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Assouad-like dimensions of a class of random Moran measures. II. Non-homogeneous Moran sets

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Abstract. In this paper, we determine the almost sure values of the Φ-dimensions of random measures μ supported on random Moran sets in $\mathbb{R}^d$ that satisfy a uniform separation condition. This paper generalizes earlier work done on random measures on homogeneous Moran sets in Hare and Mendivil (2022) to the case of unequal scaling factors. The Φ-dimensions are intermediate Assouad-like dimensions with the (quasi-)Assouad dimensions and the θ-Assouad spectrum being special cases.

The almost sure value of $\dim_\Phi \mu$ exhibits a threshold phenomenon, with one value for "large" Φ (with the quasi-Assouad dimension as an example of a "large" dimension) and another for "small" Φ (with the Assouad dimension as an example of a "small" dimension). We give many applications, including both where the scaling factors are fixed and the probabilities are uniformly distributed, and also where the probabilities are fixed and the scaling factors are uniformly distributed.

1. Introduction

A dimension provides a way of quantifying the size of a set. In the context of subsets of a metric space, there are many different dimensions that have been defined and each describes slightly different geometric properties of the subset. Two well-known examples of this are the Hausdorff and box-counting dimensions, which are both global measures of the geometry of the given subset. It is also of substantial interest to understand the local variation in the geometry and for this other dimensions have been introduced including the (upper and lower) Assouad dimensions and variations. The Assouad dimensions [1, 6, 21, 22], the less extreme quasi-Assouad dimensions [3, 11, 23], the θ-Assouad spectrum [10], and (the most general of these) the intermediate Assouad-like Φ-dimensions [10, 13] all quantify various aspects of the "thickest" and "thinnest" parts of the set. These same Assouad-like dimensions are all also available to quantify Borel measures on metric spaces [7, 16, 17].

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The Φ-dimensions range between the box dimensions and the Assouad dimensions and are also locally defined. However, they differ in the depth of scales that they consider and thus can provide precise information about the set or measure (see Section 2 for definitions). In this paper, we extend the investigation of the Φ-dimensions of random 1-variable measures on homogeneous Moran sets [18] to the case of random measures supported on random Moran sets with multiple scaling factors for the similarities.

The study of the dimensional properties of random fractal objects is well established, with some early papers investigating the almost sure Hausdorff dimension [5, 14], while more recently the Assouad and related dimensions have also been investigated [8, 9, 12, 18, 25, 26].

By a random Moran measure we mean a random Borel probability measure supported on a random Moran construction in $\mathbb{R}^D$ (see Section 3 for the precise details of the construction); our construction can also be described as a random 1-variable fractal measure. The support of the measure is constructed by a random iterative procedure, where at each stage we replace each component of the set with a random (but uniformly bounded) number of randomly scaled, separated, compact, and similar subsets. A random Borel probability measure is then defined on the random limiting set by a similar iterative process which subdivides the total mass by randomly choosing a set of probabilities at each step. The process produces a 1-variable fractal measure since at each level we make one random choice and use that same choice for all subdivisions on that level. Specifically, at level n we choose K_n random geometric scaling factors for the similarities and K_n random probabilities to use in subdividing the mass and use these $2K_n$ choices for every subdivision at that level. This is in contrast with the stochastically self-similar (or ∞-variable) construction where the choice is made independently for each subdivision. We make a blanket separation assumption which can be thought of as a uniform strong convex separation condition.

For any dimension function Φ, the Φ-dimension of the resulting random measure μ_ω is almost surely constant and this value depends on how Φ compares to the threshold function $\Psi(t) = \log|\log t|/|\log t|$ near 0; this behaviour is similar to what was seen in [12, 18, 25]. For $\Phi \ll \Psi$[1] (the "small" dimension functions Φ, such as the Assouad dimension), the computations of the almost sure values of the upper and lower Φ-dimensions of the random measure μ_ω are quite similar to the homogeneous (same scaling factor for all children) case dealt with in [18]. These computations involve the essential supremum (or essential infimum) of ratios of the logarithm of a probability to the logarithm of a scaling factor (see Section 5.1). Furthermore,

[1]For $f, g > 0$, we will write $g \ll f$ if there is a function A and $\delta > 0$ such that $f(t) \geq A(t)g(t)$ for all $0 < t < \delta$ and $A(t) \to \infty$ as $t \to 0^+$.

the almost sure value of the Φ-dimension is the same for all small dimension functions. It is natural to ask if there is a choice of probabilities such that the almost sure Φ-dimension of the associated random measures coincides with the almost sure Φ-dimension of the underlying sets, as is true in the homogeneous case. In Section 5.2 we show that this need not be true in the more general situation.

In contrast, for $\Phi \gg \Psi$ (the "large" dimension functions, such as the quasi-Assouad dimension), the computations are significantly different in the current situation of different scaling factors. Roughly, the reason for this is that the choice of the extremal branch down the tree of subdivisions depends on what exponent (dimension) one thinks is the correct one. Thus, the computation of the Φ-dimension involves solving an equation of the form $G(\theta) = \theta$ to find the correct exponent. The function G is a ratio of expected values of logarithms of probabilities to logarithms of scaling ratios (see Section 4.1 for details). Again, the almost sure value of the Φ-dimension is the same for all large dimension functions. One special case we examine carefully is when the set is deterministic with two scaling ratios, a and b, and the probabilities are uniformly chosen. Setting $a = b^\gamma$, the dimension is the root, θ, of $b^\theta + b^{\gamma\theta} = e^{-1}$. Notice that this is a polynomial in b^θ if γ is an integer. It is interesting to note that the dimension of the support (the Cantor-like set) in this case is the root of $b^\theta + b^{\gamma\theta} = 1$. Another special case we examine is again when the set is deterministic, but now the "left" probability is chosen randomly from the two possibilities p or $1 - p$ (for a fixed value of p). In this case the almost sure Φ-dimension of μ_ω is given explicitly as one of two values where the one to use depends on the relationship between a and b and also between p and $1 - p$. All of these examples are discussed in Section 4.4. It is an open problem if the probabilities can be chosen so that the almost sure Φ-dimension of the random measures coincides with that of the random sets.

The definition and basic properties of the Φ-dimensions are given in Section 2 and the details of the random construction are given in Section 3. Section 4 contains our results for large Φ and Section 5 those for small Φ.

We present most of our discussion in the context of random subsets of $\mathbb{R}$ where at each stage we split each component into two "children". This is done for simplicity of exposition only and in Section 4.6 we briefly indicate what changes are necessary to accommodate random subsets of $\mathbb{R}^D$ with a random (but uniformly bounded) number of children at each level.

It is important to note that we always assume that the scaling ratios are uniformly bounded away from 0. It is certainly possible to remove this assumption, but this seems to require some delicate technical arguments and we leave this case for future work.

2. Assouad-like dimensions

There are many ways to quantify the ‘size’ of subsets of metric spaces and Borel probability measures on these metric spaces. The so-called Φ-dimensions provide refined information on the local size of a set or concentration of a measure. To define these, we first recall some standard notation and define what we mean by a dimension function.

Notation 1. *We will write $B(x, R)$ for the open ball centred at x belonging to the bounded metric space X and radius R. By $N_r(E)$ we mean the least number of open balls of radius r required to cover $E \subseteq X$.*

Definition 2. A *dimension function* is a map Φ: $(0, 1) \to \mathbb{R}^+$ with the property that $t^{1+\Phi(t)}$ decreases to 0 as t decreases to 0.

Examples include the constant functions $\Phi(t) = \delta \geq 0$, the function $\Phi(t) = 1/|\log t|$ and the function $\Phi(t) = \log|\log t|/|\log t|$. The latter will be of particular interest in this paper.

Definition 3. We will say that a dimension function Φ is *large* if

$$\Phi(t) = H(t)\frac{\log|\log t|}{|\log t|},$$

where $H(t) \to \infty$ as $t \to 0$ and *small* if (with the same notation) $H(t) \to 0$ as $t \to 0$.

Definition 4. Let μ be a measure on X and Φ be a dimension function. The *upper* and *lower Φ-dimensions* of μ are given, respectively, by

$$\overline{\dim}_\Phi \mu = \inf\Big\{d\colon (\exists C_1, C_2 > 0)(\forall 0 < r < R^{1+\Phi(R)} \leq R \leq C_1)$$
$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_2\Big(\frac{R}{r}\Big)^d \ \forall x \in \operatorname{supp}\mu\Big\}$$

and

$$\underline{\dim}_\Phi \mu = \sup\Big\{d\colon (\exists C_1, C_2 > 0)(\forall 0 < r < R^{1+\Phi(R)} \leq R \leq C_1)$$
$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq C_2\Big(\frac{R}{r}\Big)^d \ \forall x \in \operatorname{supp}\mu\Big\}.$$

These dimensions were introduced in [16] and were motivated, in part, by the Φ-dimensions of sets, introduced in [10] and thoroughly studied in [13]. We recall the definition.

Definition 5. The *upper* and *lower* Φ*-dimensions* of $E \subseteq X$ are given, respectively, by

$$\overline{\dim}_\Phi E = \inf \Big\{ d : (\exists C_1, C_2 > 0)(\forall 0 < r \le R^{1+\Phi(R)} < R < C_1) \\ N_r(B(z,R) \cap E) \le C_2 \Big(\frac{R}{r}\Big)^d \ \forall z \in E \Big\}$$

and

$$\underline{\dim}_\Phi E = \sup \Big\{ d : (\exists C_1, C_2 > 0)(\forall 0 < r \le R^{1+\Phi(R)} < R < C_1) \\ N_r(B(z,R) \cap E) \ge C_2 \Big(\frac{R}{r}\Big)^d \ \forall z \in E \Big\}.$$

Remark 1. (i) In the special case of $\Phi = 0$, these dimensions are known as the *upper* and *lower Assouad dimensions* of the measure or set. For measures, these dimensions are also known as the upper and lower regularity dimensions and were studied by Käenmäki et al. in [20, 21] and Fraser and Howroyd in [7]. The upper and lower Assouad dimensions of the measure μ are denoted $\dim_A \mu$ and $\dim_L \mu$ respectively, and are important because the measure μ is doubling if and only if $\dim_A \mu < \infty$ ([7]) and uniformly perfect if and only if $\dim_L \mu > 0$ ([20]).

(ii) If we put $\Phi_\theta = 1/\theta - 1$ for $0 < \theta < 1$, then $\overline{\dim}_{\Phi_\theta} \mu$ and $\underline{\dim}_{\Phi_\theta} \mu$ are (basically) the *upper* and *lower* θ*-Assouad spectrum* introduced in [10]. The *upper* and *lower quasi-Assouad dimensions* of μ, developed in [17, 19], are given by

$$\dim_{qA} \mu = \lim_{\theta \to 1} \overline{\dim}_{\Phi_\theta} \mu \quad \text{and} \quad \dim_{qL} \mu = \lim_{\theta \to 1} \underline{\dim}_{\Phi_\theta} \mu.$$

Here are some basic relationships between these dimensions; for proofs see [10, 13, 16] and the references cited there.

Proposition 6. *Let* Φ, Ψ *be dimension functions and* μ *be a measure.*

(i) *If* $\Phi(t) \le \Psi(t)$ *for all* $t > 0$, *then* $\overline{\dim}_\Psi \mu \le \overline{\dim}_\Phi \mu$ *and* $\underline{\dim}_\Phi \mu \le \underline{\dim}_\Psi \mu$.

(ii) *We have that*

$$\dim_A \mu \ge \overline{\dim}_\Phi \mu \ge \overline{\dim}_\Phi \operatorname{supp} \mu \ge \dim_H \operatorname{supp} \mu$$

and $\dim_L \mu \le \underline{\dim}_\Phi \mu$. *If* μ *is doubling, then* $\underline{\dim}_\Phi \mu \le \underline{\dim}_\Phi \operatorname{supp} \mu$.

(iii) *If* $\Phi(t) \to 0$ *as* $t \to 0$, *then* $\underline{\dim}_\Phi \mu \le \dim_{qL} \mu$ *and* $\dim_{qA} \mu \le \overline{\dim}_\Phi \mu$.

(iv) *If* $\Phi(t) \le 1/|\log t|$ *for* t *near* 0, *then* $\overline{\dim}_\Phi \mu = \dim_A \mu$ *and* $\underline{\dim}_\Phi \mu = \dim_L \mu$.

(v) *For any set* E,

$$\dim_L E \le \underline{\dim}_\Phi E \le \underline{\dim}_B E \le \overline{\dim}_B E \le \overline{\dim}_\Phi E \le \dim_A E.$$

(Here $\underline{\dim}_B$ *and* $\overline{\dim}_B$ *are the lower and upper box dimensions.)*

3. Random Moran sets and measures

3.1. Definition of random Moran sets $\mathcal{C}_\omega$ and random measures μ_ω

For the majority of this paper we describe our results in the simple context of subsets of [0, 1] with two "children" at each "level". We do this for clarity and to highlight the important features of the construction. However, in Section 4.6 we briefly indicate the natural extension to compact subsets of $\mathbb{R}^D$ with an arbitrary (but uniformly bounded) number of children at each level. All of our proofs are given so that they can be easily modified for the more general situation.

Let $(\Omega, \mathcal{P})$ be a probability space. Fix $0 < 2A \leq B < 1$ and choose independently and identically distributed random variables

$$(a_n(\omega), b_n(\omega), p_n(\omega)) \in \{(x, y, z) \in [0,1]^3 \colon A \leq \min\{x, y\} < x + y \leq B\}.$$

We assume that $\mathbb{E}(e^{-t \log p_n}) = \mathbb{E}(p_n^{-t}) < \infty$ and $\mathbb{E}(e^{-t \log(1-p_n)}) = \mathbb{E}((1-p)^{-t}) < \infty$ for some $t > 0$. Note that this implies that the probability that $p_n = 0$ or $p_n = 1$ is zero. Note also that since $A > 0$, we have $\mathbb{E}(e^{-t \log a_n}) = \mathbb{E}(a_n^{-t}) < \infty$ and $\mathbb{E}(e^{-t \log b_n}) = \mathbb{E}(b_n^{-t}) < \infty$ for all $t > 0$.

