we have*

$$\dim_H\big(\{(q, y) \colon y \in \mathbb{R}\} \cap B\big) = \dim_H \mathcal{U}_q^2.$$

Moreover, if $\{(q, y) \colon y \in \mathbb{R}\} \cap B = \emptyset$, then $\mathcal{U}_q^2 = \emptyset$.

Proof. Fix $q \in J$. We begin by remarking that

$$\{(q, y) \colon y \in \mathbb{R}\} \cap B = \bigcup_{m=0}^{M-1} \bigcup_{\substack{a,b \in U_{q_0} \\ \pi_q(a) = \pi_q(b)+1}} (q, \pi_q(ma)).$$

Moreover, because $q \in J$ and $U_{q_0} = U_q$ for all $q \in J$, we in fact have

$$\{(q, y) \colon y \in \mathbb{R}\} \cap B = \bigcup_{m=0}^{M-1} \bigcup_{\substack{a,b \in U_q \\ \pi_q(a) = \pi_q(b)+1}} (q, \pi_q(ma)).$$

Therefore,

$$\dim_H (\{(q, y) \colon y \in \mathbb{R}\} \cap B) = \dim_H C_q, \tag{10}$$

where

$$C_q := \bigcup_{m=0}^{M-1} \bigcup_{\substack{a,b \in U_q \\ \pi_q(a) = \pi_q(b)+1}} \pi_q(ma).$$

If $a, b \in U_q$ are such that $\pi_q(a) = \pi_q(b) + 1$, then $\pi_q(ma) = \pi_q((m + 1)b)$ for each $0 \leq m < M$. Therefore, by Lemma 10 we may conclude that $\pi_q(ma) \in \mathcal{U}_q^2$ for all

$0 \le m < M$ and

$$\dim_H C_q \le \dim_H \mathcal{U}_q^2.$$

Now we prove that $\dim_H C_q \ge \dim_H \mathcal{U}_q^2$. Suppose $x \in \mathcal{U}_q^2$, then by Lemma 10 again, there exists a finite word $w \in \bigcup_{n=0}^{\infty}\{0,\dots,M\}^n$, $0 \le m < M$, and $a, b \in U_q$ such that $x = \pi_q(wma) = \pi_q(w(m+1)b)$. Therefore,

$$x \in \pi_q(w0^\infty) + \frac{1}{q^n} C_q,$$

and so

$$\mathcal{U}_q^2 \subseteq \bigcup_{n=0}^{\infty} \bigcup_{w\in\{0,\dots,M\}^n} \Big(\pi_q(w0^\infty) + \frac{1}{q^n} C_q\Big).$$

Since $\mathcal{U}_q^2$ is covered by countably many sets, each of which is an affine image of C_q, we conclude that $\dim_H \mathcal{U}_q^2 \le \dim_H C_q$. We have shown that $\dim_H \mathcal{U}_q^2 = \dim_H C_q$. Combining this with (10) we conclude that $\dim_H(\{(q,y)\colon y \in \mathbb{R}\} \cap B) = \dim_H \mathcal{U}_q^2$, as required.

It remains to verify that if $\{(q,y)\colon y \in \mathbb{R}\} \cap B = \emptyset$, then $\mathcal{U}_q^2 = \emptyset$. Now, if $\{(q,y)\colon y \in \mathbb{R}\} \cap B = \emptyset$, then C_q as defined above is also empty. The fact $\mathcal{U}_q^2$ is covered by countably many affine images of C_q holds even if C_q is empty. Therefore, if $\{(q,y)\colon y \in \mathbb{R}\} \cap B$ is empty, then so is $\mathcal{U}_q^2$. ■

What remains of our proof of Theorem 1 focuses on the properties of a map $f\colon A \to B$ defined below. To make sure this map is well defined, it is necessary to assume that $A_i \cap A_j = \emptyset$ for $i \neq j$. The following lemma allows us to make such an assumption.

Lemma 15. *The intervals $\{J_i\}$ can be chosen to be sufficiently small such that the set $A_i \cap A_j = \emptyset$ for $i \neq j$.*

Proof. Consider the interval

$$\left[\frac{i+1}{q_0}, \frac{i(q_0-1)+M}{q_0^2-q_0}\right].$$

If $i \neq j$, then Lemma 9 tells us that

$$\left[\frac{i+1}{q_0}, \frac{i(q_0-1)+M}{q_0^2-q_0}\right] \cap \left[\frac{j+1}{q_0}, \frac{j(q_0-1)+M}{q_0^2-q_0}\right] = \emptyset.$$

Moreover, there exists $\delta > 0$ (for example, we could take $\delta := \frac{1}{3}\frac{2q_{\mathrm{KL}}-2-M}{q_{\mathrm{KL}}^2-q_{\mathrm{KL}}}$) depending only upon M such that

$$\left[\frac{i+1}{q_0} - \delta, \frac{i(q_0-1)+M}{q_0^2-q_0} + \delta\right] \cap \left[\frac{j+1}{q_0} - \delta, \frac{j(q_0-1)+M}{q_0^2-q_0} + \delta\right] = \emptyset. \quad (11)$$

