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A combinatorial Fredholm module on self-similar sets built on n-cubes

Takashi Maruyama and Tatsuki Seto

Abstract. We construct a Fredholm module on self-similar sets such as the Cantor dust, the Sierpiński carpet and the Menger sponge. Our construction is a higher dimensional analogue of Connes' combinatorial construction of the Fredholm module on the Cantor set. We also calculate the Dixmier trace of two operators induced by the Fredholm module.

Introduction

In the 1990s, A. Connes [3, Chapter IV] introduced the quantized calculus based on the Fredholm modules. A Fredholm module on an involutive algebra $\mathcal{A}$ is a pair (H, F) of a Hilbert space H and a bounded operator F such that $\mathcal{A}$ acts on H and $a(F - F^*), a(F^2 - 1), [F, a] \in \mathcal{K}(H)$ for any $a \in \mathcal{A}$. The commutator $[F, a]$ is called a quantized differential of a. The notion and calculus of Fredholm modules provide many techniques in studying various spaces. Such examples are noncompact spaces, foliated spaces, noncommutative spaces, and fractal spaces, to name a few. In the present paper, we study Fredholm modules on a special class of fractal spaces called self-similar sets.

The first study of quantized calculus on self-similar sets is given by Connes [3, Chapter IV]. Connes defined the Fredholm module (H, F) on $C(CS)$, where CS is the Cantor set realized in the interval $[0, 1]$, by using vertices of the removed intervals. Specifically, he set $H_I = \ell^2(\{a\}) \oplus \ell^2(\{b\})$ for an open interval $I = (a, b)$ and $F_I = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ on H_I, and constructed (H, F) by taking a direct sum of (H_I, F_I) on all removed open intervals which appear along the construction of CS. The Fredholm module (H, F) defines an element in $K^0(C(CS))$. Connes also calculated the non-vanishing Dixmier trace $\mathrm{Tr}_\omega(|[F, x]|^{\dim_H(CS)})$. Here x is the coordinate function on $\mathbb{R}$ (we consider x as a multiplication operator) and $\dim_H(CS)$ is the Hausdorff dimension of CS. We call $|[F, x]|^{\dim_H(CS)}$ the quantized volume measure on CS and

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$\mathrm{Tr}_\omega(|[F,x]|^{\dim_H(CS)})$ the quantized volume on CS since the commutator $[F,x]$ is a quantized differential of x.

In this paper, we generalize the quantized volume measure and the quantized volume to higher dimensional self-similar sets. For the generalization, we construct a higher dimensional analogue of Connes' Fredholm module. We now present what we mean by the generalization of Connes' quantized volume $\mathrm{Tr}_\omega(|[F,x]|^{\dim_H(CS)})$. When we have the Fredholm module (H_K, F_K) on a fractal set $K \subset \mathbb{R}^n$ such that an algebra of functions on $\mathbb{R}^n$ acts on H_K, a commutator of operators $[F_K, x^\alpha]$ $(\alpha = 1,\dots,n)$ for the α-th coordinate function x^α on $\mathbb{R}^n$ is obtained. The commutator $[F_K, x^\alpha]$ is a quantized differential of x^α, hence we say the operator

$$|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p$$

is a quantized volume measure of the volume measure $dx^1dx^2\cdots dx^n$ on $\mathbb{R}^n$. Here $p \in \mathbb{R}$ is defined by a fractal dimension on K. Then the value

$$\mathrm{Tr}_\omega(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p)$$

may also be called a generalization of Connes' quantized volume on K.

Let us explain some examples that motivate us to conduct this work. We construct a Fredholm module on a self-similar set K built on the square in $\mathbb{R}^2$. When we adopt a standard way to construct Fredholm modules on more general K (see [5, section 2]), it suffices to choose a subset $S \subset K$ for (H_S, F_S). As constructed in [4], when we choose $S = \{a,b\}$ (2 points), we have the same Fredholm module (H_S, F_S) as Connes' one. The Fredholm module (H_S, F_S) gives rise to a Fredholm module (H_K, F_K) composed by the direct sum over all steps in the construction of K. Then the commutator $[F_K, x]$ (resp. $[F_K, y]$) is essentially given by the length of the projection of a segment ab to the x-axis (resp. y-axis), and we can calculate the value $\mathrm{Tr}_\omega(|[F_K,x][F_K,y]|^p)$. However, the value may vanish: suppose that the vertices of the square are numbered counterclockwise in the order v_0, v_1, v_2, v_3. When K is the Cantor dust (see Figure 5) and every edge of the square is parallel to either x- or y-axis, we have $[F_K,x][F_K,y] = 0$ if $S = \{v_i, v_j\}$ is the boundary of an edge of the square. On the other hand, if we choose $S = \{v_0, v_2\}$ to be the boundary of a diagonal line of the square, we have the non-trivial value $\mathrm{Tr}_\omega(|[F_K,x][F_K,y]|^{\dim_H(K)/2})$. Therefore, the subset $S = \{v_0, v_2\}$ may look like an appropriate choice for the Cantor dust. However, the value $\mathrm{Tr}_\omega(|[F_K,x][F_K,y]|^{\dim_H(K)/2})$ for $S = \{v_0, v_2\}$ is not preserved under the rotation of the square. In fact, for a self-similar set K obtained by rotation of the Cantor dust with rotation angle $\pi/4$ around v_0, we have $[F_K,x][F_K,y] = 0$ for $S = \{v_0, v_2\}$. Thus, the choice of $S = \{a,b\}$ giving a non-trivial $\mathrm{Tr}_\omega(|[F_K,x][F_K,y]|^p)$ depends on K. In this paper, we also present a way to construct a Fredholm module for K that specifies a unified choice of S (not necessarily 2 points) and show that the Fredholm module induces a non-trivial higher dimensional quantized volume measure which is invariant under the Euclidean isometries in $\mathbb{R}^n$.

The outline of our construction of the Fredholm module (H_K, F_K) on K is the following. Let $\gamma_n = [0,1]^n$ be the n-cube and $\{f_s : \gamma_n \to \gamma_n\}$ $(s = 1, 2, \ldots, N)$ be similitudes with the similarity ratio $0 < r_s < 1$. We note that we do not require the open set condition. We now have a decreasing sequence of compact sets $K_j = \bigcup_{(s_1,\ldots,s_j)} f_{s_1} \circ \cdots \circ f_{s_j}(\gamma_n)$ in which each $f_{s_1} \circ \cdots \circ f_{s_j}(\gamma_n)$ is a small copy of the n-cube. Then, the sequence gives rise to the limiting set $K = \bigcap_{j=0}^{\infty} K_j$. Our construction of (H_K, F_K) is made of 2 steps: the first step is the construction of the Fredholm module $(\mathcal{H}, F_n)$ on the n-cube; see Section 1.1. In our construction, we use *all* vertices (instead of 2 points) of n-cubes, that is, we set $\mathcal{H} = \ell^2(\{\text{vertices}\})$ with a suitable $\mathbb{Z}_2$-grading. In the definition of F_n, we use induction on the dimension n. The resulting Fredholm module represents the Kasparov product (n-times) of Connes' Fredholm module on an interval. The second step is taking the direct sum of $(\mathcal{H}, F_n)$ on all the copies of n-cubes; see Section 2.1. Our Fredholm module $(\mathcal{H}_K, F_K)$ is defined over $C(V_K)$, where we denote by V_K the closure of the vertices of all n-cubes $f_{s_1} \circ \cdots \circ f_{s_j}(\gamma_n)$. Note that, in general, V_K includes K properly, but V_K coincides with K for some important examples such as the Cantor dust, the Sierpiński carpet and the Menger sponge. Dividing by the length of edges of each n-cube, we get the Dirac operator D_K on K and the spectral triple on K.

The main results in the paper are basically twofold: our first result is the construction of a higher dimensional analogue of the Connes' Fredholm module. This Fredholm module is also non-trivial in the K^0 group under additional assumptions, which are given in Theorem 2.5 as a part of other properties of the Fredholm module delved in Section 2. The second result is the derivation of concrete values for higher dimensional variants of the quantized volume measure and the quantized volume for some self-similar sets. The results are given in Section 3. The calculation is based on a Clifford algebra's relation which the commutators $[F_n, x^\alpha]$ $(\alpha = 1, \ldots, n)$ generally satisfy for the α-th coordinate functions x^α on $\mathbb{R}^n$. The Clifford algebra's relation is quantization of the relation of the exterior differentials dx^α; see Propositions 2.8 and 2.9 for the details.

Fredholm modules on self-similar sets were constructed by various researchers and studied from various aspects. F. Cipirani–J. Sauvageot [2] constructed Fredholm modules on post critically finite fractals (p.c.f. fractals) by regular harmonic structures. M. Ionescu–L. Rogers–A. Teplyaev [7] studied weakly summable Fredholm modules in the cases of some finitely and infinitely ramified fractals. As an unbounded picture of Fredholm modules, spectral triples on some self-similar sets also have been extensively investigated. E. Christensen–C. Ivan–L. Lapidus [1] defined a spectral triple on the Sierpiński gasket $\mathcal{SG}$, which in turn defines an element in $K^1(C(\mathcal{SG}))$, by using the Dirac operator on the circle. D. Guido–T. Isola [4] defined a spectral triple on self-similar sets with the open set condition in higher dimension by using Connes' Fredholm module on an interval. Guido–Isola [5] also defined a spectral triple on nested fractals by using Connes' Fredholm module on an interval. See Introduction in [5] for more related literatures.

Let us compare our spectral triple with Guido–Isola's triples. First, our Fredholm module cannot be constructed on self-similar sets on arbitrary subsets in $\mathbb{R}^n$, but on n-cubes. Our construction also does not require the open set condition. An example of the case for a self-similar set without the open set condition is given in Section 4.5. Second, our triple and the triple in [4] are not constructed on the algebra $C(K)$ of the continuous functions on K. Our algebra $C(V_K)$ coincides with $C(K)$ for some important examples such as the Cantor dust, the Sierpiński carpet and the Menger sponge. The calculation of the value $\mathrm{Tr}_\omega(|D_K|^{-p})$ for our Dirac operator is also given in Section 3.1. The triple in [5] is defined on $C(K)$ for the class of nested fractals, but the examples mentioned above are not the case.

	Ours	G-I's [4]	G-I's [5]
space	self-similar set on n-cube	self-similar set on $\mathbb{R}^n$	nested fractal
algebra	$C(V_K)$	$C(C)$	$C(K)$

We are going to study more noncommutative geometry of our Fredholm module $(\mathcal{H}_K, F_K)$ and the corresponding spectral triple $(\mathcal{H}_K, D_K)$ in future papers.

1. Fredholm module on n-cubes

1.1. Definition of Fredholm module

In this section, we construct a "good" Fredholm module on n-cubes γ_n. For the simplicity, we set $\gamma_n = [0,e]^n$ in $\mathbb{R}^n$ with the length of edge $e > 0$; the following construction applies to any n-cubes.

Let V be the set of vertices of γ_n:

$$V = \{(a_1, \dots, a_n) \in \mathbb{R}^n; a_i = 0 \text{ or } e \quad (i = 1, 2, \dots, n)\}.$$

We give a number of vertices in V inductively. For $n = 1$, an interval $\gamma_1 = [0,e]$ has two vertices 0 and e. Set $v_0 = 0$ and $v_1 = e$. For an arbitrary n, we assume that we have a number of vertices of γ_{n-1}. Then a number of vertices of γ_n is as follows:

(1) $v_i = (a_1, \dots, a_{n-1}, 0) = (a_1, \dots, a_{n-1})$ $(0 \le i \le 2^{n-1} - 1)$ under the inclusion $\gamma_{n-1} \to \gamma_{n-1} \times \{0\} \subset \gamma_n$.