Let L denote the minimal positive integer such that

$$2B^{L-1} \leq 1 - B.$$

To create the random Moran set, $\mathcal{C}_\omega$, we begin with the closed interval [0, 1] and then at step one form the set $\mathcal{C}_\omega^{(1)}$ by keeping the outermost left subinterval of length $a_1(\omega)$ and the outermost right subinterval of length $b_1(\omega)$. Having inductively created $\mathcal{C}_\omega^{(n-1)}$, a union of 2^{n-1} closed intervals $\{I_j(\omega)\}_{j=1}^{2^{n-1}}$ (which we call the Moran intervals of step (or level) $n-1$), we let $\mathcal{C}_\omega^{(n)} = \bigcup_{j=1}^{2^{n-1}} (I_j^{(1)} \cup I_j^{(2)})$ where $I_j^{(1)} = I_j^{(1)}(\omega)$ is the outermost left closed subinterval of $I_j = I_j(\omega)$ of length $|I_j^{(1)}| = a_n(\omega)|I_j|$ and $I_j^{(2)} = I_j^{(2)}(\omega)$ is the outermost right closed subinterval of I_j of length $|I_j^{(2)}| = b_n(\omega)|I_j|$. We call $I_j^{(1)}$ the left child of I_j and $I_j^{(2)}$, the right child. The *random Moran set* $\mathcal{C}_\omega$ is the compact set

$$\mathcal{C}_\omega = \bigcap_{n=1}^{\infty} \mathcal{C}_\omega^{(n)}.$$

It can be convenient to label the Moran intervals of step N as $I_{v_1 \cdots v_N}$ with $v_j \in \{0, 1\}$, where $I_{v_1 \cdots v_{N-1} 0}$ is the left child of $I_{v_1 \cdots v_{N-1}}$ and $I_{v_1 \cdots v_{N-1} 1}$ is the right child. When we write $I_N(x)$ we mean the Moran interval of step N containing the element $x \in \mathcal{C}_\omega$.

Notice that any Moran interval of step N has length between A^N and B^N and

$$A^k \leq \frac{|I_{N+k}(x)|}{|I_N(x)|} \leq B^k$$

for any N, x. In particular, this means that none of the intervals disappear.

The *random measure* μ_ω is defined by the rule that $\mu_\omega([0,1]) = 1$ and if I_N is a Moran interval of step N, then (with the notation as above)

$$\mu_\omega(I_N^{(1)}) = p_{N+1}(\omega)\mu_\omega(I_n) \quad \text{and} \quad \mu_\omega(I_N^{(2)}) = (1 - p_{N+1}(\omega))\mu_\omega(I_N).$$

For each ω, this uniquely determines a probability measure on $\mathcal{C}_\omega$. In addition, for almost all ω the support is all of $\mathcal{C}_\omega$. For those familiar with V-variable fractals (see [2]), we mention that our construction produces a random 1-variable fractal measure. Our entire random model can also be viewed as sampling from the product space

$$\Omega = \prod_{n=1}^{\infty} \Big(\{(x, y, z) \colon A \leq \min\{x, y\} \leq x + y \leq B, 0 \leq z \leq 1\} \Big),$$

where we use the product measure on Ω induced by a given probability measure on each factor.

Notice that we allow the possibility that a_n, b_n and p_n can be dependent or independent of each other; we only assume that (a_n, b_n, p_n) is independent of (a_m, b_m, p_m) when $n \neq m$.

We remark that $\mathcal{C}_\omega$ has a "uniform separation" property in the sense that the distance between the two children of I_N is at least $(1 - B)|I_N|$. This fact allows us to prove the following simple, but useful, relationship between Moran intervals of various levels and balls.

Lemma 7. *Given $\omega \in \Omega$, $x \in \mathcal{C}_\omega$ and $0 < R < 1$, choose the integer $N = N(\omega, x)$ such that $|I_N(x)| \leq R < |I_{N-1}(x)|$. Then*

$$I_N(x) \cap \mathcal{C}_\omega \subseteq B(x, R) \cap \mathcal{C}_\omega \subseteq I_{N-L}(x) \cap \mathcal{C}_\omega.$$

Proof. The proof is similar to [18, Lemma 1], but we include it here for completeness. Obviously, $I_N(x)$ is contained in $B(x, R)$.

Assume I'_N is another Moran interval of step N which intersects $B(x, R)$ and suppose $I_{N-k}(x)$ is the common ancestor of $I_N(x)$ and I'_N with k minimal. Then the two level N intervals $I_N(x)$ and I'_N must be separated by a distance of at least $|I_{N-k}(x)|(1 - B)$ and at most $2R$. If $k \geq L$, the definition of L gives

$$\begin{aligned} |I_{N-k}(x)|(1 - B) &\leq 2R < 2|I_{N-1}(x)| \leq 2B^{k-1}|I_{N-k}(x)| \\ &\leq 2B^{L-1}|I_{N-k}(x)| \leq |I_{N-k}(x)|(1 - B), \end{aligned}$$

which is a contradiction. Hence, all step N Moran intervals intersecting $B(x, R)$ are contained in $I_{N-L}(x)$ and that implies $B(x, R) \cap \mathscr{C}_\omega \subseteq I_{N-L}(x)$. ■

Our next lemma shows that the dimension of μ_ω is completely determined by the lengths and measures of the Moran intervals. While this result is not surprising because of our separation assumption, it is very useful to make it explicit.

Lemma 8. *Let*

$$\Delta_\omega = \inf\Big\{d \colon (\exists c_1, c_2 > 0)(\forall\ I_n(\omega) \subseteq I_N(\omega), |I_N| \leq c_1, |I_n| < |I_N|^{1+\Phi(|I_N|)}) \\ \frac{\mu_\omega(I_N)}{\mu_\omega(I_n)} \leq c_2\Big(\frac{|I_N|}{|I_n|}\Big)^d\Big\}.$$

Then $\Delta_\omega = \overline{\dim}_\Phi \mu_\omega$. *A similar statement holds for the lower* Φ*-dimension.*

Proof. We fix an $\omega \in \Omega$ for the rest of the proof and simplify our notation by removing any explicit mention of the dependence on ω.

Let $\varepsilon > 0$ and get constants c_1, c_2 such that

$$\frac{\mu(I_N)}{\mu(I_n)} \leq c_2\Big(\frac{|I_N|}{|I_n|}\Big)^{\Delta+\varepsilon}$$

whenever $I_n \subseteq I_N$ with $|I_N| \leq c_1$ and $|I_n| < |I_N|^{1+\Phi(|I_N|)}$. Choose N_0 so that all Moran intervals of level $N_0 - L$ have diameter at most c_1. Choose $x \in \mathscr{C}_\omega$, and suppose $R \leq A^{N_0}$ and $0 < r < R^{1+\Phi(R)}$. Obtain $n \geq N \geq N_0$ such that

$$|I_N(x)| \leq R < |I_{N-1}(x)| \leq |I_{N-L}(x)| \leq c_1 \quad \text{and} \quad |I_n(x)| \leq r < |I_{n-1}(x)|.$$

By Lemma 7, $B(x, r) \supseteq I_n(x)$ and $B(x, R) \cap \mathscr{C}_\omega \subseteq I_{N-L}(x)$.

As the function $t^{1+\Phi(t)}$ is decreasing as $t \downarrow 0$, it follows that $|I_n(x)| \leq r < R^{1+\Phi(R)} \leq |I_{N-L}|^{1+\Phi(|I_{N-L}|)}$. Hence,

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{\mu(I_{N-L})}{\mu(I_n)} \leq c_2\Big(\frac{|I_{N-L}|}{|I_n|}\Big)^{\Delta+\varepsilon} \leq c_2\Big(\frac{A^{-L}|I_N|}{A|I_{n-1}|}\Big)^{\Delta+\varepsilon} \leq C_2\Big(\frac{R}{r}\Big)^{\Delta+\varepsilon}$$

for $C_2 = c_2 A^{-(L+1)(\Delta+\varepsilon)}$ and consequently, $\overline{\dim}_\Phi \mu \leq \Delta$.

The opposite inequality is similar. Let $D = \overline{\dim}_\Phi \mu$ and given $\varepsilon > 0$ choose C_1, C_2 such that

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_2\Big(\frac{R}{r}\Big)^{D+\varepsilon}$$

whenever $r < R^{1+\Phi(R)} \leq R \leq C_1$ and $x \in \mathscr{C}_\omega$. Suppose that $I_n \subseteq I_N$ with $|I_N| \leq C_1$ and $|I_n| < |I_N|^{1+\Phi(|I_N|)}$. Choose $x \in \mathscr{C}_\omega$ such that $I_n = I_n(x)$ and $I_N = I_N(x)$. Let $R = |I_N(x)| \leq C_1$ and $r = |I_n(x)|(1 - B)$. As the distance from $I_n(x)$ to the nearest

Moran interval of level n is at most r, $B(x,r) \cap \mathcal{C}_\omega \subseteq I_n(x)$. Clearly, $B(x,R) \supseteq I_N(x)$ and $r < R^{1+\Phi(R)}$. Thus,

$$\frac{\mu(I_N)}{\mu(I_n)} \le \frac{\mu(B(x,R))}{\mu(B(x,r))} \le C_2\Big(\frac{R}{r}\Big)^{D+\varepsilon} = C_2(1-B)^{-(D+\varepsilon)}\Big(\frac{|I_N|}{|I_n|}\Big)^{D+\varepsilon},$$

which proves $\Delta \le D$. ∎

Using this lemma it is simple to show that the Φ-dimensions of μ_ω are almost surely constant.

Proposition 9. *For any dimension function Φ, the upper and lower Φ-dimensions are almost surely constant functions of ω.*

Proof. We show that $\omega \mapsto \dim_\Phi \mu_\omega$ is a permutable random variable (meaning that it is invariant under any finite permutation of the levels) and thus is almost surely constant by the Hewitt–Savage zero–one law [4]. To see this, let ω be fixed and $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation that fixes all but finitely many values. Suppose that N_0 is the largest such value. We use I to denote a Moran interval from the unpermuted construction and J for a Moran interval from the permuted construction. Then for any $n > N_0$ and choice $v_1, v_2, \ldots, v_n \in \{0,1\}$, it is clear from the description of the construction that $|I_{v_1 v_2 \cdots v_n}| = |J_{v_1 v_2 \ldots v_n}|$. Thus, the proposition follows from Lemma 8. ∎

4. Dimension results for large Φ

In this section we continue to use the notation and assumptions from Section 3.

4.1. Statement of the dimension theorem for large Φ and preliminary results

When computing the Φ-dimension of μ we need to compare ratios of lengths to ratios of mass under μ (as in equation (4.3)). The definitions of the random variables Y, Z, and G (given next) can be understood using this, as will be clear from the work in this section.

Notation 10. *Given $\theta \ge 0$, we define the iid random variables $Y_n(\theta), Z_n(\theta) : \Omega \to \mathbb{R}$ by*

$$Y_n(\theta)(\omega) = \begin{cases} \log p_n(\omega) & \text{if } p_n(\omega) \le \frac{a_n^\theta(\omega)}{a_n^\theta(\omega)+b_n^\theta(\omega)}, \\ \log(1-p_n(\omega)) & \text{if } p_n(\omega) > \frac{a_n^\theta(\omega)}{a_n^\theta(\omega)+b_n^\theta(\omega)} \end{cases}$$

and

$$Z_n(\theta)(\omega) = \begin{cases} \log a_n(\omega) & \text{if } p_n \le \frac{a_n^\theta(\omega)}{a_n^\theta(\omega)+b_n^\theta(\omega)}, \\ \log b_n(\omega) & \text{if } p_n > \frac{a_n^\theta(\omega)}{a_n^\theta(\omega)+b_n^\theta(\omega)}. \end{cases}$$

Random variables Y_n', Z_n' *are defined similarly, but with the relationship between* p_n *and* $a_n^\theta/(a_n^\theta + b_n^\theta)$ *interchanged. Put*

$$G(\theta) = \frac{\mathbb{E}_\omega(Y_1(\theta)(\omega))}{\mathbb{E}_\omega(Z_1(\theta)(\omega))} \quad \text{and} \quad G'(\theta) = \frac{\mathbb{E}_\omega(Y_1'(\theta)(\omega))}{\mathbb{E}_\omega(Z_1'(\theta)(\omega))}. \tag{4.1}$$

We have written $\mathbb{E}_\omega$ to emphasise that the expectation is taken over the variable ω.

The condition $p \le a^\theta/(a^\theta + b^\theta)$ is relevant because it is equivalent to $a^\theta/p \ge b^\theta/(1-p)$, an inequality very important for computing these dimensions.

It would be interesting to explore the properties of the functions $G(\theta)$ and $G'(\theta)$, and better understand them as objects in their own right. However, in this paper we mainly view these functions as technical tools that we use in our proofs. We do provide some discussion in Section 4.4.4 and plots for a few examples in Appendix A. We have also explicitly computed $G(\theta)$ in a few of the examples in Section 4.4.

With this notation we can now state our main result for large dimension functions Φ.

Theorem 11. *The following statements hold.*

(i) *Suppose* $G(\psi) < \psi$. *There is a set* $\Gamma_\psi \subseteq \Omega$, *of full measure in* Ω, *such that*

$$\overline{\dim}_\Phi \mu_\omega \le \psi$$

for all large dimension functions Φ *and all* $\omega \in \Gamma_\psi$.