Let $a, b \in U_{q_0}$ be such that there exists $q^* \in J$ for which $\pi_{q^*}(a) = \pi_{q^*}(b) + 1$. Then for each $0 \le i < M$ we have $\pi_{q^*}(ia) = \pi_{q^*}((i+1)b)$ and

$$\pi_{q^*}(ia) \in \left[\frac{i+1}{q^*}, \frac{i(q^*-1)+M}{(q^*)^2 - q^*}\right]$$

by Lemma 9. By continuity, we can assume that our intervals $\{J_i\}$ were chosen to be sufficiently small such that

$$\left[\frac{i+1}{q^*}, \frac{i(q^*-1)+M}{(q^*)^2 - q^*}\right] \subset \left(\frac{i+1}{q_0} - \delta, \frac{i(q_0-1)+M}{q_0^2 - q_0} + \delta\right)$$

for any $q^* \in J$. Therefore, by continuity we can choose our $\{J_i\}$ to be sufficiently small such that

$$\pi_{q_0}(ia), \pi_{q_0}((i+1)b) \in \left[\frac{i+1}{q_0} - \delta, \frac{i(q_0-1)+M}{q_0^2 - q_0} + \delta\right]. \tag{12}$$

Combining (11) and (12) we may conclude that $A_i \cap A_j$ for $i \neq j$. ∎

For each $0 \le m < M$ we define $f_m\colon A_m \to B_m$ by

$$f_m\left((\pi_{q_0}(ma), \pi_{q_0}((m+1)b))\right) = (q^*(a,b), \pi_{q^*}(ma)).$$

Since we can assume the A_m are disjoint by Lemma 15, we can define

$$f\colon A \to B \tag{13}$$

by $f|_{A_m} = f_m$. Importantly, it follows from the definitions of A, B, and f that $f(A) = B$. Our goal now is to prove that the function $f\colon A \to B$ is Lipschitz. The following three lemmas are all technical results that will allow us to establish this fact.

Lemma 16. *Let $q \in (q_{\mathrm{KL}}, M+1)$ and $(a_i), (b_i) \in U_q$. If*

$$\pi_q((a_i)) = \pi_q((b_i)) + 1, \tag{14}$$

then for all $n \ge 1$ we have

(i) $(a_i) \succeq \alpha(q) \succ \alpha(q_{\mathrm{KL}})$ *and* $\pi_q(a_1 \cdots a_n 0^\infty) \ge \pi_q(\alpha_1(q_{\mathrm{KL}}) \cdots \alpha_n(q_{\mathrm{KL}}) 0^\infty)$.

(ii) $(b_i) \preceq \overline{\alpha(q)} \prec \overline{\alpha(q_{\mathrm{KL}})}$ *and* $\pi_q(b_1 \cdots b_n M^\infty) \le \pi_q(\overline{\alpha_1(q_{\mathrm{KL}}) \cdots \alpha_n(q_{\mathrm{KL}})} M^\infty)$.

Proof. Let $(a_i), (b_i) \in U_q$ satisfy (14). Equation (14) implies that $\pi_q((a_i)) \ge 1$. Since $(a_i) \in U_q$, it must be the greedy expansion of some element in I_q greater than or equal to 1. By Lemma 4 (ii) we have

$$(a_i) \succeq \alpha(q).$$

Moreover, because $q \in (q_{\mathrm{KL}}, M+1)$, we know by Lemma 4 (i) that $\alpha(q) \succ \alpha(q_{\mathrm{KL}})$. This implies the first part of (i).

Now we show the second part of (i). We claim that the two sequences $a_1 \cdots a_n 0^\infty$ and $\alpha_1(q_{\mathrm{KL}}) \cdots \alpha_n(q_{\mathrm{KL}})0^\infty$ are greedy q-expansions. This is true because of the characterisation of greedy expansions in base q provided by Lemma 4 (ii), and because both (a_i) and $\alpha(q_{\mathrm{KL}})$ are greedy expansions in base q. It now follows from $(a_i) \succ \alpha(q_{\mathrm{KL}})$ that

$$a_1 \cdots a_n 0^\infty \succeq \alpha_1(q_{\mathrm{KL}}) \cdots \alpha_n(q_{\mathrm{KL}})0^\infty.$$

Thus, the second part of (i) follows from Lemma 4 (iii).