(2) $v_{2^n-1-i} = (a_1, \dots, a_{n-1}, e)$ $(0 \le i \le 2^{n-1} - 1)$ if $v_i = (a_1, \dots, a_{n-1}, 0)$.

Example 1.1. We here provide examples of the numbering of vertices for $n = 2, 3$.

(1) When $n = 2$, the numbering of vertices is given by $v_0 = (0, 0)$, $v_1 = (e, 0)$, $v_2 = (e, e)$, $v_3 = (0, e)$; see Figure 1.

(2) When $n = 3$, the numbering of vertices is given by

$$\begin{aligned} &v_0 = (0,0,0), \quad v_1 = (e,0,0), \quad v_2 = (e,e,0), \quad v_3 = (0,e,0), \\ &v_4 = (0,e,e), \quad v_5 = (e,e,e), \quad v_6 = (e,0,e), \quad v_7 = (0,0,e). \end{aligned}$$

See Figure 2.

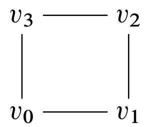

Figure 1. $n = 2$.

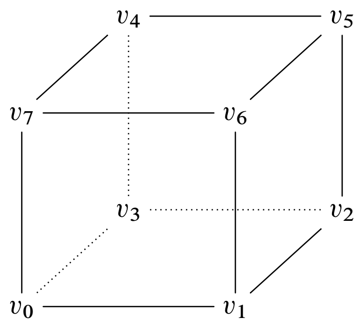

Figure 2. $n = 3$.

Set $V_0 = \{v_i; i = \text{ even}\}$ and $V_1 = \{v_i; i = \text{ odd}\}$, so we have $V = V_0 \cup V_1$. Set also

$$\begin{aligned} \mathcal{H}^+ &= \ell^2(V_0) = \ell^2(v_0) \oplus \ell^2(v_2) \oplus \cdots \oplus \ell^2(v_{2^n-2}), \\ \mathcal{H}^- &= \ell^2(V_1) = \ell^2(v_1) \oplus \ell^2(v_3) \oplus \cdots \oplus \ell^2(v_{2^n-1}) \end{aligned}$$

and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. The vector space $\mathcal{H} (\cong \mathbb{C}^{2^n})$ is a Hilbert space of dimension 2^n with an inner product

$$\langle f, g \rangle = \sum_{i=0}^{2^n-1} f(v_i)\overline{g(v_i)}.$$

We assume that $\mathcal{H}$ is $\mathbb{Z}_2$-graded with the grading $\varepsilon = \pm 1$ on $\mathcal{H}^{\pm}$, respectively. The C^*-algebra $C(V)$ of continuous functions on V acts on $\mathcal{H}$ by multiplication:

$$\rho(f) = (f(v_0) \oplus f(v_2) \oplus \cdots \oplus f(v_{2^n-2})) \oplus (f(v_1) \oplus f(v_3) \oplus \cdots \oplus f(v_{2^n-1})).$$

A Fredholm operator F_n on $\mathcal{H}$ is also defined inductively. We set

$$X_1 = 1 \quad \text{and} \quad X_n = \begin{bmatrix} O & X_{n-1} \\ X_{n-1} & O \end{bmatrix} \in M_{2^{n-1}}(\mathbb{C}) \quad (n \geq 2),$$

$$G_1 = 1 \quad \text{and} \quad G_n = \begin{bmatrix} G_{n-1} & -X_{n-1} \\ X_{n-1} & G_{n-1} \end{bmatrix} \in M_{2^{n-1}}(\mathbb{C}) \quad (n \geq 2),$$

$$\text{and} \quad U_n = \frac{1}{\sqrt{n}} G_n \quad (n \geq 1).$$

Proposition 1.2. *U_n is a unitary matrix.*

Proof. Firstly, we have

$$\begin{aligned} & X_n G_n^* - G_n X_n \\ &= \begin{bmatrix} O & X_{n-1} \\ X_{n-1} & O \end{bmatrix} \begin{bmatrix} G_{n-1}^* & X_{n-1} \\ -X_{n-1} & G_{n-1}^* \end{bmatrix} - \begin{bmatrix} G_{n-1} & -X_{n-1} \\ X_{n-1} & G_{n-1} \end{bmatrix} \begin{bmatrix} O & X_{n-1} \\ X_{n-1} & O \end{bmatrix} \\ &= X_2 \otimes (X_{n-1} G_{n-1}^* - G_{n-1} X_{n-1}) \\ &= \cdots = X_n \otimes (X_1 G_1^* - G_1 X_1) = O. \end{aligned}$$

We prove $U_n U_n^* = E_{2^n}$ by induction. Clearly, $U_1 = 1$ is unitary. Assume that U_{n-1} is a unitary matrix. Then we have

$$G_{n-1} G_{n-1}^* + X_{n-1}^2 = (n-1) E_{2^{n-2}} + E_{2^{n-2}} = n E_{2^{n-2}}.$$

Thus, we obtain

$$\begin{aligned} G_n G_n^* &= \begin{bmatrix} G_{n-1} & -X_{n-1} \\ X_{n-1} & G_{n-1} \end{bmatrix} \begin{bmatrix} G_{n-1}^* & X_{n-1} \\ -X_{n-1} & G_{n-1}^* \end{bmatrix} \\ &= \begin{bmatrix} G_{n-1} G_{n-1}^* + X_{n-1}^2 & G_{n-1} X_{n-1} - X_{n-1} G_{n-1}^* \\ X_{n-1} G_{n-1}^* - G_{n-1} X_{n-1} & X_{n-1}^2 + G_{n-1} G_{n-1}^* \end{bmatrix} \\ &= \begin{bmatrix} G_{n-1} G_{n-1}^* + X_{n-1}^2 & (X_{n-1} G_{n-1}^* - G_{n-1} X_{n-1})^* \\ X_{n-1} G_{n-1}^* - G_{n-1} X_{n-1} & X_{n-1}^2 + G_{n-1} G_{n-1}^* \end{bmatrix} = n E_{2^{n-1}}. \end{aligned}$$

Therefore, $U_n = \frac{1}{\sqrt{n}} G_n$ is a unitary matrix. ■

Set $F_n = \begin{bmatrix} & U_n^* \\ U_n & \end{bmatrix} \in M_{2^n}(\mathbb{C})$. By Proposition 1.2, we have $F_n^2 = E_{2^n}$ and $F_n^* = F_n$. We consider that F_n is a bounded operator on a finite dimensional Hilbert space

$$\mathcal{H} = (\ell^2(v_0) \oplus \ell^2(v_2) \oplus \ldots \oplus \ell^2(v_{2^n-2})) \oplus (\ell^2(v_1) \oplus \ell^2(v_3) \oplus \ldots \oplus \ell^2(v_{2^n-1})) \cong \mathbb{C}^{2^n}$$

by the left multiplication of a matrix F_n. Because of $F_n\varepsilon + \varepsilon F_n = O$, $(\mathcal{H}, F_n)$ is an even Fredholm module on $C(V)$.

Example 1.3. We here provide examples of Fredholm operators for $n = 1, 2, 3$.

(1) When $n = 1$, we have $F_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is introduced by [3, Chapter IV. 3. ε].

(2) When $n = 2$, we have

$$G_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad F_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} & & 1 & 1 \\ & & -1 & 1 \\ 1 & -1 & & \\ 1 & 1 & & \end{bmatrix}.$$

(3) When $n = 3$, we have

$$G_3 = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad U_3 = \frac{1}{\sqrt{3}}G_3 \quad \text{and} \quad F_3 = \begin{bmatrix} & U_3^* \\ U_3 & \end{bmatrix}.$$

Remark 1.4. The components of G_n correspond to the following orientation of edges, where the correspondence is similar to the adjacency matrices of oriented graphs. When $n = 1$, the orientation of the graph $\gamma_1 = [0, e]$ is from $v_0 = 0$ to $v_1 = e$; we denote such an orientation by $v_0 \to v_1$. Assume that we have the orientation of the edges of γ_{n-1}.

(1) Assume $0 \le i, j \le 2^{n-1} - 1$. The orientation in γ_n is from v_i to v_j; $v_i \to v_j$, when the orientation in γ_{n-1} is from v_i to v_j. Here we consider that γ_{n-1} is a subset in γ_n under the inclusion $\gamma_{n-1} \to \gamma_{n-1} \times \{0\} \subset \gamma_n$.

(2) $v_i \to v_{2^n-1-i}$ $(0 \le i \le 2^{n-1} - 1)$, which translates to $(a_1, \dots, a_{n-1}, 0) \to (a_1, \dots, a_{n-1}, e)$.

(3) $v_{2^n-1-i} \leftarrow v_{2^n-1-j}$ if $v_i \to v_j$ $(0 \le i, j \le 2^{n-1} - 1)$.

See Figure 3 (resp. Figure 4) for $n = 2$ (resp. $n = 3$). Then the (i, j)-component g_{ij} $(1 \le i, j \le 2^{n-1})$ of G_n is as follows.

(1) $g_{ij} = 1$ when $v_{2j-2} \to v_{2i-1}$.

(2) $g_{ij} = -1$ when $v_{2j-2} \leftarrow v_{2i-1}$.

(3) $g_{ij} = 0$ when v_{2j-2} and v_{2i-1} do not connect by an edge.

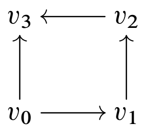

Figure 3. Orientation of edges of γ_2

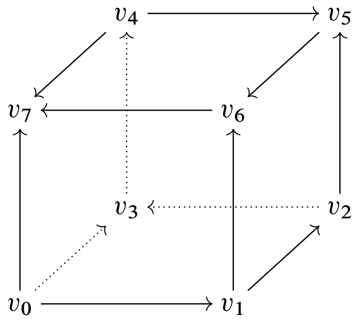

Figure 4. Orientation of edges of γ_3

1.2. Calculation of the quantized differential form

In this section we calculate an operator $[F_n, x^\alpha]$ for the coordinate function x^α on $\mathbb{R}^n$ ($\alpha = 1, 2, \ldots, n$). We also show that they satisfy a relation of the Clifford algebra on the Euclidean vector space of dimension n.

Set $d_n f = [F_n, f] = \begin{bmatrix} & d_n^- f \\ d_n^+ f & \end{bmatrix}$. Then

$$\begin{aligned} d_n^+ f &= Uf^+ - f^- U \\ d_n^- f &= U^* f^- - f^+ U^* = -(U\bar{f}^+ - \bar{f}^- U)^* = -{}^t d_n^+ f, \end{aligned}$$

where $f^+ = f|_{V_0}$ and $f^- = f|_{V_1}$. Denote by $A \circ B = [a_{ij} b_{ij}]$ the Hadamard product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size.

