

Quotients of the curve complex

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Abstract. We consider three kinds of quotients of the curve complex, which are obtained by coning off uniformly quasiconvex subspaces: symmetric curve sets, non-maximal train track sets, and compression body disc sets. We show that the actions of the mapping class group on those quotients are strongly weakly properly discontinuously (WPD), which implies that the actions are non-elementary and those quotients are of infinite diameter.

1. Introduction

If X is a Gromov hyperbolic space, and $Y = \{Y_i\}$ is a collection of uniformly quasiconvex subsets of X , then the electrification X_Y is Gromov hyperbolic. Furthermore, if a group G acts on X by isometries, and Y is G -equivariant, then G also acts on the electrification X_Y by isometries.

However, if the action of G on X has some useful property (e.g., non-elementary, acylindrical, WPD), then it is not known whether the corresponding action on X_Y has this property. In particular, in general, X_Y need not be of infinite diameter.

We shall consider the action of the mapping class group of a finite type orientable surface on the curve complex of the surface $\mathcal{C}(S)$. Our results apply to the following families of uniformly quasiconvex subsets.

The symmetric curve sets: If $q: S \rightarrow S'$ is an orbifold quotient of S , then we shall denote by $C_q(S)$ the collection of all q -equivariant curves in S . The collection $\{\mathcal{C}_q(S)\}$, as q runs over all orbifold quotients of S , is a mapping class group invariant, uniformly quasiconvex family of subsets of $\mathcal{C}(S)$ [23]. We shall denote the electrification by $\mathcal{C}_q(S)$.

Non-maximal train track sets: A train track τ on S is *maximal* if every complementary region is a triangle or a monogon containing a single puncture. Given a train track τ , we shall write $\mathcal{C}(\tau)$ for all simple closed curves carried by τ . The set $\{\mathcal{C}(\tau)\}$, as τ runs over all non-maximal train tracks in S , is a mapping class group invariant, uniformly quasiconvex family of subsets of $\mathcal{C}(S)$ [19]. We shall denote the electrification by $\mathcal{C}_\tau(S)$.

Compression body disc sets: Let V be a compression body with boundary S , and let $D(V)$ be the collection of isotopy classes of essential simple closed curves in S which bound discs in V . The set $\{D(V)\}$, as V runs over all compression bodies, forms a mapping

class group invariant, uniformly quasiconvex family of subsets of $\mathcal{C}(S)$ [19]. We write $\mathcal{C}_D(S)$ for the electrification, which is quasi-isometric to the compression body graph.

We shall show the following.

Theorem 1.1. *All $\mathcal{C}_q(S)$, $\mathcal{C}_\tau(S)$ and $\mathcal{C}_D(S)$ are infinite diameter Gromov hyperbolic spaces, and furthermore, the action of the mapping class group on these spaces is strongly WPD.*

See Sections 3, 4 and 2.4 for more detailed statements. In the case of $\mathcal{C}_\tau(S)$, this is a special case of a much more general result of Hamenstädt, which has not yet appeared in print. In the case of $\mathcal{C}_D(S)$, the compression body graph was previously shown to have infinite diameter in [15].

For simplicity, in this introduction we only consider random walks on the mapping class group $\text{MCG}(S)$ whose transition probability has the support generating the whole $\text{MCG}(S)$. In [16], Maher and Tiozzo proved that in our situation, any random walk gives rise to a loxodromic element with asymptotic probability one. Hence, in particular, the following two previously known results are now immediate corollaries of our results. Furthermore, this gives largely independent proofs of these results. More precisely, Theorem 3.7 ($C_q(S)$ part of Theorem 1.1) together with [16] implies the following.

Theorem 1.2 ([17]). *Let S be a surface of finite type of genus g with n punctures and with $2g + n > 4$. Then a random walk on the mapping class group of S gives rise to an asymmetric pseudo-Anosov element with asymptotic probability one.*

We remark that here we say a pseudo-Anosov map on S is *asymmetric* if it is not a lift of a pseudo-Anosov map on a surface S' which admits an orbifold cover $p: S \rightarrow S'$. This cover need not be regular, so this is a more general condition than requiring some power of the pseudo-Anosov element commute with a finite order element. The condition $2g + n > 4$ includes all surfaces which support a pseudo-Anosov map, but excludes the four exceptional surfaces $S_{0,4}$, $S_{1,1}$, $S_{1,2}$ and $S_{2,0}$. The result does not hold for these surfaces, as all pseudo-Anosov maps on these surfaces are symmetric with respect to the hyperelliptic involutions on the surfaces.

We say a pseudo-Anosov map has *generic singularities* if its invariant foliations have 1-prong singularities at each puncture, and all other singularities are trivalent. The following is now a consequence of Theorem 4.13 ($C_\tau(S)$ part of Theorem 1.1) and [16].

Theorem 1.3 ([9]). *Let S be a surface of finite type of genus g with n punctures and with $2g + n > 3$. Then a random walk on the mapping class group of S gives rise to a pseudo-Anosov element whose invariant foliations have generic singularities with asymptotic probability one.*

The condition $2g + n > 3$ includes all finite type surfaces which support a pseudo-Anosov map with the exception of $S_{1,1}$ for which the result does not hold, see Remark 4.1.

Furthermore, our result also has an application to the Poisson boundary. In [11], it is proved that the space of projective measured foliations \mathcal{PMF} has a unique harmonic

measure ν , and the measure space (\mathcal{PMF}, ν) is a Poisson boundary of the random walk on $\text{MCG}(S)$. Maher and Tiozzo [16] proved that if a group acts acylindrically on a Gromov hyperbolic space, then the Gromov boundary equipped with the hitting (harmonic) measure is a Poisson boundary. By the work of Osin [20], any WPD action has a quotient with acylindrical action. Combining these with a characterization of the boundary of quotient (Proposition 2.7 below), Theorem 1.1 gives the following.

Corollary 1.4. *Let S be a surface of finite type of genus g with n punctures and with $2g + n > 4$. Let ν be the harmonic measure arising from a random walk on $\text{MCG}(S)$. Then the harmonic measure ν on $\mathcal{PMF}(S)$ is concentrated on asymmetric and maximal foliations.*

2. Quotients of hyperbolic graphs

Given a metric space (X, d) and a collection of subsets $Y = \{Y_i\}_{i \in I}$, the *electrification* of X with respect to Y is the metric space X_Y obtained by adding a new vertex y_i for each set Y_i , and coning off Y_i by attaching edges of length $\frac{1}{2}$ from each $y \in Y_i$ to y_i . The metric on X_Y is the induced path metric, and so the image of each set Y_i in X_Y has diameter at most one.

We say the family of sets Y is *G-equivariant* if for all $g \in G$ and each $Y_i \in Y$, we have that gY_i is also in Y .

Theorem 2.1. *Let X be a Gromov hyperbolic space and let $Y = \{Y_i\}_{i \in I}$ be a collection of uniformly quasiconvex subsets of X . Then X_Y , the electrification of X with respect to Y , is Gromov hyperbolic.*

Furthermore, if a group G acts on X by isometries, and Y is G -equivariant, then G also acts on X_Y by isometries.

The first statements are due to Bowditch [3, Proposition 7.12], and Kapovich and Rafi [12, Proposition 2.6]. The second statement is immediate.

2.1. Weakly properly discontinuous action on quotients

We recall the definition of weakly properly discontinuous action defined by Bestvina and Fujiwara [1].

Definition 2.2. Let X be a connected metric graph and G a group of isometries of X . An element $g \in G$ acts *loxodromically*, or is *loxodromic* if the map $\mathbb{Z} \rightarrow X$ given by $n \mapsto g^n x$ is a quasi-isometric embedding for any $x \in X$. A loxodromic element $g \in G$ is said to act *weakly properly discontinuously* (WPD) or is *WPD* if for any $r > 0$ and $x \in X$, there exists $N \in \mathbb{N}$ such that

$$\#\{h \in G \mid d(x, hx) < r \text{ and } d(g^N x, hg^N x) < r\} < \infty.$$

We say that the action of G on X is *WPD* if

- G is not virtually cyclic,
- there is at least one loxodromic element in G which is WPD.

We say that G is *strongly WPD* if every loxodromic element is WPD.

Note that if there is a loxodromic element acting on a graph, then the graph must have infinite diameter.

The action of G on a Gromov hyperbolic space X is *non-elementary* if G contains a pair of loxodromic isometries with disjoint fixed points in the Gromov boundary ∂X . Bestvina and Fujiwara observed the following.

Proposition 2.3 ([1, Proposition 6]). *Suppose the action of G on a Gromov hyperbolic space X is strongly WPD, then the action of G on X is non-elementary.*

We now show that under certain conditions, if g acts WPD on a Gromov hyperbolic metric graph, then g is also WPD on the quotient graph we get by coning off.

Theorem 2.4. *Let the group G act by isometries on the hyperbolic metric space X , and let Y be a family of uniformly quasiconvex G -equivariant subsets of X .*

Let $g \in G$ act loxodromically on X , and let α be a quasigeodesic axis of g . We denote by π_α the nearest projection map to α on X . Suppose

- (1) *the action of g on X is WPD, and*
- (2) *there is a constant $C > 0$ such that for every element $Y_i \in Y$,*

$$\text{diam}_X(\pi_\alpha(Y_i)) < C.$$

Then the action of g on X_Y is also WPD.