Statement (ii) is proved similarly, this time exploiting the fact that if $(a_i), (b_i) \in U_q$ satisfy (14), then we must have $\pi_q((b_i)) \leq M/(q-1) - 1$. ■

Lemma 17. *We can choose our $\{J_i\}$ in such a way that the following is true. There exists $C_0 > 0$ that does not depend upon our choice J, such that if $(a_i), (b_i) \in U_{q_0}$ are such that*

$$\pi_q((a_i)) = \pi_q((b_i)) + 1, \tag{15}$$

for some $q \in J$, then for all n sufficiently large we have

$$\inf_{q_1, q_2 \in J} \left| \frac{1}{q_1 q_2} + \sum_{i=1}^{n} \frac{(b_i - a_i)(q_2^i + q_2^{i-1} q_1 + \cdots + q_1^i)}{q_1^{i+1} q_2^{i+1}} \right| \geq C_0.$$

Proof. If we set $q_2 = q_1$ in the above expression, we obtain

$$\inf_{q_1 \in J} \left| \frac{1}{q_1^2} + \sum_{i=1}^{n} \frac{(b_i - a_i)(i+1)}{q_1^{i+2}} \right|.$$

Now suppose that we have shown that there exists $C_1 > 0$ independent of J, such that if $(a_i), (b_i) \in U_{q_0}$ satisfy (15) for some $q \in J$, then for all n sufficiently large we have

$$\inf_{q_1 \in J} \left| \frac{1}{q_1^2} + \sum_{i=1}^{n} \frac{(b_i - a_i)(i+1)}{q_1^{i+2}} \right| \geq C_1. \tag{16}$$

Then by a continuity and compactness argument, if the $\{J_i\}$ are chosen sufficiently small, then

$$\inf_{q_1, q_2 \in J} \left| \frac{1}{q_1 q_2} + \sum_{i=1}^{n} \frac{(b_i - a_i)(q_2^i + q_2^{i-1} q_1 + \cdots + q_1^i)}{q_1^{i+1} q_2^{i+1}} \right| \geq \frac{C_1}{2}$$

for all n sufficiently large. Therefore, to complete our proof we only need to show that inequality (16) is satisfied for all n sufficiently large.

We can choose our $\{J_i\}$ to be sufficiently small so that for all n sufficiently large, we have by (15) that

$$\sum_{i=1}^{n} \frac{a_i}{q_1^i} - \sum_{i=1}^{n} \frac{b_i}{q_1^i} \geq 1 - \omega \tag{17}$$

for all $q_1 \in J$. Here we fix $\omega = 0.01$.

Using (17) for n sufficiently large we have

$$\frac{1}{q_1^2}+\sum_{i=1}^{n}\frac{(b_i-a_i)(i+1)}{q_1^{i+2}}=\frac{1}{q_1^2}+2\sum_{i=1}^{n}\frac{b_i-a_i}{q_1^{i+2}}+\sum_{i=1}^{n}\frac{(b_i-a_i)(i-1)}{q_1^{i+2}}$$
$$\leq\frac{1}{q_1^2}(-1+2\omega)+\sum_{i=1}^{n}\frac{(b_i-a_i)(i-1)}{q_1^{i+2}}. \tag{18}$$

We now proceed via a case analysis.

Case 1. $M=1$. It follows from Lemma 16 and (4) that

$$(a_i)\succ\alpha(q_{\mathrm{KL}})=11010011\cdots \text{ and } (b_i)\prec\overline{\alpha(q_{\mathrm{KL}})}=00101100\cdots, \tag{19}$$

then we obtain

$$b_1-a_1=-1, b_2-a_2=-1 \text{ and } b_i-a_i\leq 1 \text{ for all } i\geq 3. \tag{20}$$

Case 1a. If $b_3-a_3\leq 0$, then by (18) and (20) we obtain

$$\frac{1}{q_1^2}+\sum_{i=1}^{n}\frac{(b_i-a_i)(i+1)}{q_1^{i+2}}\leq\frac{1}{q_1^2}(-1+2\omega)+\sum_{i=1}^{n}\frac{(b_i-a_i)(i-1)}{q_1^{i+2}}$$
$$<\frac{1}{q_1^2}(-1+2\omega)-\frac{1}{q_1^4}+\frac{0}{q_1^5}+\sum_{i=4}^{\infty}\frac{i-1}{q_1^{i+2}}$$
$$=\frac{1}{q_1^2}(-1+2\omega)-\frac{2}{q_1^4}-\frac{2}{q_1^5}+\frac{1}{q_1^2(q_1-1)^2}.$$

The last equality follows from

$$\frac{1}{(q_1-1)^2}=\sum_{i=1}^{\infty}\frac{i-1}{q_1^i}.$$

A quick computer inspection verifies that

$$\frac{1}{q_1^2}(-1+2\omega)-\frac{2}{q_1^4}-\frac{2}{q_1^5}+\frac{1}{q_1^2(q_1-1)^2}<0$$

for all $q_1\in(q_{\mathrm{KL}},2)$. Therefore, C_1 exists in this case.