Proposition 1.5. *For any $f \in C(V)$, we set $f_{a,b} = f(v_a) - f(v_b)$ and*

$$\Delta_n f = [f_{2j,2i+1}]_{i,j=0,1,\ldots,2^{n-1}-1} \in \mathcal{B}(\ell^2(V_0), \ell^2(V_1)) \cong M_{2^{n-1}}(\mathbb{C}).$$

We have

$$d_n f = \frac{1}{\sqrt{n}} \begin{bmatrix} & -{}^t(\Delta_n f \circ G_n) \\ \Delta_n f \circ G_n & \end{bmatrix}.$$

Proof. As in Remark 1.4, we denote $G_n = [g_{ij}]$. We have

$$\begin{aligned}\sqrt{n}d_n^+ f &= G_n \begin{bmatrix} f(v_0) & & & \\ & f(v_2) & & \\ & & \ddots & \\ & & & f(v_{2^n-2}) \end{bmatrix} - \begin{bmatrix} f(v_1) & & & \\ & f(v_3) & & \\ & & \ddots & \\ & & & f(v_{2^n-1}) \end{bmatrix} G_n \\ &= \left[g_{ij} f(v_{2j})\right] - \left[f(v_{2i-1}) g_{ij}\right] \\ &= \left[f_{2j,2i-1} g_{ij}\right] \\ &= \Delta_n f \circ G_n. \end{aligned}$$

■

Thus, an (i, j)-component of $d_n^+ f$ is 0 if v_{2i-1} and v_{2j} do not connect by an edge.

Proposition 1.6. *For the coordinate function x^α on $\mathbb{R}^n$ ($\alpha = 1, 2, \ldots, n$), we set $e_{(n)}^\alpha = \frac{\sqrt{n}}{e} d_n x^\alpha$. We have*

$$e_{(n)}^\alpha = \frac{\sqrt{n}}{e} d_n x^\alpha = \begin{bmatrix} & E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\ -E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \end{bmatrix} \otimes X_\alpha. \quad (1.1)$$

Here $E_{1/2} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = 1.$

Proof. By using $\Delta_n x^n \circ G_n = -eX_n$ and Proposition 1.5, we have $e_{(n)}^n = \begin{bmatrix} & X_n \\ -X_n & \end{bmatrix}$.

Next, we calculate $e_{(n)}^{n-1+} = \frac{\sqrt{n}}{e} d_n^+ x^{n-1}$. By the definition of the numbering of vertices and the orientation of edges of γ_n, for $0 \le i, j \le 2^n - 1$, "$v_i \to v_j$ is positive (resp. negative) with x^{n-1} direction" if and only if "$v_{i+2^{n-1}} \to v_{j+2^{n-1}}$ is negative (resp. positive) with x^{n-1} direction". So we have

$$e_{(n)}^{n-1+} = \begin{bmatrix} e_{(n-1)}^{n-1+} & \\ & -e_{(n-1)}^{n-1+} \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \otimes (-X_{n-1}).$$

This implies

$$e_{(n)}^{n-1} = \begin{bmatrix} & E_1 \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\ -E_1 \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \end{bmatrix} \otimes X_{n-1}.$$

We calculate $e^\alpha_{(n)}$ ($\alpha = 1, 2, \ldots, n-2$) by induction on $n \geq 3$. Note that the calculation of $e^\alpha_{(n)}$ for $n = 1, 2$ is already done. Namely, the beginning of the induction is the following:

$$e^1_{(1)} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad e^1_{(2)} = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix}, \quad e^2_{(2)} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

Assume that equation (1.1) holds for $n-1$. By the definition of the numbering of vertices and the orientation of edges of γ_n, for $1 \leq \alpha \leq n-2$, "$i \to j$ is positive (resp. negative) with x^α direction" if and only if "$v_{i+2^{n-1}} \to v_{j+2^{n-1}}$ is positive (resp. negative) with x^α direction". So we have

$$e^{\alpha+}_{(n)} = \begin{bmatrix} e^{\alpha+}_{(n-1)} & \\ & e^{\alpha+}_{(n-1)} \end{bmatrix} = E_2 \otimes e^{\alpha+}_{(n-1)} = -E_2 \otimes \left(E_{2^{n-1-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \otimes X_\alpha \right)$$
$$= -E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \otimes X_\alpha.$$

Therefore, we have

$$e^\alpha_{(n)} = \begin{bmatrix} & E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\ -E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \end{bmatrix} \otimes X_\alpha \quad (\alpha = 1, 2, \ldots, n-2).$$

Equation (1.1) follows from the above calculations for any n and $\alpha = 1, 2, \ldots, n$. ■

By the explicit formula of $e^\alpha_{(n)}$ in Proposition 1.6, we have a Clifford relation of $d_n x^\alpha$.

Proposition 1.7. *We have*

$$e^\alpha_{(n)} e^\beta_{(n)} = \begin{cases} -e^\beta_{(n)} e^\alpha_{(n)} & (\alpha \neq \beta) \\ -E_{2^n} & (\alpha = \beta) \end{cases}.$$

By $d_n x^\alpha = \frac{e}{\sqrt{n}} e^\alpha_{(n)}$, *we have*

$$d_n x^\alpha d_n x^\beta = \begin{cases} -d_n x^\beta d_n x^\alpha & (\alpha \neq \beta) \\ -\frac{e^2}{n} E_{2^n} & (\alpha = \beta) \end{cases}.$$

Proof. Firstly, we have

$$e^{\alpha}_{(n)}e^{\alpha}_{(n)} = \begin{bmatrix} -E^2_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}^2 & \\ & -E^2_{2^{n-\alpha-1}} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}^2 \end{bmatrix} \otimes X^2_{\alpha} = -E_{2^n}.$$

Set $k = \alpha - \beta > 0$, then we have $X_\alpha = X_{k+1} \otimes X_\beta$. We can rewrite $e^{\alpha}_{(n)}$ and $e^{\beta}_{(n)}$ as follows:

$$e^{\alpha}_{(n)} = \begin{bmatrix} & E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} X_{k+1} & \\ & -X_{k+1} \end{bmatrix} \\ -E_{2^{n-\alpha-1}} \otimes \begin{bmatrix} X_{k+1} & \\ & -X_{k+1} \end{bmatrix} & \end{bmatrix} \otimes X_\beta,$$

$$e^{\beta}_{(n)} = \begin{bmatrix} & E_{2^{n-\alpha-1}} \otimes E_{2^k} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\ -E_{2^{n-\alpha-1}} \otimes E_{2^k} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \end{bmatrix} \otimes X_\beta.$$

Now, we set $\varepsilon_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, and we have

$$\begin{bmatrix} X_{k+1} & \\ & -X_{k+1} \end{bmatrix} \left(E_{2^k} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \right) = \begin{bmatrix} X_k \otimes (X_2\varepsilon_1) & \\ & -X_k \otimes (X_2\varepsilon_1) \end{bmatrix} \text{ and}$$

$$\left(E_{2^k} \otimes \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \right) \begin{bmatrix} X_{k+1} & \\ & -X_{k+1} \end{bmatrix} = \begin{bmatrix} X_k \otimes (\varepsilon_1 X_2) & \\ & -X_k \otimes (\varepsilon_1 X_2) \end{bmatrix}.$$

Thus, the relation $e^{\alpha}_{(n)}e^{\alpha}_{(n)} = -e^{\alpha}_{(n)}e^{\alpha}_{(n)}$ $(\alpha \neq \beta)$ holds since $X_2\varepsilon_1 + \varepsilon_1 X_2 = O$. ■

Remark 1.8. When we take the limit as the length of edges tends to 0, that is $e \to 0$, we have

$$d_n x^\alpha d_n x^\alpha = -\frac{e^2}{n} E_{2^n} \to O.$$

Thus, we regard $d_n x^\alpha$ as a quantization of the ordinal exterior differential dx^α on $\mathbb{R}^n$.

Remark 1.9. For any unitary matrix $U \in U(2^{n-1})$, since an odd matrix $F = \begin{bmatrix} & U^* \\ U & \end{bmatrix}$ defines an operator on $\mathcal{H}$, F defines a Fredholm module on $C(V)$. Moreover, since any F is homotopic to F_n, it defines the same K-homology class in $K^0(C(V))$. However, the general F sometimes does not have good properties. For example, we have

$[F, x^\alpha] = O$ for $\alpha = 2, 3, \dots, n$ when we assume $U = E_{2^{n-1}}$. Thus, in this case, we cannot regard $[F, x^\alpha]$ as a quantization of the ordinal exterior differential dx^α on $\mathbb{R}^n$.

By Proposition 1.7, we have the volume element $\omega_n = e^1_{(n)} e^2_{(n)} \cdots e^n_{(n)}$ in the Clifford algebra. We can easily calculate its absolute value $|\omega_n|$. We do not use $|\omega_n|$ directly, but we use $|d_n x^1 d_n x^2 \cdots d_n x^n|$, which is a constant multiple of $|\omega_n|$; see also Section 3.2.

Proposition 1.10. *We have* $|[F_n, x^1] \cdots [F_n, x^n]| = \frac{e^n}{n^{n/2}} E_{2^n}$. *By the definition of* $e^\alpha_{(n)}$, *we also have* $|\omega_n| = E_{2^n}$.

Proof. Because of $[F_n, x^\alpha]^* [F_n, x^\alpha] = \frac{e^2}{n} e^{\alpha *}_{(n)} e^\alpha_{(n)} = \frac{-e^2}{n} (e^\alpha_{(n)})^2 = \frac{e^2}{n} E_{2^n}$, we have

$$\begin{aligned} |[F_n, x^1] \cdots [F_n, x^n]|^2 &= ([F_n, x^1] \cdots [F_n, x^n])^* [F_n, x^1] \cdots [F_n, x^n] \\ &= [F_n, x^n]^* \cdots [F_n, x^1]^* [F_n, x^1] \cdots [F_n, x^n] \\ &= \left(\frac{e^2}{n} \right)^n E_{2^n}. \end{aligned}$$

This implies the claim. ∎

2. Fredholm module on self-similar sets built on *n*-cubes

2.1. Fredholm module and spectral triple

In this section we construct a Fredholm module and a spectral triple on self-similar sets built on any n-cubes γ_n. For the simplicity, we assume that the length of edges of γ_n equals 1. Let $f_s \colon \gamma_n \to \gamma_n$ $(s = 1, \dots, N)$ be similitudes. We define the similarity ratio of f_s to be

$$r_s = \frac{\| f_s(x) - f_s(y) \|_{\mathbb{R}^n}}{\| x - y \|_{\mathbb{R}^n}} \quad (< 1) \quad (x \neq y).$$

An iterated function system (IFS) $(\gamma_n, S = \{1, \dots, N\}, \{f_s\}_{s \in S})$ defines the unique non-empty compact set $K = K(\gamma_n, S = \{1, \dots, N\}, \{f_s\}_{s \in S})$ called the self-similar set such that $K = \bigcup_{s=1}^N f_s(K)$. We use $\dim_S(K)$ to denote the similarity dimension of K, that is, the number d that satisfies

$$\sum_{s=1}^N r_s^d = 1.$$

If an IFS $(\gamma_n, S, \{f_s\}_{s \in S})$ satisfies the open set condition, $\dim_S(K)$ turns out to be equal to the Hausdorff dimension $\dim_H(K)$ of K.