Proof. For notational simplicity, let $Z := X_Y$. We first observe that α has infinite diameter in Z . Suppose not, then there is a D such that any pair of points x and y in α are distance at most D apart in Z . Then there is a path ℓ in X connecting x to y with designated subsegments σ_i of the Y_i such that the number of these subsegments plus the length of their complement $\ell \setminus \bigcup_i \sigma_i$ is at most D . By assumption (2), the nearest point projection of each path to α in X has diameter at most C . Note also that $\pi_\alpha: X \rightarrow \alpha$ is Lipschitz, and hence the total length of $\pi_\alpha(\ell \setminus \bigcup_i \sigma_i)$ is at most D . Thus the distance between x and y in X is at most CD , a contradiction.

Let z be a point in X , and hence also a point in Z , and let $B_Z(z, r)$ denote the ball of radius r centered at z in Z . We denote by $\widehat{B}_Z(z, r)$ the restriction of $B_Z(z, r)$ to X . Then by assumption (2), we see that

$$\text{diam}_X(\pi_\alpha(\widehat{B}_Z(z, r))) < Cr. \tag{1}$$

Recall that in a Gromov hyperbolic space, if the nearest projections to a quasigeodesic α of given two points x, y are reasonably far apart, then the geodesic connecting x and y

must have a subarc fellow traveling with α . Therefore, for large enough N , there are a subarc $\beta \subset \alpha$ and $D_1 > 0$ such that any geodesic in X connecting a point in $\widehat{B}_Z(z, r)$ and a point in $\widehat{B}_Z(g^N z, r)$ must contain β in its D_1 -neighbourhood.

Suppose there is $h \in G$ such that $d_Z(z, hz) < r$ and $d_Z(g^N z, hg^N z) < r$. Let γ be an arbitrarily chosen geodesic in X connecting z and $g^N z$. Then γ and $h(\gamma)$ contain β in their D_1 -neighbourhoods. Estimate (1) implies that h coarsely translate β at most $Cr + 2D_1$ along α . Hence we have

$$d_X(b, hb) \leq Cr + 2D_1$$

for any $b \in \beta$. Note that the constant $Cr + 2D_1$ is independent of N , and we may take β as long as we need by taking large N . Therefore, by taking N large enough, we may apply the fact that g acts WPD on X to conclude that g acts WPD on Z . ■

2.2. The Gromov boundary of quotients

In this subsection, we discuss the boundary of the quotient graph we obtain by coning off. We first recall the work of Dowdall and Taylor.

Let X be a Gromov hyperbolic graph and Z a quotient graph obtained by coning off a family $Y = \{Y_i\}_{i \in I}$ of quasiconvex subsets of X . Recall that the Gromov boundary of a Gromov hyperbolic space X is defined as the space of equivalence classes of quasigeodesic rays. Given a quasigeodesic ray γ in X , let $\gamma(\infty) \in \partial X$ denote the corresponding point in the Gromov boundary ∂X .

Let $\text{proj}: X \rightarrow Z$ denote the map given by the inclusion $X \subset Z$. To give a description of the Gromov boundary of Z , Dowdall and Taylor define a space $\partial_Z X$ by

$$\{\gamma(\infty) \in \partial X \mid \gamma: \mathbb{R}_+ \rightarrow X \text{ is a quasigeodesic ray with } \text{diam}_Z(\text{proj}(\text{Im } \gamma)) = \infty\}.$$

Then the work of Dowdall and Taylor, which we need here, can be stated as follows.

Proposition 2.5 ([4, Theorem 3.2]). *Let X be a Gromov hyperbolic space, and let $Y = \{Y_i\}_{i \in I}$ be a collection of uniformly quasiconvex subsets of X . Let Z be the space formed by coning off X with respect to Y . Then the Gromov boundary of Z is homeomorphic to $\partial_Z(X)$.*

Remark 2.6. In [4, Theorem 3.2], the result is proved under the assumption of ‘‘alignment preserving map’’ (see [4] for the definition). In our situation, that electrification is alignment preserving is proved [12, Proposition 2.6].

We shall define ∂Y to be the union of the limit sets of each Y_i , i.e., $\partial Y = \bigcup_{i \in I} \partial Y_i$. This need not be equal to the limit set of the union of the Y_i , which will often be all of ∂X if, for example, G acts coarsely transitively on X .

Proposition 2.7. *Let X be a Gromov hyperbolic space, and let $Y = \{Y_i\}_{i \in I}$ be a collection of uniformly quasiconvex subsets of X . Let Z be the space formed by coning off X with respect to Y . Let $\partial Y := \bigcup_i \partial Y_i$. Then the space $\partial_Z(X)$ is contained in $\partial X \setminus \partial Y$.*

Proof. Let γ be a quasigeodesic ray such that $\gamma(\infty) \in \partial Y_i$ for some $Y_i \in Y$. Since Y_i is quasiconvex, we see that γ eventually fellow travels with Y_i , and thus the diameter of the image of γ in Z is finite. ■

2.3. Quotients and qi-embedded subgroups

We consider qi(quasi-isometrically)-embedded subgroups.

Definition 2.8. Let X be a path-connected hyperbolic space on which G acts isometrically. Let $H < G$ be a finitely generated subgroup. Then

- H is said to *qi-embed* into X if $H \ni h \mapsto h \cdot p \in X$ defines a quasi-isometric embedding $H \rightarrow X$ for any $p \in X$.
- *The limit set in X of H* is the set of accumulation points in ∂X of any orbit of H in X .

Remark 2.9. If H qi-embeds into a geodesic Gromov hyperbolic space, then H is also Gromov hyperbolic.

We now characterize qi-embedded subgroups in the quotients. Furthermore, if the sets Y satisfy an extra condition, which we shall call *uniform finiteness*, then if H qi-embeds into X , then it also qi-embeds into X_Y .

Definition 2.10. We say that the sets Y are *uniformly finite* if for any constant $D_0 > 0$, there are constants D_1 and D_2 such that for any x and y in X with $d_X(x, y) \geq D_1$, the number of sets Y_i in Y such that $d_X(x, Y_i) \leq D_0$ and $d_X(y, Y_i) \leq D_0$ is at most D_2 .

Some condition on the sets Y is required for Theorem 2.12 (c) to hold, as shown by the following example.

Example 2.11. Let X be the Cayley graph of the free group on two generators with presentation $G = \langle a, b \mid \rangle$, and let Y_1 be a bi-infinite geodesic with base point y which contains a translate of every finite length geodesic on either side of y . Let Y consist of all translates of Y_1 by elements of G . All non-trivial elements of G qi-embed in X and have limit sets contained in $\partial X \setminus \partial Y$, but as X_Y has finite diameter, they do not qi-embed in X_Y .

We say that the sets Y are *H-equivariant* if for all $h \in H$ and all $Y_i \in Y$, hY_i is also in Y .

Theorem 2.12. Let $X, Y, Z := X_Y$, and let G be as in Theorem 2.1. Let H be a finitely generated subgroup of G . Then

- (a) If H qi-embeds into Z , then H qi-embeds into X with limit set contained in $\partial X \setminus \partial Y$ in X .
- (b) If H qi-embeds into X with limit set contained in $\partial X \setminus \partial Y$, then no element of infinite order in H preserves any $Y_i \in Y$ setwise.
- (c) If H qi-embeds into X with limit set contained in $\partial X \setminus \partial Y$, and the sets Y are uniformly finite and H -equivariant, then H qi-embeds into Z .

Proof. Suppose H qi-embeds into Z , then H must qi-embed into X . Furthermore, if the limit set contains an element λ in ∂Y , then there must be a sequence $(g_i)_{i \geq 0}$ in H such that $i \mapsto g_i x$ is a quasi-isometric embedding $\mathbb{Z} \rightarrow X$ and $g_i x \rightarrow \lambda \in \partial X$ for some $x \in X$. As $\lambda \in \partial Y$, the orbit $g_i x$ must fellow travel with some $Y' \in Y$. But this means that $g_i x$ is in a bounded region in Z , which contradicts the assumption that H qi-embeds into Z .

If H contains an element of infinite order which preserves $Y' \in Y$, then the limit set in X must contain a point in $\partial Y'$. Thus (b) follows.

Finally, suppose H qi-embeds into X , the limit set is contained in $\partial X \setminus \partial Y$ in X , and the sets in Y are uniformly finite and H -equivariant. It suffices to show that for a fixed $x \in X$, there is a constant D such that for any pair of group elements h_1 and h_2 in H , the diameter of the nearest point projection of any set Y_i onto a geodesic $[h_1 x, h_2 x] \subset X$ has diameter at most D .

Suppose not, then there is a sequence of geodesics $[h_{1,n} x, h_{2,n} x] \subset X$, and sets $Y_n \in Y$, such that the diameter of the nearest point projection of Y_n to $[h_{1,n} x, h_{2,n} x]$ has diameter at least n . As the sets Y are H -equivariant, up to translation by elements of H , we may assume that the sequence of geodesics has the form $[x, h_n x]$. Also, as H is locally finite, we may pass to a subsequence of common initial segments, so that the elements h_n lie on a common geodesic ray γ in H . As H is qi-embedded in X , the image $\gamma_X \subset X$ of $\gamma \subset H$ obtained by translating x is a quasigeodesic in X . We may also assume that $h_n x$ is contained in the nearest point projection of Y_n .