(ii) *Suppose* $G(\psi) \ge \psi$. *There is a set* $\Gamma_\psi \subseteq \Omega$, *of full measure in* Ω, *such that*

$$\overline{\dim}_\Phi \mu_\omega \ge \psi$$

for all large dimension functions Φ *and all* $\omega \in \Gamma_\psi$.

(iii) *Suppose* $G'(\psi) > \psi$. *There is a set* $\Gamma_\psi \subseteq \Omega$, *of full measure in* Ω, *such that*

$$\underline{\dim}_\Phi \mu_\omega \ge \psi$$

for all large dimension functions Φ *and all* $\omega \in \Gamma_\psi$.

(iv) *Suppose* $G'(\psi) \le \psi$. *There is a set* $\Gamma_\psi \subseteq \Omega$, *of full measure in* Ω, *such that*

$$\underline{\dim}_\Phi \mu_\omega \le \psi$$

for all large dimension functions Φ *and all* $\omega \in \Gamma_\psi$.

An immediate corollary is as follows. Again, there is a corresponding statement for G' and the lower Φ-dimensions.

Corollary 12. *Suppose there is a choice of α such that $G(\alpha) = \alpha$ and $G(\psi) < \psi$ if $\psi > \alpha$. Then there is a set $\Gamma \subseteq \Omega$, of full measure in Ω, such that*

$$\overline{\dim}_\Phi \mu_\omega = \alpha$$

for all large dimension functions Φ and all $\omega \in \Gamma$.

Proof. From part (i) of Theorem 11 for each rational $q > \alpha$ we have a set Γ_q of full measure so that for all large dimension functions Φ and $\omega \in \Gamma_q$ we have $\overline{\dim}_\Phi \mu_\omega \leq q$. From part (ii) of the same theorem there is a set Γ_α of full measure so that for all large dimension functions Φ and $\omega \in \Gamma_\alpha$ we have $\overline{\dim}_\Phi \mu_\omega \geq \alpha$. Let

$$\Gamma = \Gamma_\alpha \cap \bigcap_{q>\alpha, q \text{ rational}} \Gamma_q,$$

which is also a subset of Ω of full measure. Then for any large dimension function Φ and $\omega \in \Gamma$, we have

$$\alpha \leq \overline{\dim}_\Phi \mu_\omega \leq \inf\{q : q > \alpha, q \text{ rational}\} = \alpha. \qquad \blacksquare$$

Of course, it is enough that $G(\psi_k) < \psi_k$ for a sequence (ψ_k) decreasing to α.

Corollary 13. *Let Φ be a large dimension function. Then $\alpha = \overline{\dim}_\Phi \mu_\omega$ almost surely if and only if $G(\psi) < \psi$ for all $\psi > \alpha$ and $G(\psi) \geq \psi$ for all $\psi < \alpha$.*

Proof. Suppose that $\alpha \geq 0$ is the almost sure value for $\overline{\dim}_\Phi \mu_\omega$ (which we know exists by Proposition 9). Take $\psi > \alpha$ and suppose that $G(\psi) \geq \psi$. Then by part (ii) of the theorem, $\overline{\dim}_\Phi \mu_\omega \geq \psi > \alpha$ almost surely, which is a contradiction. Thus, in fact $G(\psi) < \psi$. Similarly, if $\psi < \alpha$ but $G(\psi) < \psi$, then $\overline{\dim}_\Phi \mu_\omega \leq \psi < \alpha$ almost surely, which is another contradiction and so $G(\psi) \geq \psi$ in this case.

For the converse, suppose $G(\psi) < \psi$ for all $\psi > \alpha$ and $G(\psi) \geq \psi$ for all $\psi < \alpha$. Then for all $\psi > \alpha$ we have $\overline{\dim}_\Phi \mu_\omega \leq \psi$ almost surely and so $\overline{\dim}_\Phi \mu_\omega \leq \alpha$ almost surely. Similarly, for all $\psi < \alpha$ we have $\overline{\dim}_\Phi \mu_\omega \geq \psi$ almost surely and so $\overline{\dim}_\Phi \mu_\omega \geq \alpha$ almost surely. $\blacksquare$

What this last corollary shows, in particular, is that there must always be such a value α where G "crosses the diagonal" since for any given large Φ it is clear that $\overline{\dim}_\Phi \mu_\omega$ must have some almost sure value.

Before proving the theorem, we introduce further notation and establish some preliminary results. Given a large dimension function Φ, assume H and t_0 satisfy

$$\Phi(t) \geq \frac{H(t) \log|\log t|}{|\log t|} \text{ for all } 0 < t \leq t_0,$$

where $H(t) \uparrow \infty$ as $t \to 0$. Set

$$\zeta_N^H = \frac{H(B^N)\log(N|\log B|)}{|\log A|}. \tag{4.2}$$

Lemma 14. (i) *If $k < \zeta_N^H$, then for N sufficiently large there are no pairs of Moran subsets $I_N(x)$, $I_{N+k}(x)$ where*

$$|I_{N+k}(x)| \le |I_N(x)|^{1+\Phi(|I_N(x)|)}.$$

(ii) *Fix $c > 0$. If H is sufficiently large near 0, then $\sum_{N=1}^{\infty} \exp(-c\zeta_N^H) < \infty$.*

Proof. (i) Choose N_0 such that $B^{N_0} \le t_0$. Assume $N \ge N_0$ and for convenience put $r = |I_{N+k}(x)|$ and $R = |I_N(x)| \le B^N \le t_0$. Then

$$\begin{aligned}\Phi(R)|\log R| &\ge H(R)\log|\log R| \ge H(B^N)\log|N\log B| \\ &= \zeta_N^H|\log A| > k|\log A|,\end{aligned}$$

so $R^{\Phi(R)} < A^k$. As $r/R \ge A^k > R^{\Phi(R)}$, this means $r > R^{1+\Phi(R)}$, hence we cannot have $|I_{N+k}(x)| \le |I_N(x)|^{1+\Phi(|I_N(x)|)}$.

(ii) A straightforward calculation shows that if $H(B^N)$ is suitably large for $N \ge N_0$, then $\exp(-c\zeta_N^H) \le N^{-2}$ and hence $\sum_{N \ge N_0} \exp(-c\zeta_N^H) < \infty$. ∎

The next lemma is the key probabilistic result. It is based on the Chernov bounds for iid random variables $X_1, X_2, \dots, X_n$.

Theorem 15 (Chernov [24, Section 2.2]). *Let X_n be iid random variables and assume that $\mathbb{E}\big(\exp\big(t(X_i - \mathbb{E}(X_i))\big)\big) \le e^{ct^2/2}$ for some $c > 0$ and $t > 0$. Then for all $0 < \eta \le nct$ we have*

$$\mathcal{P}\left(\left|\sum_{i=1}^{n} X_i - n\mathbb{E}(X_i)\right| \ge \eta\right) \le 2e^{\eta^2/(2nc)}.$$

Lemma 16 (Probabilistic Result). *Fix any $\theta \ge 0$ and $\delta > 0$. If the constant function H is sufficiently large and ζ_N^H is defined as in (4.2), then*

$$\mathcal{P}\left\{\omega\colon \exists m \ge \zeta_N^H \text{ with } \left|\frac{\sum_{n=N+1}^{N+m} Y_n(\theta)(\omega)}{\sum_{n=N+1}^{N+m} Z_n(\theta)(\omega)} - \frac{\mathbb{E}(Y_1(\theta))}{\mathbb{E}(Z_1(\theta))}\right| > \delta \text{ i.o.}\right\} = 0.$$

A similar statement holds for Y_n', Z_n'.

Proof. Since the function $f(x, y) = x/y$ is continuous at $(\mathbb{E}(Y_1(\theta)), \mathbb{E}(Z_1(\theta)))$, for the given $\delta > 0$ there is some $\eta = \eta(\delta) > 0$ such that when both inequalities

$$\left|\frac{1}{m}\sum_{n=N+1}^{N+m} Y_n(\theta)(\omega) - \mathbb{E}(Y_1(\theta))\right| \le \eta \quad \text{and} \quad \left|\frac{1}{m}\sum_{n=N+1}^{N+m} Z_n(\theta)(\omega) - \mathbb{E}(Z_1(\theta))\right| \le \eta$$

hold, then

$$\left|\frac{\sum_{n=N+1}^{N+m} Y_n}{\sum_{n=N+1}^{N+m} Z_n} - \frac{\mathbb{E}(Y_1)}{\mathbb{E}(Z_1)}\right| \leq \delta.$$

Since we have assumed $\mathbb{E}(e^{-t\log p_n}), \mathbb{E}(e^{-t\log(1-p_n)}) < \infty$, Chernov's inequality implies there are constants C and $c > 0$ such that for all m,

$$\mathcal{P}\left\{\omega: \left|\frac{1}{m}\sum_{n=N+1}^{N+m} Y_n - \mathbb{E}(Y_1)\right| > \eta\right\} \leq Ce^{-cm}.$$

Applying Lemma 14 (ii), we know $\sum_N e^{-c\zeta_N^H} < \infty$ if H is sufficiently large. Thus, if we let

$$\Gamma_{N,\eta} = \left\{\omega: \exists m \geq \zeta_N^H \text{ with } \left|\frac{1}{m}\sum_{n=N+1}^{N+m} Y_n - \mathbb{E}(Y_1)\right| > \eta\right\},$$

then for a new constant C_1,

$$\sum_{N=1}^{\infty} \mathcal{P}(\Gamma_{N,\eta}) \leq \sum_N \sum_{m=\zeta_N^H}^{\infty} Ce^{-cm} \leq \sum_N C_1 e^{-c\zeta_N^H} < \infty.$$

By the Borel–Cantelli lemma this means $\mathcal{P}(\Gamma_{N,\eta} \text{ i.o.}) = 0$.

Similarly, if we let $\Gamma'_{N,\eta} = \{\omega: \exists m \geq \zeta_N^H \text{ with } |\sum_{n=N+1}^{N+m} Z_n - \mathbb{E}(Z_1)| > \eta\}$, then for a suitable choice of H we have $\mathcal{P}(\Gamma'_{N,\eta} \text{ i.o.}) = 0$.

Hence, there is a set $\Omega(\eta)$, of full measure, with the property that for each $\omega \in \Omega(\eta)$ there is some $N_\eta = N_\eta(\omega)$ such that for all $N \geq N_\eta$ and all $m \geq \zeta_N^H$, we have both

$$\left|\frac{1}{m}\sum_{n=N+1}^{N+m} Y_n - \mathbb{E}(Y_1)\right| \leq \eta \quad \text{and} \quad \left|\frac{1}{m}\sum_{n=N+1}^{N+m} Z_n - \mathbb{E}(Z_1)\right| \leq \eta.$$

Therefore,

$$\left|\frac{\sum_{n=N+1}^{N+m} Y_n}{\sum_{n=N+1}^{N+m} Z_n} - \frac{\mathbb{E}(Y_1)}{\mathbb{E}(Z_1)}\right| \leq \delta.$$

That completes the proof. ■

4.2. Proof of Theorem 11

Proof of Theorem 11. (i) For each positive integer j, let

$$\Phi_j(t) = \frac{j\log|\log t|}{|\log t|} \quad \text{and} \quad \zeta_N^j = \frac{j\log(N|\log B|)}{|\log A|}.$$

Consider any $N, m \in \mathbb{N}$, $\psi > 0$, Moran interval $I_N(\omega)$ and descendent interval $I_{N+m}(\omega)$. If $I_N = I_v$ for $v = v_1 \cdots v_N$ with $v_i \in \{0, 1\}$ and $I_{N+m} = I_{vv_{N+1}\cdots v_{N+m}}$, then

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_{N+m})} = \left(\prod_{\substack{v_{N+i}=0,\\ i=1,\ldots,m}} p_{N+i}(\omega) \cdot \prod_{\substack{v_{N+i}=1,\\ i=1,\ldots,m}} (1 - p_{N+i}(\omega)) \right)^{-1}$$

and

$$\frac{|I_N|}{|I_{N+m}|} = \left(\prod_{\substack{v_{N+i}=0,\\ i=1,\ldots,m}} a_{N+i}(\omega) \cdot \prod_{\substack{v_{N+i}=1,\\ i=1,\ldots,m}} b_{N+i}(\omega) \right)^{-1}.$$

Thus, for any ψ,

$$\begin{aligned} \frac{\frac{\mu_\omega(I_N)}{\mu_\omega(I_{N+m})}}{\left(\frac{|I_N|}{|I_{N+m}|}\right)^\psi} &= \left(\prod_{\substack{v_{N+i}=0,\\ i=1,\ldots,m}} \frac{a_{N+i}^\psi}{p_{N+i}} \right) \left(\prod_{\substack{v_{N+i}=1,\\ i=1,\ldots,m}} \frac{b_{N+i}^\psi}{1 - p_{N+i}} \right) \\ &\leq \prod_{i=N+1}^{N+m} \max\left(\frac{a_i^\psi(\omega)}{p_i(\omega)}, \frac{b_i^\psi(\omega)}{1 - p_i(\omega)} \right). \end{aligned} \tag{4.3}$$

Now

$$\frac{a_i^\psi}{p_i} \geq \frac{b_i^\psi}{1 - p_i} \text{ if and only if } p_i \leq \frac{a_i^\psi}{a_i^\psi + b_i^\psi},$$

hence

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_{N+m})} \leq \left(\frac{|I_N|}{|I_{N+m}|} \right)^\psi$$

if

$$\left(\prod_{\substack{i=N+1,\ldots,N+m;\\ p_i \leq a_i^\psi/(a_i^\psi + b_i^\psi)}} \frac{a_i^\psi}{p_i} \right) \left(\prod_{\substack{i=N+1,\ldots,N+m;\\ p_i > a_i^\psi/(a_i^\psi + b_i^\psi)}} \frac{b_i^\psi}{(1 - p_i)} \right) \leq 1. \tag{4.4}$$