Set $f_{\boldsymbol{s}} = f_{s_1} \circ \cdots \circ f_{s_j}$ for $\boldsymbol{s} = (s_1, \ldots, s_j) \in S^\infty = \bigcup_{j=0}^\infty S^{\times j}$ and $f_\emptyset = \mathrm{id}$. For simplicity, we will use i to express the vertex $f_{\boldsymbol{s}}(v_i)$ of an n-cube $f_{\boldsymbol{s}}(\gamma_n)$ and write $V_{\boldsymbol{s}}$ as the vertices of an n-cube $f_{\boldsymbol{s}}(\gamma_n)$. We also denote the length of the edge of $f_{\boldsymbol{s}}(\gamma_n)$ by $e_{\boldsymbol{s}}$. As introduced in Section 1.1, we define the Hilbert space $\mathcal{H}_{\boldsymbol{s}} = \ell^2(V_{\boldsymbol{s}})$ on an n-cube of the length $e_{\boldsymbol{s}}$ that consists of the positive part $\mathcal{H}_{\boldsymbol{s}}^+$ and the negative part $\mathcal{H}_{\boldsymbol{s}}^-$. By taking the direct sum on all n-cubes, we define the following data:

$$\mathcal{H}_K = \bigoplus_{\boldsymbol{s}\in S^\infty} \mathcal{H}_{\boldsymbol{s}}, \quad F_K = \bigoplus_{\boldsymbol{s}\in S^\infty} F_n, \quad D_K = \bigoplus_{\boldsymbol{s}\in S^\infty} \frac{1}{e_{\boldsymbol{s}}} F_n.$$

Let V_K be the closure of the set of vertices of all n-cubes $f_{\boldsymbol{s}}(\gamma_n) \subset \mathbb{R}^n$. That is, V_K is the closure of $\bigcup_{\boldsymbol{s}\in S^\infty} V_{\boldsymbol{s}}$. Then, if $V \subset \bigcup_{s=1}^N f_s(V)$ holds, we have $V_K = K$. If not, V_K equals the union of $\bigcup_{\boldsymbol{s}\in S^\infty} V_{\boldsymbol{s}}$ and K. We also let $\mathcal{A}_K$ be the Banach algebra of Lipschitz functions $\mathrm{Lip}(V_K)$ on V_K with the norm $\|a\|_{\mathcal{A}_K} = \|a\|_\infty + \mathrm{Lip}(a)$, where the second term is the Lipschitz constant of a Lipschitz function a. The Banach algebra $\mathcal{A}_K$ acts on $\mathcal{H}_K$ by

$$\rho_K \colon \mathcal{A}_K \to \mathcal{B}(\mathcal{H}_K); \qquad \rho_K(a)\Big(\bigoplus \xi_{\boldsymbol{s}}\Big) = \bigoplus (a|_{V_{\boldsymbol{s}}}) \cdot \xi_{\boldsymbol{s}}.$$

Lemma 2.1. *Define*

$$\mathcal{H}_K^1 = \left\{ \bigoplus_{\boldsymbol{s}\in S^\infty} \xi_{\boldsymbol{s}} \in \mathcal{H}_K; \|\bigoplus \xi_{\boldsymbol{s}}\|_{\mathcal{H}_K^1}^2 = \sum_{\boldsymbol{s}\in S^\infty} \frac{1}{e_{\boldsymbol{s}}^2} \sum_{i=0}^{2^n-1} |\xi_{\boldsymbol{s}}(i)|^2 < \infty \right\}.$$

Then, D_K is a self-adjoint operator of $\mathrm{dom}(D_K) = \mathcal{H}_K^1$.

Proof. By the inclusions $\{\bigoplus \xi_{\boldsymbol{s}} \in \mathcal{H}_K; \xi_{\boldsymbol{s}} = 0 \text{ except finite } \boldsymbol{s}\} \subset \mathcal{H}_K^1 \subset \mathcal{H}_K$, $\mathcal{H}_K^1$ is a dense subset in $\mathcal{H}_K$.

On each n-cube $f_{\boldsymbol{s}}(\gamma_n)$, we have

$$\|F_n \xi_{\boldsymbol{s}}\|_{\ell^2}^2 = \|U_n \xi_{\boldsymbol{s}}^+\|_{\ell^2}^2 + \|U_n^* \xi_{\boldsymbol{s}}^-\|_{\ell^2}^2 = \|\xi_{\boldsymbol{s}}^+\|_{\ell^2}^2 + \|\xi_{\boldsymbol{s}}^-\|_{\ell^2}^2 = \sum_{i=0}^{2^n-1} |\xi_{\boldsymbol{s}}(i)|^2$$

for any function $\xi_{\boldsymbol{s}}$ on $V_{\boldsymbol{s}}$, where $\xi_{\boldsymbol{s}}^\pm$ denote the $\mathcal{H}_{\boldsymbol{s}}^\pm$ parts of $\xi_{\boldsymbol{s}}$, respectively. Then, we have

$$\|D_K(\bigoplus \xi_{\boldsymbol{s}})\|_{\mathcal{H}_K}^2 = \sum_{\boldsymbol{s}\in S^\infty} \frac{1}{e_{\boldsymbol{s}}^2} \sum_{i=0}^{2^n-1} |\xi_{\boldsymbol{s}}(i)|^2 = \|\bigoplus \xi_{\boldsymbol{s}}\|_{\mathcal{H}_K^1}^2$$

for $\bigoplus \xi_{\boldsymbol{s}} \in \mathcal{H}_K$. Thus, we have $D_K(\mathcal{H}_K^1) \subset \mathcal{H}_K$, and D_K is a symmetric operator with domain $\mathcal{H}_K^1$.

On the other hand, we set $\bigoplus \eta_s = \bigoplus e_s F_n \xi_s$ for any $\bigoplus \xi_s \in \mathcal{H}_K$. Then, $\bigoplus \eta_s \in \mathcal{H}_K^1$ since

$$\|\bigoplus \eta_s\|_{\mathcal{H}_K^1}^2 = \sum_{s \in S^\infty} \|F_n \xi_s\|_{\ell^2}^2 = \sum_{s \in S^\infty} \|\xi_s\|_{\ell^2}^2 = \|\bigoplus \xi_s\|_{\mathcal{H}_K}^2 < \infty.$$

This implies $D_K(\mathcal{H}_K^1) \supset \mathcal{H}_K$. Thus, we have $D_K(\mathcal{H}_K^1) = \mathcal{H}_K$. Therefore, D_K is a self-adjoint operator of the stated domain. ∎

Note that we have $\rho_K(\mathcal{A}_K)(\mathcal{H}_K^1) \subset \mathcal{H}_K^1$ and $F_K = D_K|D_K|^{-1}$. We now prove some regularity of F_K and D_K.

Lemma 2.2. *We have the following regularity properties:*

(1) $[F_K, a] \in \mathcal{K}(\mathcal{H}_K)$ *for any* $a \in C(V_K)$.

(2) $[D_K, a] \in \mathcal{B}(\mathcal{H}_K)$ *for any* $a \in \mathcal{A}_K$.

(3) $|D_K|^{-1} \in \mathcal{K}(\mathcal{H}_K)$.

(4) $(D_K^2 + 1)^{-1/2} \in \mathcal{K}(\mathcal{H}_K)$.

(5) $|D_K|^{-p} \in \mathcal{L}^1(\mathcal{H}_K) \iff p > \dim_S(K)$, *where* $\mathcal{L}^1(\mathcal{H}_K)$ *is the set of trace class operators on* $\mathcal{H}_K$.

(6) $(D_K^2 + 1)^{-p/2} \in \mathcal{L}^1(\mathcal{H}_K) \iff p > \dim_S(K)$.

Proof. (1) First, we take $a \in \mathcal{A}_K$. For any $s \in S^{\times j}$, we have

$$[F_K, a]|_{\mathcal{H}_s} = \frac{1}{\sqrt{n}} \begin{bmatrix} & -{}^t(\Delta_n a \circ G_n) \\ \Delta_n a \circ G_n & \end{bmatrix}.$$

Therefore, the operator norm $\|[F_K, a]|_{\mathcal{H}_s}\|$ is less than

$$\mathrm{Lip}(a) \cdot e_s = \mathrm{Lip}(a) \cdot \prod_{k=1}^{j} r_{s_k}.$$

Thus, $[F_K, a]$ is compact for $a \in \mathcal{A}_K$ since we have $\prod_{k=1}^{j} r_{s_k} \leq \max_{s \in S} r_s^j \to 0$ as $j \to \infty$. The case for any continuous function is proved by the denseness of $\mathcal{A}_K$ in $C(V_K)$.

(2) For any $s \in S^{\times j}$, we have

$$[D_K, a]|_{\mathcal{H}_s} = \frac{1}{\sqrt{n}} \left(\prod_{k=1}^{j} r_{s_k} \right)^{-1} \begin{bmatrix} & -{}^t(\Delta_n a \circ G_n) \\ \Delta_n a \circ G_n & \end{bmatrix}.$$

So the operator norm $\|[D_K, a]|_{\mathcal{H}_s}\|$ is less than $\mathrm{Lip}(a)$, which is independent of j. Therefore, $[D_K, a]$ is bounded on $\mathcal{H}_K$.

(3) Because of $|D_K| = \bigoplus_{s\in S^\infty} \frac{1}{e_s} E_{2^n}$, we have

$$|D_K|^{-1} = \bigoplus_{j=0}^{\infty} \bigoplus_{s\in S^{\times j}} \left(\prod_{k=1}^{j} r_{s_k}\right) E_{2^n}.$$

Thus, $|D_K|^{-1}$ is compact since we have $\prod_{k=1}^{j} r_{s_k} \to 0$ as $j \to \infty$.

(4) Because of $D_K^2 + 1 = \bigoplus_{s\in S^\infty} \left(\frac{1}{e_s^2} + 1\right) E_{2^n}$, we have

$$(D_K^2 + 1)^{-1/2} = \bigoplus_{j=0}^{\infty} \bigoplus_{s\in S^{\times j}} \left(\prod_{k=1}^{j} r_{s_k}^{-2} + 1\right)^{-1/2} E_{2^n}.$$

Thus, $(D_K^2 + 1)^{-1/2}$ is a compact operator.

(5) Because of $|D_K|^{-p} = \bigoplus_{j=0}^{\infty} \bigoplus_{s\in S^{\times j}} \left(\prod_{k=1}^{j} r_{s_k}^p\right) E_{2^n}$, we have

$$\mathrm{Tr}(|D_K|^{-p}) = \sum_{j=0}^{\infty} \sum_{s\in S^{\times j}} 2^n \prod_{k=1}^{j} r_{s_k}^p = 2^n \sum_{j=0}^{\infty} \left(\sum_{s=1}^{N} r_s^p\right)^j.$$

Thus we have

$$|D_K|^{-p} \in \mathcal{L}^1(\mathcal{H}_K) \iff \sum_{s=1}^{N} r_s^p < 1.$$

This implies part (5).

(6) Because of

$$(D_K^2 + 1)^{-p/2} = \bigoplus_{j=0}^{\infty} \bigoplus_{s\in S^{\times j}} \left(\prod_{k=1}^{j} r_{s_k}^{-2} + 1\right)^{-p/2} E_{2^n},$$

we have

$$\mathrm{Tr}((D_K^2 + 1)^{-p/2}) = \sum_{j=0}^{\infty} \sum_{s\in S^{\times j}} 2^n \left(\prod_{k=1}^{j} r_{s_k}^{-2} + 1\right)^{-p/2}.$$

Thus we have

$$\sum_{j=0}^{\infty} \sum_{s\in S^{\times j}} 2^{n-p/2} \prod_{k=1}^{j} r_{s_k}^p \le \mathrm{Tr}((D_K^2 + 1)^{-p/2}) \le \sum_{j=0}^{\infty} \sum_{s\in S^{\times j}} 2^n \prod_{k=1}^{j} r_{s_k}^p,$$

that is, we have

$$2^{n-p/2} \sum_{j=0}^{\infty} \left(\sum_{s=1}^{N} r_{s_k}^p\right)^j \le \mathrm{Tr}((D_K^2 + 1)^{-p/2}) \le 2^n \sum_{j=0}^{\infty} \left(\sum_{s=1}^{N} r_{s_k}^p\right)^j.$$

This implies

$$(D_K^2+1)^{-p/2}\in\mathcal{L}^1(\mathcal{H}_K) \iff \sum_{s=1}^{N} r_{s_k}^p < 1 \iff p > \dim_S(K). \qquad \blacksquare$$

Theorem 2.3. *The pair* $(\mathcal{H}_K, F_K)$ *is an even Fredholm module over* $C(V_K)$ *with the* $\mathbb{Z}_2$*-grading* $\varepsilon_K = \bigoplus_{s\in S^\infty}\varepsilon$. *The pair* $(\mathcal{H}_K, F_K)$ *is a* $([\dim_S(K)]+1)$*-summable even Fredholm module over* $\mathcal{A}_K$. *In particular, if we have* $\dim_S(K) < n$*, and the operator*

$$[F_K, a^1][F_K, a^2]\cdots[F_K, a^n]$$

is of trace class for any $a^1, a^2, \ldots, a^n \in \mathcal{A}_K$.