As Y is uniformly finite, we may choose N with $d_X(x, h_N x)$ sufficiently large, so that there is a finite subset $F \subset Y$ such that every set in Y which passes close to both x and $h_N x$ lies in F . As the sets Y_n are uniformly quasiconvex, every set Y_n with $n \geq N$ passes close to both x and $h_N x$, and so every set Y_n in the sequence constructed above lies in F . As F is finite, we may pass to a constant subsequence, and so there is a single set Y' in Y such that Y' passes close to x and $h_n x$ for infinitely many n . But then the limit point of γ_X in X is contained in $\partial Y' \subset \partial Y$, a contradiction. ■

Corollary 2.13. *Let $X, Y, Z := X_Y$, and let G be as in Theorem 2.12 (c). Suppose $g \in G$ is loxodromic and its quasigeodesic axis $\alpha \subset X$ has limit points in $\partial X \setminus \partial Y$. Then condition (2) of Theorem 2.4 is satisfied for g .*

Proof. Let the qi-embedding $\mathbb{Z} \ni i \mapsto g^i \cdot x \in X$ be H in the condition of statement (c) of Theorem 2.12. Then, we have already shown in the proof of Theorem 2.12 that the projection of every $Y_n \in Y$ to α is uniformly bounded. ■

2.4. The compression body graph

Finally, we observe that these results apply in the case of $\mathcal{C}_D(S)$, the electrification of the curve complex along the discs sets of compression bodies. These sets are uniformly quasiconvex [19], the electrification has infinite diameter, and there are pseudo-Anosov elements which act loxodromically on $\mathcal{C}_D(S)$ [15]. It remains to verify condition (2) from Theorem 2.4, which is a corollary of the following stability result.

Theorem 2.14 ([15, Theorem 4.2]). *Given a closed orientable surface S and a constant k , there is a constant $k' > k$ with the following property. Suppose that γ is a geodesic ray in $\mathcal{C}(S)$, and V_i is a sequence of compression bodies such that, for all i , the segment $\gamma|_{[0, i]}$ lies in a k -neighbourhood of $D(V_i)$. Then there is a non-trivial compression body W , contained in infinitely many of the V_i , such that γ is contained in a k' -neighbourhood of $D(W)$.*

We now verify condition (2) from Theorem 2.4 in this setting.

Corollary 2.15. *Let S be a closed orientable surface, and let g be a pseudo-Anosov map which acts loxodromically on $\mathcal{C}_D(S)$, and let α be an axis for g in $\mathcal{C}(S)$. Then there is a constant C such that for any disc set $D(V)$, the nearest point projection of $D(V)$ to α has diameter at most C .*

Proof. Suppose there is a sequence of compression bodies V_i such that $\text{diam}(\pi_\alpha(D(V_i)))$ tends to infinity. As the $D(V_i)$ are quasiconvex, and g acts coarsely transitively on α , there is a constant k such that we may translate each $D(V_i)$ by a power n_i of g so that $\alpha|_{[0, t_i]}$ is contained in a k -neighbourhood of $g^{n_i}D(V_i)$, and t_i tends to infinity as i tends to infinity. Theorem 2.14 then implies that there is a k' such that α is contained in a k' -neighbourhood of a single disc set $D(W)$. This implies that the image of α in $C_D(S)$ is finite, contradicting the fact that g acts loxodromically on $C_D(S)$. ■

3. The co-symmetric curve graph

We introduce the co-symmetric curve graph. Let $S = S_{g,n}$ be a surface of finite type, where g is the genus and n is the number of punctures of S . If $3g - 3 + n > 0$, then the mapping class group of S contains pseudo-Anosov elements. There is a finite list of exceptional cases in which the mapping class group contains a hyperelliptic involution which acts trivially on the curve graph and the Teichmüller space of the surface, and for which every pseudo-Anosov map is symmetric with respect to this involution. We shall call these special cases the *exceptional* surfaces, and they consist precisely of the surfaces $S_{0,4}$, $S_{1,1}$, $S_{1,2}$ and $S_{2,0}$, see, for example, [7, Section 12.1]. We may exclude precisely these surfaces from the collection of surfaces satisfying $3g - 3 + n > 0$, by replacing the condition $3g - 3 + n > 0$ with the condition $2g + n > 4$.

By the word *curve* on S , we mean an isotopy class of essential simple closed curves on S . The curve graph, denoted by $\mathcal{C}(S)$, of S is the graph each of whose vertices corresponds to a curve, and two vertices are connected by an edge of length 1 if corresponding curves can be realized disjointly. We also need to consider orbifolds covered by S , so from now on, spaces that S finitely covers may be an orbifold. For such an orbifold, we define the curve graph as the one for the surface we obtain by puncturing all the orbifold points. Given a finite covering of surfaces, there is a map between the corresponding curve graphs.

Definition 3.1. Let $p: S \rightarrow S'$ be a finite covering. We denote by $\Pi: \mathcal{C}(S') \rightarrow \mathcal{C}(S)$ the associated (one-to-finite) map on the curve graphs, which is defined so that $a \in \Pi(b)$ if and only if the image of a by p equals b , that is, $p(a) = b$. We call $\Pi(\mathcal{C}(S'))$ the subspace of symmetric curves with respect to $p: S \rightarrow S'$.

By the work of Rafi and Schleimer, the subspaces of symmetric curves are quasi-isometrically embedded.

Theorem 3.2 ([23]). *The map $\Pi: \mathcal{C}(S') \rightarrow \mathcal{C}(S)$ is a qi-embedding with quasi-isometry constants depending only on S and the degree of the finite covering $p: S \rightarrow S'$.*

Remark 3.3. In a Gromov hyperbolic space, every quasi-isometrically embedded subspace is quasiconvex, and the quasiconvexity constant depends only on the quasi-isometry constants and the hyperbolicity constant.

The main object we consider in this section is the following.

Definition 3.4. The *co-symmetric* curve graph $\mathcal{C}_q(S)$ is the graph obtained by coning off the family $\Pi := \{\Pi(\mathcal{C}(S'))\}$ of subspaces of symmetric curves, where we consider all finite (possibly orbifold) coverings of type $p: S \rightarrow S'$.

Proposition 3.5. *The co-symmetric curve graph $\mathcal{C}_q(S)$ is Gromov hyperbolic, and $\text{MCG}(S)$ acts isometrically on $\mathcal{C}_q(S)$.*

Proof. Given a finite covering $p: S \rightarrow S'$, the map $p \circ g^{-1}$ is also a finite covering for any $g \in \text{MCG}(S)$. Therefore, the family Π is setwise $\text{MCG}(S)$ -invariant. Furthermore, as there is a lower bound of Euler characteristics of 2-dimensional orbifolds, the degree of the coverings from S is bounded. Hence by Theorem 3.2 and Remark 3.3, the quasiconvexity constants of the elements in the family Π is bounded. Hence, we get the conclusion by Proposition 2.1. ■

Note that we have not yet eliminated the possibility that $\mathcal{C}_q(S)$ has bounded diameter. To discuss further properties of $\mathcal{C}_q(S)$, the following terminology is useful.

Definition 3.6. Let A be a mapping class, a foliation, or a lamination. Then A is said to be *symmetric* if it is a lift with respect to some finite covering $p: S \rightarrow S'$. If A is not symmetric, it is said to be *asymmetric*.

In Section 3.1, we discuss the $\text{MCG}(S)$ -action on $\mathcal{C}_q(S)$. By using Theorem 2.4, we will prove the following theorem which is a detailed version of Theorem 1.1 for $\mathcal{C}_q(S)$.

Theorem 3.7. *Let S be an orientable surface of genus g with n punctures. Suppose that $2g + n > 4$. For the action of $\text{MCG}(S)$ on $\mathcal{C}_q(S)$, we have*

- (a) $g \in \text{MCG}(S)$ is loxodromic if and only if g is asymmetric and pseudo-Anosov,
- (b) the action of $\text{MCG}(S)$ on $\mathcal{C}_q(S)$ is strongly WPD.

In particular, the action of $\text{MCG}(S)$ on $\mathcal{C}_q(S)$ is non-elementary.

We now give a characterization of $\partial\mathcal{C}_q(S)$. Recall that by the work of Klarreich [14], the boundary of $\mathcal{C}(S)$ is identified with the space of ending laminations, denoted by $\mathcal{EL}(S)$. We call an ending lamination *symmetric* if it is a lift of an ending lamination with respect to a finite covering $p: S \rightarrow S'$. An ending lamination is called *asymmetric* if it is not symmetric. Let $\mathcal{AEL}(S)$ denote the subspace of $\mathcal{EL}(S)$ consisting of all asymmetric ending laminations. Then by Propositions 2.5 and 2.7, we have the following.

Theorem 3.8. *The Gromov boundary $\partial\mathcal{C}_q(S)$ is contained in $\mathcal{AEL}(S)$.*

In work [8], Farb and Mosher defined the notion of convex cocompact subgroups of $\text{MCG}(S)$ as an analogue of convex cocompact Kleinian groups. Instead of giving the definition, we regard the following theorem due to Kent and Leininger as a definition of convex cocompactness.

Theorem 3.9 ([13, Section 7]). *A finitely generated subgroup $H < \text{MCG}(S)$ is convex cocompact if and only if H qi-embeds into $\mathcal{C}(S)$.*

We call a subgroup $H < \text{MCG}(S)$ *purely asymmetric* if every element in H of infinite order is asymmetric. This is equivalent to saying that every element in H of infinite order does not preserve any symmetric subspaces. Then by Theorem 2.12, we have the following assertion.