Taking logarithms, we see that (4.4) is equivalent to the statement

$$\psi \geq \frac{\sum_{i=N+1}^{N+m} Y_i(\psi)(\omega)}{\sum_{i=N+1}^{N+m} Z_i(\psi)(\omega)}.$$

Finally, assume $G(\psi) < \psi$, say $G(\psi) \leq \psi - 2\delta$ for some $\delta > 0$. According to the probabilistic result, Lemma 16, there is a set $\Omega_{j,\psi}$, depending on both j and ψ and of full measure in Ω, such that for each $\omega \in \Omega_{j,\psi}$ there is some integer $N_j = N_j(\omega)$ such that for all $N \geq N_j$ and all $m \geq \zeta_N^j$,

$$\left| \frac{\sum_{n=N+1}^{N+m} Y_n(\psi)(\omega)}{\sum_{n=N+1}^{N+m} Z_n(\psi)(\omega)} - \frac{\mathbb{E}(Y_1(\psi))}{\mathbb{E}(Z_1(\psi))} \right| = \left| \frac{\sum_{n=N+1}^{N+m} Y_n(\psi)(\omega)}{\sum_{n=N+1}^{N+m} Z_n(\psi)(\omega)} - G(\psi) \right| \leq \delta.$$

Consequently,

$$\frac{\sum_{n=N+1}^{N+m} Y_n(\psi)(\omega)}{\sum_{n=N+1}^{N+m} Z_n(\psi)(\omega)} \leq G(\psi) + \delta \leq \psi - \delta < \psi.$$

Thus, our previous observations imply that for each $\omega \in \Omega_{j,\psi}$ there is an integer N_j such that for all $N \geq N_j$ and all $m \geq \zeta_N^j$,

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_{N+m})} \leq \left(\frac{|I_N|}{|I_{N+m}|}\right)^{\psi}. \tag{4.5}$$

Next, suppose $\omega \in \Omega_{j,\psi}$, $N_j = N_j(\omega)$ is as above and $x \in \mathcal{C}_\omega$. Choose $N \geq N_j$ and m so that $|I_N(x)| < A^{N_j+L}$ and $|I_{N+m}(x)| \leq |I_N(x)|^{1+\Phi(|I_N(x)|)}$. Then Lemma 14 (i) implies $m \geq \zeta_N^j$ and so by (4.5) and Lemma 8 we know that $\overline{\dim}_\Phi \mu_\omega \leq \psi$ for all $\omega \in \Omega_{j,\psi}$.

Now, let Φ be any large dimension function and

$$\omega \in \Gamma_\psi = \bigcap_{j=1}^{\infty} \Omega_{j,\psi},$$

again a set of full measure. There exists j such that $\Phi(t) \geq \Phi_j(t)$ for t sufficiently close to 0. As $\omega \in \Omega_{j,\psi}$, $\overline{\dim}_\Phi \mu_\omega \leq \overline{\dim}_{\Phi_j} \mu_\omega \leq \psi$. It follows that $\overline{\dim}_\Phi \mu_\omega \leq \psi$ for all $\omega \in \Gamma_\psi$ and all large dimension functions Φ.

(ii) Given ω, consider the Moran intervals which arise by choosing the left child at step n if

$$\frac{a_n^\psi(\omega)}{p_n(\omega)} = \max\left(\frac{a_n^\psi(\omega)}{p_n(\omega)}, \frac{b_n^\psi(\omega)}{1 - p_n(\omega)}\right)$$

and the right child otherwise. Call the interval at step n which arises by this construction $I_n = I_n(\psi, \omega)$. These form a nested sequence of Moran intervals.

For $n > N$,

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_n)} = \prod_{\substack{i=N+1 \\ p_i \leq a_i^\psi/(a_i^\psi + b_i^\psi)}}^{n} p_i^{-1} \prod_{\substack{i=N+1 \\ p_i > a_i^\psi/(a_i^\psi + b_i^\psi)}}^{n} (1 - p_i)^{-1}$$

and

$$\frac{|I_N|}{|I_n|} = \prod_{p_i \leq a_i^\psi/(a_i^\psi + b_i^\psi)} a_i^{-1} \prod_{p_i > a_i^\psi/(a_i^\psi + b_i^\psi)} b_i{}^{-1}.$$

Thus, for any $\beta > 0$,

$$\frac{\mu_\omega(I)}{\mu_\omega(I_n)} \geq \left(\frac{|I_N|}{|I_n|}\right)^{\beta}$$

if and only if

$$\sum_{i=N+1}^{n} Y_i(\psi)(\omega) \leq \beta \sum_{i=N+1}^{n} Z_i(\psi)(\omega)$$

if and only if (writing $n = N + m$)

$$\frac{\sum_{i=N+1}^{N+m} Y_i(\psi)(\omega)}{\sum_{i=N+1}^{N+m} Z_i(\psi)(\omega)} \geq \beta.$$

Fix $\delta > 0$ and choose the constant function $H = H(\delta)$ so large that Lemma 16 guarantees that there is a set $\Omega_{\delta,\psi}$, of full measure, such that for all $\omega \in \Omega_{\delta,\psi}$ and N sufficiently large,

$$\left| \frac{\sum_{i=N+1}^{N+m} Y_i(\psi)(\omega)}{\sum_{i=N+1}^{N+m} Z_i(\psi)(\omega)} - G(\psi) \right| \leq \delta \text{ for all } m \geq \zeta_N^H,$$

and hence

$$\frac{\sum_{i=N+1}^{N+m} Y_i}{\sum_{i=N+1}^{N+m} Z_i} \geq G(\psi) - \delta \geq \psi - \delta.$$

It follows that

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_{N+m})} \geq \left(\frac{|I_N|}{|I_{N+m}|} \right)^{\psi-\delta}$$

for all $\omega \in \Omega_{\delta,\psi}$, $m \geq \zeta_N^H$ and N sufficiently large.

Now, take $\delta_j = 1/j$, let $H(\delta_j) = H_j$ and $\Omega_j = \Omega_{\delta_j,\psi}$. Let $\Gamma_\psi = \bigcap_j \Omega_j$, a set of full measure. As Φ is a large dimension function, for any j there exists $t_j > 0$ such that $\Phi(t) = H(t) \log|\log t|/|\log t|$ where $H(t) \geq H_j$ for $t \leq t_j$. Consequently, for large N, $\zeta_N^H \geq \zeta_N^{H_j}$. If $\omega \in \Gamma_\psi$, then $\omega \in \Omega_j$ and therefore

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_{N+\zeta_N^H})} \geq \left(\frac{|I_N|}{|I_{N+\zeta_N^H}|} \right)^{\psi-1/j}$$

for all N sufficiently large. It follows that for all $\omega \in \Gamma_\psi$, $\overline{\dim}_\Phi \mu_\omega \geq \psi - 1/j$ and since this is true for all j, we must have $\overline{\dim}_\Phi \mu_\omega \geq \psi$ as claimed.

The arguments for the lower Φ-dimension are very similar, but rather than considering $\max\{a_n^\psi/p_n, b_n^\psi/(1-p_n)\}$, we study $\min\{a_n^\psi/p_n, b_n^\psi/(1-p_n)\}$. Thus, the functions Y_n', Z_n' and G' arise in place of Y_n, Z_n and G. The details are left for the reader. ∎

4.3. Consequences of Theorem 11

We continue to use the notation introduced earlier. In particular, G is as defined in (4.1). Since positive constant functions are large dimension functions, the following corollary follows directly from the theorem.

Corollary 17. (i) *If* $G(\psi) < \psi$, *then* $\dim_{qA} \mu_\omega \leq \psi$ *a.s.*

(ii) *If* $G(\psi) \geq \psi$, *then* $\dim_{qA} \mu_\omega \geq \psi$ *a.s.*

Similar statements hold for G' and the quasi-lower Assouad dimension.

A useful fact, which we show below, is that continuous functions G (or G') typically satisfy the hypotheses of Corollaries 12 and 13. This is often the situation, cf. (4.7), where it is shown that G is even differentiable when $a_n = a$, $b_n = b$ and p_n is uniformly distributed over [0, 1]. More generally, G is continuous if p_n has a density distribution of the form $f(t)dt$, where $f(t)\log t$ and $f(t)\log(1-t)$ are integrable over [0, 1], such as when f is bounded.

Lemma 18. *Assume* $|\mathbb{E}(\log p_1)|, |\mathbb{E}(\log(1-p_1))| < \infty$. *If* $G(\theta)$ *is continuous, then there is a unique choice of* α *such that* $G(\alpha) = \alpha$ *and* $G(\psi) < \psi$ *if* $\psi > \alpha$.

Proof. We will assume that $\mathcal{P}(a_n = b_n) = 0$ and leave the contrary case to the reader. Note that as $\theta \to \infty$,

$$\frac{a^\theta}{a^\theta + b^\theta} \to \begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a > b \end{cases}$$

and therefore

$$Y_1(\theta)(\omega) \to \begin{cases} \log(1-p_1(\omega)) & \text{if } a_1(\omega) < b_1(\omega) \\ \log p_1(\omega) & \text{if } a_1(\omega) > b_1(\omega) \end{cases} \quad \text{as } \theta \to \infty$$

and

$$Z_1(\theta)(\omega) \to \begin{cases} \log b_1(\omega) & \text{if } a_1(\omega) < b_1(\omega) \\ \log a_1(\omega) & \text{if } a_1(\omega) > b_1(\omega) \end{cases} \quad \text{as } \theta \to \infty.$$

Hence,

$$G(\theta) \to \frac{\mathcal{P}(a_1 < b_1)\mathbb{E}(\log(1-p_1)) + \mathcal{P}(a_1 > b_1)\mathbb{E}(\log p_1)}{\mathcal{P}(a_1 < b_1)\mathbb{E}(\log b_1) + \mathcal{P}(a_1 > b_1)\mathbb{E}(\log a_1)} \text{ as } \theta \to \infty.$$

In particular, G approaches a (finite) constant as $\theta \to \infty$.

On the other hand,

$$G(0) = \frac{\mathbb{E}(\log p_1|_{p_1 \leq 1/2}) + \mathbb{E}(\log(1-p_1)|_{p_1 > 1/2})}{\mathcal{P}(p_1 \leq 1/2)\mathbb{E}(\log a_1) + \mathcal{P}(p_1 > 1/2)\mathbb{E}(\log b_1)} \; 0.$$

Since G is continuous, $G(0) > 0$ and eventually $G(\theta) < \theta$, there must be a unique choice of α such that $G(\alpha) = \alpha$ and if $\psi > \alpha$, then $G(\psi) < \psi$. ■

Corollary 19. *Suppose* $|\mathbb{E}(\log p_1)|$, $|\mathbb{E}(\log(1-p_1))| < \infty$ *and* $G(\theta)$ *is continuous. Then* $\overline{\dim}_\Phi \mu_\omega = \alpha$ *a.s., where* $G(\alpha) = \alpha$ *and* $G(\psi) < \psi$ *for all* $\psi > \alpha$.

The upper Φ-dimension of μ is always an upper bound for the upper local dimension of μ at any point x, where the latter is defined by

$$\overline{\dim}_{\mathrm{loc}} \mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}. \tag{4.6}$$

Similarly, the lower Φ-dimension of μ is always a lower bound for the lower local dimension (defined as in (4.6) but using a liminf). However, in general it is possible for $\underline{\dim}_\Phi \mu < \inf_x \underline{\dim}_{\mathrm{loc}} \mu(x)$ and $\sup_x \overline{\dim}_{\mathrm{loc}} \mu(x) < \overline{\dim}_\Phi \mu$. (See [16] for proofs of these statements.) In the case of our random measures, there is no gap for either inequality.

Proposition 20. *Assume* $G(\psi) < \psi$ *for all* $\psi > \theta$ *and* $G(\theta) = \theta$. *Then for any large dimension function* Φ *and almost all* ω *we have that*

$$\sup_x \overline{\dim}_{\mathrm{loc}} \mu_\omega(x) = \theta = \overline{\dim}_\Phi \mu_\omega.$$

Similarly, if $G'(\psi) > \psi$ *for all* $\psi < \theta'$ *and* $G'(\theta') = \theta'$, *then for any large dimension function* Φ *and almost all* ω *we have that*

$$\inf_x \underline{\dim}_{\mathrm{loc}} \mu_\omega(x) = \theta' = \underline{\dim}_\Phi \mu_\omega.$$

Proof. Put $v_j = 0$ if $a_j^\theta / p_j = \max(a_j^\theta / p_j, b_j^\theta/(1-p_j))$ and $v_j = 1$ else. Let $x \in \bigcap_{n=1}^\infty I_{v_1,\dots,v_n}$, so that $I_n(x) = I_{v_1\dots,v_n}$ for each n. Given any small $r > 0$, choose n such that $|I_n(x)| \le r < |I_{n-1}(x)|$, so that $I_n(x) \subseteq B(x,r) \le I_{n-L}(x)$.