Proof. By the definition of F_K, we have $F_K^2 = 1$, $F_K^* = F_K$, and $F_K\varepsilon_K + \varepsilon_K F_K = 0$. $[F_K, a]$ is also a compact operator by Lemma 2.2. Therefore, $(\mathcal{H}_K, F_K)$ is an even Fredholm module over $C(V_K)$.

Next we prove summability of the Fredholm module $(\mathcal{H}_K, F_K)$ over $\mathcal{A}_K$. Since $[D_K, a]$ is a bounded operator for $a\in\mathcal{A}_K$ and $|D_K|^{-([\dim_S(K)]+1)}$ is of trace class, we have

$$\begin{aligned}
&[F_K, a^1][F_K, a^2]\cdots[F_K, a^{[\dim_S(K)]+1}]\\
=&[D_K, a^1]|D_K|^{-1}[D_K, a^2]|D_K|^{-1}\cdots[D_K, a^{[\dim_S(K)]+1}]|D_K|^{-1}\\
=&[D_K, a^1][D_K, a^2]\cdots[D_K, a^{[\dim_S(K)]+1}]|D_K|^{-([\dim_S(K)]+1)} \in \mathcal{L}^1(\mathcal{H}_K)
\end{aligned}$$

for $a^1, a^2, \ldots, a^{[\dim_S(K)]+1}\in\mathcal{A}_K$. Here we have $[|D_K|^{-1}, T] = 0$ if $T\in\mathcal{B}(\mathcal{H}_K)$ is a direct sum of operators on all n-cubes $f_{\boldsymbol{s}}(\gamma_n)$. Therefore we conclude that $(\mathcal{H}_K, F_K)$ is a $([\dim_S(K)]+1)$-summable even Fredholm module. ■

Theorem 2.4. *The triple* $(\mathcal{A}_K, \mathcal{H}_K, D_K)$ *is an even* QC^∞*-spectral triple of spectral dimension* $\dim_S(K)$.

Proof. By the definition of D_K and Lemma 2.2, $(\mathcal{A}_K, \mathcal{H}_K, D_K)$ is an even spectral triple of spectral dimension $\dim_S(K)$. $(\mathcal{A}_K, \mathcal{H}_K, D_K)$ is also of QC^∞-class since we have $[|D_K|, T] = 0$ for an operator $T\in\mathcal{B}(\mathcal{H}_K)$ of the direct sum of operators on n-cubes $f_{\boldsymbol{s}}(\gamma_n)$. ■

We next prove a non-vanishing property of the K^0-class of the Fredholm module $(\mathcal{H}_K, F_K)$.

Theorem 2.5. *Denote by* $X_1, \ldots, X_k$ *the connected components of* $V\cup\bigcup_{s\in S} f_s(\gamma_n)$. *Then, if there is a set* X_i *such that*

$$\#(V_0\cap X_i) \neq \#(V_1\cap X_i),$$

then the Connes–Chern character $\mathrm{Ch}_*(\mathcal{H}_K, F_K) \in H_\lambda^{\mathrm{even}}(\mathcal{A}_K)$ *induces a non-zero additive map* $K_0(C(V_K)) \cong K_0(\mathcal{A}_K) \to \mathbb{C}$ *by the Connes pairing. Moreover,* $[\mathcal{H}_K, F_K] \in K^0(C(V_K))$ *is not trivial.*

Proof. Set

$$d_0 = \#(V_0 \cap X_i), \quad d_1 = \#(V_1 \cap X_i)$$

and

$$p(x) = \begin{cases} 1 & x \in X_i, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in V_K$. Then p is a continuous function, and we have

$$\begin{aligned} \mathrm{index}(pF_K^+ p \colon p\mathcal{H}_K^+ \to p\mathcal{H}_K^-) &= \mathrm{index}(pU_n p \colon p\ell^2(V_0) \to p\ell^2(V_1)) \\ &= d_0 - d_1 \neq 0. \end{aligned}$$

Therefore, we have $\mathrm{Ch}_*(\mathcal{H}_K, F_K) \neq 0$ on $K_0(C(V_K))$. ∎

Remark 2.6. The assumption in Theorem 2.5 does not hold for some examples such as the Sierpiński carpet (see Section 4.3) and the n-cube γ_n. In these cases, the Connes–Chern character induces the 0-map on $K_0(\mathcal{A}_K)$.

Remark 2.7. As mentioned in Remark 1.9, we can define a Fredholm module on $C(V)$ by using any unitary matrix U instead of U_n. All properties in Section 2.1 hold without changing the proofs in such a situation.

2.2. Quantized differential form on self-similar sets

Note that all similitudes on γ_n take the form $f_s(\boldsymbol{x}) = r_s T_s \boldsymbol{x} + \boldsymbol{b}_s$ for an orthogonal matrix $T_s \in O(n)$ and $\boldsymbol{b}_s \in \mathbb{R}^n$. It is easy to calculate the quantum differential form $[F_K, x^\alpha]$ in the case for $\gamma_n = [0,1]^n$ and $T_s = E_n$ (for any $s \in S$), which is the direct sum of the matrix $d_n x^\alpha$; see Proposition 1.6. We can also express $[F_K, x^\alpha]$ explicitly for the general case and show that they satisfy "a variation" of the Clifford relation.

Proposition 2.8. *We have*

$$[F_K, x^\alpha][F_K, x^\beta] = \begin{cases} -[F_K, x^\beta][F_K, x^\alpha] & \alpha \neq \beta, \\ -\bigoplus_{s \in S^\infty} \frac{e_s^2}{n} E_{2^n} & \alpha = \beta. \end{cases}$$

Proof. Take an orthogonal matrix $T_{\boldsymbol{s}} = [t_{ij}]_{i,j} \in O(n)$ and a vector $\boldsymbol{b_s} \in \mathbb{R}^n$ such that the image of the affine transformation $g_{\boldsymbol{s}}(\boldsymbol{x}) = e_{\boldsymbol{s}} T_{\boldsymbol{s}} \boldsymbol{x} + \boldsymbol{b_s}$ of $[0,1]^n$ equals $f_{\boldsymbol{s}}(\gamma_n)$, and $g_s(\boldsymbol{x})$ preserves the numbering of the vertices of $[0,1]^n$ and $f_s(\gamma_n)$. If we assume $\gamma_n = [0,1]^n$, we have $f_s = g_s$. Note that we have

$$[F_K, x^\alpha]|_{\mathcal{H}_s} = \frac{1}{\sqrt{n}} \begin{bmatrix} & -{}^t(\Delta_n x^\alpha \circ G_n) \\ \Delta_n x^\alpha \circ G_n & \end{bmatrix}.$$

Recall that $v_{2j} - v_{2i-1} = \pm e_s T_s \boldsymbol{e}_k$ when $g_s^{-1}(v_{2j})$ is connecting $g_s^{-1}(v_{2i-1})$ by an edge of the n-cube $[0,1]^n$ parallel with x^k-direction and $T_s \boldsymbol{e}_k = \sum_{\alpha=1}^n t_{\alpha k} \boldsymbol{e}_\alpha$, and we have

$$[F_K, x^\alpha]|_{\mathcal{H}_s} = \frac{e_s}{\sqrt{n}} \sum_{j=1}^n t_{\alpha j} e_{(n)}^j.$$

Thus,

$$\begin{aligned} [F_K, x^\alpha][F_K, x^\beta]|_{\mathcal{H}_s} &= \frac{e_s^2}{n} \Big(\sum_{j=1}^n t_{\alpha j} e_{(n)}^j\Big)\Big(\sum_{j=1}^n t_{\beta k} e_{(n)}^k\Big) = \frac{e_s^2}{n} \sum_{j,k} t_{\alpha j} t_{\beta k} e_{(n)}^j e_{(n)}^k \\ &= \frac{e_s^2}{n} \sum_{j \neq k} t_{\alpha j} t_{\beta k} e_{(n)}^j e_{(n)}^k - \frac{e_s^2}{n} \sum_{j=1}^n t_{\alpha j} t_{\beta j} \\ &= \begin{cases} \frac{e_s^2}{n} \sum_{j \neq k} t_{\alpha j} t_{\beta k} e_{(n)}^j e_{(n)}^k & (\alpha \neq \beta), \\ -\frac{e_s^2}{n} E_{2^n} & (\alpha = \beta). \end{cases} \end{aligned}$$

Therefore, we have the claim proven. ∎

By Proposition 2.8, we get an explicit formula for $|[F_K, x^1] \cdots [F_K, x^n]|$.

Proposition 2.9. *We have*

$$|[F_K, x^1] \cdots [F_K, x^n]| = \bigoplus_{s \in S^\infty} \frac{e_s^n}{n^{n/2}} E_{2^n}.$$

Proof. Similar to the proof of Proposition 1.10. ∎

Remark 2.10. Setting $e_K^\alpha = \bigoplus_{s \in S^\infty} e_{(n)}^\alpha$, we have the Clifford relation

$$e_K^\alpha e_K^\beta = \begin{cases} -e_K^\beta e_K^\alpha & (\alpha \neq \beta), \\ -\mathrm{id}_{H_K} & (\alpha = \beta). \end{cases}$$

Thus, we can regard e_K^α as a 0-Q-form in the sense of [8].

3. Dixmier traces

In this section we calculate the Dixmier trace of two operators. In general, the value for the second operator changes if the Fredholm operator F_n changes to a different Fredholm operator.

3.1. Dixmier trace of $|D_K|^{-p}$

In this section we calculate the Dixmier trace of $|D_K|^{-p}$. This is given by the residue at the pole of the zeta function $\zeta_{D_K}(s) = \mathrm{Tr}(|D_K|^{-s})$.

Theorem 3.1. *For any $p \geq \dim_S(K)$, we have $|D_K|^{-p} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_K)$ and*

$$\mathrm{Tr}_\omega(|D_K|^{-p}) = \begin{cases} -2^n\Big(\dim_S(K)\sum_{s=1}^N r_s^{\dim_S(K)}\log r_s\Big)^{-1} & \textit{for } p = \dim_S(K), \\ 0 & \textit{for } p > \dim_S(K). \end{cases}$$

Thus we have

$$\mathrm{Tr}_\omega(f|D_K|^{-\dim_S(K)}) = -2^n\Big(\dim_S(K)\sum_{s=1}^N r_s^{\dim_S(K)}\log r_s\Big)^{-1}\int_K f|_K\, d\Lambda$$

for any $f \in C(V_K)$ by the Riesz–Markov–Kakutani representation theorem. Here Λ is the $\dim_S(K)$-dimensional Hausdorff probability measure of K.