Case 1b. If $b_3-a_3=1$, that is $b_3=1, a_3=0$. By (19) we must have $b_4=0$ and $a_4=1$. Then it follows from (20) again that

$$\frac{1}{q_1^2}+\sum_{i=1}^{n}\frac{(b_i-a_i)(i+1)}{q_1^{i+2}}\leq\frac{1}{q_1^2}(-1+2\omega)+\sum_{i=1}^{n}\frac{(b_i-a_i)(i-1)}{q_1^{i+2}}$$
$$<\frac{1}{q_1^2}(-1+2\omega)-\frac{1}{q_1^4}+\frac{2}{q_1^5}-\frac{3}{q_1^6}+\sum_{i=5}^{\infty}\frac{i-1}{q_1^{i+2}}$$

$$= \frac{1}{q_1^2}(-1+2\omega) - \frac{2}{q_1^4} - \frac{6}{q_1^6} + \frac{1}{q_1^2(q_1-1)^2}.$$

A quick computer inspection verifies that

$$\frac{1}{q_1^2}(-1+2\omega) - \frac{2}{q_1^4} - \frac{6}{q_1^6} + \frac{1}{q_1^2(q_1-1)^2} < 0$$

for all $q_1 \in (q_{\mathrm{KL}}, 2)$. Therefore, C_1 exists in this case.

Case 2. $M \geq 2$. Let $\alpha(q_1) = (\alpha_i)$. Our assumption (J-iii) on the collection of intervals $\{J_i\}$ implies that $q_1 \in J \subset (p, p+1)$ for some $p \leq M$, since $p \in V$ for any $p \in (q_{\mathrm{KL}}, M+1) \cap \mathbb{N}$. This implies that $\alpha_1 \leq p$. We will now prove that

$$b_i \leq p \text{ for all } i \geq 1. \tag{21}$$

If $p = M$, then (21) is trivially true. We now prove it for $p < M$.

Combining $\alpha_1 \leq p$ and Lemma 4 (i), we have $\alpha(q_1) \preceq p^\infty$. By Lemma 16 we have $(b_i) \preceq \overline{\alpha(q_1)}$. Since $\alpha(q_1) \succeq \overline{\alpha(q_1)}$ for $q \geq q_{\mathrm{KL}}$ we have $(b_i) \preceq \alpha(q_1) \preceq p^\infty$. Therefore, $b_1 \leq p$ and $b_1 < M$. Moreover, since (b_i) is the greedy expansion of some real number, Lemma 4 (iii) implies that

$$(b_{n+i}) \prec \alpha(q_1)$$

for all $n \geq 1$. This implies that $b_i \leq p$ for all $i \geq 2$. This completes our proof of (21).

Using (18) and (21), for n sufficiently large we have

$$\begin{aligned}\frac{1}{q_1^2} + \sum_{i=1}^{n} \frac{(b_i - a_i)(i+1)}{q_1^{i+2}} &= \frac{1}{q_1^2} + 2\sum_{i=1}^{n} \frac{b_i - a_i}{q_1^{i+2}} + \sum_{i=1}^{n} \frac{(b_i - a_i)(i-1)}{q_1^{i+2}} \\ &\leq \frac{1}{q_1^2}(-1+2\omega) + \sum_{i=1}^{n} \frac{(b_i - a_i)(i-1)}{q_1^{i+2}} \\ &\leq \frac{1}{q_1^2}(-1+2\omega) + p\sum_{i=1}^{\infty} \frac{(i-1)}{q_1^{i+2}} \\ &= \frac{1}{q_1^2}\left(-1+2\omega + \frac{p}{(q_1-1)^2}\right).\end{aligned}$$

Recall that $\omega = 0.01$. To complete our proof, it suffices to show that there exists $C_1 > 0$ such that

$$-1 + 2\omega + \frac{p}{(q_1-1)^2} < -C_1.$$

Suppose $p \geq 3$. We must have $q_1 > p$ since $J \subset (p, p+1)$. Therefore,

$$-1 + 2\omega + \frac{p}{(q_1-1)^2} \leq -1 + 2\omega + \frac{p}{(p-1)^2}.$$

For $p \geq 3$ the right-hand side of this expression can be uniformly bounded from above by a negative number. Therefore, C_1 exists in this case. If $p = 2$, then we are either in the case where $M = 2$ or $M = 3$, since it follows from (3) that $2.5 < q_{\mathrm{KL}} < 3$ only when $M = 2$ or $M = 3$. It can be verified using a computer calculation that

$$-1 + 2\omega + \frac{2}{(q_1 - 1)^2} < 0$$

for any $q_1 \geq 2.43$ for $M = 2$ or $M = 3$. The existence of C_1 follows. ■

Lemma 18. *There exists $C > 0$ depending only upon J, such that for any $q \in J$ if $x = \pi_q((a_i)), y = \pi_q((b_i)) \in \mathcal{U}_q$ satisfies*

$$|x - y| \leq C q^{-n},$$

then $b_i = a_i$ for all $1 \leq i \leq n$.

Proof. Let

$$C := \min_{q \in J} \frac{1}{2}\left(\frac{M}{q-1} - 1\right).$$

We proceed by induction on n. For $n = 1$ suppose $|x - y| \leq C q^{-1}$, we now prove $b_1 = a_1$. If $b_1 \neq a_1$, then it follows from Lemma 8 that

$$|x - y| > \frac{M}{q^2 - q} - \frac{1}{q} > C q^{-1}.$$

Which is a contradiction.