Theorem 3.10. *Let S be as in Theorem 3.7. Let H be a finitely generated subgroup of $\text{MCG}(S)$. Then*

- (i) *The subgroup H qi-embeds into $\mathcal{C}_q(S)$ if and only if H is convex cocompact with limit set of H in $\partial\mathcal{C}(S)$ asymmetric, i.e., contained in $\mathcal{AEL}(S)$.*
- (ii) *If H is convex cocompact, and the limit set of H in $\partial\mathcal{C}(S)$ is asymmetric, then H is purely asymmetric.*

Proof. The “only if” part of (i) follows from Theorem 2.12 (a).

Let us postpone the proof of the fact that the family Π of symmetric subspaces is uniformly finite until Corollary 3.14 below. If Π is uniformly finite, the “if” part of (i) follows from (c) of Theorem 2.12.

The part (ii) follows from the part (b) of Theorem 2.12. ■

For any group with non-elementary action by isometries on a Gromov hyperbolic space, Taylor and Tiozzo [25] proved that random subgroups given by independent random walks are qi-embedded free groups. Putting Theorems 3.7 and 3.10 and [25] together, we get the following.

Corollary 3.11. *There are infinitely many purely asymmetric subgroups in $\text{MCG}(S)$.*

3.1. The $\text{MCG}(S)$ action on $\mathcal{C}_q(S)$ is weakly properly discontinuous

In this subsection, we prove Theorem 3.7. First, we prepare the following.

Lemma 3.12. *Let $g \in \text{MCG}(S)$ be an asymmetric pseudo-Anosov element with quasi-geodesic axis α . We denote by π_α the nearest projection map to α on $\mathcal{C}(S)$. Then there exists $C > 0$ such that for any finite covering $p: S \rightarrow S'$ and the corresponding symmetric subspace $\Pi(\mathcal{C}(S'))$, we have $\text{diam}_{\mathcal{C}}(\pi_\alpha(\Pi(\mathcal{C}(S')))) < C$, where $\text{diam}_{\mathcal{C}}$ is the diameter in $\mathcal{C}(S)$.*

A key ingredient of a proof of Lemma 3.12 is the uniform finiteness of symmetric subspaces, which is proved in [17].

Lemma 3.13 ([17, Lemma 4.5]). *Let $p: S \rightarrow S'$ be a finite covering. Denote the corresponding symmetric subspace by $\Pi(\mathcal{C}(S'))$. Then for any $D_0 > 0$, there exist $D_1, D_2 > 0$ which depend only on S and D_0 such that for any $a, b \in \mathcal{C}(S)$ with $d_{\mathcal{C}}(a, b) > D_1$, the number of distinct elements in*

$$\{g\Pi(\mathcal{C}(S')) \mid d_{\mathcal{C}}(a, g\Pi(\mathcal{C}(S'))) < D_0 \text{ and } d_{\mathcal{C}}(b, g\Pi(\mathcal{C}(S'))) < D_0\}$$

is bounded from above by D_2 . Here we count the number of distinct images, i.e., if

$$g_1\Pi(\mathcal{C}(S')) = g_2\Pi(\mathcal{C}(S'))$$

as subsets, we just count one time.

As an immediate consequence, we have the following.

Corollary 3.14. *The family Π of symmetric subspaces is uniformly finite.*

Proof. Since in any finitely generated group, the number of subgroups of a given index is finite, the number of coverings from S up to conjugacy is finite. Hence Lemma 3.13 implies that Π is uniformly finite. ■

We need the following corollary to Lemma 3.13 to prove Lemma 3.12. This corollary may be of independent interest.

Corollary 3.15. *Let $g \in \text{MCG}(S)$ be pseudo-Anosov. Then g is symmetric if and only if one of its stable or unstable foliation is symmetric.*

Proof. Let $\lambda_+(g)$ and $\lambda_-(g)$ denote the stable foliation and the unstable foliation of g , respectively. If g is symmetric, then both $\lambda_+(g)$ and $\lambda_-(g)$ must be symmetric. To prove the converse, we first remark that in [18, Section 3] it is proved that if both $\lambda_+(g)$, $\lambda_-(g)$ are symmetric with respect to the same covering, then g must be symmetric. Hence it suffices to prove that if $\lambda_-(g)$ is symmetric with respect to a finite covering $p: S \rightarrow S'$, then $\lambda_+(g)$ must be symmetric with respect to p . Suppose $\lambda_-(g) \in \partial\Pi(\mathcal{C}(S'))$. Since $\lambda_-(g)$ is fixed by g and $g^n(a) \rightarrow \lambda_+(g)$ for any $a \in \mathcal{C}(S)$, translates $g^n(\Pi(\mathcal{C}(S')))$ for large

enough $n \gg 1$ must fellow travel each other. If $\{g^n(\Pi(\mathcal{C}(S')))\}_{n \in \mathbb{N}}$ are all distinct, then this contradicts Lemma 3.13. Hence for some $i \neq j$, we have

$$g^i(\Pi(\mathcal{C}(S'))) = g^j(\Pi(\mathcal{C}(S'))).$$

Then we see that $g^{i-j}(\Pi(\mathcal{C}(S'))) = \Pi(\mathcal{C}(S'))$, and hence $\lambda_+(g) \in \partial\Pi(\mathcal{C}(S'))$. This completes the proof. ■

Now we prove Lemma 3.12.

Proof of Lemma 3.12. By Corollary 3.14, Π is uniformly finite. Also, Corollary 3.15 implies that if g is asymmetric and pseudo-Anosov, then the limit points of its axis are asymmetric. Hence by Corollary 2.13, we have the conclusion. ■

Finally, we verify the existence of asymmetric pseudo-Anosov maps. We remark that this result does not hold for the exceptional surfaces, as every map is a lift with respect to the covering given by a hyperelliptic involution, and we now enumerate them for the convenience of the reader. Given an orbifold cover $p: S \rightarrow S'$, we shall write S' as $S'_{g'}(d_1, \dots, d_{n'})$, where g' is the genus of S' , n' is the number of orbifold points, and the $d_i \geq 2$ are the orders of the orbifold points, so each orbifold point has metric angle $2\pi/d_i$. We allow $d_i = \infty$, corresponding to a puncture. With this notation, the exceptional cases are

- $p: S_{0,4} \rightarrow S'_0(2, 2, \infty, \infty)$,
- $p: S_{1,1} \rightarrow S'_0(2, 2, 2, \infty)$,
- $p: S_{1,2} \rightarrow S'_0(2, 2, 2, 2, \infty)$,
- $p: S_{2,0} \rightarrow S'_0(2, 2, 2, 2, 2, 2)$.

Recall that the dimension of the Teichmüller space of an orbifold $S'_{g'}(d_1, \dots, d_{n'})$ is $6g' - 6 + 2n'$, and so the dimension only depends on the number of orbifold points, not on their orders. This is because any two orbifold points have small neighbourhoods which are conformally equivalent. Therefore, the set of conformal structures on $S'_{g'}(d_1, \dots, d_{n'})$ is equal to the set of conformal structures on $S_{g',n'}$, and due to work of Picard [22] in the closed case, and Heins [10] for the punctured case, given a choice of order d_i at each orbifold point, there is a unique hyperbolic metric with cone points of that order in a given conformal class.

The following result is well known, but we provide the details for convenience.

Proposition 3.16. *Let S be a surface of genus g with n punctures and with $2g + n > 4$, and let $p: S \rightarrow S'$ be an orbifold cover. Then $\dim(\mathcal{T}(S')) < \dim(\mathcal{T}(S))$.*

Proof. The hyperbolic metric on S' lifts to a hyperbolic metric on S , and this gives an isometric embedding $\mathcal{T}(S') \rightarrow \mathcal{T}(S)$; see, for example, [23, Section 7]. In particular, if the dimension of $\mathcal{T}(S')$ is equal to the dimension of $\mathcal{T}(S)$, then the image of $\mathcal{T}(S')$ is equal to $\mathcal{T}(S)$.

First consider the case in which the cover is regular, with finite deck transformation group $F < \text{MCG}(S)$. The image of $\mathcal{T}(S')$ in $\mathcal{T}(S)$ is the set of points fixed by all elements of F , and so the fixed point set of F is all of $\mathcal{T}(S)$. However, the mapping class group $\text{MCG}(S)$ acts faithfully on $\mathcal{T}(S)$, except for the finite list of exceptional cases $S_{0,4}$, $S_{1,1}$, $S_{1,2}$ and $S_{2,0}$ in which there are hyperelliptic involutions which act trivially on $\mathcal{T}(S)$; see, for example, [7, Section 12.1]. However, these are precisely the surfaces we have excluded with the condition $2g + n > 4$.

Now suppose that the cover $p: S \rightarrow S'$ is not regular. Let S' have genus g' , and n' orbifold points. The generalization of Royden's theorem [24] due to Earle and Kra [5] states that if $\mathcal{T}(S)$ is isometric to $\mathcal{T}(S')$, then S is homeomorphic to S' , unless $2g + n \leq 4$, but this is precisely the exceptional cases we have excluded. This implies that $g = g'$ and $n = n'$. We now show that, in fact, there may be no such orbifold cover with these properties.