In particular, if all similarity ratios r_s are equal, we have

$$\mathrm{Tr}_\omega(|D_K|^{-\dim_S(K)}) = \frac{2^n}{\log N}.$$

Proof. By the proof of Lemma 2.2, we have

$$\mathrm{Tr}(|D_K|^{-p}) = 2^n\sum_{j=0}^\infty\Big(\sum_{s=1}^N r_s^p\Big)^j = 2^n\Big(1-\sum_{s=1}^N r_s^p\Big)^{-1}.$$

Thus we have

$$\begin{aligned}(z-1)\,\mathrm{Tr}(|D_K|^{-zp}) &= 2^n\frac{z-1}{1-\sum_{s=1}^N r_s^{zp}} = 2^n\frac{z-1}{\sum_{s=1}^N\Big(r_s^{\dim_S(K)}-r_s^{zp}\Big)} \\ &= 2^n\Big(\sum_{s=1}^N\frac{r_s^{\dim_S(K)}-r_s^{zp}}{z-1}\Big)^{-1},\end{aligned}$$

and the value

$$\mathrm{Tr}_\omega(|D_K|^{-p}) = \lim_{z\to+1}(z-1)\,\mathrm{Tr}(|D_K|^{-zp}) = 2^n\Big(\sum_{s=1}^N\lim_{z\to+1}\frac{r_s^{\dim_S(K)}-r_s^{zp}}{z-1}\Big)^{-1}$$

converges for $p \geq \dim_S(K)$. Finally, we get

$$\begin{aligned}
\mathrm{Tr}_\omega(|D_K|^{-p}) &= -2^n \Big(\sum_{s=1}^{N} \frac{d}{dz}\Big|_{z=1} r_s^{z \dim_S(K)} \Big)^{-1} \\
&= -2^n \Big(\dim_S(K) \sum_{s=1}^{N} r_s^{\dim_S(K)} \log r_s \Big)^{-1} \quad \text{for } p = \dim_S(K)
\end{aligned}$$

$$\text{and} \quad \mathrm{Tr}_\omega(|D_K|^{-p}) = 0 \qquad \text{for } p > \dim_S(K). \quad \blacksquare$$

3.2. Dixmier trace of $|[F_K, x^1] \cdots [F_K, x^n]|^p$

In this section we calculate the Dixmier trace of $|[F_K, x^1] \cdots [F_K, x^n]|^p$ by using Proposition 2.9.

Theorem 3.2. *We have* $|[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]|^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_K)$ *for any* $p \geq \dim_S(K)/n$. *Moreover, we have*

$$\begin{aligned}
&\mathrm{Tr}_\omega(|[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]|^p) = \frac{1}{n^{np/2}} \mathrm{Tr}_\omega(|D_K|^{-np}) \\
&= \begin{cases} \frac{-2^n}{n^{\dim_S(K)/2}} \Big(\dim_S(K) \sum_{s=1}^{N} r_s^{\dim_S(K)} \log r_s \Big)^{-1} & \text{for } p = \frac{1}{n}\dim_S(K), \\ 0 & \text{for } p > \frac{1}{n}\dim_S(K). \end{cases}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathrm{Tr}_\omega(f |[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]|^{\frac{1}{n}\dim_S(K)}) \\
&= \frac{-2^n}{n^{\dim_S(K)/2}} \Big(\dim_S(K) \sum_{s=1}^{N} r_s^{\dim_S(K)} \log r_s \Big)^{-1} \int_K f|_K \, d\Lambda \\
&= \frac{1}{n^{\dim_S(K)/2}} \mathrm{Tr}_\omega(|D_K|^{-\dim_S(K)}) \int_K f|_K \, d\Lambda
\end{aligned}$$

for any $f \in C(V_K)$ *by the Riesz–Markov–Kakutani representation theorem. Here* Λ *is the* $\dim_H(K)$*-dimensional Hausdorff probability measure of* K.

Proof. By Proposition 2.9 we have

$$|[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]|^p = \bigoplus_{s \in S^\infty} \frac{e_s^{np}}{n^{np/2}} E_{2^n}.$$

Therefore, we get

$$\begin{aligned}\operatorname{Tr}(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p) &= 2^n \sum_{j=0}^{\infty} \sum_{(s_1,\ldots,s_j)\in S^j} \frac{1}{n^{np/2}} \prod_{k=1}^{j} r_{s_k}^{np} \\ &= \frac{2^n}{n^{np/2}} \sum_{j=0}^{\infty} \Big(\sum_{s=1}^{N} r_s^{np} \Big)^j\end{aligned}$$

and the following condition

$$\operatorname{Tr}(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p) < \infty \iff p > \frac{1}{n} \dim_S(K).$$

If p satisfies the above condition, the LHS can be written as

$$\operatorname{Tr}(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p) = \frac{2^n}{n^{np/2}} \Big(1 - \sum_{s=1}^{N} r_s^{np} \Big)^{-1}.$$

Therefore, a proof similar to that of Theorem 3.1 implies

$$|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_K)$$

for $p \geq \frac{1}{n} \dim_S(K)$. Moreover, we get

$$\begin{aligned}&\operatorname{Tr}_\omega(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p) \\ &\quad = \lim_{z\to+1} (z-1) \operatorname{Tr}(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^{zp}) \\ &\quad = \frac{2^n}{n^{\dim_S(K)/2}} \Big(\sum_{s=1}^{N} \lim_{z\to+1} \frac{r_s^{\dim_S(K)} - r_s^{z\dim_S(K)}}{z-1} \Big)^{-1} \\ &\quad = -\frac{2^n}{n^{\dim_S(K)/2}} \Big(\dim_S(K) \sum_{s=1}^{N} r_s^{\dim_S(K)} \log r_s \Big)^{-1} \quad \text{for } p = \frac{1}{n}\dim_S(K)\end{aligned}$$

and $\operatorname{Tr}_\omega(|[F_K, x^1][F_K, x^2]\cdots[F_K, x^n]|^p) = 0 \quad \text{for } p > \frac{1}{n}\dim_S(K)$ ■

4. Examples

In this section we apply arguments of Sections 2 and 3 to some examples.

4.1. Cantor dust

The Cantor dust is a generalization of the middle third Cantor set to a higher dimension. Let $\mathcal{CD}_n$ be the Cantor dust defined on $\gamma_n = [0,1]^n$ and the similitudes be

$$f_s(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x} + \frac{2}{3}\sum_{\alpha=1}^{n} a_\alpha \boldsymbol{e}_\alpha \quad (\boldsymbol{x} \in \gamma_n,\ s = 0, 1, 2, \ldots, 2^n - 1).$$

Here we write $a_n a_{n-1} \cdots a_2 a_1$ as a number s in binary, and $\boldsymbol{e}_\alpha$ is the standard basis of $\mathbb{R}^n$. See Figure 5 for $n = 2$. Since $\mathcal{CD}_n$ satisfies the open set condition, we have $\dim_H(\mathcal{CD}_n) = \dim_S(\mathcal{CD}_n) = n \log_3 2$. We also have $V_{\mathcal{CD}_n} = \mathcal{CD}_n$ since $V \subset \bigcup_{s=0}^{2^n-1} f_s(V)$. Then we get

$$\mathcal{A}_{\mathcal{CD}_n} = \mathrm{Lip}(\mathcal{CD}_n) \text{ and } C(V_{\mathcal{CD}_n}) = C(\mathcal{CD}_n).$$

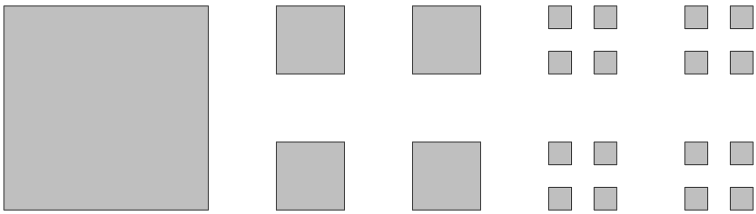

Figure 5. The first 3 steps of the construction of $\mathcal{CD}_2$.

Since all $f_s(\gamma_n)$ are disconnected from each other and also $\#(V_0 \cap f_1(\gamma_n)) = 1$ and $\#(V_1 \cap f_1(\gamma_n)) = 0$, the K^0-class of $(\mathcal{H}_{\mathcal{CD}_n}, F_{\mathcal{CD}_n})$ in $K^0(C(\mathcal{CD}_n))$ does not vanish by Theorem 2.5.

Theorem 4.1. *The Connes–Chern character*

$$\mathrm{Ch}_*(\mathcal{H}_{\mathcal{CD}_n}, F_{\mathcal{CD}_n}) \in H_\lambda^{\mathrm{even}}(\mathrm{Lip}(\mathcal{CD}_n))$$

induces a non-zero additive map $K_0(C(\mathcal{CD}_n)) \to \mathbb{C}$. *In particular,* $[\mathcal{H}_{\mathcal{CD}_n}, F_{\mathcal{CD}_n}]$ *is not trivial in* $K^0(C(\mathcal{CD}_n))$.

Since $\dim_S(\mathcal{CD}_n) = n \log_3 2$, we also get the next results.

Corollary 4.2. *The following properties hold.*

(1) $(\mathcal{H}_{\mathcal{CD}_n}, F_{\mathcal{CD}_n})$ *is a* $([n \log_3 2] + 1)$*-summable even Fredholm module over* $\mathrm{Lip}(\mathcal{CD}_n)$.

(2) $(\mathrm{Lip}(\mathcal{CD}_n), \mathcal{H}_{\mathcal{CD}_n}, D_{\mathcal{CD}_n})$ *is a* QC^∞*-spectral triple of spectral dimension* $n \log_3 2$.

Corollary 4.3. *We have the following.*

(1) $\mathrm{Tr}(|D_{\mathcal{CD}_n}|^{-p}) = \frac{2^n \cdot 3^p}{3^p - 2^n}$ *for any* $p > n\log_3 2$.

(2) $\mathrm{Tr}_\omega(|D_{\mathcal{CD}_n}|^{-n\log_3 2}) = \frac{2^n}{n\log 2}$.

(3) $\mathrm{Tr}_\omega(f|D_{\mathcal{CD}_n}|^{-n\log_3 2}) = \frac{2^n}{n\log 2}\int_{\mathcal{CD}_n} f\,d\Lambda$ *for any* $f \in C(\mathcal{CD}_n)$. *Here* Λ *is the* $(n\log_3 2)$*-dimensional Hausdorff probability measure of* $\mathcal{CD}_n$.

Corollary 4.4. *The operator* $|[F_{\mathcal{CD}_n}, x^1]\,[F_{\mathcal{CD}_n}, x^2]\,\cdots\,[F_{\mathcal{CD}_n}, x^n]|^{\log_3 2}$ *is of* $\mathcal{L}^{(1,\infty)}$*-class, and we have*

$$\mathrm{Tr}_\omega(|[F_{\mathcal{CD}_n}, x^1][F_{\mathcal{CD}_n}, x^2]\cdots[F_{\mathcal{CD}_n}, x^n]|^{\log_3 2}) = \frac{2^n}{n^{(2+n\log_3 2)/2}\log 2}.$$

Thus we have

$$\mathrm{Tr}_\omega(f|[F_{\mathcal{CD}_n}, x^1][F_{\mathcal{CD}_n}, x^2]\cdots[F_{\mathcal{CD}_n}, x^n]|^{\log_3 2}) = \frac{2^n}{n^{(2+n\log_3 2)/2}\log 2}\int_{\mathcal{CD}_n} f\,d\Lambda$$

for any $f \in C(\mathcal{CD}_n)$. *Here* Λ *is the* $(n\log_3 2)$*-dimensional Hausdorff probability measure of* $\mathcal{CD}_n$.