Assume that if $|x - y| \leq C q^{-n+1}$, then $b_i = a_i$ for all $1 \leq i \leq n - 1$. Using this assumption we now prove that if $|x - y| \leq C q^{-n}$, then $b_i = a_i$ for all $1 \leq i \leq n$. Since $|x - y| \leq C q^{-n} < C q^{-n+1}$, our assumption on $n - 1$ implies that $b_i = a_i$ for all $1 \leq i \leq n - 1$. Now we prove $b_n = a_n$. Assume that $b_n \neq a_n$. Then

$$C q^{-1} \geq q^{n-1}|x - y| = q^{n-1}\left|\sum_{i=1}^{\infty} \frac{a_i}{q^i} - \sum_{i=1}^{\infty} \frac{b_i}{q^i}\right| = \left|\sum_{i=1}^{\infty} \frac{a_{n-1+i}}{q^i} - \sum_{i=1}^{\infty} \frac{b_{n-1+i}}{q^i}\right|. \tag{22}$$

Because $a_n \neq b_n$, Lemma 8 implies that

$$\left|\sum_{i=1}^{\infty} \frac{a_{n-1+i}}{q^i} - \sum_{i=1}^{\infty} \frac{b_{n-1+i}}{q^i}\right| > C q^{-1}.$$

This contradicts (22) and so completes our proof. ■

Equipped with Lemmas 17 and 18, we are now in a position to prove that f is Lipschitz. To prove this statement, it is convenient to use the infinity norm on $\mathbb{R}^2$ which we denote by $\|\cdot\|_\infty$ (i.e., $\|(x, y)\|_\infty := \max\{|x|, |y|\}$).

Proposition 19. *Let A and f be defined as in (9) and (13). Then there exists $C' > 0$ depending only upon J such that*

$$\|f(x_1, y_1) - f(x_2, y_2)\|_\infty \leq C'\|(x_1, y_1) - (x_2, y_2)\|_\infty$$

for all $(x_1, y_1), (x_2, y_2) \in A$.

Proof. To prove that such a C' exists, it suffices to consider $(x_1, y_1), (x_2, y_2) \in A$ for which $\|(x_1, y_1) - (x_2, y_2)\|_\infty$ is small. As such, by Lemma 15 we can restrict our attention to those $(x_1, y_1), (x_2, y_2) \in A$ for which there exists a unique m satisfying

$$(x_1, y_1), (x_2, y_2) \in \frac{\mathcal{U}_{q_0} + m}{q_0} \times \frac{\mathcal{U}_{q_0} + m + 1}{q_0}.$$

Moreover, by the definition of A there exist $(a_i^1), (a_i^2), (b_i^1), (b_i^2) \in U_{q_0}$ such that

$$x_1 = \pi_{q_0}(m(a_i^1)), \qquad x_2 = \pi_{q_0}(m(a_i^2)),$$
$$y_1 = \pi_{q_0}((m+1)(b_i^1)), \qquad y_2 = \pi_{q_0}((m+1)(b_i^2)),$$

and $q_1, q_2 \in J$ for which

$$\pi_{q_1}(m(a_i^1)) = \pi_{q_1}((m+1)(b_i^1)) \quad \text{and} \quad \pi_{q_2}(m(a_i^2)) = \pi_{q_2}((m+1)(b_i^2)). \tag{23}$$

By the definition of f, we have

$$f((x_1, y_1)) = (q_1, \pi_{q_1}(m(a_i^1))) \quad \text{and} \quad f((x_2, y_2)) = (q_2, \pi_{q_2}(m(a_i^2))).$$

Therefore,

$$\begin{aligned}\|f(x_1, y_1) - f(x_2, y_2)\|_\infty &= \left\|\left(q_1, \pi_{q_1}(m(a_i^1))\right) - \left(q_2, \pi_{q_2}(m(a_i^2))\right)\right\|_\infty \\ &= \max\left\{|q_1 - q_2|, |\pi_{q_1}(m(a_i^1)) - \pi_{q_2}(m(a_i^2))|\right\}.\end{aligned}$$

We now bound the two terms appearing in this maximum accordingly. We begin by showing that there exists C' such that

$$|q_1 - q_2| < C'\|(x_1, y_1) - (x_2, y_2)\|_\infty.$$

As stated above, it is sufficient to consider those $(x_1, y_1), (x_2, y_2) \in A$ for which the norm $\|(x_1, y_1) - (x_2, y_2)\|_\infty$ is small. In particular, we can take $(x_1, y_1), (x_2, y_2) \in A$ such that the unique n satisfying

$$Cq_0^{-n-2} < \|(x_1, y_1) - (x_2, y_2)\|_\infty \le Cq_0^{-n-1} \tag{24}$$

is sufficiently large so that Lemma 17 applies. Here C is as in Lemma 18. Equation (24) implies that

$$|\pi_{q_0}((a_i^1)) - \pi_{q_0}((a_i^2))| \le Cq_0^{-n} \text{ and } |\pi_{q_0}((b_i^1)) - \pi_{q_0}((b_i^2))| \le Cq_0^{-n}.$$

Therefore, Lemma 18 implies that $a_i^1 = a_i^2$ and $b_i^1 = b_i^2$ for $1 \le i \le n$.