Recall that the Euler characteristics of the two surfaces are given by

$$\chi(S) = 2 - 2g - n \quad \text{and} \quad \chi(S') = 2 - 2g' - \sum_{i=1}^{n'} \left(1 - \frac{1}{d_i}\right).$$

The order of an orbifold point satisfies $d_i \geq 2$, and so as $g = g'$ and $n = n'$:

$$\chi(S) - \chi(S') \geq -\frac{1}{2}n.$$

The Euler characteristic is multiplicative under covers, so $\chi(S') = \chi(S)/d$, where d is the degree of the covering map

$$\chi(S) - \frac{1}{d}\chi(S) \geq -\frac{1}{2}n.$$

This gives

$$\left(1 - \frac{1}{d}\right)(2 - 2g - n) \geq -\frac{1}{2}n,$$

which we may rewrite as

$$\left(1 - \frac{1}{d}\right)(2 - 2g) \geq \left(\frac{1}{2} - \frac{1}{d}\right)n.$$

As we are assuming that the cover is irregular, $d \geq 3$. The only possible non-negative integer solutions have $g = 0$ or $g = 1$. If $g = 1$, then $n = 0$, but the torus does not have a hyperbolic metric. If $g = 0$, then

$$n \leq \frac{4d - 4}{d - 2}.$$

For $d \geq 7$, this implies that $n < 5$, so the only possibility is the exceptional surface $S_{0,4}$. The remaining finite list of possibilities consists of

$$S_{0,8}, d = 3, \quad S_{0,7}, d = 3, \quad S_{0,6}, d = 3, 4, \quad S_{0,5}, d = 3, 4, 5, 6.$$

We now consider these in turn.

$d = 3$: First consider the case in which the cover is degree $d = 3$, and the surface S is $S_{0,n}$ with $5 \leq n \leq 8$. The Euler characteristic of S is $\chi(S) = 2 - n$, so $\chi(S') = (2 - n)/3$. Let S' have a orbifold points of order 3 and $n - a$ orbifold points of order ∞ . Then

$$\chi(S') = 2 - \frac{2}{3}a - (n - a) = \frac{2 - n}{3},$$

which implies that $a = 2n - 4$. However, $a \leq n$, which implies that $n \leq 4$, a contradiction.

$d = 4$: Now consider the case in which the cover is degree $d = 4$, and the surface S is $S_{0,n}$ with $5 \leq n \leq 6$. The Euler characteristic of S is $\chi(S) = 2 - n$, so $\chi(S') = (2 - n)/4$. Let S' have a orbifold points of order 2, b orbifold points of order 4 and $n - a - b$ orbifold points of order ∞ . Then

$$\chi(S') = 2 - \frac{1}{2}a - \frac{3}{4}b - (n - a - b) = \frac{2 - n}{4},$$

which we may rewrite as

$$2a + b = 3n - 6.$$

When $n = 5$, we obtain $2a + b = 9$, but $a + b \leq 5$, so the only possible solution with non-negative integers is $a = 4$ and $b = 1$, but this means that S' has no punctures, a contradiction.

When $n = 6$, we obtain $2a + b = 12$, but $a + b \leq 6$, so the only possible solution with non-negative integers is $a = 6$ and $b = 0$, but again, this means that S' has no punctures, a contradiction.

$S = S_{0,5}$: Finally, consider possible quotients of the surface $S = S_{0,5}$ with degree $d = 5$ or $d = 6$.

When $d = 5$, as $\chi(S) = -3$, $\chi(S') = -3/5$. Let S' have a orbifold points of order 5 and $5 - a$ orbifold points of order ∞ . Then

$$\chi(S') = 2 - \frac{4}{5}a - (5 - a) = -\frac{3}{5},$$

which implies that $a = 12$, a contradiction.

When $d = 6$, as $\chi(S) = -3$, $\chi(S') = -1/2$. Let S' have a orbifold points of order 2, b orbifold points of order 3, c orbifold points of order 6 and $5 - a - b - c$ orbifold points of order ∞ . Then

$$\chi(S') = 2 - \frac{1}{2}a - \frac{2}{3}b - \frac{5}{6}c - (5 - a - b - c) = -\frac{1}{2},$$

which we may rewrite as

$$3a + 2b + c = 15.$$

However, $a + b + c = 5$, so the only possible solution with non-negative integers is $a = 5$, $b = 0$ and $c = 0$. This implies that the number of punctures for S' is also zero, a contradiction. This completes the proof of Proposition 3.16. ■

Proposition 3.17. *Let S be a surface of genus g with n punctures. Suppose $2g + n > 4$. Then there exists an asymmetric pseudo-Anosov element in $\text{MCG}(S)$.*

Proof. An orbifold cover $p: S \rightarrow S'$ gives an isometric embedding of $\mathcal{T}(S')$ in $\mathcal{T}(S)$, and as we are not in one of the exceptional cases, by Proposition 3.16, the dimension of $\mathcal{T}(S')$ is strictly less than the dimension of $\mathcal{T}(S)$.

There is a finite index subgroup $H < \text{MCG}(S')$ consisting of mapping classes which lift to maps of S . By abuse of notation, we shall also refer to the corresponding subgroup in $\text{MCG}(S)$ as H , and this subgroup preserves the image of $\mathcal{T}(S')$ in $\mathcal{T}(S)$.

Let $B(x, r)$ be a ball in $\mathcal{T}(S)$, then there is an $\varepsilon > 0$ such that $B(x, r)$ lies in the ε -thick part of $\mathcal{T}(S)$. Pick a basepoint on $\mathcal{T}(S')$. The subgroup H acts coarsely transitively on the ε -thick part of $\mathcal{T}(S')$, and so if some translate $g\mathcal{T}(S')$, for g in $\text{MCG}(S)$, intersects $B(x, r)$, then there is a translate of the basepoint at a bounded distance from x . As $\text{MCG}(S)$ acts properly on $\mathcal{T}(S)$, there are only finitely many distinct translates of $\mathcal{T}(S')$ intersecting $B(x, r)$.

Since there are only finitely many quotients $p: S \rightarrow S'$ up to covering space isomorphism, there are only finitely many isometrically embedded symmetric subsets $\mathcal{T}(S')$ intersecting $B(x, r)$. As we are in the non-exceptional case, they all have strictly smaller dimension than $\mathcal{T}(S)$. In particular, there is a ball $B(x', r') \subseteq B(x, r)$ in $\mathcal{T}(S)$ disjoint from all of the symmetric subsets (i.e., the union of the $g\mathcal{T}(S')$ over all g and all covers).

Periodic geodesics are dense in $\mathcal{T}(S)$, see, for example, [6, Section 6]. Therefore, there is a pseudo-Anosov mapping class with geodesic axis intersecting $B(x', r')$, and this pseudo-Anosov mapping class is asymmetric, as required. ■

Proof of Theorem 3.7. By Lemma 3.12, we see that assumption (2) of Theorem 2.4 is satisfied for $X = \mathcal{C}(S)$, $Y = \{\Pi(\mathcal{C}(S'))\}$ and an asymmetric pseudo-Anosov g . In this case, X_Y in the statement of Theorem 2.4 is $\mathcal{C}_q(S)$. As every pseudo-Anosov element is WPD on $\mathcal{C}(S)$ [1, Proposition 11], every asymmetric pseudo-Anosov is also WPD on $\mathcal{C}_q(S)$, especially if it is loxodromic. On the other hand, note that if an element in $\text{MCG}(S)$ acts loxodromically on $\mathcal{C}_q(S)$, then it must act loxodromically on $\mathcal{C}(S)$. Since every symmetric pseudo-Anosov element must fix corresponding symmetric subspaces, it cannot be loxodromic. Thus (a) follows.

We now prove (b). If $2g + n > 4$, then $\text{MCG}(S)$ contains a free group of rank 2, and in particular, it is not virtually cyclic. Then by Proposition 3.17, at least one loxodromic element exists. We have already seen that every loxodromic element acts WPD, hence the action of $\text{MCG}(S)$ on $\mathcal{C}_q(S)$ is strongly WPD. That the action is non-elementary is the consequence of Proposition 2.3. ■

4. Non-maximal train tracks

We say a pair of curves a and b on a surface is *filling* if when put in minimal position, there is no essential closed curve disjoint from $a \cup b$. In particular, this implies that every

complementary region is a disc containing at most one puncture. We say a pair of curves is *maximally filling* if when put in minimal position, every complementary region is a square, hexagon or a bigon containing a single puncture.

We say a pair of curves a and b is *K -maximally filling*, if for any pair of curves a' and b' with $d_{\mathcal{E}(S)}(a, a') \leq K$ and $d_{\mathcal{E}(S)}(b, b') \leq K$, a' and b' are maximally filling.

In this section, we will consider surfaces $S_{g,n}$ which support a pair of curves which are maximally filling, and which also support a pair of curves which fill the surface, but are *not* maximally filling. This consists of all surfaces which support a pseudo-Anosov map, with the exceptions of $S_{0,4}$ and $S_{1,1}$. The four-punctured sphere $S_{0,4}$ has the property that all filling curves are maximally filling. The punctured torus $S_{1,1}$ has the property that there are no maximally filling pairs of curves, as any pair of curves in the minimal position may be realised as a pair of geodesics in a flat metric on the torus with a marked point, and the marked point is then contained in a square, not a bigon. We shall therefore consider surfaces $S_{g,n}$ with $3g + n > 4$, as this condition includes all surfaces which support a pseudo-Anosov map, excluding $S_{0,4}$ and $S_{1,1}$.