We have

$$\mu_\omega(B(x,r)) \le \mu_\omega(I_{n-L}(x)) = \prod_{j=1; v_j=0}^{n-L} p_j \prod_{j=1; v_j=1}^{n-L} (1-p_j),$$

so

$$|\log \mu_\omega(B(x,r))| \ge \left| \sum_{i=1}^{n-L} Y_i(\theta) \right|.$$

Similarly,

$$r \ge \prod_{\substack{j=1;\\ v_j=0}}^{n} a_j \prod_{\substack{j=1;\\ v_j=1}}^{n} b_j = \left(\prod_{\substack{j=1;\\ v_j=0}}^{n-L} a_j \prod_{\substack{j=1;\\ v_j=1}}^{n-L} b_j \right) \left(\prod_{\substack{j=n-L+1;\\ v_j=0}}^{n} a_j \prod_{\substack{j=n-L+1;\\ v_j=1}}^{n} b_j \right).$$

As a_j, b_j are bounded away from 0 and L is fixed, there is some constant $C > 0$ such that

$$|\log r| \leq \left| \sum_{i=1}^{n-L} Z_i(\theta) \right| + C.$$

Hence,

$$\frac{|\log \mu_\omega(B(x,r))|}{|\log r|} \geq \frac{\left|\sum_{i=1}^{n-L} Y_i(\theta)\right|}{\left|\sum_{i=1}^{n-L} Z_i(\theta)\right| + C}.$$

Fix $\varepsilon > 0$ and choose a set of full measure, Ω_ε, such that

$$\frac{\sum_{i=1}^{m} Y_i(\theta)(\omega)}{\sum_{i=1}^{m} Z_i(\theta)(\omega)} \geq G(\theta) - \varepsilon$$

for $m \geq m_\omega$ and each $\omega \in \Omega_\varepsilon$. Further, as $|\sum_{i=1}^{m} Z_i| \to \infty$ as $m \to \infty$, given any $\delta > 0$ we can choose m_0 sufficiently large so that we have $C|\sum_{i=1}^{m} Z_i(\theta)|^{-1} \leq \delta$ for all $m \geq m_0$. Thus, for all $\omega \in \Omega_\varepsilon$ and all $n \geq \max(m_0, m_\omega) + L$,

$$\frac{\left|\sum_{i=1}^{n-L} Y_i(\theta)(\omega)\right|}{\left|\sum_{i=1}^{n-L} Z_i(\theta)(\omega)\right| + C} \geq \frac{G(\theta) - \varepsilon}{1 + C\left|\sum_{i=1}^{n-L} Z_i(\theta)(\omega)\right|^{-1}} \geq \frac{\theta - \varepsilon}{1 + \delta}.$$

If we make the choice of δ sufficiently small, depending on θ, then we can conclude that

$$\frac{|\log \mu_\omega(B(x,r))|}{|\log r|} \geq \theta - 2\varepsilon$$

for sufficiently small r. By choosing the sequence $\varepsilon = 1/k$ and putting $\Omega_0 = \bigcap_{k=1}^{\infty} \Omega_{1/k}$, we deduce that for all $\omega \in \Omega_0$, a set of full measure

$$\overline{\dim}_{\mathrm{loc}} \mu_\omega(x) = \limsup_{r \to 0} \frac{|\log \mu(B(x,r))|}{|\log r|} \geq \theta.$$

The claim follows since it is always true that the supremum of the upper local dimensions is dominated by $\overline{\dim}_\Phi \mu$ for any dimension function Φ (see [16]) which, according to Corollary 12, is equal to θ almost everywhere.

The statement about the lower local dimension and G' is proved in an analogous manner. ∎

4.4. Example: The deterministic Moran set $\mathcal{C}_{ab}$

Consider the deterministic Moran set $\mathcal{C}_{ab}$, which can be viewed as a random Moran set where (a_n, b_n) is chosen from the singleton $\{(a, b)\}$.

4.4.1. Choosing p_n uniformly over [0, 1]. Suppose p_n has the uniform distribution over [0, 1]. Then, we have

$$\begin{aligned}\mathbb{E}(Y(\theta)(\omega)) &= \int\limits_{p_n(\omega)\le a^\theta/(a^\theta+b^\theta)} \log p_n(\omega)d\mathbb{P}(\omega) + \int\limits_{p_n(\omega)>a^\theta/(a^\theta+b^\theta)} \log(1-p_n(\omega))d\mathbb{P}(\omega)\\ &= \frac{a^\theta}{a^\theta+b^\theta}\log\Big(\frac{a^\theta}{a^\theta+b^\theta}\Big) + \frac{b^\theta}{a^\theta+b^\theta}\log\Big(\frac{b^\theta}{a^\theta+b^\theta}\Big) - 1\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(Z(\theta)(\omega)) &= (\log a)\mathcal{P}\big(p_n \le a^\theta/(a^\theta+b^\theta)\big) + (\log b)\mathcal{P}\big(p_n > a^\theta/(a^\theta+b^\theta)\big)\\ &= \frac{a^\theta}{a^\theta+b^\theta}\log a + \frac{b^\theta}{a^\theta+b^\theta}\log b.\end{aligned}$$

Consequently,

$$G(\theta) = \frac{a^\theta\log\Big(\frac{a^\theta}{a^\theta+b^\theta}\Big) - a^\theta + b^\theta\log\Big(\frac{b^\theta}{a^\theta+b^\theta}\Big) - b^\theta}{a^\theta\log a + b^\theta\log b}. \tag{4.7}$$

One can clearly see that G is a continuous function (even differentiable) and so Corollary 19 applies.

Choose γ so that $a = b^\gamma$. Then $G(\theta) = \theta$ if and only if

$$\begin{aligned}-(b^{\theta\gamma}+b^\theta)\log(1+b^{\theta(1-\gamma)}) - (b^{\theta\gamma}+b^\theta) + \theta b^\theta(1-\gamma)\log b\\ = \theta(\log b)(\gamma b^{\theta\gamma}+b^\theta)\end{aligned}$$

if and only if

$$-(b^{\theta\gamma}+b^\theta)\log(1+b^{\theta(1-\gamma)}) - (b^{\theta\gamma}+b^\theta) = \theta\gamma(\log b)(b^{\theta\gamma}+b^\theta).$$

Dividing through by $b^{\theta\gamma}+b^\theta$, this is equivalent to the statement

$$\log(1+b^{\theta(1-\gamma)}) + 1 = -\gamma\theta\log b.$$

Taking the exponential of both sides, it follows that $G(\theta) = \theta$ if and only if

$$b^\theta + b^{\theta\gamma} - e^{-1} = 0.$$

Example 1. Suppose $\mathcal{C}_{ab}$ is the deterministic Moran set with $a = b^2$, p_n is uniformly distributed over [0, 1] and μ is the corresponding random measure. The analysis above shows that $G(\theta) = \theta$ if and only if

$$b^{2\theta} + b^\theta - e^{-1} = 0,$$

equivalently, $b^\theta = \big(-1 \pm \sqrt{1+4e^{-1}}\big)/2$. Hence, according to Corollary 19, for all large Φ,

$$\overline{\dim}_\Phi \mu = \frac{\log\Big(\frac{\sqrt{1+4e^{-1}}-1}{2}\Big)}{\log b} \text{ a.s.}$$

For example, if $b = 1/2$ and $a = 1/4$, then $\overline{\dim}_\Phi \mu \approx (1.25)/\log 2$.

It is interesting that the ratio of $\overline{\dim}_\Phi \mu$ to $\dim_H \mathcal{C}_{ab}$ is constant (and approximately 2.60) for these measures. We see this since

$$\dim_H \mathcal{C}_{ab} = \frac{\log(\sqrt{5}/2 - 1/2)}{\log b}$$

is the non-negative solution to $b^{2d} + b^d = 1$. (We note that because of self-similarity and the separation condition, all the "usual" dimensions of $\mathcal{C}_{ab}$ agree with the similarity dimension.)

Example 2. We continue with $\mathcal{C}_{ab}$ as the deterministic Moran set with p_n being drawn uniformly from [0, 1] and with μ as the corresponding random measure. In Figure 1 (obtained by numerically solving $G(\theta) = \theta$) we show the almost sure upper Φ-dimension (for large Φ) of μ on $\mathcal{C}_{ab}$ as a function of (a, b) where, for this figure, we draw (a, b) from the set

$$\Lambda = \big\{(a, b)\colon 1/50 \leq \min\{a, b\} \leq a + b \leq 49/50\big\}.$$

It is notable that the dimension is a continuous function of $(a, b) \in \Lambda$, and it appears to increase as either $a \to 1$ or $b \to 1$ and vanish as a and b both tend to 0. In fact, that is indeed the case as we now argue.

For our discussion, let D_{ab} be the almost sure upper Φ-dimension of the random measure μ for the large Φ case. What we wish to show is that $D_{ab} \to 0$ as a and b tend to 0 and $D_{ab} \to \infty$ as a or b tend to 1.

Proof. From (4.7) we have

$$G(\theta) = \frac{a^\theta \log\Big(\frac{a^\theta}{a^\theta+b^\theta}\Big) - a^\theta + b^\theta \log\Big(\frac{b^\theta}{a^\theta+b^\theta}\Big) - b^\theta}{a^\theta \log a + b^\theta \log b}$$

and thus $G(\theta) \geq \theta$ if and only if

$$a^\theta \log\Big(\frac{a^\theta}{a^\theta + b^\theta}\Big) - a^\theta + b^\theta \log\Big(\frac{b^\theta}{a^\theta + b^\theta}\Big) - b^\theta \leq \theta\big(a^\theta \log a + b^\theta \log b\big).$$

After some simplification, we see that this happens precisely when

$$(a^\theta + b^\theta)\big(1 + \log(a^\theta + b^\theta)\big) \geq 0$$

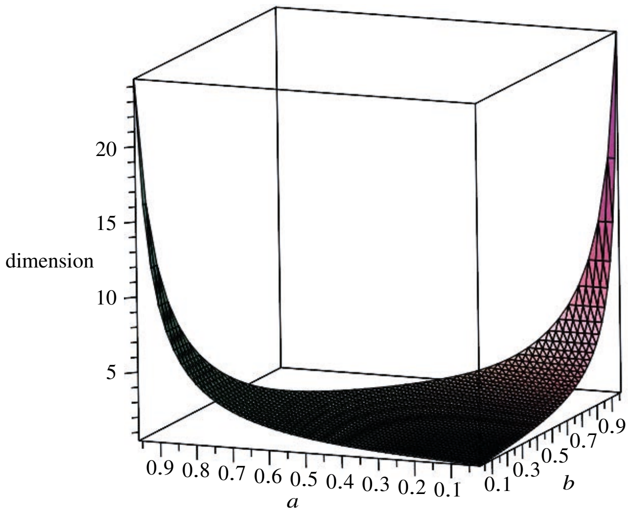

Figure 1. $\overline{\dim}_\Phi \mu$ as a function of (a, b) for μ on C_{ab} with $p \sim U[0, 1]$.

which, since $a^\theta + b^\theta > 0$, is equivalent to

$$a^\theta + b^\theta \geq e^{-1}.$$

Suppose $\theta < \infty$. Then this inequality will clearly hold once either a or b is sufficiently close to 1. Consequently, Theorem 11 (ii) implies $D_{ab} \geq \theta$ if either a or b is sufficiently large. Since θ was arbitrary, it follows that D_{ab} tends to infinity as either a or b tend to 1.

On the other hand, if $\theta > 0$ and a, b are both sufficiently small, then $a^\theta + b^\theta < e^{-1}$ and hence $G(\theta) < \theta$. Consequently, Theorem 11 (i) implies that $D_{ab} < \theta$ and hence $D_{ab} \to 0$ as both $a, b \to 0$. ∎

4.4.2. Choosing p_n from the two-element set $\{p, 1 - p\}$ for fixed $0 < p < 1/2$. For our next example, we consider the deterministic Moran set $\mathcal{C}_{ab}$, but with p_n chosen from a two-element set.

Example 3. Consider the deterministic Moran set $\mathcal{C}_{ab}$ with $a < b$, but in this case let μ be the random measure with probability p_n chosen uniformly from the two values p or $1 - p$, where $0 < p < 1/2$ is fixed. Define β and η by $a^\beta = b$ and $(1 - p)^\eta = p$, so $\beta < 1$ and $\eta > 1$. We claim

$$\overline{\dim}_\Phi \mu_\omega = \begin{cases} \frac{\log p + \log(1-p)}{2 \log b} & \text{if } \eta + 1 + \beta - 3\eta\beta \geq 0, \\ \frac{\log p}{\frac{1}{2}(\log a + \log b)} & \text{if } \eta + 1 + \beta - 3\eta\beta < 0 \end{cases} \quad \text{a.s.} \tag{4.8}$$

Proof. Let $c(\theta) = a^\theta/(a^\theta + b^\theta)$. Note that $c(\theta)$ is a decreasing function and for $\theta \geq 0$, $c(\theta) \leq 1/2$. In particular, there is no non-negative solution to $c(\theta) = 1 - p > 1/2$. Let θ_0 satisfy $c(\theta_0) = p$, so

$$\theta_0 = \frac{\log((1-p)/p)}{\log(b/a)} = \frac{(\eta - 1)\log(1-p)}{(1-\beta)\log a}.$$

If $\theta \geq \theta_0$, then both $p, 1-p \geq c(\theta)$, so $Y_n = \log(1-p_n)$ and $Z_n = \log b$.

If $0 \leq \theta < \theta_0$, then $p \leq c(\theta) < 1-p$, hence if $p_n = p$, then $Y_n = \log p_n = \log p$ and $Z_n = \log a$, while if $p_n = 1-p$, then $Y_n = \log(1-p_n) = \log p$ and $Z_n = \log b$.