It follows from (23) that

$$\frac{1}{q_1}+\sum_{i=1}^{\infty}\frac{b_i^1-a_i^1}{q_1^{i+1}}=0 \quad\text{and}\quad \frac{1}{q_2}+\sum_{i=1}^{\infty}\frac{b_i^2-a_i^2}{q_2^{i+1}}=0.$$

Combining these two equations, we have

$$\frac{1}{q_1}+\sum_{i=1}^{\infty}\frac{b_i^1-a_i^1}{q_1^{i+1}}-\frac{1}{q_2}-\sum_{i=1}^{\infty}\frac{b_i^2-a_i^2}{q_2^{i+1}}=0. \tag{25}$$

Using the fact $a_i^1=a_i^2$ and $b_i^1=b_i^2$ for $1\le i\le n$, we see that (25) yields

$$\begin{aligned}&\left|\frac{1}{q_1}+\sum_{i=1}^{n}\frac{b_i^1-a_i^1}{q_1^{i+1}}-\frac{1}{q_2}-\sum_{i=1}^{n}\frac{b_i^1-a_i^1}{q_2^{i+1}}\right|\\ &\quad=\left|\sum_{i=n+1}^{\infty}\frac{b_i^2-a_i^2}{q_2^{i+1}}-\sum_{i=n+1}^{\infty}\frac{b_i^1-a_i^1}{q_1^{i+1}}\right|\\ &\quad\le\sum_{i=n+1}^{\infty}\frac{2M}{q_0^{i+1}}=\frac{2M}{q_0^2-q_0}q_0^{-n}<C_1q_0^{-n}.\end{aligned}$$

Here $C_1:=2M/(q_{\mathrm{KL}}^2-q_{\mathrm{KL}})$. In the final line we used that $q_0=\min\{q\colon q\in J\}$. It now follows from (24) that

$$\left|\frac{1}{q_1}+\sum_{i=1}^{n}\frac{b_i^1-a_i^1}{q_1^{i+1}}-\frac{1}{q_2}-\sum_{i=1}^{n}\frac{b_i^1-a_i^1}{q_2^{i+1}}\right|\le C_2\|(x_1,y_1)-(x_2,y_2)\|_\infty. \tag{26}$$

Here $C_2:=C_1M^2/C$.

We now focus on removing a $|q_1-q_2|$ term from the left-hand side of (26):

$$\begin{aligned}&\left|\frac{1}{q_1}+\sum_{i=1}^{n}\frac{b_i^1-a_i^1}{q_1^{i+1}}-\frac{1}{q_2}-\sum_{i=1}^{n}\frac{b_i^1-a_i^1}{q_2^{i+1}}\right|\\ &\quad=\left|\frac{q_2-q_1}{q_1q_2}+\sum_{i=1}^{n}\frac{(b_i^1-a_i^1)(q_2^{i+1}-q_1^{i+1})}{q_1^{i+1}q_2^{i+1}}\right|\\ &\quad=\left|\frac{q_2-q_1}{q_1q_2}+(q_2-q_1)\sum_{i=1}^{n}\frac{(b_i^1-a_i^1)(q_2^i+q_2^{i-1}q_1+\cdots+q_1^i)}{q_1^{i+1}q_2^{i+1}}\right|\\ &\quad=|q_2-q_1|\left|\frac{1}{q_1q_2}+\sum_{i=1}^{n}\frac{(b_i^1-a_i^1)(q_2^i+q_2^{i-1}q_1+\cdots+q_1^i)}{q_1^{i+1}q_2^{i+1}}\right|.\end{aligned}$$

Lemma 17 tells us that the second term in the product in the final line can be bounded from below by a constant. Therefore, if we combine Lemma 17 and (26) with the above, we can assert that

$$|q_2-q_1|\le C_3\|(x_1,y_1)-(x_2,y_2)\|_\infty. \tag{27}$$

Here $C_3:=C_2/C_0$ and C_0 is as in Lemma 17.