Remark 4.1. In the case that the surface S is either $S_{0,4}$ or $S_{1,1}$, the Teichmüller space $\mathcal{T}(S)$ has complex dimension 1, so there is a single stratum of quadratic differentials, and all the pseudo-Anosov maps have invariant foliations with the same collection of singularities. In the case of $S_{0,4}$, the invariant foliation is the quotient of an invariant foliation for an Anosov map on a torus by the hyperelliptic involution, with the order two cone points replaced by punctures. This foliation then has exactly four 1-prong singularities, and so Theorem 1.3 holds in this case. However, in the case of $S_{1,1}$, Theorem 1.3 does not hold, as the invariant foliations are preserved by the hyperelliptic involution, which fixes the puncture, and so the foliations cannot have 1-prong singularities at the puncture.

Lemma 4.2. *Let $S = S_{g,n}$ be a surface of finite type of genus g with n punctures with $3g + n > 4$. Let g be a pseudo-Anosov element on S whose invariant foliations have generic singularities, and let α be an axis for g . Then for any K , there is an R such that any two curves on α distance at least R apart are K -maximally filling.*

We shall prove this result in this section. We start by reviewing some useful properties of train tracks.

4.1. Train tracks

We briefly recall some of the results we use about train tracks on surfaces. For more details, see, for example, Penner and Harer [21].

Recall that a train track on a surface S is a smoothly embedded graph, such that the edges at each vertex are all mutually tangent, and there is at least one edge in each of the two possible directed tangent directions. Furthermore, there are no complementary regions which are nullgons, monogons, bigons or annuli. The vertices are commonly referred to as *switches* and the edges as *branches*. We will always assume that all switches have valence at least three. A trivalent switch is illustrated below in Figure 1.

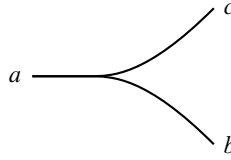


Figure 1. A trivalent switch for a train track.

An assignment of non-negative numbers to the branches of τ , known as *weights*, satisfies the *switch equality* if the sum of weights in each of the two possible directed tangent directions is equal: that is $a = b + c$ in Figure 1 above. A train track with integral weights defines a simple closed multicurve on the surface, and we say that this curve is *carried* by τ . We shall write $\mathcal{C}(\tau)$ for the subset of the curve complex consisting of the simple closed curves carried by τ .

More generally, for weights with non-integral values, a weighted train track defines a measured foliation on the surface. We say that the corresponding foliation is *carried* by the train track. We shall write $\mathcal{MF}(S)$ for the space of measured laminations on the surface S , and $P(\tau)$ for the set of measured foliations carried by τ . The set $P(\tau)$ is projectively invariant, and we shall write $\overline{P(\tau)}$ for its projectivization in $\mathcal{PMF}(S)$, the space of projective measured foliations on the surface. The set $\overline{P(\tau)}$ is a polytope in $\mathcal{PMF}(S)$. Let $V(\tau)$ be the set of vertices of $\overline{P(\tau)}$. Every $v \in V(\tau)$ gives a *vertex cycle*: a simple closed curve, carried by τ , that puts weight at most two on each branch of τ . By abuse of notation, we shall also write $V(\tau)$ for the corresponding set of simple closed curves in $\mathcal{C}(S)$.

We say a train track τ is *maximal* if every complementary region is a triangle or a monogon containing a single puncture. Every surface which contains a maximally filling pair of curves also contains a maximal train track, as given a choice of orientations on the curves, one may smooth the intersections compatibly, so that the hexagons become triangles, the squares become bigons and the punctured bigons become monogons. The bigons may then all be collapsed to produce a maximal train track; see, for example, [15, Proposition A.2] for full details.

4.2. Non-classical interval exchange transformations

Train tracks are closely related to non-classical interval exchanges, which we now describe; see, for example, [2].

Definition 4.3. A *non-classical interval exchange* consists of the following:

- (1) Given positive numbers δ and ε , let B_0 be a Euclidean rectangle $[0, \delta] \times [0, \varepsilon]$ which we shall call the *base rectangle*. We will refer to the sides of length ε as the *vertical sides* and the sides of length δ as the *horizontal sides*. We shall call $\{0\} \times [0, \varepsilon]$ the *initial vertical side* and $\{\delta\} \times [0, \varepsilon]$ the *terminal vertical side*. We shall call the horizontal side $[0, \delta] \times \{\varepsilon\}$ the *upper side* and label it I_+ , and the other side $[0, \delta] \times \{0\}$ the *lower side* and label it I_- .

(2) Let B_1, \dots, B_n be a finite collection of metric Euclidean rectangles, which we shall call *bands*. Each band B_i has one pair of opposite sides called the horizontal sides, and one pair called the vertical sides. The length of the horizontal sides of the band B_i is called the *width* w_i of the band B_i , and the total width of the bands $\sum w_i$ is equal to δ .

(3) For each horizontal side of a band, there is a Euclidean isometry from the horizontal side to the disjoint union $I_+ \sqcup I_-$, with the following properties:

- (a) The images of the interiors of any two horizontal sides are disjoint.
- (b) The quotient space obtained from gluing a band to the base rectangle along its two horizontal sides is orientable, i.e., homeomorphic to an annulus and not a Möbius band.

If a band has one horizontal side mapped to I_+ and the other mapped to I_- , then we call it an *orientation preserving band*. Otherwise, we call it an *orientation reversing band*.

Every generalized interval exchange transformation gives rise to a measured train track with a single vertex, by collapsing the base rectangle to a vertex, and each band to an edge of weight w_i . As the total width of the bands $\sum w_i$ is equal to the length δ of the base rectangle, the images of the horizontal sides cover $I_+ \sqcup I_-$, and the measured train track satisfies the switch condition at the vertex.

In the description above, we attached the bands to the horizontal sides of the rectangle, but we could instead have attached them all to the vertical sides, though in this case they would need to have total width ε .

4.3. Quadratic differentials and transversals

In this section, we describe how a transversal for a quadratic differential gives rise to a non-classical interval exchange. To simplify the discussion, we shall assume that the quadratic differential is determined by a pseudo-Anosov map g with generic singularities.

A pseudo-Anosov map g preserves a unique geodesic axis γ in Teichmüller space. Choose a unit speed parameterization γ_t such that $g(\gamma_0) = \gamma_t$ for some positive t . Let q be the quadratic differential at γ_0 determined by the geodesic γ . The quadratic differential q determines a flat structure on S which we shall denote by S_q . Let F_+ be the vertical measured foliation determined by S_q , whose projectivization \overline{F}_+ is the stable invariant projective measured foliation for g . Similarly, let F_- be the horizontal measured foliation for S_q , whose projectivization \overline{F}_- is the unstable invariant foliation for g .

A *transversal* t for a foliation F in S is an embedded arc in S which is disjoint from the singularities of F , and is transverse to the leaves of F . We say a transversal for F_+ is *horizontal* if it is a horizontal geodesic in the flat metric S_q corresponding to q . Similarly, we say a transversal for F_- is *vertical* if it is a vertical geodesic.

The transversal t determines a non-classical interval exchange. This may be thought of as arising by cutting the surface along the singular flow lines, and then completing

each maximal open interval of non-singular flow lines by adding a pair of non-singular edges at each end. We now give a detailed description of this construction for horizontal transversals for the vertical foliation F_+ .

Let t be a horizontal transversal for the vertical foliation F_+ . As t is a horizontal geodesic disjoint from the singular set of F , there is an $\varepsilon > 0$ such that the Euclidean rectangle $t \times [-\varepsilon, \varepsilon]$ is also disjoint from the singular set. We shall choose this to be the base rectangle B_0 in the non-classical interval exchange map. Recall that one horizontal side of the base rectangle is called I_+ and the other horizontal side is called I_- .

A *flow line* ℓ is the closure of a connected component of a leaf of the foliation F in $S \setminus B_0$. A flow line is *non-singular* if it is a properly embedded arc with distinct endpoints in the horizontal boundary of B_0 . A flow line is *singular* if it contains a singular point of the foliation F . A *tripod* is the topological space homeomorphic to the connected and simply connected graph consisting of three edges meeting at a common vertex. The foliation F_+ contains finitely many singular points, all of which are trivalent, so there are finitely many singular flow lines, which are tripods, properly embedded with distinct endpoints in the horizontal boundary of B_0 . All other flow lines are non-singular, and this determines a first return map on $I_+ \sqcup I_-$, defined on all but finitely many points, which is an isometry on connected intervals for which it is defined. Furthermore, as S is orientable, any subsurface consisting of B_0 and any collection of non-singular flow lines is orientable, so if we choose a band for each maximal subinterval of $I_+ \sqcup I_-$ on which the first return map is defined, this is a non-classical interval exchange transformation.

The same construction applies to a vertical transversal for the horizontal foliation F_- , but with the vertical and horizontal directions swapped.

4.4. Rauzy induction

We say a transversal $t = [t_0, t_1]$ for a foliation F is *admissible* if it determines a non-classical interval exchange such that the terminal point t_1 is the endpoint of a singular flow line. In this case, let $t'_1 \in [t_0, t_1)$ be the closest endpoint of a singular flow line to t_1 . The transversal $[t_0, t'_1]$ is a subset of $[t_0, t_1]$ and determines a new non-classical interval exchange, again with the property that the terminal endpoint meets a singular flow line. An example of this is illustrated in Figure 2.