It is easy to see from these observations that

$$\mathbb{E}(Y_1) = \begin{cases} \frac{1}{2}(\log p + \log(1-p)) & \text{if } \theta \geq \theta_0, \\ \log p & \text{if } 0 \leq \theta < \theta_0 \end{cases}$$

and

$$\mathbb{E}(Z_1) = \begin{cases} \log b & \text{if } \theta \geq \theta_0, \\ \frac{1}{2}(\log a + \log b) & \text{if } 0 \leq \theta < \theta_0. \end{cases}$$

Hence,

$$G(\theta) = \begin{cases} \frac{\log p + \log(1-p)}{2\log b} & \text{if } \theta \geq \theta_0, \\ \frac{\log p}{\frac{1}{2}(\log a + \log b)} & \text{if } 0 \leq \theta < \theta_0. \end{cases}$$

Replacing p by $(1-p)^\eta$ and b by a^β, this is the same as stating

$$G(\theta) = \begin{cases} \frac{(\eta+1)\log(1-p)}{2\beta\log a} & \text{if } \theta \geq \frac{(\eta-1)\log(1-p)}{(1-\beta)\log a}, \\ \frac{\eta\log(1-p)}{\frac{1}{2}(\beta+1)\log a} & \text{if } 0 \leq \theta < \frac{(\eta-1)\log(1-p)}{(1-\beta)\log a}. \end{cases}$$

It is easy to check that if $\eta + 1 + \beta - 3\eta\beta < 0$, then

$$\frac{\eta\log(1-p)}{\frac{1}{2}(1+\beta)\log a} < \frac{(\eta-1)\log(1-p)}{(1-\beta)\log a},$$

so $G(\alpha) = \alpha$ for

$$\alpha = \frac{\eta\log(1-p)}{\frac{1}{2}(\beta+1)\log a} = \frac{\log p}{\frac{1}{2}(\log a + \log b)}.$$

If $\alpha < \psi < \theta_0$, then obviously $G(\psi) = \alpha < \psi$. If $\psi \geq \theta_0 > \alpha$, then one can also check that $\eta + 1 + \beta - 3\eta\beta < 0$ implies

$$\frac{(\eta+1)\log(1-p)}{2\beta\log a} < \frac{\eta\log(1-p)}{\frac{1}{2}(\beta+1)\log a},$$

so again we have $G(\psi) < \alpha < \psi$.

Similarly, if $\eta + 1 + \beta - 3\eta\beta \geq 0$, then

$$\frac{(\eta+1)\log(1-p)}{2\beta\log a} \geq \frac{(\eta-1)\log(1-p)}{(1-\beta)\log a},$$

hence $G(\alpha) = \alpha$ for

$$\alpha = \frac{(\eta+1)\log(1-p)}{2\beta\log a} = \frac{\log p + \log(1-p)}{2\log b},$$

and if $\psi > \alpha$, then $G(\psi) = \alpha < \psi$.

It follows from Theorem 11 that $\overline{\dim}_\Phi \mu$ is as claimed in (4.8). ■

4.4.3. Taking p_n deterministic and (a, b) random. For our last two examples we now take $p_n = 1/2$ and choose the scalings a_n and b_n randomly.

Example 4. Consider the random Moran set, $\mathcal{C}_\omega$, where a_n, b_n are chosen independently from $\{A, B\}$ with equal likelihood and $0 < A < B < 1/2$. Let μ_ω be the random measure supported on $\mathcal{C}_\omega$, where $p_n = 1/2$ for all n. Obviously, $Y_n(\theta) = \log 1/2$ for all n and all θ. Note that the condition $p_n \leq a_n^\theta/(a_n^\theta + b_n^\theta)$ simply reduces to the inequality $a_n \geq b_n$ and this is true whenever $a_n = A$ or $a_n = b_n = A$. Thus, for all n and θ,

$$\begin{aligned}\mathbb{E}(Z_n(\theta)) &= \int_{\{a_n=B, a_n=b_n=A\}} \log a_n + \int_{\{a_n=A, b_n=B\}} \log b_n \\ &= \frac{1}{2}\log B + \frac{1}{4}\log A + \frac{1}{4}\log B = \frac{3}{4}\log B + \frac{1}{4}\log A.\end{aligned}$$

It follows from Theorem 11 that for all large Φ,

$$\overline{\dim}_\Phi \mu_\omega = \frac{4\log 1/2}{3\log B + \log A} \text{ a.s.}$$

Example 5. Fix $0 < 2A < B < 1$ and consider the random Moran set with (a_n, b_n) chosen uniformly over the set

$$\Lambda = \{(x, y) \colon A \leq \min\{x, y\} \leq x + y \leq B\}.$$

Let μ_ω be the random measure, where $p_n = 1/2$ for all n. Similarly to the previous example, for all n and θ, $Y_n(\theta) = \log 1/2$ and

$$Z_n(\theta) = \begin{cases} \log a_n & \text{if } a_n \geq b_n, \\ \log b_n & \text{if } a_n < b_n. \end{cases}$$

Thus,

$$
\begin{aligned}
&\mathbb{E}(Z_n)\\
&\quad = \int_{a_n \geq b_n} \log a_n + \int_{a_n < b_n} \log b_n = 2 \int_{a_n \geq b_n} \log a_n \\
&\quad = \frac{4}{(B-2A)^2}\left(\int_A^{B/2}\left(\int_A^x \log x\, dy\right) dx + \int_{B/2}^{B-A}\left(\int_A^{B-x} \log x\, dy\right) dx\right)\\
&\quad = \frac{4}{(B-2A)^2}\left(\int_A^{B/2} (x-A)\log x\, dx + \int_{B/2}^{B-A} (B-A-x)\log x\, dx\right)\\
&\quad = \frac{2(B-A)^2\log(B-A) + B^2\log(2) + 2A^2\log(A) - B^2\log(B) - 6(B/2-A)^2}{(B-a2A)^2}.
\end{aligned}
$$

Hence, for all large Φ, almost surely we have

$$
\begin{aligned}
&\overline{\dim}_\Phi \mu_\omega\\
&\quad = \frac{(B-2A)^2\log(1/2)}{2(B-A)^2\log(B-A) + B^2\log(2) + 2A^2\log(A) - B^2\log(B) - 6(B/2-A)^2}.
\end{aligned}
$$

4.4.4. Further remarks on $G(\theta)$. For a fixed A, B with $0 < A < B < 1$, set

$$\Lambda = \{(a,b,z) \colon A \leq \min\{a,b\} \leq a+b \leq B, 0 \leq z \leq 1\}$$

as our parameter space. Then each point $(a, b, p) \in \Lambda$ defines an iterated function system with probabilities (IFSP). If this configuration (scalings a, b and probabilities p, $1-p$) is chosen at every level, the resulting (deterministic) Moran set and measure are both self-similar. The associated function $G(\theta)$ is piecewise constant with at most one discontinuity. It is possible to show that the location of the discontinuity cannot be between the two values of $G(\theta)$. Using this it is not difficult to see that there is a unique solution to $G(\theta) = \theta$.

In terms of our random model we can identify this single IFSP with a probability measure on Λ which is a point-mass at the point (a, b, p). If, instead, we take a probability measure on Λ which is a combination of N point masses, then this is identified with a finite collection of different IFSPs from which we randomly choose at each level, with the choice independent from level to level. This time the function $G(\theta)$ has at most N points of discontinuity and hence at most a finite number of solutions to $G(\theta) = \theta$. It would be very interesting to know if it were possible to construct an explicit example where $G(\theta) = \theta$ has no solutions; this would happen if a point of discontinuity of $G(\theta)$ coincided with a jump in the value from $G(\psi) > \psi$ to $G(\psi) < \psi$. For a single IFSP this is not possible, but it is unclear if this might be possible for a collection of IFSPs.

On the other hand, if we begin with a probability measure η on Λ which is absolutely continuous with respect to Lebesgue measure, then $G(\theta)$ is a continuous function of θ and so Corollary 19 applies. It is worth pausing for a moment to contemplate why this is the case. For each fixed value of $\theta > 0$, the set Λ is partitioned into the two regions

$$\left\{p \leq \frac{a^\theta}{a^\theta + b^\theta}\right\} \quad \text{and} \quad \left\{p > \frac{a^\theta}{a^\theta + b^\theta}\right\}$$

and the boundary between these regions is a smooth function of (a, b, p) and also of θ. The values of $Y(\theta)$ and $Z(\theta)$ depend entirely on which of these two sets the particular (random) choice of (a, b, p) belongs to, and thus the expected values of Y and Z are given by the distribution of η over these two sets. Since the boundary is a smooth surface, if η is absolutely continuous, changing θ moves the boundary smoothly and thus changes $G(\theta)$ in a continuous way.

4.5. Relating $\dim_\Phi \mu$ to $\dim_\Phi \mathscr{C}_\omega$

It is known that $\overline{\dim}_\Phi \mu \geq \overline{\dim}_\Phi \operatorname{supp} \mu$ for any measure μ and if μ is doubling, then we also have $\underline{\dim}_\Phi \mu \leq \underline{\dim}_\Phi \operatorname{supp} \mu$ (see [16, Prop 2.9]). This leads us to ask if we can arrange for an almost sure equality, that is, can we choose the $p_n(\omega)$ in such a way that $\overline{\dim}_\Phi \mu_\omega = \overline{\dim}_\Phi \mathscr{C}_\omega$ for ω in a set of full measure.

There is a standard, and "natural", way of doing this for a single IFS with probabilities. Given scaling factors a and b, we set $p = a^d$ and $1 - p = b^d$, where $d > 0$ is the solution to the Moran equation $a^x + b^x = 1$. This choice of p will "balance" the scaling of the lengths with the redistribution of the mass to ensure that $\dim \mu = \dim \mathscr{C}$, with all the large Φ-dimensions coinciding with the Hausdorff dimension. In fact, in this particular case it is easy to see that $G'(\theta) = G(\theta) = d$ for all θ.

However, even in the next simplest case of randomly choosing between two IFSPs, this "natural" choice of probabilities does not typically give $\overline{\dim}_\Phi \mu_\omega = \overline{\dim}_\Phi \mathscr{C}_\omega$ almost surely. As an example, take the IFSP $\{x/3, x/9 + 8/9\}$ with probabilities $p, 1 - p$ and a second IFSP $\{x/4, x/16 + 15/16\}$ with corresponding probabilities $q, 1 - q$. To get our random $\mathscr{C}_\omega$ and μ_ω we will choose equally likely between these two IFSs at each level.

The Moran equation for the first IFS is $3^{-x} + 3^{-2x} = 1$, whose solution is

$$d_1 = \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{-\ln(3)}, \text{ with corresponding } p = 3^{-d_1} = \frac{\sqrt{5} - 1}{2}.$$

Similarly, the Moran equation for the second is $4^{-x} + 4^{-2x} = 1$, with solution

$$d_2 = \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{-\ln(4)}, \text{ and corresponding } q = 4^{-d_2} = \frac{\sqrt{5} - 1}{2}.$$

Using these choices for p and q, elementary computations show that

$$G(\theta) = \begin{cases} \frac{2\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{4}\right)+\ln\left(\frac{1}{3}\right)} & \text{if } \theta \le \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{4}\right)}, \\ \frac{3\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{4}\right)+\ln(\frac{1}{9})} & \text{if } \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{4}\right)} < \theta \le \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{3}\right)}, \\ \frac{2\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{4}\right)+\ln\left(\frac{1}{3}\right)} & \text{if } \theta > \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{3}\right)}. \end{cases}$$

From this we can deduce that

$$\overline{\dim}_{\Phi}\mu_{\omega} = \frac{3\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln\left(\frac{1}{4}\right)+\ln\left(\frac{1}{9}\right)} \approx 0.402 \text{ almost surely.}$$

The almost sure Hausdorff dimension of $\mathcal{C}_{\omega}$ is given by the solution $D > 0$ to $(3^{-x}+3^{-2x})^{1/2}(4^{-x}+4^{-2x})^{1/2} = 1$ (see [15]). It is conjectured in [25] that for all large Φ we also have

$$\overline{\dim}_{\Phi}\mathcal{C}_{\omega} = D \approx 0.388 \text{ almost surely.}$$

On the other hand, if there is a $d > 0$ so that $a_n(\omega)^d + b_n(\omega)^d = 1$ for all ω and n (i.e., for all possible IFS in the given model), then choosing $p_n(\omega) = a_n(\omega)^d$ will result in $d = \overline{\dim}_{\Phi}\mu_{\omega} = \overline{\dim}_{\Phi}\mathcal{C}_{\omega}$ almost surely. This is because in this very special situation we will have $G(\theta) = d$ for all θ. We conjecture that, other than in this special case, generically the "natural" choice of p_n will result in $\overline{\dim}_{\Phi}\mu_{\omega} > \overline{\dim}_{\Phi}\mathcal{C}_{\omega}$ almost surely. Verifying this by explicit computations seems to be exceedingly complicated.

4.6. Comments on a more general construction

In this short subsection we briefly indicate how we can modify our construction so that it works in $\mathbb{R}^D$ and with the possibility of more than two children per parent. We can also allow the number of children at each level to be random and change from level to level. None of these significantly change anything as long as the number of children is uniformly bounded. To describe the generalization, we first need to establish some notation and definitions.

For $I \subset \mathbb{R}^d$, we denote by $\operatorname{diam}(I)$ the diameter of I. Given $r > 0$, we say $J \subseteq I$ is an *r-similarity* of I if there is a similarity S such that $J = S(I)$ and $\operatorname{diam}(J) = r \cdot \operatorname{diam}(I)$. A collection of r_j-similarities, $J_1, J_2, \ldots, J_k$, (possibly of distinct contraction factors) is *τ-separated* if $d(J_i, J_j) \ge \tau \cdot \operatorname{diam}(I)$ for all $i \ne j$. If such a collection exists, we say that I has the *(k,τ)-separation property*.

In the event that the interior of I is non-empty, then for a given k and small enough $\tau > 0$, it is easy to see that I will have the (k,τ)-separation property for any $r_j \le \rho_k$,

$j = 1, 2, \dots, k$, for a suitably small $\rho_k > 0$. For example, if $I = [0, 1]$ and $\tau k < 1$, then $\rho_k = (1 - (k - 1)\tau)/k$ will work. We can view the (k, τ)-separation condition as a uniform strong separation condition.

Lemma 7, which relates balls with level n sets and thus contains the essential geometric result, is changed very little in the more general setup. We redefine L by the condition that

$$2B^{L-1} \le \tau$$

and replace $1 - B$ with τ in the proof and everything else is the same.