4.2. Middle third Cantor set, revisited

In this section we focus on the middle third Cantor set $\mathcal{CS} = \mathcal{CD}_1$.

First, we see a relationship between our Fredholm module and Connes' Fredholm module defined in [3, Chapter IV. 3. ε]. We recall Connes' Fredholm module (H, F) on $C(\mathcal{CS})$. Let $I_{i,j} = (a_{i,j}, b_{i,j})$ $(i \in \mathbb{N},\ j = 1, 2, \ldots, 2^i)$ be open intervals in $[0, 1]$ which are defined as

$$I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right) \quad \text{and} \quad I_{i+1,j} = \left(\frac{2b_{i,j-1} + a_{i,j}}{3}, \frac{b_{i,j-1} + 2a_{i,j}}{3}\right).$$

Here we set $b_{i,0} = 0$ and $a_{i,i+1} = 1$. The middle third Cantor set satisfies $\mathcal{CS} = [0,1] \setminus \bigcup_{i,j} I_{i,j}$. Connes defined

$$H = \bigoplus_{i,j} \ell^2(\{a_{i,j}, b_{i,j}\}) \text{ and } F = \bigoplus_{i,j} F_1.$$

Note that $H \oplus \ell^2(\{0,1\}) \cong \mathcal{H}_{\mathcal{CS}}$ as Hilbert spaces.

Lemma 4.5. *Let* $a < b < c$ *be real numbers. We assume*

$$[\ell^2(\{a,b\}), F_1], [\ell^2(\{b,c\}), F_1], [\ell^2(\{a,c\}), F_1] \in K^0(C(\{a,b,c\}))$$

under homomorphisms

$$K^0(C(\{a,b\})) \to K^0(C(\{a,b,c\})),$$
$$K^0(C(\{b,c\})) \to K^0(C(\{a,b,c\})),$$
$$K^0(C(\{a,c\})) \to K^0(C(\{a,b,c\})),$$

defined by inclusions $\{a,b\} \to \{a,b,c\}$, $\{b,c\} \to \{a,b,c\}$ *and* $\{a,c\} \to \{a,b,c\}$, *respectively. Then we have*

$$[\ell^2(\{a,b\}), F_1] + [\ell^2(\{b,c\}), F_1] = [\ell^2(\{a,c\}), F_1] \text{ in } K^0(C(\{a,b,c\})).$$

Proof. Set $b = b_1 = b_2$, $\{a,b\} = \{a,b_1\}$, and $\{b,c\} = \{b_2,c\}$. We have

$$\begin{aligned}[\ell^2(\{a,b_1\}), F_1] + [\ell^2(\{b_2,c\}), F_1] &= [\ell^2(\{a,b_1\}) \oplus \ell^2(\{b_2,c\}), F_1 \oplus F_1] \\ &= \left[\ell^2(\{a,c\}) \oplus \ell^2(\{b_1,b_2\}), \begin{bmatrix} & E_2 \\ E_2 & \end{bmatrix}\right].\end{aligned}$$

Here the $\mathbb{Z}_2$-grading operator of the last Fredholm module is defined by $\tilde{\varepsilon} = \varepsilon \oplus (-\varepsilon)$. Set

$$T_t = \begin{bmatrix} F_1 \cos t & \sin t \\ \sin t & -F_1 \cos t \end{bmatrix}$$

on $\ell^2(\{a,c\}) \oplus \ell^2(\{b_1,b_2\})$. Then we have $T_t\tilde{\varepsilon} + \tilde{\varepsilon}T_t = 0$, $T_0 = F_1 \oplus (-F_1)$, and $T_{\pi/2} = \begin{bmatrix} & E_2 \\ E_2 & \end{bmatrix}$. Thus we get

$$\begin{aligned}[\ell^2(\{a,b_1\}), F_1] + [\ell^2(\{b_2,c\}), F_1] &= \left[\ell^2(\{a,c\}) \oplus \ell^2(\{b_1,b_2\}), F_1 \oplus (-F_1)\right] \\ &= [\ell^2(\{a,c\}), F_1] - [\ell^2(\{b_1,b_2\}), F_1] \\ &= [\ell^2(\{a,c\}), F_1].\end{aligned}$$

Here the last equality is given by $b = b_1 = b_2$. ■

By Lemma 4.5, we have

$$[H, F] + [\mathcal{H}_{\mathcal{CS}}, F_{\mathcal{CS}}] = [H_{\mathcal{CS}}, F_{\mathcal{CS}}] + [\ell^2(\{0,1\}), F_1].$$

Therefore, we have $[H, F] = [\ell^2(\{0,1\}), F_1]$ in $K^0(C(\mathcal{CS}))$. On the other hand, if we set

$$p_k(x) = \begin{cases} 1 & x \in [0, 1/3^k] \cap \mathcal{CS}, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathcal{CS}$, then we get $\langle [H_{\mathcal{CS}}, F_{\mathcal{CS}}], [p_k] \rangle = k$ and $\langle [\ell^2(\{0,1\}), F_1], [p_k] \rangle = 1$ by the index pairing between K-homology and K-theory. Thus, a pair $([H_{\mathcal{CS}}, F_{\mathcal{CS}}], [H, F])$ is linearly independent on $\mathbb{Z}$ in $K^0(C(\mathcal{CS}))$.

Second, we set similitudes

$$f_1(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x}, \quad f_2(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x} + \frac{2}{3}\boldsymbol{e}_1$$

for $\boldsymbol{x} \in \gamma_2$, and denote by K the self-similar set defined by the IFS $(\gamma_2, \{f_1, f_2\})$. Then we get $K = \mathcal{CS} \times \{0\}$ as sets (see Figure 6). So the Fredholm module $(\mathcal{H}_K, F_K)$ is a novel Fredholm module of the middle third Cantor set. Note that we have $V_K \neq K$ and $\left(\bigcup_{s \in S^\infty} V_s\right) \cap K \neq \emptyset$ in this case.

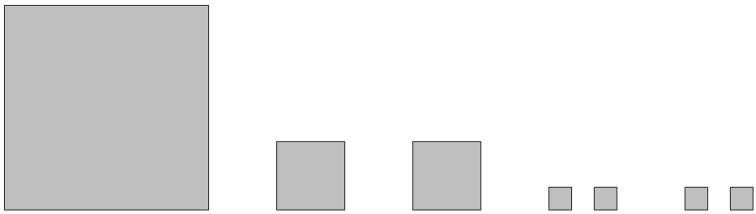

Figure 6. The first 3 steps of construction of K.

4.3. Sierpiński carpet and its higher dimensional analogue

The Sierpiński carpet is another generalization of the middle third Cantor set to a "2-dimensional space". The Menger sponge is also an analogue of the Sierpiński carpet but in a "3-dimensional space". In this section we delve into such self-similar sets in n-dimensional spaces ($n \geq 2$). Let $S_n \subset \mathbb{N} \cup \{0\}$ be the index set defined by

$$S_n = \{s \in \mathbb{N} \cup \{0\}\,;\, 0 \leq s \leq 3^n - 1 \text{ and at most one of its digits equals 1 in ternary expression of } s\}.$$

For example, for $n = 2, 3$, we have $S_2 = \{0, 1, 2, 3, 5, 6, 7, 8\}$ and

$$S_3 = S_2 \cup \{9, 11, 15, 17, 18, 19, 20, 21, 23, 24, 25, 26\}.$$

Define similitudes $f_s \colon \gamma_n \to \gamma_n$ for $s \in S_n$ by

$$f_s(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x} + \frac{1}{3}\sum_{\alpha=1}^{n} a_\alpha \boldsymbol{e}_\alpha.$$

Here we use a number s to express $a_n a_{n-1} \cdots a_2 a_1$ in ternary. We write $\mathcal{SC}_n$ for the self-similar set on the IFS $(\gamma_n, S_n, \{f_s\}_{s \in S_n})$. When $n = 2$ and 3, $\mathcal{SC}_2$ is the Sierpiński carpet (see Figure 7) and $\mathcal{SC}_3$ is the Menger sponge. Since $\mathcal{CD}_n$ satisfies the open set condition, we have $\dim_H(\mathcal{CD}_n) = \dim_S(\mathcal{CD}_n) = \log_3(\#S_n) = \log_3(2^{n-1}(n+2))$. We have $V_{\mathcal{SC}_n} = \mathcal{SC}_n$ since we have $V \subset \bigcup_{s\in S_n} f_s(V)$. Then, we get

$$\mathcal{A}_{\mathcal{SC}_n} = \mathrm{Lip}(\mathcal{SC}_n), \quad C(V_{\mathcal{SC}_n}) = C(\mathcal{SC}_n).$$

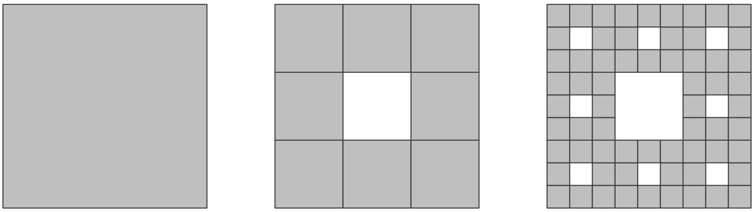

Figure 7. The first 3 steps of construction of $\mathcal{SC}_2$.

Since $X = V \cup \bigcup_{s\in S_n} f_s(\gamma_n)$ is connected, we have $\#(V_0 \cap X) = \#(V_1 \cap X)$; the assumption in Theorem 2.5 does not hold.

Remark 4.6. The Sierpiński carpet $\mathcal{SC}_2$ is a compact set in $\mathbb{R}^2$. Furthermore, we have $K_0(C(\mathcal{SC}_2)) = \mathbb{Z}$ which is generated by (matrix valued) constant functions on $\mathcal{SC}_2$, and the index pairing between K-theory and K-homology induces the 0-map $K_0(C(\mathcal{SC}_2)) \to \mathbb{Z}$. Therefore we get $[\mathcal{H}_{\mathcal{SC}_2}, F_{\mathcal{SC}_2}] = 0$ in $K^0(C(\mathcal{SC}_2))$ by [6, Theorem 7.5.5].

On the other hand, we can construct a non-trivial Fredholm module corresponding to the Sierpiński carpet in a manner similar to the construction shown in Section 4.2. Define $z\colon \gamma_1 \to \gamma_1$ by $z(t) = t/3$ and $\tilde{f_s} = (f_s, z)\colon \gamma_3 \to \gamma_3$ for $s \in S_2$. Then we get a new IFS $(\gamma_3, S_2, \{\tilde{f_s}\}_{s\in S_2})$. Denote by $\widetilde{\mathcal{SC}}_2$ the self-similar set on the new IFS, and we get $\widetilde{\mathcal{SC}}_2 = \mathcal{SC}_2 \times \{0\}$. The corresponding Fredholm module $(\mathcal{H}_{\widetilde{\mathcal{SC}}_2}, F_{\widetilde{\mathcal{SC}}_2})$ represents a non-trivial element in $K^0(C(V_{\widetilde{\mathcal{SC}}_2}))$.