It remains to show that there exists $C' > 0$ such that

$$|\pi_{q_1}(m(a_i^1)) - \pi_{q_2}(m(a_i^2))| < C'\|(x_1, y_1) - (x_2, y_2)\|_\infty.$$

Note that

$$\pi_{q_1}(m(a_i^1)) = \frac{m}{q_1} + \sum_{i=1}^{\infty} \frac{a_i^1}{q_1^{i+1}} \quad \text{and} \quad \pi_{q_2}(m(a_i^2)) = \frac{m}{q_2} + \sum_{i=1}^{\infty} \frac{a_i^2}{q_2^{i+1}},$$

and $a_i^1 = a_i^2$ for $1 \leq i \leq n$. Therefore,

$$\begin{aligned}
&\left|\pi_{q_1}(m(a_i^1)) - \pi_{q_2}(m(a_i^2))\right| \\
&\quad \leq \left|\frac{m}{q_1} - \frac{m}{q_2}\right| + \left|\sum_{i=1}^{n} \frac{a_i^1}{q_1^{i+1}} - \sum_{i=1}^{n} \frac{a_i^1}{q_2^{i+1}}\right| + \left|\sum_{i=n+1}^{\infty} \frac{a_i^1}{q_1^{i+1}} - \sum_{i=n+1}^{\infty} \frac{a_i^2}{q_2^{i+1}}\right| \\
&\quad = \left|\frac{m}{q_1} - \frac{m}{q_2}\right| + \left|\sum_{i=1}^{n} \frac{a_i^1(q_2^{i+1} - q_1^{i+1})}{q_1^{i+1}q_2^{i+1}}\right| + \left|\sum_{i=n+1}^{\infty} \frac{a_i^1}{q_1^{i+1}} - \sum_{i=n+1}^{\infty} \frac{a_i^2}{q_2^{i+1}}\right| \\
&\quad = |q_2 - q_1|\left(\left|\frac{m}{q_1 q_2}\right| + \left|\sum_{i=1}^{n} \frac{a_i^1(q_2^i + q_2^{i-1}q_1 + \cdots + q_1^i)}{q_1^{i+1}q_2^{i+1}}\right|\right) \\
&\qquad + \left|\sum_{i=n+1}^{\infty} \frac{a_i^1}{q_1^{i+1}} - \sum_{i=n+1}^{\infty} \frac{a_i^2}{q_2^{i+1}}\right|. \qquad (28)
\end{aligned}$$

There exists $C_4 := M\left(q_{\mathrm{KL}}^{-2} + (q_{\mathrm{KL}} - 1)^{-2}\right) > 0$ and $C_5 := M/(q_0^2 - q_0) > 0$ such that

$$\begin{aligned}
\left|\frac{m}{q_1 q_2}\right| + \left|\sum_{i=1}^{n} \frac{a_i^1(q_2^i + q_2^{i-1}q_1 + \cdots + q_1^i)}{q_1^{i+1}q_2^{i+1}}\right| &\leq M\left(\frac{1}{q_{\mathrm{KL}}^2} + \sum_{i=1}^{n} \frac{i+1}{q_{\mathrm{KL}}^{i+2}}\right) \\
&\leq M\left(\frac{1}{q_{\mathrm{KL}}^2} + \sum_{i=0}^{\infty} \frac{i+1}{q_{\mathrm{KL}}^{i+2}}\right) = C_4 \qquad (29)
\end{aligned}$$

and

$$\left|\sum_{i=n+1}^{\infty} \frac{a_i^1}{q_1^{i+1}} - \sum_{i=n+1}^{\infty} \frac{a_i^2}{q_2^{i+1}}\right| \leq \sum_{i=n+1}^{\infty} \frac{M}{q_0^{i+1}} = C_5 q_0^{-n}.$$

Using the line above and (24), it follows that there exists $C_6 := C_5 M^2/C > 0$ such that

$$\left|\sum_{i=n+1}^{\infty} \frac{a_i^1}{q_1^{i+1}} - \sum_{i=n+1}^{\infty} \frac{a_i^2}{q_2^{i+1}}\right| \leq C_6\|(x_1, y_1) - (x_2, y_2)\|_\infty. \qquad (30)$$

Using (27), (29), and (30), we see that (28) implies that there exists $C_7 := C_3 C_4 + C_6 > 0$ such that

$$|\pi_{q_1}(m(a_i^1)) - \pi_{q_2}(m(a_i^2))| \leq C_7\|(x_1, y_1) - (x_2, y_2)\|_\infty. \qquad (31)$$

Equations (27) and (31) together imply that there exists $C' := \max\{C_3, C_7\} > 0$ such that

$$\|f(x_1, y_1) - f(x_2, y_2)\|_\infty \leq C'\|(x_1, y_1) - (x_2, y_2)\|_\infty. \qquad \blacksquare$$

We have now proved all of the technical results we need to prove Theorem 1. We just require the following well-known theorem due to Marstrand and a lemma which both can be found in the book by Bishop and Peres [7].

Theorem 20 (Marstrand slicing theorem). *Let* $E \subset \mathbb{R}^2$.

(i) *If* $\dim_H E < 1$, *then*

$$\{(q, y) \colon y \in \mathbb{R}\} \cap E = \emptyset$$

for Lebesgue almost every q.

(ii) *If* $\dim_H E \geq 1$, *then*

$$\dim_H(\{(q, y) \colon y \in \mathbb{R}\} \cap E) \leq \dim_H E - 1$$

for Lebesgue almost every q.