Let I_0 be a fixed compact subset of $\mathbb{R}^D$ with non-empty interior and diameter one. Fix $\tau \in (0, 1)$, $K \ge 2$ and let $B_i \in (0, 1)$, $i = 2, \dots, K$, be such that I_0 has the (i, τ)-separation property for all $r_j \le B_i$. We again let $A \in (0, \min_i B_i)$. For each ω and step n in the construction, we take the random variables $K_n = k_n(\omega) \in \{2, 3, \dots, K\}$ and $a_n^{(1)}(\omega), \dots, a_n^{(K_n)}(\omega)$ where $a_n^{(j)} = a_n^{(j)}(\omega) \ge A$ for each $j = 1, 2, \dots, K_n$ and also $a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(K_n)} \le B_{K_n}$; these determine the relative sizes of the children at step n. Specifically, the children $J_j(\omega)$ of the parent $I_n(\omega) = I_n$ are $a_n^{(j)}$-similarities of I_n, for $j = 1, 2, \dots, K_n$, which are τ-separated. The random Moran set $\mathcal{C}_\omega$ is then defined (as usual) to be

$$\mathcal{C}_\omega = \bigcap_{n=1}^{\infty} \mathcal{M}_n(\omega),$$

where $\mathcal{M}_n(\omega)$ is the union of the step n children.

Define a random measure μ_ω supported on this Moran set $\mathcal{C}_\omega$ by the rule that if the children of I_n are labelled $I_n^{(j)}$, $j = 1, \dots, K_n$, then $\mu_\omega(I_n^{(j)}) = p_n^{(j)} \mu_\omega(I_n)$, where the random variables $p_n^{(j)}(\omega) \ge 0$ satisfy $\sum_{j=1}^{K_n} p_n^{(j)} = 1$ for all n. We assume that $\mathbb{E}((a_n^{(j)})^{-t}) < \infty$ and $\mathbb{E}((p_n^{(j)})^{-t}) < \infty$ for some $t > 0$ and all $j = 1, \dots, K_n$ and n.

Define

$$\begin{aligned} Y_n(\theta)(\omega) &= \log p_n^{(m)}(\omega) \\ Z_n(\theta)(\omega) &= \log a_n^{(m)}(\omega) \end{aligned} \qquad \text{where} \qquad \frac{a_n^{(m)\theta}}{p_n^{(m)}} = \max\left\{ \frac{a_n^{(k)\theta}}{p_n^{(k)}} : k = 1, \dots, K_n \right\}$$

and, as before, define

$$G(\theta) = \frac{\mathbb{E}_\omega(Y_1(\theta)(\omega))}{\mathbb{E}_\omega(Z_1(\theta)(\omega))}.$$

Essentially the same arguments as before show that Theorem 11 holds in this case as well.

Example 6. Suppose $K_n = 3$ (the same for all n) and the ratios are $a_n^{(1)} = 1/4$, $a_n^{(2)} = a_n^{(3)} = 1/16$ for all ω. Assume the probabilities $p_n^{(j)}$ are $1/2, 1/4, 1/4$ with $1/2$ assigned to position j with equal likelihood. Note that if $p_n^{(1)} = 1/2$, then

$\max(a_n^{(k)\theta}/p_n^{(k)}) = a_n^{(1)\theta}/p_n^{(1)}$ if $\theta \geq 1/2$ and $a_n^{(2)\theta}/p_n^{(2)}$ otherwise. If $p_n^{(j)} = 1/2$ for $j = 2, 3$, then $\max(a_n^{(k)\theta}/p_n^{(k)}) = a_n^{(1)\theta}/p_n^{(1)}$ for all $\theta \geq 0$. One can check that

$$\mathbb{E}(Y_1(\theta)) = \begin{cases} \log(1/4) & \text{if } \theta < 1/2, \\ \frac{5}{3}\log(1/2) & \text{if } \theta \geq 1/2 \end{cases}$$

and

$$\mathbb{E}(Z_1(\theta)) = \begin{cases} \frac{4}{3}\log(1/4) & \text{if } \theta < 1/2, \\ \log(1/4) & \text{if } \theta \geq 1/2. \end{cases}$$

Thus,

$$G(\theta) = \begin{cases} 3/4 & \text{if } \theta < 1/2, \\ 5/6 & \text{if } \theta \geq 1/2 \end{cases}$$

and consequently, for all large dimension functions Φ, $\overline{\dim}_\Phi \mu = 5/6$ a.s.

5. Dimension results for small Φ

We now move to a discussion of the "small" dimension functions Φ. Recall that this means that $\Phi \ll \log|\log t|/|\log t|$. We again restrict our discussion to the case of two children per parent interval for the sake of clarity. The modifications necessary for the more general case are straightforward.

5.1. The dimension theorem for small Φ

Put

$$\alpha = \max\left\{\operatorname{ess\,sup}\left(\frac{\log p_1(\omega)}{\log a_1(\omega)}\right), \operatorname{ess\,sup}\left(\frac{\log(1-p_1(\omega))}{\log b_1(\omega)}\right)\right\},$$
$$\beta = \min\left\{\operatorname{ess\,inf}\left(\frac{\log p_1(\omega)}{\log a_1(\omega)}\right), \operatorname{ess\,inf}\left(\frac{\log(1-p_1(\omega))}{\log b_1(\omega)}\right)\right\}.$$

Theorem 21. *There is a set Γ of full measure, such that $\overline{\dim}_\Phi \mu_\omega = \alpha$ and $\underline{\dim}_\Phi \mu_\omega = \beta$ for all $\omega \in \Gamma$ and for all small dimension functions Φ.*

Proof. We will begin by verifying that $\overline{\dim}_\Phi \mu \leq \alpha$ a.s. (and this will hold for all choices of Φ, not just small Φ). Of course, this is obvious if $\alpha = \infty$. Otherwise, consider the Moran interval $I_v(\omega) = I_{v_1 \ldots v_N}$ and descendent interval $I_u(\omega) = I_{v_1 \ldots v_n}$ where $|I_u| \leq |I_v|^{1+\Phi(|I_v|)}$. Then

$$\frac{\mu_\omega(I_v)}{\mu_\omega(I_u)} = \prod_{\substack{j=N+1 \\ v_j=0}}^{n} p_j^{-1} \prod_{\substack{j=N+1 \\ v_j=1}}^{n} (1-p_j)^{-1}$$

and
$$\frac{|I_v|}{|I_u|} = \prod_{\substack{j=N+1\\ v_j=0}}^{n} a_j^{-1} \prod_{\substack{j=N+1\\ v_j=1}}^{n} b_j^{-1},$$
so
$$\frac{\frac{\mu_\omega(I_v)}{\mu_\omega(I_u)}}{\left(\frac{|I_v|}{|I_u|}\right)^\alpha} = \prod_{\substack{j=N+1\\ v_j=0}}^{n} \frac{p_j^{-1}}{a_j^{-\alpha}} \prod_{\substack{j=N+1\\ v_j=1}}^{n} \frac{(1-p_j)^{-1}}{b_j^{-\alpha}}.$$

Almost surely, $\alpha \geq \log p_j / \log a_j$ and $\alpha \geq \log(1-p_j)/\log b_j$ for all j, hence $a_j^{-\alpha} \geq p_j^{-1}$ and $b_j^{-\alpha} \geq (1-p_j)^{-1}$ a.s. Thus,
$$\frac{\mu_\omega(I_v)}{\mu_\omega(I_u)} \leq \left(\frac{|I_v|}{|I_u|}\right)^\alpha \text{ a.s.}$$
and consequently, $\overline{\dim}_\Phi \mu \leq \alpha$ a.s.

For the reverse inequality, first suppose $\alpha < \infty$. Fix $i \in \mathbb{N}$. Without loss of generality, we will assume
$$\alpha = \operatorname{ess\,sup}\left(\frac{\log p_1}{\log a_1}\right) = \operatorname{ess\,sup}\left(\frac{\log p_1^{-1}}{\log a_1^{-1}}\right).$$
From the definition of the essential supremum, there must be some $0 < \delta_i < 1$ such that
$$\mathcal{P}\left(\omega\colon \frac{\log p_1^{-1}}{\log a_1^{-1}} \geq \alpha - \frac{1}{2i}\right) \geq \delta_i.$$

Let
$$J_i = \frac{|\log B|}{2|\log \delta_i|} \quad \text{and} \quad \Phi_i(t) = \frac{J_i \log|\log t|}{|\log t|}.$$
For each positive integer N, let
$$\chi_{N,i} = \frac{J_i \log(N|\log A|)}{|\log B|}.$$
Clearly, $\chi_{N,i} \to \infty$ as $N \to \infty$ and the definitions ensure that $\delta_i^{\chi_{N,i}} \geq 1/N$ for large enough N. Set
$$\Gamma_{N,i} = \left\{\omega\colon \frac{\log p_j^{-1}}{\log a_j^{-1}} \geq \alpha - \frac{1}{2i} \text{ for } j = N+1,\dots,N+\chi_{N,i}\right\}.$$
As the tuples (p_n, a_n, b_n) are independent,
$$\mathcal{P}(\Gamma_{N,i}) = \prod_{j=N+1}^{N+\chi_{N,i}} \mathcal{P}\left(\omega\colon \frac{\log p_j^{-1}}{\log a_j^{-1}} \geq \alpha - \frac{1}{2i}\right) = \delta_i^{\chi_{N,i}} \geq \frac{1}{N}.$$

Thus, if we let $N_k = k \log k$, then for some suitably large K_0,

$$\sum_k \mathcal{P}(\Gamma_{N_k,i}) \geq \sum_{k \geq K_0} \frac{1}{k \log k} = \infty.$$

As we can replace δ_i with any smaller, strictly positive number, there is no loss of generality in assuming it is so small that $N_{k+1} > N_k + \chi_{N_k,i}$. Hence, the events $\Gamma_{N_k,i}$ are independent and thus the Borel–Cantelli lemma implies that $\mathcal{P}(\Gamma_{N_k,i} \text{ i.o.}) = 1$ for each (fixed) i. Let Γ_i be this set of full measure.

Take any $\omega \in \Gamma_i$ and consider any Moran interval, $I_N(\omega)$, of step N. Let $I_n(\omega)$ be the left-most descendent of I_N at level $n = N + \chi_{N,i}$ (where we make the choice of the left descendent since $\alpha = \operatorname{ess\,sup}(\log p_1/\log a_1)$). Since $|I_N| \geq A^N$ and the function $t^{\Phi_i(t)}$ decreases as t decreases to 0, the choice of $\chi_{N,i}$ ensures

$$|I_N|^{\Phi_i(|I_N|)} \geq A^{N\Phi_i(A^N)} = A^{\frac{J_i \log(N|\log A|)}{|\log A|}} \geq B^{\chi_{N,i}} \geq \frac{|I_n|}{|I_N|}.$$

Hence, $|I_n| \leq |I_N|^{1+\Phi_i(|I_N|)}$. As $\omega \in \Gamma_i$, it follows that for infinitely many N,

$$\log p_j^{-1} \geq (\log a_j^{-1})\Big(\alpha - \frac{1}{2i}\Big) \text{ for } j = N+1, \ldots, N+\chi_{N,i},$$

equivalently,

$$p_j^{-1} \geq a_j^{-(\alpha - 1/(2i))}.$$

Thus,

$$\frac{p_j^{-1}}{a_j^{-(\alpha-1/i)}} \geq a_j^{-1/(2i)} \geq B^{-(1/(2i))} \text{ for } j = N+1, \ldots, N+\chi_{N,i}.$$

Consequently, for each fixed i,

$$\frac{\frac{\mu_\omega(I_N)}{\mu_\omega(I_n)}}{\left(\frac{|I_N|}{|I_n|}\right)^{\alpha-1/i}} = \prod_{j=N+1}^{N+\chi_{N,i}} \frac{p_j^{-1}}{a_j^{-(\alpha-1/i)}} \geq (B^{-1/(2i)})^{\chi_{N,i}}$$

and since $(B^{-1/(2i)})^{\chi_{N,i}}$ as $N \to \infty$ it follows that there can be no constant C such that

$$\frac{\mu_\omega(I_N)}{\mu_\omega(I_n)} \leq C\left(\frac{|I_N|}{|I_n|}\right)^{\alpha-1/i}$$

for all such N, n. By Lemma 8 that implies $\overline{\dim}_{\Phi_i}\mu_\omega \geq \alpha - 1/i$ for all $\omega \in \Gamma_i$.

Let $\Gamma = \bigcap_{i=1}^{\infty} \Gamma_i$, a set of full measure, and assume Φ is any small dimension function. Then there is some function $H(t) \to 0$ as $t \to 0$ so that

$$\Phi(t) \leq \frac{H(t)\log|\log t|}{|\log t|} \text{ for all } t \leq t_0.$$

Consequently, for each i there is some $t_i > 0$ such that $\Phi(t) \leq \Phi_i(t)$ for all $t \leq t_i$. This property and our observations above ensure that $\overline{\dim}_{\Phi}\mu_\omega \geq \overline{\dim}_{\Phi_i}\mu_\omega \geq \alpha - 1/i$ for all i and all $\omega \in \Gamma$. We conclude that $\overline{\dim}_{\Phi}\mu_\omega \geq \alpha$ for all $\omega \in \Gamma$, as we desired to show.

If $\alpha = \infty$, replacing '$\alpha - 1/(2i)$' in the arguments with '$2i$', in the same manner we deduce that for every $i \in \mathbb{N}$ and infinitely many N,

$$\frac{\frac{\mu_\omega(I_N)}{\mu_\omega(I_n)}}{\left(\frac{|I_N|}{|I_n|}\right)^i} \geq B^{-i\chi_{N,i}}$$

and, of course, this tends to infinity as $N \to \infty$. It follows that $\overline{\dim}_{\Phi_i}\mu_\omega \geq i$ for all $\omega \in \Gamma_i$, a set of full measure and with similar reasoning to above, we deduce that $\overline{\dim}_{\Phi}\mu_\omega = \infty$ a.s.