Remark 4.7. The construction of the IFS in Remark 4.6 can be generalized. Let $(\gamma_n, S, \{f_s\}_{s\in S})$ be an IFS and K its self-similar set. Then $(\gamma_{n+1}, S, \{(f_s, z)\}_{s\in S})$ is a new IFS and the corresponding self-similar set is denoted by $\widetilde{K}$ for which $\widetilde{K} = K \times \{0\}$ and $[\mathcal{H}_{\widetilde{K}}, F_{\widetilde{K}}] \neq 0$ in $K^0(C(V_{\widetilde{K}}))$.

Since $\dim_S(\mathcal{SC}_n) = \log_3(2^{n-1}(n+2))$, we get the next results.

Corollary 4.8. *The following properties hold.*

(1) $(\mathcal{H}_{\mathcal{SC}_n}, F_{\mathcal{SC}_n})$ *is a* $([\log_3(2^{n-1}(n+2))]+1)$*-summable even Fredholm module over* $\mathrm{Lip}(\mathcal{SC}_n)$.

(2) $(\mathrm{Lip}(\mathcal{SC}_n), \mathcal{H}_{\mathcal{SC}_n}, D_{\mathcal{SC}_n})$ *is a* QC^∞*-spectral triple of spectral dimension* $\log_3(2^{n-1}(n+2))$.

Corollary 4.9. *We have the following.*

(1) $\mathrm{Tr}(|D_{\mathcal{SC}_n}|^{-p}) = \frac{2^n \cdot 3^p}{3^p - 2^{n-1}(n+2)}$ *for any* $p > \log_3(2^{n-1}(n+2))$.

(2) $\mathrm{Tr}_\omega(|D_{\mathcal{SC}_n}|^{-\log_3(2^{n-1}(n+2))}) = \frac{2^n}{\log(2^{n-1}(n+2))}$.

(3) $\mathrm{Tr}_\omega(f|D_{\mathcal{SC}_n}|^{-\log_3(2^{n-1}(n+2))}) = \frac{2^n}{\log(2^{n-1}(n+2))} \int_{\mathcal{SC}_n} f\, d\Lambda$ *for* $f \in C(\mathcal{SC}_n)$. *Here* Λ *is the* $(\log_3(2^{n-1}(n+2)))$*-dimensional Hausdorff probability measure of* $\mathcal{SC}_n$.

Corollary 4.10. *For* $d = (\log_3(2^{n-1}(n+2)))/n$, *we have*

$$|[F_{\mathcal{SC}_n}, x^1][F_{\mathcal{SC}_n}, x^2] \cdots [F_{\mathcal{SC}_n}, x^n]|^d \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_{\mathcal{SC}_n})$$

and

$$\mathrm{Tr}_\omega(|[F_{\mathcal{SC}_n}, x^1][F_{\mathcal{SC}_n}, x^2] \cdots [F_{\mathcal{SC}_n}, x^n]|^d) = \frac{2^n}{n^{nd/2} \log(2^{n-1}(n+2))}.$$

Thus we have

$$\mathrm{Tr}_\omega(f|[F_{\mathcal{CD}_n}, x^1][F_{\mathcal{CD}_n}, x^2] \cdots [F_{\mathcal{CD}_n}, x^n]|^d) = \frac{2^n}{n^{nd/2} \log(2^{n-1}(n+2))} \int_{\mathcal{SC}_n} f\, d\Lambda$$

for any $f \in C(\mathcal{SC}_n)$. *Here* Λ *is the* $(\log_3(2^{n-1}(n+2)))$*-dimensional Hausdorff probability measure of* $\mathcal{SC}_n$.

4.4. With rotations

Let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be a rotation matrix. Let also f_1, f_2, f_3, f_4 be four similitudes defined by

$$f_s(\boldsymbol{x}) = \frac{1}{2\sqrt{2}} R\left(\boldsymbol{x} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \boldsymbol{b}_s.$$

Here we set

$$\boldsymbol{b}_1 = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_3 = \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{b}_4 = \frac{1}{4}\begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The IFS $(\gamma_2, \{f_1, f_2, f_3, f_4\})$ is defined by using a rotation of angle θ. We get the self-similar set K on the IFS $(\gamma_2, \{f_1, f_2, f_3, f_4\})$ that satisfies the open set condition (see Figure 8). Then we have $V_K \neq K$ and $\left(\bigcup_{\boldsymbol{s} \in \{1,2,3,4\}^\infty} V_{\boldsymbol{s}}\right) \cap K = \emptyset$. Since $\{(0,0)\}$ is a connected component of $V \cup \bigcup_{s \in \{1,2,3,4\}} f_s(\gamma_2)$, the Fredholm module $(\mathcal{H}_K, F_K)$ defines a non-trivial element in $K^0(C(V_K))$.

Since $\dim_S(K) = \log_{2\sqrt{2}} 4 = 4/3$, we get the next results.

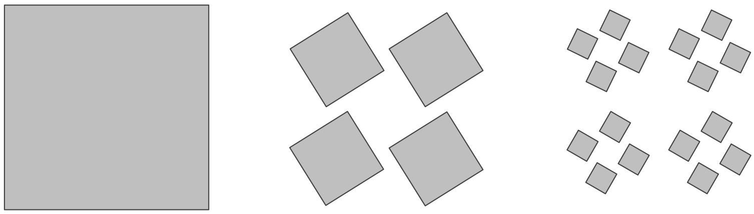

Figure 8. The first 3 steps of construction of K.

Corollary 4.11. *The following properties hold.*

(1) $(\mathcal{H}_K, F_K)$ *is a* 2*-summable even Fredholm module over* $\mathcal{A}_K$.

(2) $(\mathcal{A}_K, \mathcal{H}_K, D_K)$ *is a* QC^∞*-spectral triple of spectral dimension* $4/3$.

Corollary 4.12. *We have the following.*

(1) $\mathrm{Tr}(|D_K|^{-p}) = 4/(2^{3p/2} - 4)$ *for any* $p > 4/3$.

(2) $\mathrm{Tr}_\omega(|D_K|^{-4/3}) = 2/(\log 2)$.

(3) $\mathrm{Tr}_\omega(f|D_K|^{-4/3}) = (2/(\log 2)) \int_K f|_K \, d\Lambda$ *for any* $f \in C(V_K)$. *Here* Λ *is the* $4/3$*-dimensional Hausdorff probability measure of* K.

By Proposition 1.5, the quantized differential forms $[F_K, x^\alpha]$ $(\alpha = 1, 2)$ are given as

$$[F_K, x^1] = \bigoplus_{j=0}^{\infty} \bigoplus_{s \in S^{\times j}} \frac{e_s}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \cos j\theta & -\sin j\theta \\ 0 & 0 & -\sin j\theta & -\cos j\theta \\ -\cos j\theta & \sin j\theta & 0 & 0 \\ \sin j\theta & \cos j\theta & 0 & 0 \end{bmatrix},$$

$$[F_K, x^2] = \bigoplus_{j=0}^{\infty} \bigoplus_{s \in S^{\times j}} \frac{e_s}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \sin j\theta & \cos j\theta \\ 0 & 0 & \cos j\theta & -\sin j\theta \\ -\sin j\theta & -\cos j\theta & 0 & 0 \\ -\cos j\theta & \sin j\theta & 0 & 0 \end{bmatrix}.$$

Thus, we have

$$|[F_K, x^1][F_K, x^2]| = \bigoplus_{s \in S^\infty} \frac{e_s^2}{2} E_4.$$

This implies the next result.

Corollary 4.13. *The operator* $|[F_K, x^1][F_K, x^2]|^{2/3}$ *is of* $\mathcal{L}^{(1,\infty)}$*-class and we have*

$$\mathrm{Tr}_\omega(|[F_K, x^1][F_K, x^2]|^{2/3}) = \frac{\sqrt[3]{2}}{\log 2}.$$

Thus we have

$$\mathrm{Tr}_\omega(f\,|[F_K,x^1][F_K,x^2]|^{2/3}) = \frac{\sqrt[3]{2}}{\log 2}\int_K f|_K\,d\Lambda$$

for any $f \in C(V_K)$. *Here* Λ *is the* $4/3$*-dimensional Hausdorff probability measure of* K.

4.5. Without the open set condition

In this section we present an example of a self-similar set that does not satisfy the open set condition. In this case, we can detect the similarity dimension by using our Fredholm module but not detect the Hausdorff dimension explicitly.

Let $(\gamma_2, S = \{1,2,3,4,5\}, \{f_s\}_{s\in S})$ be the IFS defined by

$$f_1(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x},\quad f_2(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x} + \frac{2}{3}\boldsymbol{e}_1,\quad f_3(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x} + \frac{2}{3}\boldsymbol{e}_2,$$
$$f_4(\boldsymbol{x}) = \frac{1}{3}\boldsymbol{x} + \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2,\quad f_5(\boldsymbol{x}) = \frac{2}{3}\boldsymbol{x} + \frac{1}{6}\boldsymbol{e}_1 + \frac{1}{6}\boldsymbol{e}_2.$$

Note that this IFS does not satisfy the open set condition. Let K be the self-similar set on this IFS. Since we have $V \subset \bigcup_{s=1}^5 f_s(V)$, we have $V_K = K$. The similarity dimension $s = \dim_S(K)$ of K is given by the identity

$$4\cdot\left(\frac{1}{3}\right)^s + \left(\frac{2}{3}\right)^s = 1.$$

We can easily check that $1 < s < 2$.

Corollary 4.14. *The following properties hold.*

(1) $(\mathcal{H}_K, F_K)$ *is a* 2*-summable even Fredholm module over* $\mathrm{Lip}(K)$.

(2) $(\mathrm{Lip}(K), \mathcal{H}_K, D_K)$ *is a* QC^∞*-spectral triple of spectral dimension* s.

Corollary 4.15. *We have the following.*

(1) $\mathrm{Tr}(|D_K|^{-p}) = \frac{4\cdot 3^p}{3^p - 2^p - 4}$ *for any* $p > s$.

(2) $\mathrm{Tr}_\omega(|D_K|^{-\dim_S(K)}) = \frac{4\cdot 3^s}{3^s s\log 3 - 2^s s\log 2}$.

(3) $\mathrm{Tr}_\omega(f\,|D_K|^{-\dim_S(K)}) = \frac{4\cdot 3^s}{3^s s\log 3 - 2^s s\log 2}\int_K f\,d\Lambda$ *for* $f \in C(K)$. *Here* Λ *is the* $\dim_H(K)$*-dimensional Hausdorff probability measure of* K.

Corollary 4.16. *The operator* $|[F_K,x^1][F_K,x^2]|^{s/2}$ *is of* $\mathcal{L}^{(1,\infty)}$*-class, and we have*

$$\mathrm{Tr}_\omega(|[F_K,x^1][F_K,x^2]|^d) = \frac{2^{2-s/2}\cdot 3^s}{3^s s\log 3 - 2^s s\log 2}.$$

Thus we have

$$\mathrm{Tr}_\omega(f\,|[F_K, x^1][F_K, x^2]|^d) = \frac{2^{2-s/2}\cdot 3^s}{3^s s\log 3 - 2^s s\log 2}\int_K f\, d\Lambda$$

for any $f \in C(K)$. *Here* Λ *is the* $\dim_H(K)$*-dimensional Hausdorff probability measure of* K.

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Takashi Maruyama
System Platform Research Laboratories, NEC Corporation, 1753 Shimonumabe, Nakahara-ku, Kawasaki, Kanagawa, Japan; 49takashi@nec.com; 49takashi@gmail.com

Tatsuki Seto
General Education and Research Center, Meiji Pharmaceutical University, 2-522-1 Noshio, Kiyose-shi, Tokyo, Japan; tatsukis@my-pharm.ac.jp