Lemma 21. *Suppose* $K \subset \mathbb{R}^d$ *and* $f \colon \mathbb{R}^d \to \mathbb{R}^n$ *is Lipschitz. Then* $\dim_H f(K) \leq \dim_H K$.

Proof of Theorem 1. As previously remarked, to prove Theorem 1, it suffices to show that for our fixed choice of ε, for Lebesgue almost every $q \in J$ we have

$$\dim_H \mathcal{U}_q^2 \leq \max\{2\dim_H \mathcal{U}_q - 1 + \varepsilon, 0\}. \tag{32}$$

The set A satisfies $A \subseteq \bigcup_{m=0}^{M-1} (\mathcal{U}_{q_0} + m)/q_0 \times (\mathcal{U}_{q_0} + m + 1)/q_0$. In [1] it was shown that $\dim_H \mathcal{U}_{q_0} = \dim_B \mathcal{U}_{q_0}$ for any $q \in (1, M+1]$. Therefore, using well-known properties of the Cartesian product of sets, see for instance [15], we know that

$$\dim_H\left(\bigcup_{m=0}^{M-1} (\mathcal{U}_{q_0} + m)/q_0 \times (\mathcal{U}_{q_0} + m + 1)/q_0\right) = 2\dim_H \mathcal{U}_{q_0}.$$

This equality and the inclusion above imply

$$\dim_H A \leq 2\dim_H \mathcal{U}_{q_0}. \tag{33}$$

Since $f(A) = B$ and f is Lipschitz by Proposition 19, (33) and Lemma 21 tell us that

$$\dim_H B \leq 2\dim_H \mathcal{U}_{q_0}. \tag{34}$$

For any $q \in J$, it follows by Lemma 14 that

$$\dim_H(\{(q, y) \colon y \in \mathbb{R}\} \cap B) = \dim_H \mathcal{U}_q^2. \tag{35}$$

Theorem 20 and (34) imply that for Lebesgue almost every $q \in J$ we have

$$\dim_H(\{(q, y) : y \in \mathbb{R}\} \cap B) \leq \max\{2\dim_H \mathcal{U}_{q_0} - 1, 0\}.$$

Therefore, (35) implies that for Lebesgue almost every $q \in J$ we have

$$\dim_H \mathcal{U}_q^2 \leq \max\{2\dim_H \mathcal{U}_{q_0} - 1, 0\}. \tag{36}$$

Let $q' \in J$ be an arbitrary element of the full measure subset of J for which (36) holds. Property (J-iv) of the collection of intervals $\{J_i\}$ tells us that

$$|\dim_H \mathcal{U}_{q'} - \dim_H \mathcal{U}_{q_0}| < \varepsilon/2.$$

Using this fact and (36), we see that q' satisfies

$$\dim_H \mathcal{U}_{q'}^2 \leq \max\{2\dim_H \mathcal{U}_{q'} - 1 + \varepsilon, 0\}.$$

Since q' was an arbitrary element of a full measure subset of J, this completes our proof that (32) holds for Lebesgue almost every $q \in J$. ∎

We now briefly explain how the argument used to prove Theorem 1 can be adapted to prove Theorem 2.

Proof of Theorem 2. We recall that

$$O := \left\{q \in (q_{\mathrm{KL}}, M+1) : \dim_H \mathcal{U}_q < \frac{1}{2}\right\}.$$

Just as in the proof of Theorem 1, we can cover O up to a set of measure zero by a collection of intervals $\{J_i\}$ satisfying properties (J-ii)–(J-iv). We can also assume that each J_i is chosen such that $q_i := \min\{q : q \in J_i\}$ satisfies $q_i \in O$.

We now fix an arbitrary J from the collection $\{J_i\}$ and let $q_0 := \min\{q : q \in J\}$. To prove Theorem 2, it suffices to show that for Lebesgue almost every $q \in J$ we have $\mathcal{U}_q^j = \emptyset$ for $j \geq 2$. Moreover, because of Lemma 6, it is in fact sufficient to show that for Lebesgue almost every $q \in J$ we have $\mathcal{U}_q^2 = \emptyset$. We now proceed as in our proof of Theorem 1 and define the sets A, B and the map f in the same way. Importantly, (34) tells us that $\dim_H B \leq 2\dim_H \mathcal{U}_{q_0}$. We chose our intervals $\{J_i\}$ in such a way that $q_i \in O$. Therefore, $q_0 \in O$ and we have $\dim_H B < 1$. Applying Theorem 20 we may conclude that $\{(q, y) : y \in \mathbb{R}\} \cap B = \emptyset$ for Lebesgue almost every $q \in J$. Our theorem now follows via an application of Lemma 14 which tells us that if $\{(q, y) : y \in \mathbb{R}\} \cap B = \emptyset$, then $\mathcal{U}_q^2 = \emptyset$. ∎

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