If, instead, $\alpha = \operatorname{ess\,sup}(\log(1 - p_1)/\log b_1)$, we consider a Moran interval of level N and its rightmost descendent at level $N + \chi_{N,i}$, and argue in a similar fashion.

The arguments to establish $\underline{\dim}_{\Phi}\mu = \beta$ a.s. are analogous and left to the reader. ∎

Corollary 22. *Almost surely,* $\dim_A \mu_\omega = \alpha$ *and* $\dim_L \mu_\omega = \beta$.

Proof. This is immediate from Theorem 21 as the constant function $\Phi = 0$ is a small dimension function. ∎

Let

$$\begin{aligned} a_0 &= \operatorname{ess\,inf} a_1(\omega), & b_0 &= \operatorname{ess\,inf} b_1(\omega), \\ A_0 &= \operatorname{ess\,sup} a_1(\omega), & B_0 &= \operatorname{ess\,sup} b_1(\omega), \\ p_0 &= \operatorname{ess\,sup} p_1(\omega), & q_0 &= \operatorname{ess\,sup}(1 - p_1(\omega)), \\ P_0 &= \operatorname{ess\,inf} p_1(\omega), & Q_0 &= \operatorname{ess\,inf}(1 - p_1(\omega)) \end{aligned}$$

and put

$$\alpha' = \max\left\{\frac{\log P_0}{\log A_0}, \frac{\log Q_0}{\log B_0}\right\}, \qquad \beta' = \min\left\{\frac{\log p_0}{\log a_0}, \frac{\log q_0}{\log b_0}\right\}.$$

Notice that if p_n is chosen independently of (a_n, b_n) for all n, then $\alpha' = \alpha$ and $\beta' = \beta$. Consequently, there is another immediate consequence of Theorem 21.

Corollary 23. *Suppose p_n is chosen independently of (a_n, b_n) for all n. There is a set Γ of full measure, such that $\overline{\dim}_{\Phi}\mu_\omega = \alpha'$ and $\underline{\dim}_{\Phi}\mu_\omega = \beta'$ for all $\omega \in \Gamma$ and for all small dimension functions Φ.*

Example 7. Consider, again, the Moran set $\mathcal{C}_{ab}$ and random measure μ_ω with probabilities chosen with equal likelihood from $\{p, 1-p\}$ with $a < b$ and $p < 1-p$, as in Example 3. Then $\overline{\dim}_\Phi \mu_\omega = \log p / \log b$ and $\underline{\dim}_\Phi \mu_\omega = \log(1-p)/\log a$ almost surely.

Example 8. In both Examples 4 and 5, it is trivial to compute the Φ-dimensions for small Φ. In both cases $a_0 = b_0 = A$, $A_0 = B_0 = B$ and $p_0 = q_0 = P_0 = Q_0 = 1/2$. Thus, $\overline{\dim}_\Phi \mu_\omega = \frac{\log 1/2}{\log B}$ and $\underline{\dim}_\Phi \mu_\omega = \frac{\log 1/2}{\log A}$ almost surely.

Remark 2. As in Section 4.6, suppose that each parent interval in the Moran set construction has $K \geq 2$ children and define a random Moran set $\mathcal{C}_\omega$ and measure μ_ω as was done there (with the same assumptions). With the notation of that subsection, for $j = 1, \ldots, K$ put

$$\alpha = \max_{j=1,\ldots,K} \operatorname{ess\,sup}\left(\frac{\log p_j(\omega)}{\log a_j(\omega)}\right), \qquad \beta = \min_{j=1,\ldots,K} \operatorname{ess\,inf}\left(\frac{\log p_j(\omega)}{\log a_j(\omega)}\right).$$

The same reasoning as in the proof of Theorem 21 shows that $\overline{\dim}_\Phi \mu_\omega = \alpha$ and $\underline{\dim}_\Phi \mu_\omega = \beta$ for almost all ω and for all small dimension functions Φ.

5.2. Relating $\dim_\Phi \mu$ to $\dim_\Phi \mathcal{C}_\omega$

As in the case of the large dimension functions, it is natural to ask if one could obtain the almost sure Φ-dimensions of these random Moran sets as the almost sure Φ-dimensions of the random measures arising from *some* choice of probabilities, as is the case when $a_n(\omega) = b_n(\omega)$ for all n and ω (see [18]). The following example shows that this need not be the case for the small dimension functions Φ when the random set is not generated by equicontractive similarities.

Example 9. Choose $0 < a_2 < a_1 < 1/2$, $0 < b_1 < b_2 < 1/2$ and consider the family of random Moran sets C_ω where we choose (a_n, b_n) independently and with equal likelihood from $\{(a_1, b_1), (a_2, b_2)\}$. We will let $\mathcal{C}_1, \mathcal{C}_2$ denote the (deterministic) Moran sets generated by (a_1, b_1) and (a_2, b_2), respectively. It is known [8, Thm. 2.6] that

$$\dim_A \mathcal{C}_\omega = \max\left\{\dim_A \mathcal{C}_1, \dim_A \mathcal{C}_2\right\} \text{ a.s.}$$

and that $\dim_A \mathcal{C}_j$ is the value of d_j satisfying $a_j^{d_j} + b_j^{d_j} = 1$ for $j = 1, 2$.

Let $0 \leq p \leq q \leq 1$ and for convenience let

$$\lambda(p, q) = \max\left\{\frac{\log p}{\log a_1}, \frac{\log(1-q)}{\log b_2}\right\}.$$

Theorem 21 shows that if we denote by $\mu_\omega = \mu_\omega(p, q)$ the random measures supported on the Moran sets $\mathcal{C}_\omega$ where we choose probabilities p_n independently and with equal likelihood from $\{p,q\}$, then for each p,q

$$\dim_A \mu_\omega(p,q) = \lambda(p,q) \text{ a.s.}$$

Notice that a compactness argument ensures that $\inf_{0\le p\le q\le 1}\lambda(p,q) = \lambda(p_0,q_0)$ for a suitable choice of p_0, q_0.

We will see that

$$\dim_A \mathcal{C}_\omega < \lambda(p_0,q_0) \text{ a.s.} \tag{5.1}$$

To prove this, we first note that at the minimal value of $\lambda(p,q)$ we must have $\log p_0/\log a_1 = \log(1-q_0)/\log b_2$. Moreover, as the function $\log p/\log a_1$ decreases as p increases and the function $\log(1-q)/\log b_2$ decreases as q decreases, the minimum value occurs when $p_0 = q_0$, hence at a choice of p_0, where

$$\frac{\log p_0}{\log a_1} = \frac{\log(1-p_0)}{\log b_2}.$$

If we suppose γ is chosen so that $b_2 = a_1^\gamma$, solving the equation above gives that p_0 satisfies $p_0^\gamma + p_0 = 1$ or, equivalently, $p_0 = (1-p_0)^{1/\gamma}$. To summarize,

$$\lambda(p_0,p_0) = \frac{\log p_0}{\log a_1} = \frac{\log(1-p_0)}{\log b_2}, \text{ where } p_0^\gamma + p_0 = 1.$$

Now assume κ is chosen so that $b_1 = a_1^\kappa$. As $b_1 < b_2$, we must have $\kappa > \gamma$. Furthermore, $d_1 = \dim_A \mathcal{C}_1$ is defined by the rule $1 = a_1^{d_1} + (a_1^{d_1})^\kappa$. Since $\kappa > \gamma$, if $a_1^{d_1} \le p_0$ we obtain the contradiction

$$1 = a_1^{d_1} + (a_1^{d_1})^\kappa \le p_0 + p_0^\kappa < p_0 + p_0^\gamma = 1.$$

Hence, $a_1^{d_1} > p_0$ and thus

$$\dim_A \mathcal{C}_1 = d_1 < \frac{\log p_0}{\log a_1}.$$

Likewise, if we assume $a_2^\eta = b_2$, then as $a_1 > a_2$ we must have $1/\eta > 1/\gamma$. And as d_2 is defined by the rule $1 = \left(b_2^{d_2}\right)^{1/\eta} + b_2^{d_2}$, it similarly follows that $b_2^{d_2} > 1 - p_0$, that is

$$\dim_A \mathcal{C}_2 = d_2 < \frac{\log(1-p_0)}{\log b_2}.$$

These observations prove (5.1).

Since $\dim_A \mu_\omega = \overline{\dim}_\Phi \mu_\omega$ a.s. for all small Φ, and $\dim_A \mathcal{C}_\omega \ge \overline{\dim}_\Phi \mathcal{C}_\omega$ for all ω and Φ, this also proves that for all small dimension functions Φ and for each p,q

$$\dim_\Phi \mathcal{C}_\omega < \frac{\log p_0}{\log a_1} \le \overline{\dim}_\Phi \mu_\omega(p,q) \text{ a.s.,}$$

where p_0 is given by the rule $p_0 + p_0^\gamma = 1$ and $\gamma = \log b_2/\log a_1$.

For an explicit example, suppose $a_1 = 1/2$, $b_1 = 1/4$, $a_2 = 1/3 = b_2$. Then

$$\dim_A \mathcal{C}_1 = \frac{\log((\sqrt{5}-1)/2)}{\log 1/2} \approx 0.69 \quad \text{and} \quad \dim_A \mathcal{C}_2 = \frac{\log 2}{\log 3} \approx 0.63$$

so

$$\dim_A \mathcal{C}_\omega = \dim_A \mathcal{C}_1 \approx 0.69 \text{ a.s.}$$

We have $b_2 = a_1^\gamma$ for $\gamma = \log 3/\log 2$ and Maple gives the approximate solution to $p_0^\gamma + p_0 = 1$ as $p_0 \approx 0.58$. Hence for each choice of p, q,

$$\dim_A \mu_\omega(p,q) \geq \frac{\log p_0}{\log 1/2} \approx 0.78 \text{ a.s.}$$

A. Appendix: Examples of $G(\theta)$

Even though the function $G(\theta)$ is only used as a technical tool for proving our results, it is helpful to see a few plots of some examples. Referring to Figure 2, we present our four examples starting with the first row and going left to right. We comment that we used Maple for both the computations and for the plots.

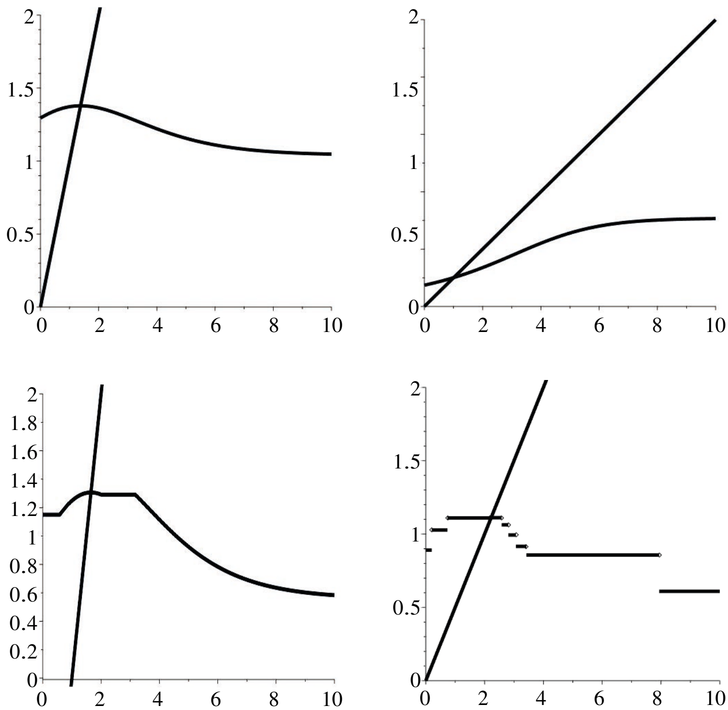

Figure 2. Four different examples of $G(\theta)$, described in the appendix.

For the first example, we set $b = 2a$ and choose a uniformly from the interval $\left[1/10, 3/10\right]$. In addition, p is chosen uniformly from $[0, 1]$. The resulting function $G(\theta)$ is clearly smooth, but not monotone.

In the second, we now set $b = 50a$ and choose a uniformly in the range $1/100 \le a \le (1 - 1/100)/51$. Again p is chosen uniformly from $[0, 1]$. This time $G(\theta)$ is monotone increasing, and still smooth.

For the third, we again set $b=2a$ and choose a uniformly from $[1/10, 3/10]$. However, this time p is chosen uniformly from the set $[0, 1/10] \cup [1/5, 2/5] \cup [7/10, 1]$. The function is continuous, but only piecewise smooth and not monotone. *Note that the vertical axis is different than in the other plots.* This was done in order to more clearly show the shape of the graph.

Finally, for our fourth example we take the 9 triples (a, b, p)

$$\left(\frac{2}{25}, \frac{18}{25}, \frac{1}{18}\right), \left(\frac{4}{25}, \frac{16}{25}, \frac{1}{8}\right), \left(\frac{6}{25}, \frac{14}{25}, \frac{3}{14}\right), \left(\frac{8}{25}, \frac{12}{25}, \frac{1}{3}\right), \left(\frac{10}{25}, \frac{10}{25}, \frac{1}{2}\right),$$
$$\left(\frac{12}{25}, \frac{8}{25}, \frac{5}{6}\right), \left(\frac{14}{25}, \frac{6}{25}, \frac{5}{7}\right), \left(\frac{16}{25}, \frac{4}{25}, \frac{5}{8}\right), \left(\frac{18}{25}, \frac{2}{25}, \frac{5}{9}\right),$$

which each define an IFS with probabilities. For our random model, at each level we choose one of these IFSPs equally likely. As is clear from the plot, in this case $G(\theta)$ is discontinuous, piecewise constant, and non-monotone. The discontinuities occur at $\theta = \log(p/(1-p))/\log(a/b)$ for the various choices of a, b and p.

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