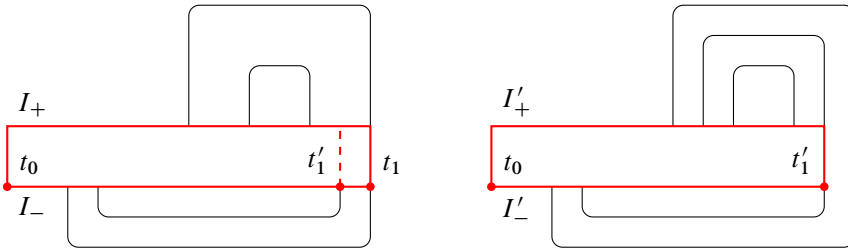


Figure 2. The Rauzy induction.

This process is called *Rauzy induction* and is a special case of splitting a train track. For general non-classical interval exchanges, the Rauzy induction need not always be defined, but the following result shows that it is always defined for the invariant foliations of a pseudo-Anosov map.

Proposition 4.4. *Let g be a pseudo-Anosov map on a surface S , let q be a choice of a quadratic differential corresponding to the invariant Teichmüller geodesic, and let S_q be the corresponding flat surface. Let $t = t^1$ be an admissible horizontal transversal. Then there is an infinite sequence of horizontal transversals $(t^n)_{n \in \mathbb{N}}$, in which t^{n+1} is obtained from t^n by the Rauzy induction. Furthermore, the length of the horizontal transversals t^n tends to zero as n tends to infinity.*

Replacing g by g^{-1} in Proposition 4.4 switches the horizontal and vertical directions, so the result also holds for a vertical transversal for the horizontal foliation F_- . Proposition 4.4 is an immediate consequence of the next two results, Lemma 4.5 and Proposition 4.6.

A *saddle connection* for a flat surface is a geodesic connecting two singular points, whose interior is disjoint from the singular set. We say a saddle connection is vertical if it is contained in the vertical foliation. The following fact is well known; see, for example, [7, Lemma 14.11].

Lemma 4.5 ([7, Lemma 14.11]). *The invariant foliations for a pseudo-Anosov map do not contain any saddle connections.*

In this case, the Rauzy induction is always defined and gives an infinite sequence of transversals t_n determining non-classical interval exchanges.

Proposition 4.6 ([2, Proposition 4.2]). *If the vertical foliation F_+ has no saddle connections, then the Rauzy induction starting at a horizontal transversal t^1 gives an infinite sequence of transversals $(t_n)_{n \in \mathbb{N}}$ determining non-classical interval exchanges, such that the horizontal length of the transversals t_n tends to zero as n tends to infinity.*

4.5. Maximally filling curves

Given a non-singular point x in the flat surface S_q , there are positive numbers δ and ε such that x is the bottom left corner of an embedded Euclidean rectangle $B_0 = s \times t$, disjoint from the singular set, where $|s| = \delta$ and $|t| = \varepsilon$. In particular, s is a horizontal transversal for the vertical foliation F_+ and t is a vertical transversal for the horizontal foliation F_- . We may therefore use B_0 as the base rectangle for a non-classical interval exchange in both the horizontal and vertical directions. Furthermore, possibly after passing to a subrectangle $[0, \delta'] \times [0, \varepsilon'] \subseteq [0, \delta] \times [0, \varepsilon]$, we may assume that both non-classical interval exchanges are admissible. By abuse of notation, we shall relabel the new rectangle as B_0 , and relabel the sides as s and t of lengths δ and ε .

For the vertical non-classical interval exchange, let d_+ be the union of the singular flow lines for the vertical foliation F_+ in $S \setminus B_0$, together with both of the flow lines incid-

ent to the initial vertical edge $\{0\} \times [0, \varepsilon]$, and both of the flow lines incident to the terminal vertical edge $\{\delta\} \times [0, \varepsilon]$. Similarly, for the horizontal non-classical interval exchange, let d_- be the union of the singular flow lines for the horizontal foliation F_- in $S \setminus B$, together with both of the flow lines incident to the initial horizontal edge $[0, \delta] \times \{0\}$, and both of the flow lines incident to the terminal horizontal edge $[0, \delta] \times \{\varepsilon\}$.

This determines a cell decomposition of the surface as follows:

- (1) The vertices V are the singular points of S_q , together with all intersection points $d_+ \cap d_-$, and all endpoints of d_+ and d_- with the base rectangle B_0 .
- (2) The edges E are the connected components of $(\partial B \cup d_+ \cup d_-) \setminus V$.
- (3) The 2-cells or faces F are the connected components of $S \setminus (\partial B \cup d_+ \cup d_-)$.

The collection of faces consists of the base rectangle B_0 , together with a collection of discs, each of which is a Euclidean rectangle in S_q with two vertical edges, each of which is contained either in d_+ or in the vertical boundary of B_0 , and two horizontal edges, each of which is contained either in d_- or in the horizontal boundary of B_0 .

The vertices are one of the following types:

- (1) Trivalent vertices in the interiors of the sides of B_0 .
- (2) 4-valent vertices corresponding to the corners of B_0 , and all non-singular intersection points of $d_+ \cap d_-$.
- (3) 2-valent and 6-valent vertices corresponding respectively to the 1-prong and trivalent singular points of S_q .

Proposition 4.7. *Let v be a simple closed curve carried by the vertical train track, which passes over every vertical band at least once. Let w be a horizontal curve carried by the horizontal train track, which passes over every horizontal band at least once. Then v and w are maximally filling.*

Proof. We may isotope v to consist of a union of vertical flow lines, together with geodesic arcs in B_0 , each of which has endpoints in the opposite horizontal sides of B_0 . Similarly, we may isotope w to consist of a union of horizontal flow lines, together with geodesic arcs in B_0 , each of which has endpoints in the opposite vertical sides of B_0 .

The intersections of v and w with each face of the cell structure for S_q are given in Figure 3.

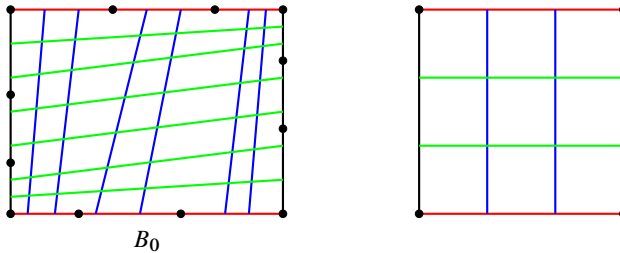


Figure 3. The curves v and w .

If a complementary region of $v \cup w$ is contained in the interior of a single face, then it is a square.

If it intersects an edge of the cell structure, but is disjoint from the vertices, then it is a square; see Figure 4.

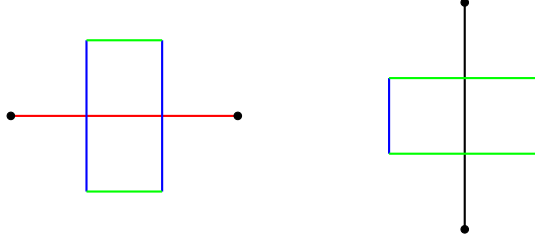


Figure 4. A complementary region intersecting an edge.

If it contains a vertex which is either trivalent or 4-valent, then the component is a square; see Figure 5.

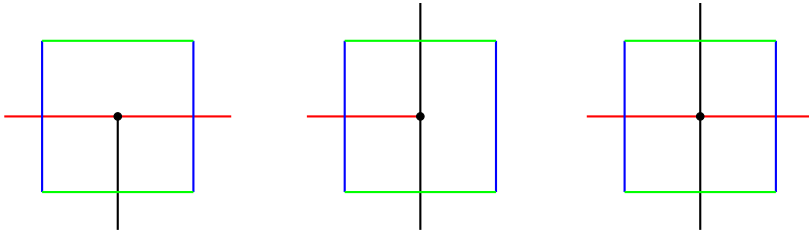


Figure 5. Complementary regions containing vertices.

Finally, if it contains a singular point of S_q , then this singular point is either a 1-prong singularity, or a trivalent singular point. In the first case, the component is a bigon containing a puncture, and in the second case, the component is a hexagon. This is illustrated in Figure 6.

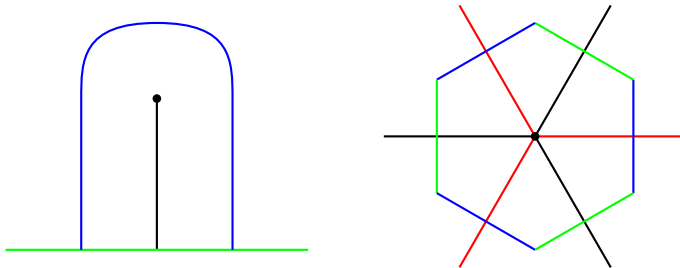


Figure 6. Complementary regions containing singularities of S_q .

This completes the proof. ■

4.6. Widely separated curves are maximally filling

It is well known that if the vertical foliation is uniquely ergodic, then the intersection of the projective measured foliations carried by the train tracks arising from the Rauzy induction is equal to the vertical foliation. We provide a proof for the convenience of the reader.

Proposition 4.8. *Let S_q be a flat surface with uniquely ergodic vertical foliation F_+ . Let t^n be a sequence of nested horizontal transversals in S_q such that the length of t^n tends to zero as n tends to infinity. Let τ_n be the train track determined by t^n . Then $\cap P(\tau_n) = \{\overline{F_+}\}$ in $\mathcal{PMF}(S)$.*

Proof. Let τ_n be the train track corresponding to t^n . Since the transversals are nested, $t^{n+1} \subset t^n$, each train track is carried by the previous one, $\tau_{n+1} \prec \tau_n$, and so $P(\tau_{n+1}) \subseteq P(\tau_n)$. Every train track τ_n in the splitting sequence carries the vertical foliation F_+ , so $F_+ \in \cap P(\tau_n)$. We now show $\cap P(\tau_n) = \{\overline{F_+}\}$.

Let $F_+(a)$ be the measure of a with respect to the vertical measured foliation F_+ , and let $F_-(a)$ be the measure of a with respect to the horizontal foliation F_- . Consider the ratio between these two measures, $\rho_q(a) = F_-(a)/F_+(a)$, which is projectively invariant, i.e., $\rho_q(\lambda a) = F_-(\lambda a)/F_+(\lambda a) = \lambda F_-(a)/\lambda F_+(a) = \rho_q(a)$, and so defines a function on $\mathcal{PMF}(S)$.

If a is carried by τ_n , then it has a piecewise geodesic representative in S_q consisting of flow lines in bands, which are vertical geodesic segments, and geodesic segments in the base rectangle B_0 , each of which has one endpoint in the lower horizontal side, and one endpoint in the upper horizontal side. Flow lines have zero vertical measure in F_+ , and positive horizontal measure in F_- . Each segment in B_0 contributes at least 2ε to the horizontal measure of a , and at most $F_+(t^n)$ to the vertical measure of a . Therefore, $\rho_q(a) \geq 2\varepsilon/F_+(t^n)$, where $F_+(t^n)$ is equal to the length of the horizontal geodesic t^n in the flat metric S_q .

Now consider elements $\overline{F} \in \mathcal{PMF}(S)$. Suppose that $(\overline{a_n})_{n \in \mathbb{N}}$ is a sequence of simple closed curves converging to \overline{F} . Then for any representative F in $\mathcal{MF}(S)$ for \overline{F} , there is a sequence of numbers λ_n such that $\lambda_n a_n \rightarrow F$. Then $F_+(\lambda_n a_n) \rightarrow F_+(F)$, where $F_+(F)$ is equal to the intersection number $i(F_+, F)$ of the two measured foliations. If $i(F_+, F) > 0$ and $F \in \cap P(\tau_n)$, then the curves a_n can be chosen so that $a_n \in P(\tau_n)$ for each n . As $F_+(t^n) \rightarrow 0$, this implies that $\rho_q(\lambda_n a_n) \rightarrow \infty$, contradicting the fact that $\rho_q(F)$ is finite. In particular, if $i(F_+, F) > 0$, then there is an n such that $F \notin P(\tau_n)$. Therefore, all foliations in $\cap P(\tau_n)$ have zero intersection number with F_+ , but as F_+ is uniquely ergodic, the set of all projective measured foliations with zero intersection number with F_+ consists just of $\{\overline{F_+}\}$. ■

We will also use the following results of Masur and Minsky, and Klarreich. The first says that the vertex sequences arising from splitting sequences of train tracks are unparameterized quasigeodesics in the curve complex.

Theorem 4.9 ([19, Theorem 1.3]). *Given a surface S , there are constants Q and c such that the vertices of a nested train-track splitting sequence form an unparameterized (Q, c) -quasigeodesic in $\mathcal{C}(S)$.*

The next says that a 1-neighbourhood of a simple closed curve which runs over every branch of a train track is in fact carried by the train track.

Lemma 4.10 ([19, Lemma 3.4]). *If τ is a maximal birecurrent train track and a is a simple closed curve which runs over every branch of τ , then $N_1(a) \subseteq P(\tau)$.*

Finally, the boundary of the curve complex is the space of minimal foliations.

Theorem 4.11 ([14, Theorem 1.3]). *The boundary at infinity of the curve complex $\mathcal{C}(S)$ is the space of minimal foliations on S .*

We may now prove Lemma 4.2.

Proof. Choose a non-singular point in S_q and construct a base rectangle $B_0 = t^1 \times s^1$ disjoint from the singular set, such that both vertical and horizontal non-classical interval exchanges are admissible. Let $(t^n)_{n \in \mathbb{N}}$ and $(s^n)_{n \in \mathbb{N}}$ be the sequences of subintervals obtained from applying the Rauzy induction to t^1 and s^1 , respectively, and let $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ be the corresponding sequence of train tracks.

Recall that the Rauzy induction moves may be realised in S_q by extending the flow lines giving the boundaries of the bands. As F_+ is minimal, every leaf is dense, and so for every τ_k there is a $k' > k$ such that every band of $\tau_{k'}$ runs at least once over every band of τ_k . In particular, there is a transversal t^{k_1} such that each band runs at least once over every band of the original non-classical interval exchange corresponding to t^1 . Similarly, after applying the Rauzy induction to s^1 , there is a transversal s^{k_2} and such that each band runs at least once over every band of the original non-classical interval exchange. To simplify notation, choose $k = \max\{k_1, k_2\}$ and consider the train track τ_k determined by t^k , and σ_k determined by s^k .

By Proposition 4.8 and Theorems 4.9 and 4.11, the vertex sequence $(V(\tau_n))_{n \in \mathbb{N}}$ is a (Q, c) -quasigeodesic, whose limit point in $\partial\mathcal{C}(S)$ is equal to $\overline{F_+}$. The positive limit point of the (Q', c') -quasigeodesic α_g is also equal to $\overline{F_+}$, so there are constants L and r_+ such that a terminal subray $\alpha_+ = \alpha_g([r_+, \infty))$ of α_g is contained in an L -neighbourhood of the $V(\tau_n)$.

Applying the Rauzy induction again to t^k , we get a transversal $t^{k'}$ with $k' > k$ such that every band of $\tau_{k'}$ runs at least once over every band of τ_k . The nesting lemma, Lemma 4.10, then implies that $N_1(\mathcal{C}(\tau_{k'})) \subseteq \mathcal{C}(\tau_k)$. By repeating this process, for any K there is a k' such that $N_K(\mathcal{C}(\tau_{k'})) \subseteq \mathcal{C}(\tau_k)$. In particular, for any $K + L$ there is a k' such that $N_{K+L}(\mathcal{C}(\tau_{k'})) \subset \mathcal{C}(\tau_k)$ in $\mathcal{C}(S)$. Let

$$\alpha_+ = \alpha_g([r_+, \infty))$$

be the terminal ray of α such that $\alpha_+ \subseteq N_L((V(\tau_n))_{n \geq k'})$. If $n \geq k'$, then $V(\tau_n) \subseteq \mathcal{C}(\tau_{k'})$, so $N_K(\alpha_+) \subseteq N_{K+L}(\mathcal{C}(\tau_{k'})) \subseteq \mathcal{C}(\tau_k)$.

We may apply exactly the same argument to the horizontal foliation, so there is an infinite initial ray $\alpha_- = \alpha_g((-\infty, r_-])$ of the (Q', c') -quasi-axis α_g such that

$$N_K(\alpha_-) \subseteq \mathcal{C}(\sigma_k).$$

As g acts by translations on α_g , given any two points v and w on α_g distance at least $R = d_{\mathcal{C}}(\alpha_-, \alpha_+) + d_{\mathcal{C}}(a, ga) + O(\delta)$ apart, we may translate them by powers of g such that, up to relabeling, v lies in α_- and w lies in α_+ . Let v' and w' be curves distance at most K from v and w respectively. Then $v' \in \mathcal{C}(\sigma_k)$ and $w' \in \mathcal{C}(\tau_k)$, so v' and w' are maximally filling, by Proposition 4.7. Therefore, v and w are K -maximally filling, as required. ■

Lemma 4.12. *Let g be a pseudo-Anosov element with maximal invariant laminations and axis α . Then there is a D such that for any non-maximal train track τ , the nearest point projection of $\mathcal{C}(\tau)$ to α has diameter at most D .*

Proof. The set $\mathcal{C}(\tau)$ is uniformly quasiconvex, so if there is a projection of size R , then there are curves a_1, a_2 in $\mathcal{C}(\tau)$ with $d_{\mathcal{C}(S)}(\alpha, a_i) \leq K$ and $d_{\mathcal{C}(S)}(a_1, a_2) \geq R - 2K - O(\delta)$, where K is a uniform constant independent of $\mathcal{C}(\tau)$.

As a_1 and a_2 are both carried by τ , they have a complementary region which is not either a triangle or a monogon with exactly one puncture, so this contradicts the lemma above. ■

Now by Lemma 4.12 combined with Theorem 2.4, we have the following theorem which corresponds to the statement for $\mathcal{C}_\tau(S)$ in Theorem 1.1.

Theorem 4.13. *Let S be an orientable surface of genus g with n punctures. Suppose $2g + n > 3$. For the action of $\text{MCG}(S)$ on $\mathcal{C}_\tau(S)$, we have*

- (a) *$g \in \text{MCG}(S)$ is loxodromic if and only if g has maximal invariant laminations, and*
- (b) *the action of $\text{MCG}(S)$ on $\mathcal{C}_\tau(S)$ is strongly WPD.*

In particular, the action of $\text{MCG}(S)$ on $\mathcal{C}_\tau(S)$ is non-elementary.

Lemma 4.12 holds for all surfaces with $3g + n > 4$. However, part (a) of this result also holds for $S_{0,4}$ by Remark 4.1. In fact, (b) also holds, as the non-maximal train tracks in $S_{0,4}$ have finite diameter in $\mathcal{C}(S_{0,4})$, and so the result holds for all surfaces with $2g + n > 3$. Among surfaces supporting pseudo-Anosov elements, this excludes only $S_{1,1}$, for which the result does not hold.

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